

Radiation Reaction at Extreme Intensity

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Abstract: The radiation reaction force is examined for an idealized short pulse of electromagnetic radiation and for a plane wave. Exact solutions (without radiation reaction) are discussed, the total radiated power is calculated. A new and simpler approach to the approximate form of the equation of motion is presented that automatically removes the runaway solutions. Finally, analytical solutions are presented for the equations of motion that include the radiation reaction forces in the very high intensity regime. A classical scattering angle is defined and it shows that the electron is scattered in a small cone in the forward direction. The radiation reaction corrections to this angle are also considered.

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1. Introduction

The equation of motion of a charged particle in an electromagnetic field has demanded our attention for over a century, and for a large part of that time the self force has been of particular theoretical interest. With current laser intensities reaching 10^{22} Wm⁻², and expectations of increasing this by two orders of magnitude in the near future,[1] there has been a renewed interest, and it is not simply theoretical anymore. The point of this article is to provide a new derivation of the approximate equation of motion with the self force, and to use these equations to describe both radiation pressure and a new classical scattering angle.

The literature is rife with references on this topic, and instead of attempting to provide an extensive list, I will refer the reader to a paper that introduces the problem and contains some important references.[2] I will however, review some basic notation, in cgs

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units. The electromagnetic field tensor is defined by[3]

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \equiv A_{\nu,\mu} - A_{\mu,\nu} \quad (1)$$

where $A_\mu = (\phi, \mathbf{A})$ is the four potential. The four velocity is defined as

$$v^\mu = \frac{dx^\mu}{d\tau} = v^0 u^\mu = \gamma u^\mu \quad (2)$$

where τ is the proper time, $u^\mu = dx^\mu/dt$, and $\gamma = 1/\sqrt{1 - u^2/c^2}$. The four acceleration is defined by $a^\mu = dv^\mu/d\tau$ and the Lorentz force is $eF^{\mu\nu}v_\nu/c$. The electromagnetic field tensor is

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \quad (3)$$

and the equation of motion is

$$\frac{dv^\mu}{d\tau} = \frac{e}{mc} F^{\mu\nu} v_\nu. \quad (4)$$

2. Relativistic Motion

This section reviews mostly known results that will be useful for the next section. Let us consider the equation of motion of an electron under the influence of a pulse of laser radiation. The exact solution (to Maxwell's equations) of such a pulse can very complicated, which prevents us from finding manageable or closed form solutions.[4] To avoid this, we will use a plane wave pulse polarized in the x direction, which is given by

$$\mathbf{E} = Eh(kz - \omega t)\hat{\mathbf{x}} \quad (5)$$

where E is constant. Since it is a function of $z - t/c$, it satisfies Maxwell equations. The magnetic field is

$$\mathbf{B} = Eh(kz - \omega t)\hat{\mathbf{y}}. \quad (6)$$

It is helpful to write the equations in non-dimensional form. We let $x^\mu \rightarrow x^\mu/L$ and $t \rightarrow tc/L$. For example, $\cos(kz - \omega t) \rightarrow \cos(z - t)$ if $L = \lambda/2\pi$. Also, $F^{\mu\nu} \rightarrow F^{\mu\nu}/E$. Now we can assume

$$h = \frac{1}{w} e^{-((z-t)/w)^2} \cos(\Omega(z - t)) \quad (7)$$

where w is a dimensionless parameter that controls the width of the Gaussian and Ω is a dimensionless parameter controlling the frequency. Using (5) and (6) in (4) yields,

$$\frac{dv^0}{d\tau} = ahv^1 \quad (8)$$

$$\frac{dv^1}{d\tau} = ah(v^0 - v^3) \quad (9)$$

$$\frac{dv^2}{d\tau} = 0 \quad (10)$$

$$\frac{dv^3}{d\tau} = ahv^1. \quad (11)$$

We note that (8) and (11) imply,

$$v^0 = 1 + v^3 \quad (12)$$

which leaves the pair

$$\frac{dv^0}{d\tau} = ahv^1 \quad (13)$$

$$\frac{dv^1}{d\tau} = ah. \quad (14)$$

These imply,

$$v^0 = 1 + (v^1)^2/2. \quad (15)$$

Using the integral of (12) in the right hand side of (14) yields

$$v^1 = a \int h(-\tau) d\tau. \quad (16)$$

This is an important result, and shows that the entire solution can be found in terms of (16), a known result.[5] The integral may be found directly by completing the square and is given by the function $\mathcal{E}(x)$,

$$v^1 = a\mathcal{E}, \quad (17)$$

where $\mathcal{E}(x)$ is given in terms of the error function:

$$\mathcal{E}(x) = \frac{\sqrt{\pi}}{4} e^{-\Lambda^2} (2 + \operatorname{erf}(\tau/w + i\Lambda) + \operatorname{erf}(\tau/w - i\Lambda)) \quad (18)$$

where $\Lambda = \Omega w/2$. This assumes the initial condition that $v^1(-\infty) = 0$. Note that this is of form $z+z^*$, assuring that it is real. From (18) alone, it might be difficult to picture what this function looks like, but we may find another representation by continued integrating by parts. Defining $\zeta = \tau/w$ and $d_n \equiv \frac{d^n}{d\zeta^n} e^{-\zeta^2}$, we may write,

$$\mathcal{E}(\tau) = \sin \Omega\tau \sum_{n,\text{even}} (-1)^{n/2} \frac{d_n}{(2\Lambda)^{n+1}} + \cos \Omega\tau \sum_{n,\text{odd}} (-1)^{(n-1)/2} \frac{d_n}{(2\Lambda)^{n+1}} \quad (19)$$

which is useful for $\Lambda > 1$. This shows that the (x component of the) velocity is sinusoidal with an exponential envelope function, a result that is not self evident from (18). This series is very robust, and gives an excellent approximation keeping only the first term in the sum when Λ is bigger than unity. For example, Fig. 1 depicts a graph comparing v^1 using (18) and (19) using only one term in the sum with $\Lambda = 5/2$. For $\Lambda = 5$ the graphs are barely discernible.

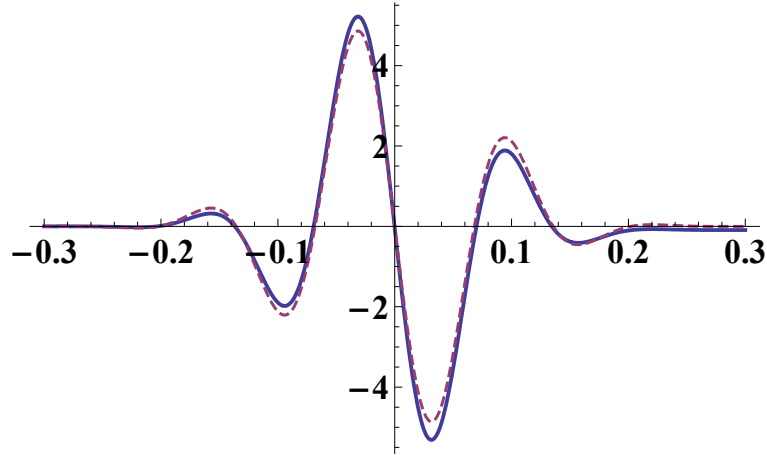


Fig. 1 v_1 using (17) (solid), and (19) (dashed), for $I = 10^{20} \text{ W cm}^{-2}$

There has been considerable interest in the Lawson-Woodward theorem,[6] which loosely states that a charged particle may not obtain a net velocity from an electromagnetic wave. An exception is the plane wave, and it is easy to see that the net x component of the velocity is

$$\Delta v^1/c = a \int_{-\infty}^{\infty} h(-\tau) d\tau = \sqrt{\pi} a e^{-\Lambda^2}. \quad (20)$$

Another way to view this is to note that the four potential is given by

$$A^\mu = (0, \phi, 0, 0) \quad (A_1 \equiv \phi) \quad \text{where}$$

$$\phi = -EL\mathcal{E}. \quad (21)$$

The kinetic energy of the particle is

$$K = (\gamma - 1)mc^2 = (v^0 - 1)mc^2 = v^3 m = \frac{m}{2}(v^1)^2 \quad (22)$$

where (12)-(15) were used. Using (17) and (21) one may also show that this may be put in the form, defining the $U = e\phi$,

$$K = \frac{1}{2}(U(\infty) - U(-\infty))^2. \quad (23)$$

In some versions of the Lawson-Woodward theorem it is stated that $U(\pm\infty) \rightarrow 0$, so that, in this more operational sense, there is no violation of the Lawson-Woodward theorem, since this condition is violated with (7).

One may also consider power radiated by the charge,

$$P = -\frac{2}{3} \frac{e^2}{c^3} \dot{v}^\nu \dot{v}_\nu \quad (24)$$

where $\dot{v}^\nu \equiv dv^\nu/d\tau$. Using the above this becomes

$$P = \frac{2}{3} \frac{e^2}{c^3} (\dot{v}^1)^2 = \frac{2e^2}{3c^3} \left(\frac{ahc}{L} \right)^2. \quad (25)$$

The average Poynting vector of the incident field is

$$\mathbf{S} = \left(\frac{c}{4\pi} \right) \mathbf{E} \times \mathbf{H}^* \quad (26)$$

from which we may obtain the ratio of the radiated power to the incident power per unit area, which yields,

$$\frac{P}{S} = \frac{8\pi}{3} a_0^2 = \sigma_T \quad (27)$$

where $a_0 = e^2/mc^2$ is the classical electron radius and σ_T is the Thomson cross section.

3. Radiation Reaction

As a charged particle accelerates, it radiates, creating an electromagnetic field that acts on the particle that created the field. This is called the self force, or radiation reaction force.[2] When the charged particle of charge e and mass m is subjected to an electromagnetic field, then the equation of motion is given by,[3]

$$\frac{dv^\sigma}{d\tau} = aF^{\sigma\mu}v_\mu + b(\ddot{v}^\sigma + Sv^\sigma) \quad (28)$$

where $S = \dot{v}^\nu \dot{v}_\nu$ and $a = eEL/(mc^2)$ and $b \equiv c\tau_0/L$, $\tau_0 = 2e^2/(3mc^3)$. This is called the Lorentz, Abraham, Dirac (LAD) equation and has well known problems, including the runaway solution and the fact that it is of third differential order. A common approach is to consider this equation with $b = 0$, and to use this value of \dot{v}^σ in the right side of (28). This gives

$$\frac{dv^\sigma}{d\tau} = aF^{\sigma\mu}v_\mu + b(a\dot{F}^{\sigma\mu}v_\mu + a^2(F^{\sigma\gamma}F_\gamma^\phi v_\phi + F^{\nu\gamma}v_\gamma F_\nu^\phi v_\phi v^\sigma)) \quad (29)$$

which is called the Landau Lifschitz (LL) equation.

The condition for validity is that the second term (ST) on the right side of (29) (i.e., the term with b) is small compared to the first term (FT) on the right side. It is not always easy to compare these, but we can do so by using the solution given above. If we take $w \sim 1$ so that $F^{\mu\nu} \sim 1$ the above shows that the LL equation is valid for, dropping indices (so exponents give the power), $av \gg ba^2v^3$. We will take the results found above, so that the velocity is either $a\mathcal{E}$ or $a^2\mathcal{E}^2$ (either the x component of the velocity or the x component). Looking at the best case (meaning most favorable to LL, using $a^2\mathcal{E}^2$ in FT and $a\mathcal{E}$ in ST), this gives $ba^2 \ll 1$ for the LL to be valid, which fails as we approach 10^{25}

W cm^{-2} . Since this is the most favorable case, more generally the LL will fail for smaller values of a . Unfortunately we are fast approaching this condition in the labs. Taking the worst case scenario, we find $ba^6 \ll 1$ for LL to be valid, which fails at $10^{20} \text{ W cm}^{-2}$.

Another approach capitalizes on the fact that b is so small ($\sim 10^{-9}$ for visible wavelengths). Let us write the LAD equation as

$$a^\sigma - b\dot{a}^\sigma = aF^{\sigma\mu}v_\mu + b\dot{v}^\nu\dot{v}_\nu v^\sigma. \quad (30)$$

Capitalizing on the smallness of b , we recognize the left side as the Taylor series of $a^\sigma(\tau-b)$ to $\mathcal{O}(b)$. We may expand the other terms to be functions of $(\tau-b)$ and find,

$$\frac{dv^\sigma}{d\tau} = aF^{\sigma\mu}v_\mu + b\left(\frac{d}{d\tau}(aF^{\sigma\mu}v_\mu) + \dot{v}^\nu\dot{v}_\nu v^\sigma\right) + \mathcal{O}(b^2). \quad (31)$$

This result requires only that b be small, so there is no restriction on the value of a . However, for small a , this may be shown to be equivalent to the LL equation, but in general it is different. To see the difference let us write the LL equation in the form

$$\frac{dv^\sigma}{d\tau} = aF^{\sigma\mu}v_\mu + b\left(\frac{d}{d\tau}(aF^{\sigma\mu}v_\mu) + a^2(F^{\sigma\gamma}F_\gamma^\phi v_\phi + F^{\nu\gamma}v_\gamma F_\nu^\phi v_\phi v^\sigma)\right). \quad (32)$$

Strictly speaking, once we decide to use $aF^{\sigma\mu}v_\mu$ on the right side of (28), as in the LL approach, we must make the replacement everywhere, and this shows that the Taylor series approach (31), which relies only on the smallness of b , is different than the LL equation (32), which relies on the smallness of ba^2 , or something more severe. We see that the last terms are different, but also, the derivative terms (on the right side) are different! This is because in the LL approach, we must replace v_μ within the derivative by the value found by setting the reaction forces to zero. For “weak” fields it does not matter, since they are so close, but for strong fields, the difference is important. In practice, (32) is often used as is. In this case, (32) becomes a postulated equation, which is justified by the smallness parameter discussed above. Even in this case (31) and (32) are different due to the last term. Nevertheless, for the weak field case, these two results are identical, but as noted, are different in the extremely high energy regime we are now approaching.

Since the only requirement for the validity of (31) is that b is small, it is natural to consider an asymptotic series of the form

$$v^\sigma = {}_0v^\sigma + b({}_1v^\sigma) + \mathcal{O}(b^2). \quad (33)$$

To lowest order we obtain (12) to (16) for ${}_0v^\sigma$. Using that, to order b we have, calling $u^\sigma \equiv {}_1v^\sigma$ and $\phi \equiv {}_0v^1$,

$$\dot{u}^0 = ah u^1 + a^2 \hbar \mathcal{E} - \frac{a^4 h^2 \mathcal{E}^2}{2} \quad (34)$$

$$\dot{u}^1 = ah(u^0 - u^3) + a\dot{\hbar} - ha^3 h^2 \mathcal{E} \quad (35)$$

$$\dot{u}^3 = ah u^1 + a^2(\dot{h}\mathcal{E} + h^2) - \frac{a^4 h^2 \mathcal{E}^2}{2}. \quad (36)$$

These equations describe the equation of motion for any h to order b . Let us first consider the case that $a \ll 1$ so that in the above only terms linear in a are retained and higher powers are ignored. (For example, $a \sim 0.8$ at $I = 10^{17}$ W cm⁻².) In this case (34) and (36) show that $u^0 = u^3$. With (35) this shows that $u^1 = ah$ so that (34) gives

$$u^0 = a^2 \int h^2 d\tau \quad (37)$$

and the problem is solved to quadrature (to order b).

As an example, let $h = \cos(z - t)$. This gives

$$u^3 = \frac{1}{2} \left(t + \frac{\sin 2t}{c} \right) \quad (38)$$

as $\tau \rightarrow t$.

Now we may consider $\langle F_z \rangle = m\dot{v}^3$ and use (33). This yields, for the z direction (the transverse direction averages to zero),

$$\langle F \rangle = \sigma_T \frac{\langle S \rangle}{c}. \quad (39)$$

where S is the z component of the Poynting vector. S/c is the force per unit area, so this equation tells us exactly what we expect from elementary considerations: The particle experiences a force equal to the Poynting vector's force per unit area, multiplied by the Thomson cross section of the electron. In fact, a more general result may be obtained by considering the four momentum p^μ . Suppose we integrate the zero component over the full time interval of the pulse. Using (28) we have

$$W = c \int d\tau \left(a F^{0\mu} v_\mu + b(\dot{v}^0 + S v^0) \right) \quad (40)$$

where W is the final energy of the particle. The first term on the right is

$$ca \int dt \mathbf{E} \cdot \mathbf{V}$$

which yields zero when the Lawson-Woodward theorem applies. The next term also integrates to zero, since the acceleration vanishes before and after the pulse, or wave train, ends. We also know that the power scalar is

$$P = m\tau_0 \dot{v}^\sigma \dot{v}_\sigma \quad (41)$$

which reduces to $P = 2e^2(\dot{v})^2/(3c^3)$ in the low velocity limit. Thus we find,

$$W = -c \int d\tau P \quad (42)$$

which says that the energy the particle gains is equal to the work done by the radiation field, or minus the power radiated. This result is limited to low velocity, and more importantly, to the case that the field is transverse.

With this approach we may also look at the “large a ” limit. From (34) - (36) we see that $u^0 \approx u^3 \gg u^1$ (which also hold for zero order), and

$$u^3 = -\frac{a^4}{2w^2} \int h^2 \mathcal{E}^2 d\tau. \quad (43)$$

This verifies that the particle is ejected in the forward direction, parallel to the direction of the beam of light.

4. Astrophysical Applications

Although the above analysis is motivated by the extremely intense laser pulses that are becoming available, another realm where intense fields must be dealt with is compact astrophysical sources such as neutron stars or magnetars, and black holes. In addition to the electromagnetic force there is also the gravitational force. The equation of motion for a charged particle in a combined gravitational and electromagnetic field with radiation reaction is (but ignoring gravitational radiation reaction), in cgs units,

$$\frac{Dv^\mu}{D\tau} = \frac{e}{mc} F^{\mu\nu} v_\nu + \tau_0 (\ddot{v}^\mu + v^\mu \dot{v}^\sigma \dot{v}_\sigma) \quad (44)$$

where the covariant derivative is defined as

$$\frac{Dv^\mu}{D\tau} = \frac{dv^\mu}{d\tau} + \{\alpha\beta\}^\mu v^\alpha v^\beta \quad (45)$$

where $\{\alpha\beta\}^\mu$ is the Christoffel symbol. The technique described above can be used here, and the result is

$$\frac{Dv^\mu}{D\tau} = \frac{e}{mc} F^{\mu\nu} v_\nu + \tau_0 \left(v^\mu \dot{v}^\sigma \dot{v}_\sigma - \{\alpha\beta\}_{,\lambda}^\mu v^\lambda v^\alpha v^\beta - 2\{\alpha\beta\}^\mu \dot{v}^\alpha v^\beta - \frac{e}{mc} (\dot{F}^{\mu\sigma} v_\sigma + F^{\mu\sigma} \dot{v}_\sigma) \right). \quad (46)$$

This formula is useful for describing charged particles as they orbit or spiral into a neutron star or magnetar, or as they fall into a black hole. In this case both the electromagnetic and the gravitation field is large, and the terms in the parenthesis in (46) may be appreciable.

5. Discussion

Several effects concerning radiation reaction forces have been discussed. It is shown how an electromagnetic pulse exerts a *net* force on an electron by calculating the effect of the radiation reaction forces. In fact, it is shown that the average constant force associated with the Poynting vector of a plane wave actually arises from the self forces. In addition, a new and simple way of treating the equation of motion with radiation reaction was developed that, to order τ_0 , automatically removes the runaway solutions. Using the equations to this order, it was shown that an electron interacting with an electromagnetic

pulse is scattered in a highly concentrated forward direction, and that the scattering angle is inversely proportional to the intensity of the wave. Finally, gravitational forces were included and applications to astrophysical sources was briefly indicated.

References

- [1] <http://www.eecs.umich.edu/CUOS>
- [2] F. Rohrlich, *Am. J. of Physics* **68**,1109 (2000); E. Poisson, arXiv gr-qc/0306052.
- [3] J. D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, New York 1975, 2nd ed.).
- [4] S. M. Sepke and D. P. Umstadter, *J. Opt. Soc. Am. B* **23**, 2295 (2006).
- [5] R. T. Hammond, “Intense ultrashort electromagnetic pulses and the equation of motion,” arXiv physics/0607285 (2006); J. H. Eberly and A Sleeper, *Phys. Rev.* **176**, 1570 (1968); E. S. Sarachik and G. T. Schappert, *Phys. Rev.* **1**, 2738 (1970); J. W. Meyer, *Phys. Rev.* **3**, 621 (1971); E. Esarey, S. K. Ride, and P. Sprangle, *Phys. Rev.* **48**, 3003.
- [6] For an introduction, see A. L. Troha and F. V. Hartemann, *Phys. Rev. E.* **65**, 028502 (2002).
- [7] L. D. Landau and E. M. Lifshitz *The Classical Theory of Fields* (Pergamon Press, Addison-Welsey, Reading, MA, 1971), equation 76.1.

