

Bosons-Parafermions Wess-Zumino Model*

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Abstract: A Wess-Zumino model in terms of bosons and parafermions of order $p = 2$ is investigated. We show that the parasupercharges associated to the parasupersymmetric transformations satisfy the $p = 2$ trilinear relations. The closure of the transformations algebra is established with a trilinear product rule for the fermionic elements. Finally, we verify that these parasupercharges are really the generators of the parasupersymmetric transformations.

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1. Introduction

Generalized statistics first was proposed by H. S. Green in 1953 [1]. Known nowadays as paraquantization, it leads to two families of generalized statistics (parabose and parafermi statistics which include the ordinary bose and fermi statistics as particular cases) which are collectively called parastatistics [2].

A fundamental question is faced respect to a spin statistics theorem which assert that integer-spin fields can not be quantized with the help of anticommutators and half-integer-spin fields can not be quantized with the help of commutators [3, 4, 5]. In the literature, A. B. Gorokov [6] published a very nice review article of a generalized statistics which consist on a modern review of different aspects of generalized statistics and first of all the paraboson and parafermion systems. Indeed, as is mentioned in [6], all particles

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which we consider to be elementary obey either bose or fermi statistics. This concerns both particles which are directly observable in our laboratories and those which can not. The reason expected for this general law of nature is hidden in deep properties of matter itself: identity and repeatability. The main idea of A. B. Govorkov [6] consist to adopt the fact that the invariance of the density matrix with respect to the permutation of all variables of identical particles as the starting definition of indistinguishability of the particles. Previously, some authors, Pauli [7], Dirac[8] and others have always underscored the possibility of describing the statistics of identical particles not only by means of symmetric and antisymmetric representations but also by multidimensional representations of the group of permutation of the coordinates and spin variables of particles, even it was proved [9] that it is impossible to establish commutation relations for operators corresponding to multidimensional representations of permutation groups. Afterwards, a theorem [10] came to reinforce the foregoing about the fact that the statistics of identical particles can be parafermi or parabose statistics. As a result of the starting definition of indistinguishability of the particles, the particles creation and annihilation operators obey to trilinear commutation relations. One principal result of this is the following [6]: Finite order parabose and parafermi statistics allow to put a definite number of particles in an antisymmetric state and a symmetric state, respectively, not exceeding a fixed number p , the so-called order of the paraquantization.

The physics of elementary particles is based on two main symmetries: Poincaré and internal symmetries. A symmetry different from all of these was brought to the attention of the particles physics community by Wess and Zumino, and goes by the name of supersymmetry, the Wess-Zumino [11, 12] model has allowed anticommutation relations of the generators of supersymmetry which transform bosons into fermions and vice versa.

Generalized statistics and supersymmetry may be unified in what is called parasupersymmetry, which is a symmetry between bosons and parafermions. The parasupersymmetry structure will of course depend on the number p of parafermions that can occupy the same state. The paraextension of the Wess-Zumino model in the formalism of the superspace is developed in [13], the aim of this work is to investigate the most immediate field theoretical realization of the parasuperPoincaré algebra without the context of the parasuperspace. Following the same procedure in [14], we demonstrate by explicit calculation that the spinor parasupercharges of the theory considered as linear operators in the Fock space satisfy the parasuperPoincaré algebra. We study the closure of the transformations algebra and verify that the parasupercharges constructed are the generators of the parasupersymmetry transformations considered.

2. The Model

Introducing parasupersymmetry in the context of a simple four dimensional massive and free field theory : paraquantum version of the Wess-Zumino model, let us assume that the model possess one parafermion Ψ^α of order $p = 2$ which is a Majorana spinor, on shell, that is $(i\partial_\mu\gamma^\mu\Psi - m) = 0$. Let us assume that the two degrees of freedom of

the parafermion Ψ^α impose the introduction of two bosonic degrees of freedom in order to form a realization of parasupersymmetry. We introduce then one ordinary real scalar field A and one ordinary real pseudoscalar field B subject to $(\square + m^2)A = (\square + m^2)B = 0$. Here we assume that the mass dimension of the parafermion Ψ is, like in the ordinary case, always $3/2$ and of course the ordinary fields A and B have the dimension 1. With an adequate symmetrization of the parafermionic field, the lagrangian density describing this simple system is given by :

$$\mathcal{L} = \frac{1}{4} [\bar{\Psi}, (i\gamma^\mu \partial_\mu - m)\Psi] + \frac{1}{2} (\partial_\mu A)^2 + \frac{1}{2} (\partial_\mu B)^2 - \frac{1}{2} m^2 A^2 - \frac{1}{2} m^2 B^2 \quad (1)$$

where all the fields have the same mass m . and $\bar{\Psi} = \Psi^T \gamma^0$. The basic commutation relations for the $p = 2$ parafermionic fields (at fixed time) are

$$\langle \Psi_i(\vec{x}, t) \bar{\Psi}_j(\vec{y}, t) \Psi_k(\vec{z}, t) \rangle = 2\delta(\vec{x} - \vec{y}) \delta_{ij} \Psi_k(\vec{z}, t) + 2\delta(\vec{y} - \vec{z}) \delta_{jk} \Psi_i(\vec{x}, t) \quad (2)$$

$$\langle \Psi_i(\vec{x}, t) \Psi_j(\vec{y}, t) \bar{\Psi}_k(\vec{z}, t) \rangle = 2\delta(\vec{y} - \vec{z}) \delta_{jk} \Psi_i(\vec{x}, t) \quad (3)$$

$$\langle \Psi_i(\vec{x}, t) \Psi_j(\vec{y}, t) \Psi_k(\vec{z}, t) \rangle = 0 \quad (4)$$

with the notation $\langle abc \rangle = abc + cba$. Those for the ordinary fields A and B are

$$[A(\vec{x}, t), \pi(\vec{y}, t)] = i\delta(\vec{x} - \vec{y}) \quad (5)$$

$$[\pi(\vec{x}, t), \pi(\vec{y}, t)] = [A(\vec{x}, t), A(\vec{y}, t)] = 0 \quad (6)$$

$$[B(\vec{x}, t), \pi'(\vec{y}, t)] = i\delta(\vec{x} - \vec{y}) \quad (7)$$

$$[\pi'(\vec{x}, t), \pi'(\vec{y}, t)] = [B(\vec{x}, t), B(\vec{y}, t)] = 0 \quad (8)$$

where

$$\pi(\vec{x}, t) = \frac{\delta \mathcal{L}}{\delta(\partial_0 A(\vec{x}, t))}, \pi'(\vec{x}, t) = \frac{\delta \mathcal{L}}{\delta(\partial_0 B(\vec{x}, t))}$$

Furthermore, bosonic and parafermionic fields are taken to commute among themselves

$$[\Psi_i, A] = [\Psi_i, B] = [A, B] = 0$$

Now, the later development of this work necessitates the plane wave expansions of the Majorana parafield Ψ^α and the ordinary fields A and B which are given by

$$\Psi^a(x) = \frac{1}{(2\pi)^{3/2}} \sum_s \int d^3p \left(\frac{m}{\omega_p}\right)^{1/2} (d(\vec{p}, s) u_a(\vec{p}, s) e^{-ipx} + d^+(\vec{p}, s) v_a(\vec{p}, s) e^{ipx}) \quad (9)$$

$$A(x) = \frac{1}{(2\pi)^{3/2}} \int d^3p \left(\frac{1}{2\omega_p}\right)^{1/2} \{a(\vec{p}) e^{-ipx} + a^+(\vec{p}) e^{ipx}\} \quad (10)$$

$$B(x) = \frac{1}{(2\pi)^{3/2}} \int d^3p \left(\frac{1}{2\omega_p}\right)^{1/2} \{b(\vec{p}) e^{-ipx} + b^+(\vec{p}) e^{ipx}\} \quad (11)$$

with the usual notations:

$$\begin{aligned}\omega_p &= (\vec{p}^2 + m^2)^{1/2} \\ (\not{p} - m)u(\vec{p}, s) &= 0 \\ (\not{p} + m)v(\vec{p}, s) &= 0\end{aligned}$$

and where, in terms of the modes, the relations (2 – 8) are rewritten in the following forms:

$$\begin{aligned}\langle d(\vec{p}, s)d^+(\vec{k}, l)d(\vec{p}', r) \rangle &= 2\delta(\vec{p} - \vec{k})\delta_{ls}d(\vec{p}', r) \\ &\quad + 2\delta(\vec{p}' - \vec{k})\delta_{lr}d(\vec{p}, s)\end{aligned}\tag{12}$$

$$\langle d^+(\vec{p}, s)d^+(\vec{k}, l)d(\vec{p}', r) \rangle = 2\delta(\vec{p}' - \vec{k})\delta_{lr}d^+(\vec{p}, s)\tag{13}$$

$$\langle d(\vec{p}, s)d(\vec{k}, l)d(\vec{p}', r) \rangle = 0\tag{14}$$

$$[a(\vec{p}), a^+(\vec{k})] = [b(\vec{p}), b^+(\vec{k})] = \delta(\vec{p} - \vec{k})\tag{15}$$

$$\begin{aligned}[a(\vec{p}), a(\vec{k})] \\ &= [b(\vec{p}), b(\vec{k})] = [a(\vec{p}), d^+(\vec{k}, l)] \\ &= [a(\vec{p}), b(\vec{k})] = [b(\vec{p}), d^+(\vec{k}, l)] = 0\end{aligned}\tag{16}$$

3. Parasupersymmetry

3.1 Parasupersymmetric Transformations

We now investigate continuous transformations of the fields A , B and the parafield Ψ^α , which will be the parasupersymmetric transformations of the theory defined by (1). We are then led to verify that the action

$$\begin{aligned}S &= \int \left\{ \frac{1}{4} [\bar{\Psi}, (i\gamma^\mu \partial_\mu - m)\Psi] + \frac{1}{2} (\partial_\mu A)^2 \right. \\ &\quad \left. + \frac{1}{2} (\partial_\mu B)^2 - \frac{1}{2} m^2 A^2 - \frac{1}{2} m^2 B^2 \right\} d^4x\end{aligned}\tag{17}$$

is left invariant. To do this, it suffices that, under these transformations, the free lagrangian density changes by a total derivative

$$\delta\mathcal{L} = \partial_\mu J^\mu\tag{18}$$

where J^μ is a conserved Noether parasupersymmetric current density. As in the ordinary Wess-Zumino model, let us consider the following variations which transform

parafermions and bosons into each other

$$\delta\Psi = (-i\gamma^\mu\partial_\mu + m)(A - i\gamma^5 B)\varepsilon \quad (19)$$

$$\delta\bar{\Psi} = \bar{\varepsilon}(A - i\gamma^5 B)(i\gamma^\mu\partial_\mu - m) \quad (20)$$

$$\delta A = \frac{1}{2} [\bar{\varepsilon}, \Psi] \quad (21)$$

$$\delta B = \frac{-i}{2} [\bar{\varepsilon}, \gamma^5\Psi] \quad (22)$$

which are rewritten with an appropriate symmetrization of the product $\bar{\varepsilon}\Psi$ and where ε is a constant Majorana spinor. It is important to notice here that, by analogy with the ordinary supersymmetric case for which ε^α is anticommuting just like Ψ^α , here, we have to take the components ε^α as paraGrassmann which obey the algebra (specific to the order $p = 2$)

$$\varepsilon^\alpha\varepsilon^\beta\varepsilon^\gamma + \varepsilon^\gamma\varepsilon^\beta\varepsilon^\alpha = 0 \quad (23)$$

Note that this implies $(\varepsilon^\alpha)^3 = 0$. the components ε^α are then assumed to have non trivial commutations relations with the parafields Ψ^α

$$[[\bar{\varepsilon}^\alpha, \Psi_i(\vec{x}, t)], \bar{\Psi}_j(\vec{y}, t)] = 2\bar{\varepsilon}^\alpha\delta_{ij}\delta(\vec{x} - \vec{y}) \quad (24)$$

$$[[\bar{\varepsilon}^\alpha, \Psi_i(\vec{x}, t)], \Psi_j(\vec{y}, t)] = 0 \quad (25)$$

$$[[\bar{\varepsilon}^\alpha, \Psi_i(\vec{x}, t)], \varepsilon^\beta] = 0 \quad (26)$$

and by analogy to the ordinary case, they are assumed to commute with A and B . One can rewrite the precedent relations as follows:

$$[[\bar{\varepsilon}^\alpha, d(\vec{p}, s)], d^+(\vec{k}, l)] = 2\bar{\varepsilon}^\alpha\delta_{ls}\delta(\vec{p} - \vec{k}) \quad (27)$$

$$[[\bar{\varepsilon}^\alpha, d(\vec{p}, s)], d(\vec{k}, l)] = 0 \quad (28)$$

$$[[\bar{\varepsilon}^\alpha, d(\vec{p}, s)], \varepsilon^\beta] = 0 \quad (29)$$

$$[a(\vec{p}), \varepsilon^\alpha] = [b(\vec{p}), \varepsilon^\alpha] = 0 \quad (30)$$

Let us now proceed to the evaluation of the Noether current J^μ . Computing the variation of the lagrangian by the use of the transformations(19 – 22) and the equations of motion of the fields, one can write

$$\delta\mathcal{L} = \partial_\mu V^\mu \quad (31)$$

where

$$V^\mu = \frac{1}{4} [\bar{\varepsilon}, \partial^\mu (A - i\gamma^5 B) \Psi] \quad (32)$$

The conserved parasupercurrent J^μ is given by the relation

$$J^\mu = V^\mu - \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\phi_i \quad (33)$$

where

$$\begin{aligned}
& \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta \phi_i \\
&= \frac{\partial \mathcal{L}}{\partial \partial_\mu A} \delta A + \frac{\partial \mathcal{L}}{\partial \partial_\mu B} \delta B + \frac{\partial \mathcal{L}}{\partial \partial_\mu \Psi} \delta \Psi \\
&= \frac{1}{4} [\bar{\varepsilon}, \partial_\mu (A - i\gamma^5 B) \Psi] \\
&\quad - \frac{i}{4} [\bar{\Psi} (i\gamma^\mu \partial_\mu + m)(A - i\gamma^5 B), \varepsilon]
\end{aligned} \tag{34}$$

so that

$$J^\mu = \frac{i}{4} [\bar{\Psi} (i\gamma^\mu \partial_\mu + m)(A - i\gamma^5 B), \varepsilon] \tag{35}$$

which can be noted as

$$J^\mu = \frac{1}{\lambda} [\bar{k}, \varepsilon]$$

where λ is a real constant which has to be determined.

3.2 Parasupersymmetric Algebra

Now, if this $p = 2$ extension of the Wess-Zumino model is a field theoretical realization of the parasupersymmetric algebra, the spinor parasupercharge

$$Q_a = \int d^3 \vec{x} k_a^o \tag{36}$$

must satisfy the commutation relations and the trilinear relations of a parasupersymmetric Poincaré algebra (in the sense of Debergh and Becker)

$$[P_\mu, P_\nu] = 0 \tag{37}$$

$$[M_{\mu\nu}, P_\rho] = 0 \tag{38}$$

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= -i (\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho}) \\
&\quad - i (-\eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho})
\end{aligned} \tag{39}$$

$$[M_{\mu\nu}, Q_a] = -(\sigma_{\mu\nu}^4)_{ab} Q_b \tag{40}$$

$$[P_\mu, Q_a] = 0 \tag{41}$$

$$\langle Q_a, \bar{Q}_b, Q_c \rangle = 4P_\mu \gamma_{ab}^\mu Q_c + 4Q_a P_\mu \gamma_{cb}^\mu \tag{42}$$

$$\begin{aligned}
\langle Q_a, \bar{Q}_b, \bar{Q}_c \rangle &= 4P_\mu \gamma_{ab}^\mu \bar{Q}_c \\
&\quad + 4Q_a (c^{-1} \gamma^\mu)_{bc} P_\mu
\end{aligned} \tag{43}$$

$$\begin{aligned}
\langle \bar{Q}_a, \bar{Q}_b, \bar{Q}_c \rangle &= 4\bar{Q}_a (c^{-1} \gamma^\mu)_{bc} P_\mu \\
&\quad + 4(c^{-1} \gamma^\mu)_{ab} P_\mu \bar{Q}_c
\end{aligned} \tag{44}$$

where

$$\sigma_{\mu\nu}^4 = \frac{i}{4} [\gamma_\mu, \gamma_\nu]$$

and c is the charge conjugation matrix. In particular, let us show that the Q_a satisfy the $p = 2$ trilinear relation (42). To do this, we have to give a representation of the para-supercharges (36) as a linear operator acting on a Fock space defined by a fundamental state $|0\rangle$ satisfying

$$a|0\rangle = b|0\rangle = 0 \quad (45)$$

and

$$d(\vec{p}, s)d^+(\vec{k}, l)|0\rangle = 2\delta_{ls}\delta(\vec{p} - k)|0\rangle \quad (46)$$

$$d(\vec{p}, s)|0\rangle = 0 \quad (47)$$

which fixes the order of the paraquantization $p = 2$. With the same calculation stapes as in [14] one can obtain these expressions for Q_a and \bar{Q}_a

$$Q_a = \frac{i\lambda}{2^{1/2}}m^{1/2}\sum_s \int d^3p(\{C(\vec{p})d^+(\vec{p}, s)v(\vec{p}, s)\}_a - \{D(\vec{p})d(\vec{p}, s)u(\vec{p}, s)\}_a) \quad (48)$$

$$\bar{Q}_a = -\frac{i\lambda}{2^{1/2}}m^{1/2}\sum_s \int d^3p(\{\bar{v}(\vec{p}, s)d(\vec{p}, s)D(\vec{p})\}_a - \{\bar{u}(\vec{p}, s)d^+(\vec{p}, s)C(\vec{p})\}_a) \quad (49)$$

where

$$C(\vec{p}) = (a - i\gamma^5 b)$$

$$D(\vec{p}) = (a^+ - i\gamma^5 b^+)$$

By the use of the same steps of calculus in [14] in the context of the paraquantization by the use of the relations, one can show that (12-16):

$$\begin{aligned}
& \langle Q_a, \bar{Q}_b, Q_c \rangle \\
&= \int d^3p \frac{\lambda^2}{2} (\{a^+(\vec{p})a(\vec{p}) + b^+(\vec{p})b(\vec{p})\} \\
&+ \left\{ 1 + \frac{1}{2} \sum_s [d^+(\vec{p}, s), d(\vec{p}, s)] \right\}) p_\mu \gamma_{ab}^\mu \\
&\left(\left\{ i\lambda \left(\frac{m}{2}\right)^{1/2} \sum_r \int d^3p' C_{c\gamma}(\vec{p}') d^+(\vec{p}', r) v_\gamma(\vec{p}', r) \right\} \right. \\
&- \left. \left\{ i\lambda \left(\frac{m}{2}\right)^{1/2} \sum_r \int d^3p' D_{c\gamma}(\vec{p}') d(\vec{p}', r) u_\gamma(\vec{p}', r) \right\} \right) \\
&+ i\lambda \left(\frac{m}{2}\right)^{1/2} \sum_s \int d^3p (\{C_{a\alpha}(\vec{p}) d^+(\vec{p}, s) v_\alpha(\vec{p}, s)\} \\
&- \{D_{a\alpha}(\vec{p}) d(\vec{p}, s) u_\alpha(\vec{p}, s)\}) \\
&(\{ \int \frac{\lambda^2}{2} d^3p' [a^+(\vec{p}')a(\vec{p}') + b^+(\vec{p}')b(\vec{p}') + 1] \} \\
&+ \left\{ \frac{1}{2} \sum_r [d^+(\vec{p}', r), d(\vec{p}', r)] p_\mu \gamma_{cb}^\mu \right\}) \\
&+ R_1
\end{aligned} \tag{50}$$

where, again by the use of the symmetry properties, the contribution of the terms proportional to $(\gamma^\mu \gamma^5)_{ab}$ in R_1 vanishes so that:

$$\begin{aligned}
R_1 &= \sum_{ls} \frac{i}{16} \lambda^3 \int d^3p \int d^3k p_\mu (\gamma^\mu c)_{ac} C_{\beta b} \bar{u}_\beta(\vec{k}, l) \\
&[d^+(\vec{p}, s) d^+(\vec{k}, l) d(\vec{p}, s) - d(\vec{p}, s) d^+(\vec{k}, l) \\
&d^+(\vec{p}, s)] - \sum_{ls} \frac{i}{16} \lambda^3 \int d^3p \int d^3k p_\mu (\gamma^\mu c)_{ac} \\
&D_{\beta b} \bar{v}_\beta(\vec{k}, l) [d(\vec{p}, s) d(\vec{k}, l) d^+(\vec{p}, s) \\
&- d^+(\vec{p}, s) d(\vec{k}, l) d(\vec{p}, s)]
\end{aligned}$$

In the other hand, one can prove that the energy momentum takes the form

$$\begin{aligned}
P_\mu &= \int d^3p (\{a^+(\vec{p})a(\vec{p}) + b^+(\vec{p})b(\vec{p}) + 1\} \\
&+ \left\{ \frac{1}{2} \sum_s [d^+(\vec{p}, s), d(\vec{p}, s)] \right\}) p_\mu
\end{aligned}$$

so that, the relation (50) can be rewritten in the form

$$\langle Q_a, \bar{Q}_b, Q_c \rangle = \frac{1}{2} \lambda^2 P_\mu \gamma_{ab}^\mu Q_c + \frac{1}{2} \lambda^2 Q_a P_\mu \gamma_{cb}^\mu + R_1$$

Thus if we choose $\lambda = (2)^{3/2}$, we obtain

$$\langle Q_a, \bar{Q}_b, Q_c \rangle = 4P_\mu \gamma_{ab}^\mu Q_c + 4Q_a P_\mu \gamma_{cb}^\mu + R_1 \quad (51)$$

a straightforward calculation permits to verify that

$$[Q_a, P_\mu] = 0 \quad (52)$$

Clearly, from the relations (51), (52), R_1 must also be conserved, in fact, one can verify that

$$[R_1, P_\mu] = 0$$

Now, and like in the work of [15], since in general R_1 is non zero, the parasupersymmetric algebra (51) is complicated. A choice of a vacuum state analogous to (42) would reduce it to

$$\langle Q_a, \bar{Q}_b, Q_c \rangle = 4P_\mu \gamma_{ab}^\mu Q_c + 4Q_a P_\mu \gamma_{cb}^\mu$$

since in the corresponding Fock space, R_1 is represented by the null operator.

Now, in the same way, one obtain

$$\begin{aligned} \langle Q_a, \bar{Q}_b, \bar{Q}_c \rangle &= 4P_\mu \gamma_{ab}^\mu \bar{Q}_c + 4Q_a (c^{-1} \gamma^\mu)_{bc} P_\mu + R_2 \\ \langle \bar{Q}_a, \bar{Q}_b, \bar{Q}_c \rangle &= 4\bar{Q}_a (c^{-1} \gamma^\mu)_{bc} P_\mu + 4(c^{-1} \gamma^\mu)_{ab} P_\mu \bar{Q}_c + R_3 \end{aligned}$$

Where

$$\begin{aligned} R_2 &= \sum_{ls} \frac{i}{16} \lambda^3 \int d^3 p \int d^3 k p_\mu (\gamma^\mu)_{ac} C_{\beta b} \bar{u}_\beta(\vec{k}, l) \\ &[d(\vec{p}, s) d^+(\vec{k}, l) d^+(\vec{p}, s) - d^+(\vec{p}, s) d^+(\vec{k}, l) \\ &d(\vec{p}, s)] - \sum_{ls} \frac{i}{16} \lambda^3 \int d^3 p \int d^3 k p_\mu (\gamma^\mu)_{ac} \\ &D_{\beta b} \bar{v}_\beta(\vec{k}, l) [d^+(\vec{p}, s) d(\vec{k}, l) d(\vec{p}, s) \\ &- d(\vec{p}, s) d(\vec{k}, l) d^+(\vec{p}, s)] \end{aligned}$$

and

$$\begin{aligned} R_3 &= - \sum_{ls} \frac{i}{16} \lambda^3 \int d^3 p \int d^3 k p_\mu (\gamma^\mu c^{-1})_{ac} C_{\beta b} \bar{u}_\beta(\vec{k}, l) \\ &[d(\vec{p}, s) d^+(\vec{k}, l) d^+(\vec{p}, s) - d^+(\vec{p}, s) d^+(\vec{k}, l) \\ &d(\vec{p}, s)] + \sum_{ls} \frac{i}{16} \lambda^3 \int d^3 p \int d^3 k p_\mu (\gamma^\mu c^{-1})_{ac} \\ &D_{\beta b} \bar{v}_\beta(\vec{k}, l) [d^+(\vec{p}, s) d(\vec{k}, l) d(\vec{p}, s) \\ &- d(\vec{p}, s) d(\vec{k}, l) d^+(\vec{p}, s)] \end{aligned}$$

Which, as in the case of R_1 , are again represented by the null operators in the Fock space.

4. Closure

Notice that, to the trilinear nature of the parasupercharge algebra (42 – 44) correspond the fact that the infinitesimal transformations(19 – 22) must close onto an algebra that involves trilinear relations. Indeed, let us calculate the term

$$\begin{aligned}\delta_3\delta_2\delta_1A &= -\frac{i}{4}\gamma_{ab}^\mu [\bar{\varepsilon}_1^a, \varepsilon_2^b] [\bar{\varepsilon}_3^c, \partial_\mu \Psi^c] \\ &\quad + \frac{i}{4}(\gamma^\mu\gamma^5)_{ab} \gamma_{c\alpha}^5 [\bar{\varepsilon}_1^a, \varepsilon_2^b] [\bar{\varepsilon}_3^c, \partial_\mu \Psi^\alpha]\end{aligned}$$

using the properties of the paraGrassmann ε^α one finds:

$$\begin{aligned}\delta_3\delta_2\delta_1A &= \partial_\mu \frac{1}{2} \left(\left[-\frac{i}{2} [\bar{\varepsilon}_1^a \gamma_{ab}^\mu, \varepsilon_2^b] \bar{\varepsilon}_3^c, \Psi^\alpha \right] \right. \\ &\quad \left. + \left[\frac{i}{2} [\bar{\varepsilon}_1^a (\gamma^\mu\gamma^5)_{ab}, \varepsilon_2^b] \bar{\varepsilon}_3^c \gamma_{c\alpha}^5, \Psi^\alpha \right] \right) \\ &\quad \partial_\mu \frac{1}{2} \left(\left[\frac{i}{2} [\bar{\varepsilon}_1^a, \varepsilon_2^a] \bar{\varepsilon}_3^b \gamma_{b\alpha}^\mu, \Psi^\alpha \right] \right. \\ &\quad \left. - \left[\frac{i}{2} [\bar{\varepsilon}_1^a (\gamma^5)_{ab}, \varepsilon_2^b] \bar{\varepsilon}_3^c (\gamma^5\gamma^\mu)_{c\alpha}, \Psi^\alpha \right] \right)\end{aligned}$$

which can be rewritten as

$$\delta_3\delta_2\delta_1A = \partial_\mu \delta_{\bar{\theta}} A$$

where the subscript $\bar{\theta}$ of the transformation δ is given by

$$\begin{aligned}\bar{\theta} &= -\frac{i}{2} [\bar{\varepsilon}_1^a \gamma_{ab}^\mu, \varepsilon_2^b] \bar{\varepsilon}_3^c + \frac{i}{2} [\bar{\varepsilon}_1^a (\gamma^\mu\gamma^5)_{ab}, \varepsilon_2^b] \bar{\varepsilon}_3^c \gamma_{c\alpha}^5 \\ &\quad + \frac{i}{2} [\bar{\varepsilon}_1^a, \varepsilon_2^a] \bar{\varepsilon}_3^b \gamma_{b\alpha}^\mu - \frac{i}{2} [\bar{\varepsilon}_1^a (\gamma^5)_{ab}, \varepsilon_2^b] \bar{\varepsilon}_3^c (\gamma^5\gamma^\mu)_{c\alpha}\end{aligned}$$

Finally, the closure of the transformations algebra takes the form

$$(\delta_3\delta_2\delta_1 + \delta_1\delta_3\delta_2 + \delta_2\delta_1\delta_3)A = \partial_\mu \delta_{\bar{\zeta}^\mu} A \quad (53)$$

where

$$\begin{aligned}(\bar{\zeta}^\mu)^\alpha &= -\frac{i}{2} [\bar{\varepsilon}_1^a \gamma_{ab}^\mu, \varepsilon_2^b] \bar{\varepsilon}_3^c + \frac{i}{2} [\bar{\varepsilon}_1^a (\gamma^\mu\gamma^5)_{ab}, \varepsilon_2^b] \bar{\varepsilon}_3^c \gamma_c^{5\alpha} \\ &\quad - \frac{i}{2} [\bar{\varepsilon}_2^a \gamma_{ab}^\mu, \varepsilon_3^b] \bar{\varepsilon}_1^c + \frac{i}{2} [\bar{\varepsilon}_2^a (\gamma^\mu\gamma^5)_{ab}, \varepsilon_3^b] \bar{\varepsilon}_1^c \gamma_c^{5\alpha} \\ &\quad - \frac{i}{2} [\bar{\varepsilon}_3^a \gamma_{ab}^\mu, \varepsilon_1^b] \bar{\varepsilon}_2^c + \frac{i}{2} [\bar{\varepsilon}_3^a (\gamma^\mu\gamma^5)_{ab}, \varepsilon_1^b] \bar{\varepsilon}_2^c \gamma_c^{5\alpha} \\ &\quad + \frac{i}{2} [\bar{\varepsilon}_1^a, \varepsilon_2^a] \bar{\varepsilon}_3^b \gamma_{b\alpha}^\mu - \frac{i}{2} [\bar{\varepsilon}_1^a (\gamma^5)_{ab}, \varepsilon_2^b] \bar{\varepsilon}_3^c (\gamma^5\gamma^\mu)_{c\alpha} \\ &\quad + \frac{i}{2} [\bar{\varepsilon}_3^a, \varepsilon_1^a] \bar{\varepsilon}_2^b \gamma_{b\alpha}^\mu - \frac{i}{2} [\bar{\varepsilon}_3^a (\gamma^5)_{ab}, \varepsilon_1^b] \bar{\varepsilon}_2^c (\gamma^5\gamma^\mu)_{c\alpha} \\ &\quad + \frac{i}{2} [\bar{\varepsilon}_3^a, \varepsilon_2^a] \bar{\varepsilon}_1^b \gamma_{b\alpha}^\mu - \frac{i}{2} [\bar{\varepsilon}_3^a (\gamma^5)_{ab}, \varepsilon_2^b] \bar{\varepsilon}_1^c (\gamma^5\gamma^\mu)_{c\alpha}\end{aligned}$$

Analogous relations for B and Ψ^α with the same parameter $\bar{\zeta}$ are derived. One can see that the algebra (42 – 44) is nothing other than the Hamiltonian rewriting of (53). Their distinctive feature is the occurrence of a trilinear product rule for the fermionic elements which is the translating of a $p = 2$ parasuperalgebra.

5. Generators of the Parasupersymmetric Transformations

Now that the required transformations are derived, one is tempted to seek whether the spinor parasupercharge Q_a defined by (48) gives the correct parasupersymmetric transformations (19 – 22) of the fields $A(x)$, $B(x)$ and $\Psi^\alpha(x)$. Indeed, it is easy to check that Q_a generates (19 – 22) through the relation :

$$\delta\phi = i [[\bar{\varepsilon}, Q], \phi]$$

where ϕ stands for the fields $A(x)$, $B(x)$ or the parafield $\Psi^\alpha(x)$.

a- Calculation of $-i [[\bar{\varepsilon}_a, Q_a], A]$

By inserting into the trilinear commutator the Fourier expansions obtained earlier for the relevant quantities, one can write :

$$\begin{aligned} & -i [[\bar{\varepsilon}_a, Q_a], A] \\ &= -i [[\bar{\varepsilon}_a, \frac{i}{2} \sum_s \int d^3p \{ C_{ab}(\vec{p}) d^+(\vec{p}, s) v_b(\vec{p}, s) \} \\ & \quad - \{ D_{ab}(\vec{p}) d(\vec{p}, s) u_b(\vec{p}, s) \}]] \\ & \quad , \frac{1}{(2\pi)^{3/2}} \int \frac{1}{(2\omega_k)^{1/2}} d^3k \{ a(\vec{k}) e^{ikx} + a^+(\vec{k}) e^{-ikx} \}] \end{aligned}$$

which is simplified as follows

$$\begin{aligned} & -i [[\bar{\varepsilon}_a, Q_a], A] \\ &= \frac{1}{2} \sum_s \int d^3p \frac{1}{(2\pi)^{3/2}} \int \left(\frac{1}{2\omega_k} \right)^{1/2} d^3k \\ & \quad ([\bar{\varepsilon}_a, d^+(\vec{p}, s) v_b(\vec{p}, s)] [C_{ab}(\vec{p}), a^+(\vec{k})] e^{-ikx} \\ & \quad - [\bar{\varepsilon}_a, d(\vec{p}, s) u_b(\vec{p}, s)] [D_{ab}(\vec{p}), a(\vec{k})] e^{ikx}) \end{aligned}$$

by the use of

$$\begin{aligned} [C_{ab}(\vec{p}), a^+(\vec{k})] &= \delta(\vec{p} - \vec{k}) \delta_{ab} \\ [D_{ab}(\vec{p}), a(\vec{k})] &= -\delta(\vec{p} - \vec{k}) \delta_{ab} \end{aligned}$$

and, working out the integral over k one finds

$$-i [[\bar{\varepsilon}_a, Q_a], A] = \frac{1}{2} [\bar{\varepsilon}_a, \Psi_a] = \delta A$$

in the same way, the use of the relations

$$\begin{aligned} [C_{ab}(\vec{p}), b^+(\vec{k})] &= -i\gamma_{ab}^5 \delta(\vec{p} - \vec{k}) \\ [D_{ab}(\vec{p}), b(\vec{k})] &= i\gamma_{ab}^5 \delta(\vec{p} - \vec{k}) \end{aligned}$$

leads to :

$$-i [[\bar{\varepsilon}_a, Q_a], B] = \delta B$$

b-Calculation of $-i [[\bar{\varepsilon}_a, Q_a], \Psi_b]$

One can write:

$$\begin{aligned}
 & -i [[\bar{\varepsilon}_a, Q_a], \Psi_b] = \\
 & \frac{1}{2} \frac{1}{(2\pi)^{3/2}} \sum_{s,l} \int d^3p \int \left(\frac{m}{2\omega_k} \right)^{1/2} d^3k \\
 & \{ [[\bar{\varepsilon}_a, d^+(\vec{p}, s)], d(\vec{k}, l)] C_{ac}(\vec{p}) v_c(\vec{p}, s) \\
 & u_b(\vec{k}, l) e^{-ikx} + [[\bar{\varepsilon}_a, d^+(\vec{p}, s)], d^+(\vec{k}, l)] C_{ac}(\vec{p}) \\
 & v_c(\vec{p}, s) v_b(\vec{k}, l) e^{ikx} - [[\bar{\varepsilon}_a, d(\vec{p}, s)], d(\vec{k}, l)] \\
 & D_{ac}(\vec{p}) u_c(\vec{p}, s) u_b(\vec{k}, l) e^{-ikx} \\
 & - [[\bar{\varepsilon}_a, d(\vec{p}, s)], d^+(\vec{k}, l)] D_{ac}(\vec{p}) \\
 & u_c(\vec{p}, s) v_b(\vec{k}, l) e^{ikx} \}
 \end{aligned}$$

Using (27) and (28), one finds

$$\begin{aligned}
 -i [[\bar{\varepsilon}_a, Q_a], \Psi_b] &= \frac{1}{2} \frac{1}{(2\pi)^{3/2}} \sum_{s,l} \int d^3p \int d^3k \left(\frac{m}{2\omega_k} \right)^{1/2} \\
 & \{ 2\delta_{ls} \delta(\vec{p} - \vec{k}) \bar{\varepsilon}_a C_{ac}(\vec{p}) v_c(\vec{p}, s) \\
 & u_b(\vec{k}, l) e^{-ikx} - 2\delta_{ls} \delta(\vec{p} - \vec{k}) \bar{\varepsilon}_a \\
 & D_{ac}(\vec{p}) u_c(\vec{p}, s) v_b(\vec{k}, l) e^{ikx} \}
 \end{aligned}$$

the relations

$$\begin{aligned}
 \sum_s u_c(\vec{p}, s) v_b(\vec{p}, s) &= \left(\frac{\not{p} + m}{2m} c \right)_{cb} \\
 \sum_s v_c(\vec{p}, s) u_b(\vec{p}, s) &= \left(\frac{\not{p} - m}{2m} c \right)_{cb}
 \end{aligned}$$

lead to

$$\begin{aligned}
 -i [[\bar{\varepsilon}_a, Q_a], \Psi_b] &= \frac{1}{(2\pi)^{3/2}} \sum_s \int d^3p \{ \bar{\varepsilon}_a C_{ac}(\vec{p}) \left(\frac{\not{p} + m}{2m} c \right)_{cb} \\
 & e^{-ipx} - \bar{\varepsilon}_a D_{ac}(\vec{p}) \left(\frac{\not{p} - m}{2m} c \right)_{cb} e^{ipx} \}
 \end{aligned}$$

which can be rewritten as

$$\begin{aligned}
 -i [[\bar{\varepsilon}_a, Q_a], \Psi_b] &= -i \gamma_{ba}^\mu \varepsilon_a \partial_\mu A + (\gamma^\mu \gamma^5)_{ba} \varepsilon_a \partial_\mu B \\
 &= \delta \Psi_b
 \end{aligned}$$

Conclusion

In this work, we have investigated the most simple paraextension of the Wess-Zumino model outside the parasuperspace formalism. This model forms a field theoretical realization of the parasuperPoincaré algebra, where, as a bosons-parafermions system, the

parasupercharges of this model satisfy the trilinear commutations relations dictated by these types of systems.

The distinctive feature of the closure is the occurrence of a trilinear product rule for the fermionic elements which is the translating of the $p = 2$ parasuperalgebra. Unlike the ordinary bilinear case, this result imply that the three times repeated applications of the transformations on a field gives rise to a translation of the transformed field. This closure is reinforced by the verification that these parasupercharges are the effective generators of these transformations.

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