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Exact Solution of The Non - Central Modified Kratzer Potential Plus a Ring - Shaped Like Potential By The Factorization Method

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Abstract: In this paper, we study the Schrödinger equation with a non - central modified Kratzer potential plus a ring - shaped like potential, which is not spherically symmetric. Thus, the standard methods for separation of variables do not quite apply. However we are able to separate variables using a simple extension of the standard method, which leads to solutions in the associated Laguerre function for the radial part and Jacobi polynomials for the polar angle part. We also introduce an interesting pair of first order ladder operators, which allow us to generate the energy eigenvalues for all states of the system. The obtained results show that the lack of spherical symmetry removes the degeneracy of second quantum number m which completely expected.

Keywords: Modified Kratzer Potential; Schrödinger Equation; Factorization Method; Raising and Lowering Operators
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1. Introduction

One of the important work in theoretical physics is to obtain exact solution of the Schrödinger equation for special potentials. It is well known that exact solution of Schrödinger equation

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are only possible for certain cases. The exact solution of Schrödinger equation for a class of non - central potential already studied in quantum chemistry [1, 2, 3]. In most of these studies, the eigenvalues and eigenfunctions are obtained by separation of variables. The path integral for particles moving in non - central potentials is evaluated to find the energy spectrum of the system [4]. In recent years, many work yields to obtain the exact solution of the ring - shaped potential [5]. This potential usually add to certain potential, for example Kratzer potential. The Kratzer potential [6, 7] have played an important role in the history of the molecular structure and interactions [8]. This potential offered one of the most important exactly models of atomic and molecular physics and quantum chemistry. It may be apply to energy spectrum for the CO diatomic molecule with different quantum numbers. Also the analytical solution of the radial Schrödinger equation is of high important in non - relativistic quantum mechanics since the wave function contains all the necessary information to describe a quantum system fully. As we know the quasi - exactly solution for the radial Schrödinger equation within a given potential is given by Ref [9]. Recently a new potential which is called the modified Kratzer's type of molecular potential proposed. Very recently Chen and Dong [10] found a new ring - shaped potential and now Chen and Dai [11] try to solve the Schrödinger equation for the modified Kratzer potential plus this new ring - shaped potential. In this paper we consider non - central modified Kratzer potential plus a ring - shaped like potential which related to the angle dependent part of Schrödinger equation. In the spherical coordinates, this potential is defined as

$$V(r,\theta) = D(\frac{r-a}{r})^2 + \frac{\beta'}{r^2\sin^2\theta} + \frac{\gamma\cos\theta}{r^2\sin^2\theta}, \tag{1}$$

where β' and γ are strictly positive constants, D is the dissociation energy and a is the equilibrium internuclear separation. It is clear that the limiting case of $\beta' = 0$ and $\gamma = 0$ reduce to the modified Kratzer potential [12]. In other word the first term of this potential is the modified Kratzer potential and other terms are the angle dependent parts. There are different methods used to obtain the exact solutions of the Schrödinger equation for the ring-shaped like potentials. The most important of this methods ,for example, are the standard methods [2], the path integral approach [4], the supersymmetric quantum mechanics and shape invariance method [13] and the Nikiforov - Uvarov (NU) method [15]. The NU method is based on solving the second - order linear differential equation by reducing to a generalized equation of hypergeometric type. But we use another way. In this paper the factorization method is used to solve the Schrödinger equation for potential introduced in equation (1). This work is organized as follows: in section 2 we introduce the factorization method. Radial part of Schrödinger equation considered in section 3. in section 4 we study the polar angle part of Schrödinger equation. Finally in section 5 we have conclusion for obtained result.

2. Factorization Method

In this section we show our way for solve Schrödinger equation. In the spherical coordinates, the Schrödinger equation is,

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\psi+V(r,\theta)\psi=E\psi, \tag{2}$$

where $V(r,\theta)$ is non - central modified Kratzer potential plus a ring - shaped like potential as equation (1). In order to find exact solution of equation (2) we give spherical total wave function as

$$\psi(r,\theta,\varphi)=R(r)\Theta(\theta)\Phi(\varphi). \tag{3}$$

Then put wave function (3) into Schrödinger equation (2). The wave equation for the non - central modified Kratzer potential plus a ring - shaped like potential separated into independent variable and the following equations are obtained,

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\frac{2\mu}{\hbar^2}\left[E-\frac{D(r-a)^2}{r}\right]R-\frac{\lambda}{r^2}R=0, \tag{4}$$

$$\frac{d^2\Theta(\theta)}{d\theta^2}+\cot\theta\frac{d\Theta(\theta)}{d\theta}+\left[\lambda-\frac{m^2}{\sin^2\theta}-\frac{2m}{\hbar^2}\left(\frac{(\beta'+\gamma\cos\theta)}{\sin^2\theta}\right)\right]\Theta(\theta)=0, \tag{5}$$

and

$$\frac{d^2\Phi(\varphi)}{d\varphi^2}+m'^2\Phi(\varphi)=0. \tag{6}$$

Now we can solve above equations separately. equations (4) and (5) are more complicated but solution of equation (6) is well known. therefor we focus on the equations (4) and (5). In the next sections we discuss about the radial part and polar angle part of Schrödinger equation corresponding to the modified Kratezr potential plus a ring - shaped like potential. In that case we choose suitable variable for obtain exact solutions. It lead to the associated Laguerre and Jacobi equations. For obtain exact solutions of the Schrödinger equation the factorization information from associated function were employed to solve the corresponding equations. It lead us to have some rising and lowering operators which are first order equations. These operators help us to obtain all quantum states and energy spectrum for different quantum number n and m.

3. The Solutions of the Radial Part of the Schrödinger Equation

In equation (4) we obtained the radial part of Schrödinger equation for non - central modified Kratzer potential plus a ring - shaped like potential which can be rewritten as,

$$r\frac{d^2R}{dr^2}+2\frac{dR}{dr}+\frac{2\mu}{\hbar^2 r}\left[(E-D)r^2+2Dar-(Da^2+\frac{\lambda\hbar^2}{2\mu})\right]R=0. \tag{7}$$

This equation can be solve by many method but in our case we choose suitable variable as follows

$$R(r) = U(r)L(r). \tag{8}$$

By using variable (8) in equation (7) we have

$$rL''(r)+\left[2r\frac{U'}{U}+2\right]L'(r)+\left[r\frac{U''}{U}+\frac{2U'}{U}+\frac{2\mu}{\hbar^2}(E-D)r+\frac{4D\mu a}{\hbar^2}-\frac{2\mu}{\hbar^2}(Da^2+\frac{\lambda\hbar^2}{\mu})\frac{1}{r}\right]R=0. \tag{9}$$

In order to obtain the parameters (D and λ), eigenfunction and eigenvalue for the non - central modified Kratzer potential plus a ring - shaped like potential we compare equation (9) with the following associated Laguerre differential equation [17, 18, 19],

$$rL''^{(\alpha,\beta)}_{n,m}+(1+\alpha-\beta r)L'^{(\alpha,\beta)(r)}_{n,m}+\left[(n-\frac{m}{2})\beta-\frac{m}{2}(\alpha+\frac{m}{2})\frac{1}{r}\right]L^{(\alpha,\beta)}_{n,m}(r)=0, \tag{10}$$

so we obtain the $U(r)$, D and λ,

$$U(r)=e^{\frac{-\beta r}{2}}r^{\frac{\alpha-1}{2}}, \tag{11}$$

$$D=\frac{\hbar^2}{4\mu a}[(n-\frac{m}{2})\beta+\frac{(\alpha+\beta)}{2}] \tag{12}$$

and

$$\lambda=\frac{1}{2}[\frac{m}{2}(\alpha+\frac{m}{2})+\frac{\alpha-1}{2}+\frac{(\alpha-1)^2}{4}-\frac{1}{2a}(n-\frac{m}{2})\beta-\frac{1}{4a}(\alpha+\beta)]. \tag{13}$$

From equations (8) and (11) one can obtain the corresponding eigenfunction,

$$R(r)=e^{\frac{-\beta r}{2}}r^{\frac{\alpha-1}{2}}L^{\alpha,\beta}_{n,m}(r). \tag{14}$$

In that case the exact energy eigenvalues of the radial part of the Schrödinger equation with the non-central modified kratzer potential plus a ring - shaped like potential are derived as,

$$E_{n,m}=\frac{\hbar^2}{4\mu a}\left[(n-\frac{m}{2})\beta+\frac{\alpha+\beta}{2}-\frac{\beta^2}{2}\right]. \tag{15}$$

In here we note that the solution associated Laguerre in the Rodrigues representation are,

$$L^{\alpha,\beta}_{n,m}(x)=\frac{a_{n,m}(\alpha,\beta)}{r^{\alpha+\frac{m}{2}}e^{-\beta r}}(\frac{d}{dx})^{n-m}(r^{n+\alpha}e^{-\beta r}), \tag{16}$$

where $a_{n,m}(\alpha,\beta)$ is the normalization coefficient, and also obtained by,

$$a_{n,m}(\alpha,\beta)=(-1)^m\sqrt{\frac{\beta^{\alpha+m+1}}{\Gamma(n-m+1)\Gamma(n+\alpha+1)}}. \tag{17}$$

In here we also discuss the raising and lowering operators which is corresponding to the redial part of modified Kratzer potential plus a ring - shaped like potential. So, we can factorize the associated Laguerre differential equation with respect to the parameters n and m as follows,

$$\begin{aligned} A^+_{n,m}(r)A^-_{n,m}(r)L^{(\alpha,\beta)}_{n,m}(r) &= (n-m)(n+\alpha)L^{(\alpha,\beta)}_{n,m}(r) \\ A^-_{n,m}(r)A^+_{n,m}(r)L^{(\alpha,\beta)}_{n-1,m}(r) &= (n-m)(n+\alpha)L^{(\alpha,\beta)}_{n-1,m}(r), \end{aligned} \tag{18}$$

where the differential operators as functions of parameters n and m are respectively :

$$\begin{aligned} A^+_{n,m}(r) &= r\frac{d}{dr} - \beta r + \frac{1}{2}(2n+2\alpha-m) \\ A^-_{n,m}(r) &= -r\frac{d}{dr} + \frac{1}{2}(2n-m). \end{aligned} \tag{19}$$

Note that the shape invariance equation (18) can also be written as the raising and lowering relation,

$$\begin{aligned} A^+_{n,m}(r)L^{(\alpha,\beta)}_{n-1,m}(r) &= \sqrt{(n-m)(n+\alpha)}L^{(\alpha,\beta)}_{n,m}(r) \\ A^-_{n,m}(r)L^{(\alpha,\beta)}_{n,m}(r) &= \sqrt{(n-m)(n+\alpha)}L^{(\alpha,\beta)}_{n-1,m}(r). \end{aligned} \tag{20}$$

So, we obtain the raising and lowering operators for the redial part of the modified Kratzer potential plus a ring - shaped like potential. These operators help us to have a bound states for that system. In order to show the effect of the modified Kratzer potential plus ring - shaped like potential on the energy spectrum, the obtained energy spectrum of radial part of Schrödinger equation is illustrated in Fig. 1, as function of quantum number n.

4. The Solutions of the Polar Angle Part of Schrödinger Equation

Now we try to obtain the eigenvalues and eigenfunctions of the polar angle of Schrödinger equation similar to the solution of the radial part. By introducing a new variable $x = \cos\theta$, we can write equation (5) as follows,

$$\frac{d^2\Theta(x)}{dx^2} - \frac{2x}{1-x^2}\frac{d\Theta(x)}{dx} + \left(\frac{\lambda(1-x^2) - m'^2 - \frac{2m'}{\hbar^2}(\beta' + \gamma x)}{(1-x^2)^2}\right)\Theta(x) = 0. \tag{21}$$

As equation (8) in previous section we choose suitable variable as,

$$\Theta(x) = U(x)P(x). \tag{22}$$

Now we put variable (22) to equation (21) and obtain,

$$\begin{aligned} &(1-x^2)P''(x) + \left[\frac{2U'(1-x^2)}{U} - 2x\right]P'(x) \\ &+ \left[(1-x^2)\frac{U''}{U} - 2x\frac{U'}{U} + \lambda - \frac{m'^2 - \frac{2m'}{\hbar^2}(\beta' + \gamma x)}{1-x^2}\right]P(x) = 0. \end{aligned} \tag{23}$$

Here, in order to obtain the exact solution of the Schrödinger equation for the non - central modified Kratzer potential plus ring - shaped like potential we compare equation (23) with the following associated Jacobi differential equation [17],

$$\begin{aligned}&(1-x^2)P''_{n,m}(x)-[\alpha-\beta+(\alpha+\beta+2)x]\,P'_{n,m}\\&+\left[n(\alpha+\beta+n+1)-\frac{m(\alpha+\beta+m+(\alpha-\beta))x}{1-x^2}\right]P_{n,m}(x)=0,\end{aligned} \tag{24}$$

where

$$m=\frac{2m'}{\hbar^2}-(\frac{\alpha+\beta}{2}). \tag{25}$$

So, the energy spectrum is,

$$E_{n,m'}=\frac{\hbar^2}{4\mu a}\left[(n-\frac{m'}{\hbar^2}+\frac{(\alpha+\beta)}{4})\beta+\frac{\alpha+\beta}{2}-\frac{\beta^2}{2}\right], \tag{26}$$

where

$$\begin{aligned}\alpha+\beta&=\beta'\\\alpha-\beta&=\gamma,\end{aligned} \tag{27}$$

The corresponding wave function are found as,

$$\Theta_{n,m}(x)=(\frac{\beta-\alpha}{2}\int(\frac{dx}{1-x^2})-\frac{(\alpha+\beta)}{2}\int\frac{x}{(1-x^2)}dx)P^{(\alpha,\beta)}_{n,m}(x). \tag{28}$$

The associated Jacobi functions $P^{(\alpha,\beta)}_{n,m}(x)$ as the solution of the differential equation have the following Rodrigues representation,

$$P^{(\alpha,\beta)}_{n,m}(x)=\frac{a_{n,m}(\alpha,\beta)}{(1-x)^{\alpha+\frac{m}{2}}(1+x)^{\beta+\frac{m}{2}}}(\frac{d}{dx})^{n-m}\left((1-x)^{\alpha+n}(1+x)^{\beta+n}\right). \tag{29}$$

Now we are going to discuss the raising and lowering operators corresponding to the polar angle part of Schrödinger equation for non - central modified Kratzer potential plus ring - shaped like potential. As mentioned in Refs [14, 17, 18], we can write the associated Jacobi differential equation (23) as the following,

$$\begin{aligned}A^+_{n,m}(x)A^-_{n,m}(x)P^{(\beta',\gamma)}_{n,m}(x)&=B_{n,m}P^{(\beta',\gamma)}_{n,m}(x)\\A^-_{n,m}(x)A^+_{n,m}(x)P^{(\beta',\gamma)}_{n-1,m}(x)&=B_{n,m}P^{(\beta',\gamma)}_{n-1,m}(x)\end{aligned} \tag{30}$$

where

$$\begin{aligned}B_{n,m}&=\frac{(n-m)(\beta'+\gamma+2n)(\beta'+m+n)}{(\beta'+2n)^2}\\A^+_{n,m}(x)&=(1-x^2)\frac{d}{dx}-(\beta'+n)x-\gamma\frac{(\beta'+n+m)}{\beta'+2n}\\A^-_{n,m}(x)&=-(1-x^2)\frac{d}{dx}-nx+\gamma\frac{(n-m)}{(\beta'+2n)}\end{aligned} \tag{31}$$

In the case of shape invariance with respect to m we have ,

$$\begin{aligned} A_m^+(x)A_m^-(x)P_{n,m}^{(\beta',\gamma)}(x) &= C_{n,m}P_{n,m}^{(\beta',\gamma)}(x) \\ A_m^-(x)A_m^+(x)P_{n,m-1}^{(\beta',\gamma)}(x) &= C_{n,m}P_{n,m-1}^{(\beta',\gamma)}(x), \end{aligned} \tag{32}$$

where,

$$C(n,m) = (n-m+1)(\beta'+n+m) \tag{33}$$

and lader operators are,

$$\begin{aligned} A_m^+ &= \sqrt{1-x^2}\frac{d}{dx} + \frac{(m-1)}{\sqrt{1-x^2}}x \\ A_m^- &= -\sqrt{1-x^2}\frac{d}{dx} + \frac{\gamma+(\beta'+m)}{\sqrt{1-x^2}}x. \end{aligned} \tag{34}$$

These operators help us to discuss about the bound states for the corresponding potential. In both part we can see that the energy spectrum obtained in equations (15) and (26) are independent of parameters β' and γ. Also we show the effect of the modified Kratzer potential plus a ring - shaped like potential on the energy spectrum obtained from the angular part of Schrödiger equation in Fig. 2, as function of quantum number m.

Conclusion

In this paper we find exact solution of the Schrödinger equation for non - central modified Kratzer potential plus a ring - shaped like potential by the factorization method. We have shown that the energy spectrum and the corresponding eigenfunctions of the Schrödinger equation with non - central potentials can be easily obtained by using the factorization method. We saw that the energy spectrum of radial part and polar angle part of Schrödinger equation are independent of parameters β' and γ. As we see in figures (1) and (2) the degeneracy of second quantum number is completely removed and also the variation of energy spectrum for the special value of n the value of $m \leq n-1$ decreases but this process for the figure (2) completely increases. These results may be in future apply to the CO like diatomic molecule for different quantum numbers. We know that it is possible to study the modified Kratzer potential plus another ring - shaped like potential [19] and to solve exactly the Schrödinger equation for this system.

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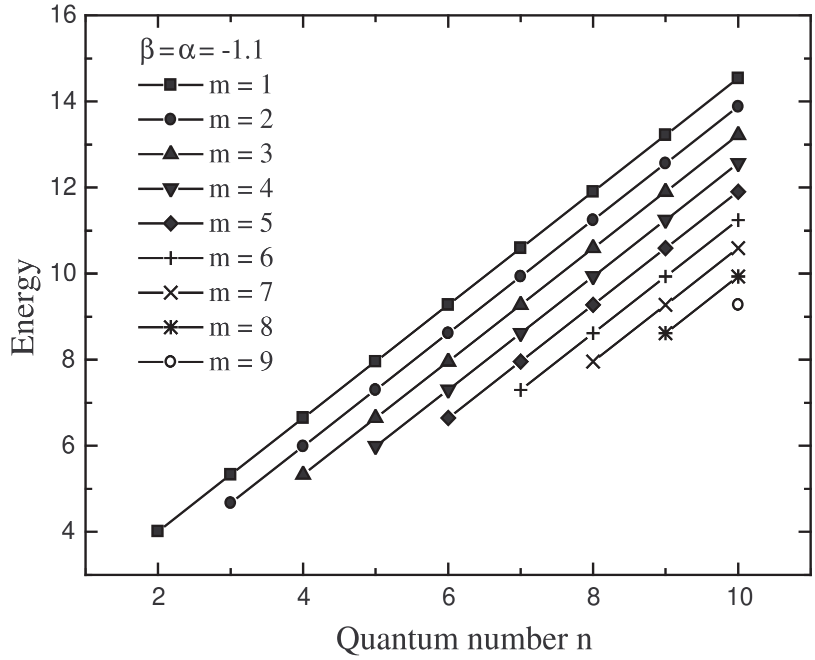

Fig. 1 The energy spectrum of solving the radial part of Schrödiger equation. β and α are arbetrery numbers.

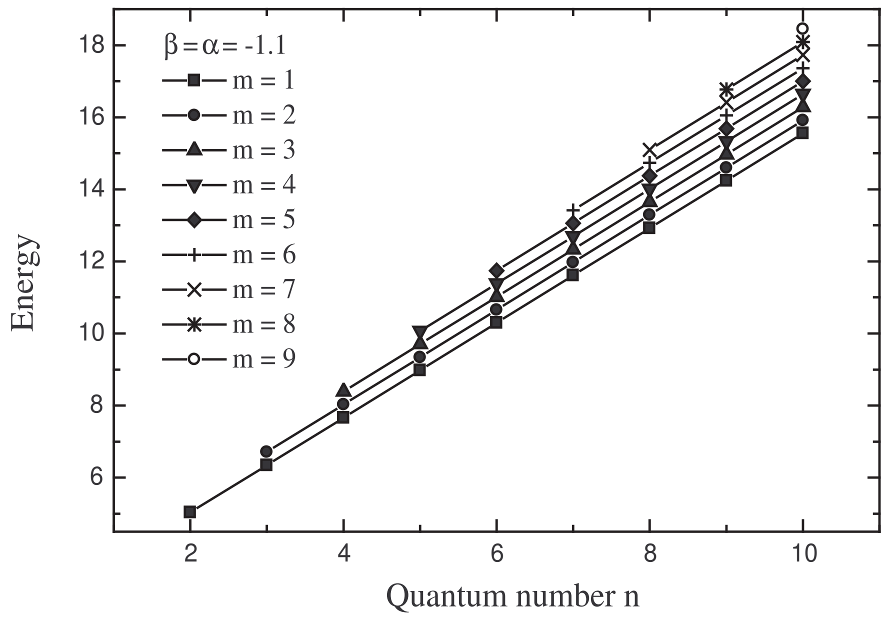

Fig. 2 The energy spectrum of solving the angular part of Schrödiger equation. β and α are arbetrery numbers.