**Editorial Board**

<table>
<thead>
<tr>
<th>Co-Editor</th>
<th>Wai-ning Mei</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Ignazio Licata.</strong></td>
<td><strong>Condensed matter Theory</strong></td>
</tr>
<tr>
<td>Foundations of Quantum Mechanics</td>
<td>Physics Department</td>
</tr>
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<td>Complex System &amp; Computation in Physics and Biology</td>
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<tr>
<td>Sicily – Italy</td>
<td>Omaha, Nebraska, USA</td>
</tr>
<tr>
<td><a href="mailto:editor@ejtp.info">editor@ejtp.info</a></td>
<td>e-mail: <a href="mailto:wmei@mail.unomaha.edu">wmei@mail.unomaha.edu</a></td>
</tr>
<tr>
<td><a href="mailto:ignazio.licata@ejtp.info">ignazio.licata@ejtp.info</a></td>
<td><a href="mailto:physmei@unomaha.edu">physmei@unomaha.edu</a></td>
</tr>
<tr>
<td><a href="mailto:ignazio.licata@ixtucyber.org">ignazio.licata@ixtucyber.org</a></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Richard Hammond</th>
<th>F.K. Diakonos</th>
</tr>
</thead>
<tbody>
<tr>
<td>General Relativity</td>
<td>Statistical Physics</td>
</tr>
<tr>
<td>High energy laser interactions with charged particles</td>
<td>Physics Department, University of Athens</td>
</tr>
<tr>
<td>Classical equation of motion with radiation reaction</td>
<td>Panepistimiopolis GR 5874 Zographos, Athens, Greece</td>
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<tr>
<td>Electromagnetic radiation reaction forces</td>
<td>e-mail: <a href="mailto:fdiakono@cc.uoa.gr">fdiakono@cc.uoa.gr</a></td>
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<tr>
<td>Department of Physics</td>
<td></td>
</tr>
<tr>
<td>University of North Carolina at Chapel Hill</td>
<td></td>
</tr>
<tr>
<td>e-mail: <a href="mailto:rhammond@email.unc.edu">rhammond@email.unc.edu</a></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Tepper L. Gill</th>
<th>J. A. Maki</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematical Physics, Quantum Field Theory</td>
<td>Applied Mathematics</td>
</tr>
<tr>
<td>Department of Electrical and Computer Engineering</td>
<td>School of Mathematics</td>
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<tr>
<td>Howard University, Washington, DC, USA</td>
<td>University of East Anglia Norwich NR4 7TJ UK</td>
</tr>
<tr>
<td>e-mail: <a href="mailto:tgill@Howard.edu">tgill@Howard.edu</a></td>
<td>e-mail: <a href="mailto:jam@cmp.uea.ac.uk">jam@cmp.uea.ac.uk</a></td>
</tr>
<tr>
<td><a href="mailto:tgill@ejtp.info">tgill@ejtp.info</a></td>
<td><a href="mailto:maki@ejtp.info">maki@ejtp.info</a></td>
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<th>Nicola Yordanov</th>
<th>S.I. Themelis</th>
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</thead>
<tbody>
<tr>
<td>Physical Chemistry</td>
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</tr>
<tr>
<td>Bulgarian Academy of Sciences, BG-1113 Sofia, Bulgaria</td>
<td>Foundation for Research and Technology - Hellas</td>
</tr>
<tr>
<td>Telephone: (+359 2) 724917 , (+359 2) 9792546</td>
<td>P.O. Box 1527, GR-711 10 Heraklion, Greece</td>
</tr>
<tr>
<td>e-mail: <a href="mailto:ndyep@ic.bas.bg">ndyep@ic.bas.bg</a></td>
<td>e-mail: <a href="mailto:stheme@iesl.forth.gr">stheme@iesl.forth.gr</a></td>
</tr>
<tr>
<td>ndyep[AT]bas.bg</td>
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<tr>
<th>T. A. Hawary</th>
<th>Arbab Ibrahim</th>
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</thead>
<tbody>
<tr>
<td>Mathematics</td>
<td>Theoretical Astrophysics and Cosmology</td>
</tr>
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<td>Department of Physics,</td>
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<tr>
<td>Mu'tah University</td>
<td>Faculty of Science</td>
</tr>
<tr>
<td>P.O.Box 6</td>
<td>University of Khartoum,</td>
</tr>
<tr>
<td>Karak- Jordan</td>
<td>P.O. Box 321, Khartoum 11115, Sudan</td>
</tr>
<tr>
<td>e-mail: <a href="mailto:drtalal@yahoo.com">drtalal@yahoo.com</a></td>
<td>e-mail: <a href="mailto:aiarbab@uofk.edu">aiarbab@uofk.edu</a></td>
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<td>e-mail: <a href="mailto:arbab_ibrahim@ejtp.info">arbab_ibrahim@ejtp.info</a></td>
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<td>Sergey Danilkin</td>
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<tr>
<td>Robert V. Gentry</td>
<td>The Orion Foundation</td>
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<td>Nonlinear phenomena, chaos and solitons in classic and quantum physics Technical Institute &quot;G. Cardano&quot; Via Alfredo Casella 3 00013 Mentana RM - ITALY e-mail: <a href="mailto:solitone@yahoo.it">solitone@yahoo.it</a></td>
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<td>Advanced Materials Division</td>
</tr>
<tr>
<td>A. Abdelkader</td>
<td>Experimental Physics</td>
</tr>
<tr>
<td>Leonardo Chiatti</td>
<td>Medical Physics Laboratory ASL VT</td>
</tr>
<tr>
<td>Zdenek Stuchlik</td>
<td>Relativistic Astrophysics</td>
</tr>
</tbody>
</table>
# Table of Contents

<table>
<thead>
<tr>
<th>No</th>
<th>Articles</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><em>A Review of Leading Quantum Gravitational Corrections to Newtonian Gravity</em></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Arif Akhundov and Anwar Shiekh</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td><em>Radiation Reaction at Extreme Intensity</em></td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>Richard T. Hammond</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td><em>Super-Light Electromagnetic Wave With Longitudinal And Transversal Modes</em></td>
<td>27</td>
</tr>
<tr>
<td></td>
<td>M.M. Kononenko</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td><em>Non Commutative Geometry Constraints and the Standard Renormalization Group Approach: Two Doublets Higgs Model as An Example.</em></td>
<td>41</td>
</tr>
<tr>
<td></td>
<td>N.Mebarki and M.Harrat</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td><em>Hamilton-Jacobi Formulation of a Non-Abelian Yang-Mills Theories</em></td>
<td>65</td>
</tr>
<tr>
<td></td>
<td>W. I. Eshraim and N. I. Farahat</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td><em>Physical Form of The Clustering Parameter And Gravitational Galaxy Clustering</em></td>
<td>73</td>
</tr>
<tr>
<td></td>
<td>Sajad Masood, Manzoor A Malik, Shakeel Ahmad And N. A. Rather</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td><em>Penrose Model Potential, Compared With Coleman-Weinberg Potential for Early Universe Scalar Evolution</em></td>
<td>91</td>
</tr>
<tr>
<td></td>
<td>A.W. Beckwith</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td><em>Increasing Effective Gravitational Constant In Fractional Add Brane Cosmology</em></td>
<td>103</td>
</tr>
<tr>
<td></td>
<td>El-Nabulsi Ahmad Rami</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td><em>A Two-Dimensional Discrete Mapping with $C^\infty$ Multifold Chaotic Attractors</em></td>
<td>107</td>
</tr>
<tr>
<td></td>
<td>Zeraouilia Elhadj and J. C. Sprott</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td><em>Bosons-Parafermions Wess-Zumino Model</em></td>
<td>121</td>
</tr>
<tr>
<td></td>
<td>L. Maghlaoui and N. Belaloui</td>
<td></td>
</tr>
</tbody>
</table>
11 Geometrodynamics of Information on Curved Statistical Manifolds and Its Applications to Chaos
C. Cafaro and S. A. Ali

12 Stochastic Measures and Modular Evolution in Non-Equilibrium Thermodynamics
Enrique Hernandez-Lemus, and Jesus K. Estrada-Gil

13 Beltrami Flow of an Unsteady Dusty Fluid between Parallel Plates in Anholonomic Co-Ordinate System
B.J.Gireesha, C.S.Bagewadi and C.S.Vishalakshi

14 Exact Solution of The Non-Central Modified Kratzer Potential Plus a Ring-Shaped Like Potential By The Factorization Method
J. Sadeghi and B. Pourhassan

15 Discrete Self-Similarity between Rr Lyrae Stars And Singly-Excited Helium Atoms
Robert L. Oldershaw

16 Brownian Dynamics of Nanoparticles Moving Near a Fluctuating Membrane
A. Bendouch, M. Benhamou, and H. Kaidi

17 Influence of Third Order Perturbation on Heisenberg Hamiltonian Of Thick Ferromagnetic Films
P. Samarasekara

18 Viscous Dusty Fluid Flow with Constant Velocity Magnitude
Siddabasappa, Venkateshappa, Rudraswamy, Gopinath

19 The Influence of Long-Range Interaction on Critical Behavior of Some Alloys
S. V. Belim

135

159

181

193

203

211

227

237

253
A Review of Leading Quantum Gravitational Corrections to Newtonian Gravity

Arif Akhundov\textsuperscript{a,b} and Anwar Shiekh\textsuperscript{c}

\textsuperscript{a} Departamento de Física Teórica and IFIC, Universidad de Valencia-CSIC, E-46100 Burjassot (Valencia), Spain
\textsuperscript{b} Institute of Physics, Azerbaijan Academy of Sciences, H. Cavid ave. 33, 370143 Baku, Azerbaijan
\textsuperscript{c} Mathematics, Science and Technology division, Diné College, Tsaile, AZ 86556, USA

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Abstract: In this review we present the theoretical background for treating General Relativity as an effective field theory and focus on the concrete results of such a treatment. As a result we present the calculations of the low-energy leading gravitational corrections to the Newtonian potential between two sources.

\begin{equation}
V(r) = -\frac{G m_1 m_2}{r} \tag{1}
\end{equation}

In contrast the theory of General Relativity [1] provides a framework for extending Newton’s theory to objects with relativistic velocities. In general relativity one solves the basic field equation:

\begin{equation}
R_{\mu\nu}(g_{\mu\nu}) - \frac{1}{2} R(g_{\mu\nu}) g_{\mu\nu} = 16\pi G T_{\mu\nu} - \Lambda g_{\mu\nu} \tag{2}
\end{equation}
where $g_{\mu\nu}$ is the gravitational metric, $R^\alpha_{\mu\nu}$ is the tensor for the curvature of space-time\(^1\) and $T_{\mu\nu}$ is the total energy-momentum tensor. The cosmological constant $\Lambda$ may be needed on cosmological scales, and is today believed to have a non-zero expectation value in the Universe. When we solve the Einstein equation we find the metric which is a local object that depends on the geometry of space-time. In this way a solution of the gravitational problem is found. Einstein’s description holds in the fully relativistic regime, and its low-energy and non-relativistic predictions match the expectations of Newtonian mechanics.

A longstanding puzzle in Modern Physics is how to wed General Relativity with the quantum theory. It is not at all obvious how this might be achieved since General Relativity and quantum mechanics seem to be based on completely different perceptions of physics – nevertheless this question is one of the most pressing questions of modern theoretical physics and has been the subject of many studies, e.g., see refs. [2, 3, 4, 5, 6, 7, 8, 9, 10, 11].

All sorts of interpretational complications arise when trying to quantize General Relativity. A possible starting point for such a theory appears to be to interpret General Relativity as a quantum field theory, to let the metric be the basic gravitational field, and to quantize the Einstein-Hilbert action:

$$S_{EH} = \int d^4x \sqrt{-g} \frac{R}{16\pi G}$$

where $g = \det(g_{\mu\nu})$ and $R$ is the scalar curvature. However the above action is not self contained under renormalization since loop diagrams will generate new terms not present in the original action refs. [10, 11, 25, 26]. This is the renowned renormalization problem that hinders the quantization of general relativity.

One of the physically interesting problem is the calculation of the leading order quantum corrections to the Newtonian potential which has been in the focus of many studies in different schemes, using Feynman diagrams for the loops in the graviton propagator [12, 13, 14, 15, 16, 17, 10], renormalizable $R^2$ gravity [18, 19, 20] and Semiclassical Gravity [21, 22, 23, 24].

After introducing an effective field theory for processes with a typical energy less the Planck mass, i.e. with $|q^2| \ll M_P^2 \approx 10^{38}$ GeV\(^2\), by Weinberg [27], the effective theory for gravity can been modeled in a manner analogous with that of Chiral Perturbation Theory [28] for QCD. This way to look at General Relativity was proposed by Shiekh [29] and Donoghue [30], and they have shown that reliable quantum predictions at the low energies can be made.

In spite of fact that unmodified General Relativity is not renormalizable, be it pure General Relativity or General Relativity coupled to bosonic or fermionic matter, see e.g. [10, 11, 31, 32], using the framework of an effective field theory, these theories do become order by order renormalizable in the low energy limit. When General Relativity is treated as an effective theory, renormalizability simply fails to be an issue. The ultraviolet

\(^{1}\left(R_{\mu\nu} = R^\beta_{\mu\nu\beta}\right), \left(R \equiv g^{\alpha\nu}R_{\mu\nu}\right)\)
divergences arising e.g. at the 1-loop level are dealt with by renormalizing the parameters of higher derivative terms in the action.

When approaching general relativity in this manner, it is convenient to use the background field method [2, 33]. Divergent terms are absorbed away into phenomenological constants which characterize the effective action of the theory. The price paid is the introduction of a set of never-ending higher order derivative couplings into the theory, unless using the approach of Shiekh [29]. The effective action contains all terms consistent with the underlying symmetries of the theory. Perturbatively only a finite number of terms in the action are required for each loop order.

In pioneering papers [30] Donoghue first has shown how to derive the leading quantum and classical relativistic corrections to the Newtonian potential of two masses. This calculation has since been the focus of a number of publications [34, 35, 36, 37, 38, 39], and this work continues, most recently in the paper [40].

Unfortunately, due to difficulty of the calculation and its myriad of tensor indices there has been some disagreement among the results of various authors. The classical component of the corrections were found long ago by Einstein, Infeld and Hoffmann [41], and by Eddington and Clark [42]. Later this result was reproduced by Iwasaki [9] by means of Feynman diagrams and has been discussed in the papers [43, 44, 45], and here there is general agreement although there exists an unavoidable ambiguity in defining the potential.

An interesting calculation has been made involving quantum gravitational corrections to the Schwarzschild and Kerr metrics of scalars and fermions [46, 47] where it is shown in detail how the higher order gravitational contributions to these metrics emerge from loop calculations. In the papers [48] and [49] have been calculated the leading post-Newtonian and quantum corrections to the non-relativistic scattering amplitude of charged scalars and spin-$\frac{1}{2}$ fermions in the combined theory of general relativity and QED. For the recent reviews of general relativity as an effective field theory, see refs. [50, 51].

Our notations and conventions on the metric tensor, the gauge-fixed gravitational action, etc. are the same as in [36], namely ($\hbar = c = 1$) as well as the Minkowski metric convention (+1, −1, −1, −1).

2. The Quantization of General Relativity

The Einstein action for General Relativity has the form:

$$S = \int d^4x \sqrt{-g} \left[ \frac{2R}{\kappa^2} + \mathcal{L}_{\text{matter}} \right]$$

where $\kappa^2 = 32\pi G$ is defined as the gravitational coupling, and the curvature tensor is defined as:

$$R^\mu_{\nu\alpha\beta} \equiv \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\sigma\alpha} \Gamma^\sigma_{\nu\beta} - \Gamma^\mu_{\sigma\beta} \Gamma^\sigma_{\nu\alpha}$$

and

$$\Gamma^\lambda_{\alpha\beta} = \frac{1}{2} g^{\lambda\sigma} (\partial_\alpha g_{\beta\sigma} + \partial_\beta g_{\alpha\sigma} - \partial_\sigma g_{\alpha\beta})$$
The term $\sqrt{-g} \mathcal{L}_{\text{matter}}$ is a covariant expression for the inclusion of matter into the theory. We can include any type of matter. As a classical theory the above Lagrangian defines the theory of general relativity.

Massive spinless matter fields interact with the gravitational field as described by the action

$$S_{\text{matter}} = \int d^4x \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right]$$

(7)

Any effective field theory can be seen as an expansion in energies of the light fields of the theory below a certain scale. Above the scale transition energy there will be additional heavy fields that will manifest themselves. Below the transition the heavy degrees of freedom will be integrated out and will hence not contribute to the physics. Any effective field theory is built up from terms with higher and higher numbers of derivative couplings on the light fields and obeying the gauge symmetries of the basic theory. This gives us a precise description of how to construct effective Lagrangians from the gauge invariants of the theory. We expand the effective Lagrangian in the invariants ordered in magnitude of their derivative contributions.

An effective treatment of pure General Relativity results in the following Lagrangian:

$$\mathcal{L}_{\text{grav}} = \sqrt{-g} \left[ \frac{2R}{\kappa^2} + c_1 R^2 + c_2 R^{\mu\nu} R_{\mu\nu} + \ldots \right]$$

(8)

where the ellipses denote that the effective action is in fact an infinite series—at each new loop order additional higher derivative terms must be taken into account. This Lagrangian includes all possible higher derivative couplings, and every coupling constant in the Lagrangian is considered to be determined empirically unless set to zero to achieve causality [29]. Similarly one must include higher derivative contributions to the matter Lagrangian in order to treat this piece of the Lagrangian as an effective field theory [30].

Computing the leading low-energy quantum corrections of an effective field theory, a useful distinction is between non-analytical and analytical contributions from the diagrams. Non-analytical contributions are generated by the propagation of two or more massless particles in the Feynman diagrams. Such non-analytical effects are long-ranged and, in the low energy limit of the effective field theory, they dominate over the analytical contributions which arise from the propagation of massive particles. The difference between massive and massless particle modes originates from the impossibility of expanding a massless propagator $\sim \frac{1}{q^2}$ while:

$$\frac{1}{q^2 - m^2} = -\frac{1}{m^2} \left(1 + \frac{q^2}{m^2} + \ldots \right)$$

(9)

No $1/q^2$ terms are generated in the above expansion of the massive propagator, thus such terms all arise from the propagation of massless modes. The analytical contributions from the diagrams are local effects and thus expandable in power series.

Non-analytical effects are typically originating from terms which in the S-matrix go as, e.g., $\sim \ln(-q^2)$ or $\sim 1/\sqrt{-q^2}$, while the generic example of an analytical contribution
is a power series in momentum $q$. Our interest is only in the non-local effects, thus we will only consider the non-analytical contributions of the diagrams.

The procedure of the background field quantization is as follows. The quantum fluctuations of the gravitational field are expanded about a smooth background metric $\bar{g}_{\mu\nu}$ [10, 11], i.e. flat space-time $\bar{g}_{\mu\nu} \equiv \eta_{\mu\nu} = diag(1, -1, -1, -1)$, and the metric $g_{\mu\nu}$ is the sum of this background part and a quantum contribution $\kappa h_{\mu\nu}$:

$$g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \kappa h_{\mu\nu} \quad (10)$$

From this equation we get the expansions for the upper metric field $g^{\mu\nu}$, and for $\sqrt{-g}$:

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \kappa h^{\mu\nu} + \ldots$$

$$\sqrt{-g} = \sqrt{-\bar{g}} \left[ 1 + \frac{1}{2} \kappa h + \ldots \right] \quad (11)$$

where $h^{\mu\nu} \equiv \bar{g}^{\alpha\beta} \bar{g}^{\mu\nu} h_{\alpha\beta}$ and $h \equiv \bar{g}^{\mu\nu} h_{\mu\nu}$.

The corresponding curvatures are given by

$$\bar{R}_{\mu\nu} = \frac{\kappa}{2} \left[ \partial_{\mu} \partial_{\nu} h + \partial_{\lambda} \partial^{\lambda} h_{\mu\nu} - \partial_{\mu} \partial_{\lambda} h^{\lambda}_{\nu} - \partial_{\nu} \partial_{\lambda} h^{\lambda}_{\mu} \right]$$

$$\bar{R} = \bar{g}^{\mu\nu} \bar{R}_{\mu\nu} = \kappa \left[ \Box h - \partial_{\mu} \partial_{\nu} h^{\mu\nu} \right] \quad (12)$$

In order to quantize the field $h_{\mu\nu}$ one needs to fix the gauge. In the harmonic (or deDonder) gauge $[10] - g^{\mu\nu} \Gamma^\lambda_{\mu\nu} = 0$—which requires, to first order in the field expansion,

$$\partial^\beta h_{\alpha\beta} - \frac{1}{2} \partial_{\nu} h = 0 \quad (13)$$

In the quantization, the Lagrangians are expanded in the gravitational fields, separated in quantum and background parts, and the vertex factors as well as the propagator are derived from the expanded action.

The expansion of the Einstein action takes the form $[10, 11]$:

$$S_{\text{grav}} = \int d^4x \sqrt{-g} \left[ \frac{2\bar{R}}{\kappa^2} + \mathcal{L}^{(1)}_g + \mathcal{L}^{(2)}_g + \ldots \right] \quad (14)$$

where the subscripts count the number of powers of $\kappa$ and

$$\mathcal{L}^{(1)}_g = \frac{h_{\mu\nu}}{\kappa} \left[ \bar{g}^{\mu\nu} \bar{R} - 2\bar{R}^{\mu\nu} \right]$$

$$\mathcal{L}^{(2)}_g = \frac{1}{2} D_\alpha h_{\mu\nu} D^\alpha h^{\mu\nu} - \frac{1}{2} D_\alpha h D^\alpha h + D_\alpha h D_\beta h^{\alpha\beta} - D_\alpha h_{\mu\beta} D^\beta h^{\mu\nu}$$

$$+ \bar{R} \left( \frac{1}{4} h^2 - \frac{1}{2} h_{\mu\nu} h^{\mu\nu} \right) + (2h^{\mu}_{\mu} h_{\nu\lambda} - hh_{\mu\nu}) \bar{R}^{\mu\nu} \quad (15)$$

where $D_\alpha$ denotes the covariant derivative with respect to the background metric.

A similar expansion of the matter action yields $[30]$:
\( S_{\text{matter}} = \int d^4x \sqrt{-g} \left[ \mathcal{L}^{(0)}_m + \mathcal{L}^{(1)}_m + \mathcal{L}^{(2)}_m + \ldots \right] \) \hspace{1cm} (16)

with

\[
\mathcal{L}^{(0)}_m = \frac{1}{2} \left( \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right) \\
\mathcal{L}^{(1)}_m = -\frac{\kappa}{2} h_{\mu\nu} T^{\mu\nu} \\
T_{\mu\nu} \equiv \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \bar{g}_{\mu\nu} \left( \partial_\lambda \phi \partial^\lambda \phi - m^2 \phi^2 \right) \\
\mathcal{L}^{(2)}_m = \kappa^2 \left( \frac{1}{2} h^{\mu\nu} h_\nu^\lambda - \frac{1}{4} h h^{\mu\nu} \right) \partial_\mu \phi \partial_\nu \phi \\
- \frac{\kappa^2}{8} \left( h^{\lambda\sigma} h_{\lambda\sigma} - \frac{1}{2} h h \right) \left[ \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right] 
\] \hspace{1cm} (17)

The background metric \( \bar{R}^{\mu\nu} \) should satisfy Einstein’s equation

\[
\bar{R}^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \bar{R} = \frac{\kappa^2}{4} T^{\mu\nu} 
\] \hspace{1cm} (18)

and the linear terms in \( h_{\mu\nu}, \mathcal{L}^{(1)}_g + \mathcal{L}^{(1)}_m \), is vanishing.

For the calculation of the quantum gravitational corrections at one loop, we need to consider the following actions:

\[
S_0 = \int d^4x \sqrt{-g} \left[ \frac{2 \bar{R}}{\kappa^2} + \mathcal{L}^{(0)}_m \right] \\
S_2 = \int d^4x \sqrt{-g} \left[ \mathcal{L}^{(2)}_g + \mathcal{L}^{(2)}_m + \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{ghost}} \right] 
\] \hspace{1cm} (19)

with the gauge fixing Lagrangian [10]

\[
\mathcal{L}_{\text{gauge}} = \left( D^\nu h_{\mu\nu} - \frac{1}{2} D_\mu h \right) \left( D_\lambda h^{\mu\lambda} - \frac{1}{2} D^\mu h \right) 
\] \hspace{1cm} (20)

and the ghost Lagrangian

\[
\mathcal{L}_{\text{ghost}} = \eta^{*\mu} \left( D_\lambda D^\lambda \eta_\mu - \bar{R}_{\mu\nu} \eta^\nu \right) 
\] \hspace{1cm} (21)

for the Faddeev-Popov field \( \eta_\mu \).

3. The Feynman Rules

From the Lagrangians (19)-(21) we can derive the list of Feynman rules [38].

- **Scalar propagator**

  The massive scalar propagator is:
Graviton propagator

The graviton propagator in harmonic gauge is:

\[
q = i q^2 - m^2 + i \epsilon
\]

where

\[
\mathcal{P}^{\alpha\beta\gamma\delta} = \frac{1}{2} \left[ \eta^{\alpha\gamma} \eta^{\beta\delta} + \eta^{\beta\gamma} \eta^{\alpha\delta} - \eta^{\alpha\beta} \eta^{\gamma\delta} \right]
\]

2-scalar-1-graviton vertex

The two scalar - one graviton vertex is:

\[
\mu \nu \rho \sigma
\]

where

\[
\tau^{\mu \nu}(p, p', m) = -\frac{i \kappa}{2} \left[ p^\mu p'^{\nu} + p^\nu p'^{\mu} - \eta^{\mu \nu} \left( (p \cdot p') - m^2 \right) \right]
\]

2-scalar-2-graviton vertex

The two scalar - two graviton vertex is

\[
\mu \nu \rho \sigma
\]

where

\[
\tau^{\eta \lambda \rho \sigma}(p, p') = i \kappa^2 \left[ \left\{ I^{\eta \lambda \rho \sigma} I^{\rho \sigma \beta} - \frac{1}{4} \left\{ \eta^{\eta \lambda} I^{\rho \sigma \alpha \beta} + \eta^{\rho \sigma} I^{\eta \lambda \alpha \beta} \right\} \right\} (p_\alpha p'_\beta + p'_\alpha p_\beta)
\]

\[
- \frac{1}{2} \left\{ I^{\eta \lambda \rho \sigma} - \frac{1}{2} \eta^{\eta \lambda} \eta^{\rho \sigma} \right\} \left[ (p \cdot p') - m^2 \right]
\]

with

\[
I_{\alpha \beta \gamma \delta} = \frac{1}{2} (\eta_{\alpha \gamma} \eta_{\beta \delta} + \eta_{\alpha \delta} \eta_{\beta \gamma})
\]

3-graviton vertex

The three graviton vertex is:

\[
\alpha \beta \eta \rho \sigma \mu \nu
\]
where

\[ \tau_{\alpha\beta\gamma\delta}^{\mu\nu}(k, q) = \frac{-ik}{2} \times \left[ P_{\alpha\beta\gamma\delta} \left[ k^\mu k^\nu + (k - q)^\mu (k - q)^\nu + q^\mu q^\nu - \frac{3}{2} \eta^{\mu\nu} q^2 \right] \right. \]

\[ + 2q\eta^{\sigma\lambda} \left( I_{\alpha\beta}^{\sigma\lambda} I_{\gamma\delta}^{\mu\nu} + I_{\gamma\delta}^{\sigma\lambda} I_{\alpha\beta}^{\mu\nu} - I_{\alpha\beta}^{\mu\sigma} I_{\gamma\delta}^{\nu\lambda} - I_{\gamma\delta}^{\mu\sigma} I_{\alpha\beta}^{\nu\lambda} \right) \]

\[ + \left[ q\eta^{\mu\nu} \left( \eta_{\alpha\beta} I_{\gamma\delta}^{\sigma\lambda} + \eta_{\gamma\delta} I_{\alpha\beta}^{\sigma\lambda} \right) + q\lambda q^{\sigma} \left( \eta_{\alpha\beta} I_{\gamma\delta}^{\mu\lambda} + \eta_{\gamma\delta} I_{\alpha\beta}^{\mu\lambda} \right) \right. \]

\[ - q^2 \left( \eta_{\alpha\beta} I_{\gamma\delta}^{\mu\nu} - \eta_{\gamma\delta} I_{\alpha\beta}^{\mu\nu} \right) - \eta^{\mu\nu} q\sigma q\lambda \left( \eta_{\alpha\beta} I_{\gamma\delta}^{\sigma\lambda} + \eta_{\gamma\delta} I_{\alpha\beta}^{\sigma\lambda} \right) \]

\[ + \left[ 2q\eta^{\sigma\lambda} I_{\gamma\delta}^{\mu\nu}(k - q)^\mu + I_{\alpha\beta}^{\lambda\sigma} I_{\gamma\delta}^{\mu\nu}(k - q)^\nu - I_{\gamma\delta}^{\lambda\sigma} I_{\alpha\beta}^{\mu\nu} k^\mu - I_{\gamma\delta}^{\lambda\sigma} I_{\alpha\beta}^{\mu\nu} k^\nu \right) \]

\[ + q^2 \left( I_{\alpha\beta}^{\mu\sigma} I_{\gamma\delta}^{\nu\varphi} + I_{\alpha\beta}^{\varphi\sigma} I_{\gamma\delta}^{\nu\mu} + \eta^{\mu\nu} q\sigma q\lambda \left( I_{\alpha\beta}^{\lambda\varphi} I_{\gamma\delta}^{\mu\sigma} + I_{\gamma\delta}^{\lambda\varphi} I_{\alpha\beta}^{\mu\sigma} \right) \right) \]

\[ \left. + \left\{ (k^2 + (k - q)^2) \left[ I_{\alpha\beta}^{\mu\sigma} I_{\gamma\delta}^{\nu\varphi} + I_{\gamma\delta}^{\mu\sigma} I_{\alpha\beta}^{\nu\varphi} - \frac{1}{2} \eta^{\mu\nu} P_{\alpha\beta\gamma\delta} \right] \right\} \right] \]

\[ + \left( I_{\gamma\delta}^{\mu\nu} \eta_{\alpha\beta} k^2 + I_{\gamma\delta}^{\nu\varphi} \eta_{\alpha\beta}(k - q)^2 \right) \]  

(23)

4. Scattering Amplitude and Potential

The general form for any diagram contributing to the scattering amplitude of gravitational interactions of two masses is:

\[ \mathcal{M} \sim \left( A + B q^2 + \ldots + C_0 \kappa^4 \frac{1}{q^2} + C_1 \kappa^4 \ln(-q^2) + C_2 \kappa^4 \frac{m}{\sqrt{-q^2}} + \ldots \right) \]  

(24)

where \( A, B, \ldots \) correspond to the local analytical interactions which are of no interest to us (these terms will only dominate in the high energy regime of the effective theory) and \( C_0, C_1, C_2, \ldots \) correspond to the non-local, non-analytical interactions.

The \( C_1 \) and \( C_2 \) terms will yield the leading quantum gravitational and relativistic post-Newtonian corrections to the Newtonian potential. The space parts of the non-analytical terms Fourier transform as:

\[ \int \frac{d^3q}{(2\pi)^3} e^{iqr} \frac{1}{|q|^2} = \frac{1}{4\pi r} \]

\[ \int \frac{d^3q}{(2\pi)^3} e^{iqr} \frac{1}{|q|} = \frac{1}{2\pi^2 r^2} \]  

\[ \int \frac{d^3q}{(2\pi)^3} e^{iqr} \ln(q^2) = \frac{-1}{2\pi r^3} \]  

(25)

so clearly these terms will contribute to the corrections.

The importance of these transforms, is that they are from non-analytic terms in momentum space and so cannot be renormalized into the original Lagrangian, and as
such one might anticipate that they are of finite magnitude. Because of this, the problem of renormalizing quantum gravity is put off.

In the quantization of General Relativity the definition of a potential is certainly not obvious. One can choose between several definitions of the potential depending on, e.g., the physical situation, how to define the energy of the fields, the diagrams included etc. The choice of potential, which includes all 1-loop diagrams [35, 52], is the simplest, gauge invariant definition of the potential.

The calculation of the non-relativistic potential using the full amplitude is as follows. First, to relate the expectation value for the $S$ matrix to the Fourier transform of the potential $\tilde{V}(q)$ in the non-relativistic limit:

$$\langle k_1, k_3 | S | k_2, k_4 \rangle = -i \tilde{V}(q)(2\pi)\delta(E_i - E_f)$$

(26)

where $k_1, k_3$ and $k_2, k_4$ are the incoming and outgoing momentum respectively, $q \equiv k_2 - k_1 = k_3 - k_4$, and $E_i - E_f$ is the energy difference between the incoming and outgoing states. The invariant matrix element $iM$ resulting from the diagrams is:

$$\langle k_1, k_3 | S | k_2, k_4 \rangle = (2\pi)^4\delta(4)(k_2 + k_4 - k_1 - k_3)(iM)$$

(27)

In the non-relativistic limit ($q = (0, q)$) we have:

$$\tilde{V}(q) = -\frac{1}{2m_1 m_2}M$$

(28)

so that

$$V(x) = -\frac{1}{2m_1 m_2} \int \frac{d^3k}{(2\pi)^3} e^{ik\cdot x} M$$

(29)

This is how we define the non-relativistic potential $V(q)$ generated by the considered non-analytic parts. In the above equation $M$ is the non-analytical part of the amplitude of the scattering process in non-relativistic limit to a given loop order [35].

5. The Contributions of Feynman Diagrams

In general, the Feynman rules are large and the tensor algebra immense. Much of the calculational simplicity should be restored by submitting this part of the complexity to the computer. However, the intermediate results can be so extensive that even a supercomputer can choke without help. For example, imagine one had the contraction of three tensors: $\alpha^{\mu\nu}\beta^\rho\gamma_{\mu\nu\rho\sigma}$ each of which consists of many terms. Then the computer, in trying to contract out the indices, tends to expand out the entire expression, which can easily lead to thousands of terms in the intermediate expression, and so overpower the computers memory. The resolution lies in asking the computer to initially expand out only $\alpha$ for example: $(\alpha_1^{\mu\nu} + \alpha_2^{\mu\nu} + \ldots)\beta^\rho\gamma_{\mu\nu\rho\sigma}$. In this way the computer is presented with several terms that can each be contracted separately. This seemingly innocuous move can make all the difference between the machine being able to perform the calculation or not. It is fine details like this that in practice can occupy much of the investigators time.
The best way to perform such kind of tensor algebra on a computer is use the Ricci package [53] under the Mathematica [54] program.

5.1 Vacuum Polarization

’t Hooft and Veltman [10] were the first to calculate the vacuum polarization diagram in gravity. For the contribution of the graviton plus ghost vacuum polarization Feynman diagrams:

\[
\Pi_{\alpha\beta\gamma\delta} = -\frac{\kappa^2}{16\pi^2} L \left[ \frac{21}{120} q^4 I_{\alpha\beta\gamma\delta} + \frac{23}{120} q^4 \eta_{\alpha\beta} \eta_{\gamma\delta} - \frac{23}{120} q^2 (\eta_{\alpha\beta} q_{\gamma} q_{\delta} + \eta_{\gamma\delta} q_{\alpha} q_{\beta}) \right. \\
\left. - \frac{21}{240} q^2 (q_{\alpha} q_{\delta} \eta_{\beta\gamma} + q_{\beta} q_{\delta} \eta_{\alpha\gamma} + q_{\alpha} q_{\gamma} \eta_{\beta\delta} + q_{\beta} q_{\gamma} \eta_{\alpha\delta}) + \frac{11}{30} q_{\alpha} q_{\delta} q_{\gamma} q_{\delta} \right]
\]

(30)

where \( L \equiv \log(-q^2) = \ln(q^2) \)

The result after contracting the various indices is [30, 36, 38, 37]:

\[
\mathcal{M}_{\text{vac}}(q) = \frac{43}{15} G^2 m_1 m_2 L
\]

(31)

The Fourier transform gives the following contribution to the scattering potential

\[
V_{\text{vac}}(r) = -\frac{43}{30\pi} G^2 \frac{m_1 m_2}{r^3}
\]

(32)

5.2 Double-seagull Contribution

The calculation of the double-seagull loop diagram:

\[
\mathcal{M}_{\text{seag}}(q) = 44 G^2 m_1 m_2 L
\]

(33)

whose Fourier transform yields the double-seagull contribution to the potential [37, 38]:

\[
V_{\text{seag}}(r) = -\frac{22}{\pi} G^2 \frac{m_1 m_2}{r^3}
\]

(34)

\[2\] The Feynman graphs have been plotted with JaxoDraw [55]
5.3 The Triangle Diagrams

The calculation of the triangle loop diagram:

\[ M_{\text{tri}}(q) = -8G^2m_1m_2 \left( \frac{7}{2}L + \frac{\pi^2m_1}{|q|} \right) \]

and the Fourier transformed result is \[30, 36, 37, 38]\):

\[ V_{\text{tri}}(r) = -4G^2\frac{m_1m_2(m_1 + m_2)}{r^2} + \frac{28}{\pi}G^2\frac{m_1m_2}{r^3} \]  

(36)

5.4 Vertex Corrections

Two classes of diagrams go into the set of vertex corrections. There are two diagrams with a massless graviton in the loop:

The calculation of these diagrams is sufficiently simple and results in:

\[ M_{\text{vert}}(q) = -\frac{52}{3}G^2m_1m_2L \]

(37)

Much more tedious is the calculation of the vertex diagrams with massive particle in the loop:

\[ M_{\text{vert}}(q) = 2G^2m_1m_2 \left( \frac{5}{3}L + \frac{\pi^2(m_1 + m_2)}{|q|} \right) \]

(38)
The vertex diagrams are among the most complicated to calculate. The first results for these diagrams date back to the original calculation of Donoghue [30] — but because of an algebraic error in the calculation, the original result was in error and despite various checks of the calculation [36, 37] the correct result has not been given until [38].

The Fourier transform yields the following result for the vertex modification of the scattering potential [38, 39]:

\[
V_{\text{vert}}^a(r) = \frac{26}{3\pi} G^2 \frac{m_1 m_2}{r^3}
\]

and

\[
V_{\text{vert}}^b(r) = G^2 \frac{m_1 m_2 (m_1 + m_2)}{r^2} - \frac{5}{3\pi} G^2 \frac{m_1 m_2}{r^3}
\]

5.5 The Box Diagrams

The contribution of the box and crossed box diagrams:

\[
\mathcal{M}_{\text{box}}(q) = \frac{94}{3} G^2 m_1 m_2 L
\]

and to the potential [37, 38]:

\[
V_{\text{box}}(r) = -\frac{47}{3\pi} G^2 \frac{m_1 m_2}{r^3}
\]

6. The Gravitational Corrections

Adding up all one-loop gravitational corrections we have the final result for the non-relativistic Newtonian potential [38]:

\[
V(r) = -G \frac{m_1 m_2}{r} \left[ 1 + 3 \frac{G(m_1 + m_2)}{c^2 r} + \frac{41}{10\pi} \frac{G\hbar}{c^3 r^2} \right]
\]

In the above expressions we have restored the appropriate physical factors \(c\) and \(\hbar\).

On the grounds of dimensional analysis alone one can anticipate this form of the lowest-order corrections to the Newtonian potential [30]. The relativistic classical corrections are proportional to \(\ell_{\text{cl}}/r\), where

\[
\ell_{\text{cl}} = \frac{Gm}{c^2}
\]

is the classical length for the mass \(m\), and the quantum corrections (also relativistic) are proportional to \(\ell_p^2/r^2\), where

\[
\ell_p = \sqrt{\frac{G\hbar}{c^3}}
\]
\[ \ell_p = 1.6 \times 10^{-35} \text{ m is the Planck length.} \]

The classical and quantum pieces of (43) arise from the same loop diagrams, and the order of magnitude of the quantum corrections \( G\hbar/c^3 r^2 \) can be derived from the classical one using [56] the concept of “zitterbewegung”. In fact, in transition from classical to quantum corrections the classical distance \( r \) between two masses \( m_1 \) and \( m_2 \) must be modified by an uncertainty of the order the Compton wavelengths of each masses:

\[
r \rightarrow r + \frac{\hbar}{m_1 c} + \frac{\hbar}{m_2 c}
\]

and

\[
\frac{1}{r} \rightarrow \frac{1}{r} - \frac{\hbar}{(m_1 + m_2) c r^2} + \ldots,
\]

so that the quantum corrections of can be understood as “zitterbewegung” effects applied to the classical distance \( r \).

It should be noticed that the classical post-Newtonian term in the expression (43) corresponds to the lowest-order scattering potential and agrees with Eq. 2.5 of Iwasaki [9]. The correct result for the quantum corrections first published in [38] and later was confirmed in [39].

### Conclusion and Outlook

The result (41) for the leading quantum corrections to the Newton law could be written in the form:

\[
V(r) = -G \frac{m_1 m_2}{r} [1 + \delta_{QC}]
\]

where

\[
\delta_{QC} = \frac{41}{10\pi} \frac{\ell_p^2}{r^2}
\]

There are also additional quantum corrections due to the contributions to the vacuum polarization by photons and massless neutrinos:

which were calculated by Radkowski [12], Capper, Duff, and Halpern [15], Capper and Duff [16], Duff and Liu [17]:

\[
\delta_{QC}^{\gamma\nu} = \frac{4 + N_\nu}{15\pi} \frac{\ell_p^2}{r^2}
\]

where \( N_\nu \) is the number of massless two-component neutrinos.

The value of the both quantum corrections are controlled by the Planck length \( \ell_p \), the corrections vanish at large values of \( r \) and it is accompanied by a very small coefficient, so even for astronomical purposes these corrections are irrelevant and unlikely to be
measured in the foreseeable future. Nevertheless, such predictions would need to be replicated by any candidate theory of high energy quantum gravity.

Only at \( r_0 \simeq \ell_p \) the quantum corrections become large. But in this regime the effective field theory approach breaks down.

However, from a cosmological viewpoint there is a cumulative effect of gravity and, given a fixed density of energy, the integration of this effect over large volumes could give an observable signal [57]. The authors of ref. [57] have found that during inflation, the quantum corrections are significant, leading to deviations from the standard inflationary expansion.

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Radiation Reaction at Extreme Intensity

Richard T. Hammond*

Department of Physics
University of North Carolina at Chapel Hill, USA

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Abstract: The radiation reaction force is examined for an idealized short pulse of electromagnetic radiation and for a plane wave. Exact solutions (without radiation reaction) are discussed, the total radiated power is calculated. A new and simpler approach to the approximate form of the equation of motion is presented that automatically removes the runaway solutions. Finally, analytical solutions are presented for the equations of motion that include the radiation reaction forces in the very high intensity regime. A classical scattering angle is defined and it shows that the electron is scattered in a small cone in the forward direction. The radiation reaction corrections to this angle are also considered.

Keywords: Radiation; Radiation Reaction; Classical Electromagnetic; Relativistic Motion; Laser Pulses; Relativistic Fields

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1. Introduction

The equation of motion of a charged particle in an electromagnetic field has demanded our attention for over a century, and for a large part of that time the self force has been of particular theoretical interest. With current laser intensities reaching $10^{22}$ Wm$^{-2}$, and expectations of increasing this by two orders of magnitude in the near future,[1] there has been a renewed interest, and it is not simply theoretical anymore. The point of this article is to provide a new derivation of the approximate equation of motion with the self force, and to use these equations to describe both radiation pressure and a new classical scattering angle.

The literature is rife with references on this topic, and instead of attempting to provide an extensive list, I will refer the reader to a paper that introduces the problem and contains some important references.[2] I will however, review some basic notation, in cgs

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* rhammond@email.unc.edu
units. The electromagnetic field tensor is defined by[3]

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} = A_{\nu,\mu} - A_{\mu,\nu}$$  \hspace{1cm} (1)

where $A_\mu = (\phi, \mathbf{A})$ is the four potential. The four velocity is defined as

$$v^\mu = \frac{dx^\mu}{d\tau} = v^0 u^\mu = \gamma u^\mu$$  \hspace{1cm} (2)

where $\tau$ is the proper time, $u^\mu = dx^\mu/dt$, and $\gamma = 1/\sqrt{1 - u^2/c^2}$. The four acceleration is defined by $a^\mu = dv^\mu/d\tau$ and the Lorentz force is $eF^{\mu\nu}v_\nu/c$. The electromagnetic field tensor is

$$F^{\mu\nu} = \begin{pmatrix}
0 & -E_x & -E_y & -E_z \\
E_x & 0 & -B_z & B_y \\
E_y & B_z & 0 & -B_x \\
E_z & -B_y & B_x & 0
\end{pmatrix}$$  \hspace{1cm} (3)

and the equation of motion is

$$\frac{dv^\mu}{d\tau} = \frac{e}{mc} F^{\mu\nu}v_\nu.$$  \hspace{1cm} (4)

2. Relativistic Motion

This section reviews mostly known results that will be useful for the next section. Let us consider the equation of motion of an electron under the influence of a pulse of laser radiation. The exact solution (to Maxwell’s equations) of such a pulse can very complicated, which prevents us from finding manageable or closed form solutions.[4] To avoid this, we will use a plane wave pulse polarized in the $x$ direction, which is given by

$$E = Eh(kz - \omega t)\hat{x}$$  \hspace{1cm} (5)

where $E$ is constant. Since it is a function of $z - t/c$, it satisfies Maxwell equations.

The magnetic field is

$$B = Eh(kz - \omega t)\hat{y}.$$  \hspace{1cm} (6)

It is helpful to write the equations in non-dimensional form. We let $x^\mu \rightarrow x^\mu/L$ and $t \rightarrow tc/L$. For example, $\cos(kz - \omega t) \rightarrow \cos(z - t)$ if $L = \lambda/2\pi$. Also, $F^{\mu\nu} \rightarrow F^{\mu\nu}/E$.

Now we can assume

$$h = \frac{1}{w} e^{-((z-t)/w)^2} \cos(\Omega(z - t))$$  \hspace{1cm} (7)

where $w$ is a dimensionless parameter that controls the width of the Gaussian and $\Omega$ is a dimensionless parameter controlling the frequency. Using (5) and (6) in (4) yields,
\[
\frac{dv_0}{d\tau} = ahv^1 \quad (8)
\]
\[
\frac{dv^1}{d\tau} = ah(v^0 - v^3) \quad (9)
\]
\[
\frac{dv^2}{d\tau} = 0 \quad (10)
\]
\[
\frac{dv^3}{d\tau} = ahv^1. \quad (11)
\]

We note that (8) and (11) imply,
\[
v^0 = 1 + v^3 \quad (12)
\]
which leaves the pair
\[
\frac{dv^0}{d\tau} = ahv^1 \quad (13)
\]
\[
\frac{dv^1}{d\tau} = ah. \quad (14)
\]
These imply,
\[
v^0 = 1 + (v^1)^2/2. \quad (15)
\]

Using the integral of (12) in the right hand side of (14) yields
\[
v^1 = a \int h(-\tau)d\tau. \quad (16)
\]

This is an important result, and shows that the entire solution can be found in terms of (16), a known result.[5] The integral may be found directly by completing the square and is given by the function \( E(x) \),
\[
v^1 = aE, \quad (17)
\]
where \( E(x) \) is given in terms of the error function:
\[
E(x) = \frac{\sqrt{\pi}}{4}e^{-\Lambda^2} \left( 2 + \text{erf}(\tau/w + i\Lambda) + \text{erf}(\tau/w - i\Lambda) \right) \quad (18)
\]
where \( \Lambda = \Omega w/2 \). This assumes the initial condition that \( v^1(-\infty) = 0 \). Note that this is of form \( z + z^* \), assuring that it is real. From (18) alone, it might be difficult to picture what this function looks like, but we may find another representation by continued integrating by parts. Defining \( \zeta = \tau/w \) and \( d_n \equiv \frac{d^n}{d\zeta^n}e^{-\zeta^2} \), we may write,
\[
E(\tau) = \sin \Omega \tau \sum_{n,\text{even}} (-1)^{n/2} \frac{d_n}{(2\Lambda)^{n+1}} + \cos \Omega \tau \sum_{n,\text{odd}} (-1)^{(n-1)/2} \frac{d_n}{(2\Lambda)^{n+1}} \quad (19)
\]
which is useful for $\Lambda > 1$. This shows that the ($x$ component of the) velocity is sinusoidal with an exponential envelope function, a result that is not self evident from (18). This series is very robust, and gives an excellent approximation keeping only the first term in the sum when $\Lambda$ is bigger than unity. For example, Fig. 1 depicts a graph comparing $v^1$ using (18) and (19) using only one term in the sum with $\Lambda = 5/2$. For $\Lambda = 5$ the graphs are barely discernible.

![Graph](image)

Fig. 1 $v_1$ using (17) (solid), and (19) (dashed), for $I = 10^{20}$ W cm$^{-2}$

There has been considerable interest in the Lawson-Woodward theorem,[6] which loosely states that a charged particle may not obtain a net velocity from an electromagnetic wave. An exception is the plane wave, and it is easy to see that the net $x$ component of the velocity is

$$\Delta v^1/c = a \int_{-\infty}^{\infty} h(-\tau) d\tau = \sqrt{\pi}ae^{-\Lambda^2}. \quad (20)$$

Another way to view this is to note that the four potential is given by

$$A^\mu = (0, \phi, 0, 0) \quad (A^1 \equiv \phi)$$

where

$$\phi = -EL\mathcal{E}. \quad (21)$$

The kinetic energy of the particle is

$$K = (\gamma - 1)mc^2 = (v_0 - 1)mc^2 = v^3m = \frac{m}{2}(v^1)^2 \quad (22)$$

where (12)-(15) were used. Using (17) and (21) one may also show that this may be put in the form, defining the $U = e\phi$,

$$K = \frac{1}{2}(U(\infty) - U(-\infty))^2. \quad (23)$$

In some versions of the Lawson-Woodward theorem it is stated that $U(\pm\infty) \rightarrow 0$, so that, in this more operational sense, there is no violation of the Lawson-Woodward theorem, since this condition is violated with (7).

One may also consider power radiated by the charge,
\[ P = -\frac{2e^2}{3c^3} \dot{v}^\nu \dot{v}_\nu \]  

where \( \dot{v}^\nu \equiv d\nu^\nu/d\tau \). Using the above this becomes

\[ P = \frac{2e^2}{3c^3}(\dot{v}^1)^2 = \frac{2e^2}{3c^3} \left( \frac{ahc}{L} \right)^2. \]  

The average Poynting vector of the incident field is

\[ \mathbf{S} = \left( \frac{c}{4\pi} \right) \mathbf{E} \times \mathbf{H}^* \]  

from which we may obtain the ratio of the radiated power to the incident power per unit area, which yields,

\[ \frac{P}{S} = \frac{8\pi}{3} a_0^2 = \sigma_T \]  

where \( a_0 = e^2/mc^2 \) is the classical electron radius and \( \sigma_T \) is the Thomson cross section.

3. Radiation Reaction

As a charged particle accelerates, it radiates, creating an electromagnetic field that acts on the particle that created the field. This is called the self force, or radiation reaction force.[2] When the charged particle of charge \( e \) and mass \( m \) is subjected to an electromagnetic field, then the equation of motion is given by,[3]

\[ \frac{dv^\sigma}{d\tau} = aF^{\sigma\mu}v_\mu + b(\dot{v}^\sigma + S v^\sigma) \]  

where \( S = \dot{v}^\nu \dot{v}_\nu \) and \( a = eEL/(mc^2) \) and \( b \equiv c\tau_0/L, \tau_0 = 2e^2/(3mc^3) \). This is called the Lorentz, Abraham, Dirac (LAD) equation and has well-known problems, including the runaway solution and the fact that it is of third differential order. A common approach is to consider this equation with \( b = 0 \), and to use this value of \( \dot{v}^\sigma \) in the right side of (28). This gives

\[ \frac{dv^\sigma}{d\tau} = aF^{\sigma\mu}v_\mu + b(aF^{\sigma\mu}v_\mu + a^2(F^{\sigma\gamma}F^\gamma_\nu v_\nu F^\mu_\nu v_\nu v^\sigma)) \]  

which is called the Landau Lifschitz (LL) equation.

The condition for validity is that the second term (ST) on the right side of (29) (i.e., the term with \( b \)) is small compared to the first term (FT) on the right side. It is not always easy to compare these, but we can do so by using the solution given above. If we take \( w \sim 1 \) so that \( F^{\mu\nu} \sim 1 \) the above shows that the LL equation is valid for, dropping indices (so exponents give the power), \( av \gg ba^2v^3 \). We will take the results found above, so that the velocity is either \( aE \) or \( a^2E^2 \) (either the \( x \) component of the velocity or the \( x \) component). Looking at the best case (meaning most favorable to LL, using \( a^2E^2 \) in FT and \( aE \) in ST), this gives \( ba^2 \ll 1 \) for the LL to be valid, which fails as we approach \( 10^{25} \)
Since this is the most favorable case, more generally the LL will fail for smaller values of $a$. Unfortunately we are fast approaching this condition in the labs. Taking the worst case scenario, we find $ba^6 \ll 1$ for LL to be valid, which fails at $10^{20}$ W cm$^{-2}$.

Another approach capitalizes on the fact that $b$ is so small ($\sim 10^{-9}$ for visible wavelengths). Let us write the LAD equation as

$$a^\sigma - ba^\sigma = aF^{\sigma \mu}v_\mu + bv^\nu v_\nu v^\sigma.$$  

(30)

Capitalizing on the smallness of $b$, we recognize the left side as the Taylor series of $a^\sigma (\tau - b)$ to $O(b)$. We may expand the other terms to be functions of $(\tau - b)$ and find,

$$\frac{dv^\sigma}{d\tau} = aF^{\sigma \mu}v_\mu + b\left(\frac{d}{d\tau}(aF^{\sigma \mu}v_\mu) + v^\nu v_\nu v^\sigma\right) + O(b^2).$$  

(31)

This result requires only that $b$ be small, so there is no restriction on the value of $a$. However, for small $a$, this may be shown to be equivalent to the LL equation, but in general it is different. To see the difference let us write the LL equation in the form

$$\frac{dv^\sigma}{d\tau} = aF^{\sigma \mu}v_\mu + b\left(\frac{d}{d\tau}(aF^{\sigma \mu}v_\mu) + 2F^{\sigma \gamma}F_\gamma v_\phi + F^{\nu \gamma}v_\gamma F_\nu v_\phi v_\sigma\right) + O(b^2).$$  

(32)

Strictly speaking, once we decide to use $aF^{\sigma \mu}v_\mu$ on the right side of (28), as in the LL approach, we must make the replacement everywhere, and this shows that the Taylor series approach (31), which relies only on the smallness of $b$, is different than the LL equation (32), which relies on the smallness of $ba^2$, or something more severe. We see that the last terms are different, but also, the derivative terms (on the right side) are different! This is because in the LL approach, we must replace $v_\mu$ within the derivative by the value found by setting the reaction forces to zero. For “weak” fields it does not matter, since they are so close, but for strong fields, the difference is important. In practice, (32) is often used as is. In this case, (32) becomes a postulated equation, which is justified by the smallness parameter discussed above. Even in this case (31) and (32) are different due to the last term. Nevertheless, for the weak field case, these two results are identical, but as noted, are different in the extremely high energy regime we are now approaching.

Since the only requirement for the validity of (31) is that $b$ is small, it is natural to consider an asymptotic series of the form

$$v^\sigma = 0v^\sigma + b(1v^\sigma) + O(b^2).$$  

(33)

To lowest order we obtain (12) to (16) for $0v^\sigma$. Using that, to order $b$ we have, calling $u^\sigma \equiv 1v^\sigma$ and $\phi \equiv 0v^1$,

$$\dot{u}^0 = ahu^1 + a^2h^2\mathcal{E} - \frac{a^4h^2\mathcal{E}^2}{2}$$  

(34)

$$\dot{u}^1 = ah(u^0 - u^3) + \dot{a}h - ha^3h^2\mathcal{E}$$  

(35)
\[
\dot{u}^3 = ahu^1 + a^2(h\mathcal{E} + h^2) - \frac{a^4h^2\mathcal{E}^2}{2}.
\] (36)

These equations describe the equation of motion for any \( h \) to order \( b \). Let us first consider the case that \( a << 1 \) so that in the above only terms linear in \( a \) are retained and higher powers are ignored. (For example, \( a \sim 0.8 \) at \( I = 10^{17} \text{ W cm}^{-2} \).) In this case (34) and (36) show that \( u^0 = u^3 \). With (35) this shows that \( u^1 = ah \) so that (34) gives

\[
u^0 = a^2 \int h^2 d\tau
\] (37)

and the problem is solved to quadrature (to order \( b \)).

As an example, let \( h = \cos(z - t) \). This gives

\[
u^3 = \frac{1}{2} \left( t + \frac{\sin 2t}{c} \right)
\] (38)

as \( \tau \to t \).

Now we may consider \( \langle F_z \rangle = m\dot{v}^3 \) and use (33). This yields, for the \( z \) direction (the transverse direction averages to zero),

\[
\langle F \rangle = \sigma_T \frac{\langle S \rangle}{c}
\] (39)

where \( S \) is the \( z \) component of the Poynting vector. \( S/c \) is the force per unit area, so this equation tells us exactly what we expect from elementary considerations: The particle experiences a force equal to the Poynting vector’s force per unit area, multiplied by the Thomson cross section of the electron. In fact, a more general result may be obtained by considering the four momentum \( p^\mu \). Suppose we integrate the zero component over the full time interval of the pulse. Using (28) we have

\[
W = c \int d\tau \left( aF^0\nu_\mu + b(\dot{v}^0 + Sv^0) \right)
\] (40)

where \( W \) is the final energy of the particle. The first term on the right is

\[
ca \int dt \mathbf{E} \cdot \mathbf{V}
\]

which yields zero when the Lawson-Woodward theorem applies. The next term also integrates to zero, since the acceleration vanishes before and after the pulse, or wave train, ends. We also know that the power scalar is

\[
P = m\tau_0 \dot{v}^\sigma \dot{v}_\sigma
\] (41)

which reduces to \( P = 2e^2(\dot{v})^2/(3c^3) \) in the low velocity limit. Thus we find,

\[
W = -c \int d\tau P
\] (42)

which says that the energy the particle gains is equal to the work done by the radiation field, or minus the power radiated. This result is limited to low velocity, and more importantly, to the case that the field is transverse.
With this approach we may also look at the “large $a$” limit. From (34) - (36) we see that $u^0 \approx u^3 >> u^1$ (which also hold for zero order), and

$$u^3 = -\frac{a^4}{2w^2} \int h^2 E^2 d\tau.$$  

(43)

This verifies that the particle is ejected in the forward direction, parallel to the direction of the beam of light.

4. Astrophysical Applications

Although the above analysis is motivated by the extremely intense laser pulses that are becoming available, another realm where intense fields must be dealt with is compact astrophysical sources such as neutron stars or magnetars, and black holes. In addition to the electromagnetic force there is also the gravitational force. The equation of motion for a charged particle in a combined gravitational and electromagnetic field with radiation reaction is (but ignoring gravitational radiation reaction), in cgs units,

$$\frac{Dv^\mu}{D\tau} = \frac{e}{mc} F^{\mu\nu} v_\nu + \tau_0 (\ddot{v}^\mu + v^\mu \dot{v}^\nu \dot{v}_\nu)$$  

(44)

where the covariant derivative is defined as

$$\frac{Dv^\mu}{D\tau} = \frac{dv^\mu}{d\tau} + \{^\mu_{\alpha\beta}\} v^\alpha v^\beta$$  

(45)

where $\{^\mu_{\alpha\beta}\}$ is the Christoffel symbol. The technique described above can be used here, and the result is

$$\frac{Dv^\mu}{D\tau} = \frac{e}{mc} F^{\mu\nu} v_\nu + \tau_0 \left( v^\mu \ddot{v}_\sigma - \{^\mu_{\alpha\beta}\} \dot{v}^\lambda v^\alpha v^\beta - 2 \{^\mu_{\alpha\beta}\} \dot{v}^\alpha v^\beta - \frac{e}{mc} (\dot{F}^{\mu\sigma} v_\sigma + F^{\mu\sigma} \ddot{v}_\sigma) \right).$$  

(46)

This formula is useful for describing charged particles as they orbit or spiral into a neutron star or magnetar, or as they fall into a black hole. In this case both the electromagnetic and the gravitational field is large, and the terms in the parenthesis in (46) may be appreciable.

5. Discussion

Several effects concerning radiation reaction forces have been discussed. It is shown how an electromagnetic pulse exerts a net force on an electron by calculating the effect of the radiation reaction forces. In fact, it is shown that the average constant force associated with the Poynting vector of a plane wave actually arises from the self forces. In addition, a new and simple way of treating the equation of motion with radiation reaction was developed that, to order $\tau_0$, automatically removes the runaway solutions. Using the equations to this order, it was shown that an electron interacting with an electromagnetic
pulse is scattered in a highly concentrated forward direction, and that the scattering angle is inversely proportional to the intensity of the wave. Finally, gravitational forces were included and applications to astrophysical sources was briefly indicated.

References

Super-light Electromagnetic Wave with Longitudinal and Transversal Modes

M. M. Kononenko*

Institute of Applied Mechanics of Russian Academy of Science
Leninskiy prosp 32A, 119991, Moscow, Russia

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Abstract: The transformation converting equations invariant under Lorentz into the equations invariant under Galileo is obtained. On this basis: (1) the super-light electromagnetic wave with longitudinal and transversal modes is found out; (2) it is shown the wave velocity coincides with that of de Broglie’s wave; (3) the connection between Maxwell’s electrodynamics and Shrödinger’s equation is established; (4) structural elements of space are discovered and “a horizon of visibility” is found. It is shown Bell’s inequalities and the principle of the light speed constancy are based on the SRT artifact and “Einstein’s local realism” is determined by the wave referred above. Objectivity of results for quantum and classical objects is discussed.

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1. Introduction

At present it is widely accepted, as a result of Bell’s theorem and the EPR experiments (e.g., [1]), that “Einstein’s local realism” must be rejected. I am going to show that this idea is erroneous. It is a consequence of fallacious understanding of the locality principle that Einstein used.

According to the postulate of the Special Relativity Theory (SRT) velocities of interactions must be limited to the light velocity and this limitation relates to the locality principle. Such an interpretation of the principle, as it is known, is the physical basis of Bell’s inequalities. In the present paper it will be shown: (i) the interpretation of the principle contradicts to the SRT; (ii) the postulate is one of artifacts of the SRT; (iii) “Einstein’s local realism” is provided with the L. de Broglie wave, [2], that is taken into

*kkononenko@mail.ru
account by Einstein’s theory in an implicit kind.

To execute the program (i) - (iii) we shall represent the relativity in terms of absolute space - time of Newton. As it is obvious, such a representation is possible only if Newton’s space - time can be considered as an abstract mathematical concept.

In the theory based on the concept as postulates we have taken: the relativity and locality principles and Maxwell’s equations. Not setting properties of space, we suppose that they should be revealed from the known fundamental equations and principles which, on their nature, embody these properties.

As a result of the premises, we have obtained the Lorentz – Galileo Transformation (L-GT), [3-5]. (Here alongside with new results the basic ones of these works will be briefly submitted.) The L-GT converts the equations invariant under Lorentz into the equations satisfying the relativity principle of Galileo-Newton. The transformation is an indisputable fact that, nevertheless, contradicts Einstein’s theory that we consider to be formally true. To find out the underlying cause of the contradiction, our theory has accepted only such initial premises that are true in the frameworks of the SRT. This fact allows us to reveal artifacts of the SRT which are in part pointed out in the program (i) - (iii).

2. Lorentz-Galileo Transformation

The Lorentz Transformation (LT) that should be obtained with the help of the absolute space-time concept we write down as follows

\[ t = t' \cos \alpha + x' \tan \alpha, \quad x = t' \tan \alpha + x' \cos \alpha, \]

\[ y = y', \quad z = z' \quad (1) \]

Here \(| \alpha | < \pi / 2\), and the system of units in which the light velocity \( c = 1 \) is used (for transition to one of customary representations of the LT it is necessary to put \( sh \varphi = \tan \alpha \) or \( v = \sin \alpha \)).

Such a record of the LT is convenient for its geometrical representation as mutually orthogonal oblique coordinates \( \{x,t\} \) and \( \{x',t'\} \) with equal scales on all axes. The angle between the axes \( x \) and \( t' \) is equal to \( \pi / 2 \) and so is the angle between the axes \( x't \) and\( t' \); the angles between the same-name axes is equal to \( \alpha \)(see Fig. 1). This representation is discussed in detail in our work, [4, 5], and the uniqueness of such a geometrical interpretation is proved ibidem.

Obtain the LT formulas (1) using the geometrical representation. For the radius-vector of an arbitrary point of coordinate systems \( \{x,t\} \) and \( \{x',t'\} \) with a common point of their origin the following equality is true

\[ r = ti_x + xi_x = t'i_{x'} + x'i_{x'} \quad (3) \]
The relative position of the coordinate axes is defined by scalar products of the unit vectors of the axes as follows:

\[(i_t, i_{t'}) = (i_x, i_{x'}) = \cos \alpha, \quad (i_x, i_t) = -\sin \alpha, \quad (i_x, i_{t'}) = \sin \alpha, \quad (i_{x'}, i_t) = (i_{x'}, i_{t'}) = 0\]

(4)

After scalar multiplication of Eq. (3) by \(i_{t'}\) and by \(i_{x'}\), taking into account Eqs. (4), ones obtain two equalities that coincide with LT (1).

As it usually is, the non-primed system of coordinates is bound up with an observer who is located at the origin of coordinates of an inertial reference system \(K\). The primed system is bound up with a material particle located at the origin of coordinates of another inertial reference system \(K'\) moving with respect to \(K\) uniformly and in straight lines along the axis \(x\). It is more correctly to speak about an inertial configuration \(KK'\) connecting two material particles with each other, but it is possible to use habitual terminology taking into account, however, that reference frames have no physical properties.

The velocity of \(K'\) with respect to \(K\) should be expressed through the parameter \(\alpha\). By the SRT definition, the system of readout \(K'\) determined in the LT by the condition \(x' = 0\) moves along the axis \(x\) with the velocity \(v = x/t = \sin \alpha\). The definition seems to be evident. However, it is necessary to perceive that it extends the velocity understanding based on kinematics of Galileo to the case when this kinematics is rejected.

A correct velocity definition can be obtained only by means of abstract concepts of temporal and spatial coordinates, for in that case the theoretical tool cannot be a source of artifacts distorting a picture of reality. Such a definition is given by the Galileo Transformation (GT):

\[\tau = \tau', \quad x = ur' + x'\]

(5)
Here $u$ is the normalized velocity (in the system of units in which $c = 1$); $\tau$ is the absolute time; the non-primed and primed coordinates are connected, correspondingly, with $K$ and $K'$.

Let’s notice such a velocity definition demands to model the ideal concepts of time and length by means of devices. However this requirement is not additional in comparison with SRT which, as it will be shown, also requires these ideal measures.

The fact of a physical interaction of two material bodies means that a change of the physical state of the first body entails the corresponding change of the second. Concretizing the locality principle, as it is accepted in [6], let’s assume that there is a signal in Nature which transfers information about parameters of such an event from the first body to the second. The hypothetical signal, on its functional role, can be named as the “correlation signal” or briefly “$B$-signal”.

This conception can be generalized to any events occurring in physical space. Let $(\tau, x)$ and $(\tau', x')$ be coordinates of an event in the $K$ and $K'$, respectively. The event has taken place on the $x$-axis segment connecting the coordinate origins of these systems (see Fig. 1), and $\tau = \tau'$ is the absolute time of the event. Such a choice of the event place is not accidental; in work [5] it has been shown that at other locations the causality principle is broken (the registration time of the event precedes the event time).

Suppose that the $B$-signal velocity is equal to $b = \text{const}$ for the given situation. In that case we obtain the time instants, $t$ and $t'$, when the signals are registered by observers in the points of origins of $K$ and $K'$, respectively, in the form

$$ t = \tau + x/b, \quad t' = \tau' - x'/b $$

Here it is taken into account that for the chosen event $x > 0$ and $x' < 0$; relationships (6) are built upon kinematics of Galileo. Below, the time $t$ and $t'$, measured on the same absolute time scale, is named as “the relative time”.

Excluding the absolute time, $\tau$ and $\tau'$, from GT (5) by means of Egs. (6), ones obtain the following transformation

$$ t = t' (1 + u/b) + x' (2 + u/b) /b, \quad x = t'u + x' (1 + u/b) $$

Its determinant is equal to unity. It is visible the transformation has the structure of LT (1) but depends on two parameters – $u$ and $b$.

The replacement of operators at transition from the non-primed system of readout to the primed system by means of transformations (2), (7) is carried out under formulas

$$ \frac{\partial}{\partial t} = (1 + \frac{u}{b}) \frac{\partial}{\partial t'} - u \frac{\partial}{\partial x'} \frac{\partial}{\partial x} = - \left( \frac{2}{b} + \frac{u}{b^2} \right) \frac{\partial}{\partial t'} + \left( 1 + \frac{u}{b} \right) \frac{\partial}{\partial x'} $$

$$ \frac{\partial}{\partial y} = \frac{\partial}{\partial y'}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z'} $$

From Maxwell’s electrodynamics and the SRT it is known that the space state depends on a motion. Therefore, a velocity definition must be coordinated with Maxwell’s equations.
In other words, the definition cannot be arbitrary; there should be a dependence $b = b(u)$ that can be obtained from the Maxwell equations. These equations may be written thus:

$$\frac{\partial}{\partial t} \mathbf{G} + i \nabla \times \mathbf{G} = 0, \quad \nabla \cdot \mathbf{G} = 0, \quad \mathbf{G} = \mathbf{E} + i \mathbf{H}$$  \hspace{1cm} (10)

Here $i$ is the imaginary unit. The time used in these equations, as it is known, is not absolute. Suppose that this time is the relative one introduced above; the sequel confirms the supposition.

To find the dependence $b = b(u)$ we write down Eqs. (10) in reference frame $K'$, using operators (8) and (9). Having done this operation, ones obtain two series of formulas for components of the primed complex vector of an electromagnetic field:

$$G'_x = G_x, \quad G'_y = \left(1 + \frac{u}{b}\right)G_y + i\left(\frac{2}{b} + \frac{u}{b^2}\right)G_z, \quad G'_z = \left(1 + \frac{u}{b}\right)G_z - i\frac{u}{b}G_y$$  \hspace{1cm} (11)

$$G'_x = G_x, \quad G'_y = \left(1 + \frac{u}{b}\right)G_y + i\frac{u}{b}G_z, \quad G'_z = \left(1 + \frac{u}{b}\right)G_z - i\left(\frac{2}{b} + \frac{u}{b^2}\right)G_y$$  \hspace{1cm} (12)

The same-name formulas from (11) and (12) coincide with each other and with the SRT results if $u$ and $b$ are connected by the following equation:

$$u = \frac{2b}{b^2 - 1}$$  \hspace{1cm} (13)

Parametrizing relationship (13) by means of ordinary trigonometric functions, the motion velocity of $K'$ with respect to $K$ and the $B$-signal speed can be written down thus:

$$u = \tan \alpha$$  \hspace{1cm} (14)

$$b = 1/\tan (\alpha/2)$$  \hspace{1cm} (15)

Then transformation (7) coincides with LT (1). Formulas (11) and (12) result in the known relationships between the primed and non-primed components of an electromagnetic field:

$$G'_x = G_x, \quad G'_y = G_y/\cos \alpha + i \tan \alpha G_z, \quad G'_z = G_z/\cos \alpha - i \tan \alpha G_y$$  \hspace{1cm} (16)

If to start from the LT, the velocity motion would be possible to define as $u = x/\ell'$. Consequently, the absolute time coincides with the proper time.

The given derivation of the LT proves the compatibility of the absolute time concept with Maxwell’s electrodynamics and shows that the concept is the basis of LT.

However, the result obtained contradicts the SRT postulates because $u \to \infty$ when $\alpha \to \pi/2$, and the $B$-signal speed always is more than unit (at $\alpha \to 0$ it can be arbitrary large).

At the same time, both the theories give the same quantitative description of most experimental results. For example, such are measurements of distances passing by unstable particles for their life-time ($\tau$). Under our theory, the distance is equal to $L_1 = u\tau$, and
within the framework of SRT – \( L_2 = v\tau / \cos \alpha \). As \( u = v / \cos \alpha \), ones obtain \( L_1 = L_2 \).

The similar result is also obtained for the relativistic definition of a momentum that has a classical form, \( p = mu \), coincided with the SRT definition, \( p = mv / \cos \alpha \), where \( m \) is the rest mass.

Let’s obtain the L-GT. For this purpose it is enough to replace in LT (1) the relative time, \( t' \), by the absolute time, \( \tau' \), using the second of Eq. (6). As a result ones obtain the L-GT for an inertial configuration \( KK' \):

\[
t = \tau' / \cos \alpha + x' \tan(\alpha/2), \quad x = \tau' \tan \alpha + x'
\]

It is self-evident that the L-GT also involves Eqs. (2). Here \( t \) and \( x \) are the parameters of the Lorentz-Poincaré space-time, and \( \tau' \), \( x' \) are that of Newton’s absolute space-time.

From (17) follows the operator expression of the L-GT:

\[
\left( \frac{\partial}{\partial t} \right)_x = \left( \frac{\partial}{\partial \tau'} \right)_{x'}, \quad \left( \frac{\partial}{\partial x} \right)_t = -\tan \alpha \left( \frac{\partial}{\partial x'} \right)_{\tau'},
\]

\[
\quad \frac{1}{\cos \alpha} \left( \frac{\partial}{\partial x'} \right)_{\tau'}
\]

The second of Eqs. (17) coincides with the \( x \)-transformation of Galileo. Therefore, spatial coordinates of the LT are expected to be absolute.

In SRT the dependence of spatial coordinates on a motion velocity arises only at using the proper time which, as it was shown, coincides with the absolute time. This is conditioned by the following circumstance. A constant motion is described by three kinematics parameters (time, distance, and velocity) and only two of them are independent from each other. The SRT introduced an absolute measure of velocities by means of the light speed postulate. Hence, another arbitrary measure (named, otherwise, the absolute one and determined by the will of a researcher and by the historical tradition) can be introduced only for one of two other parameters. Within the framework of the SRT the arbitrary measure is introduced either for the time or for the distance. The choice depends on a specific character of interpretation. Thus the SRT also uses absolute coordinates: absolute temporal and relative spatial coordinates or absolute spatial and relative temporal coordinates.

It is easy to check up that the spatial coordinates of the LT are absolute. If it so, the GT should be deduced from the LT. With this purpose we exclude \( t \) and \( t' \) from LT (1) with the help of Eqs. (6), identifying the absolute spatial coordinates from Eqs. (6) with the corresponding spatial coordinates of the LT. As the result of elementary calculations we obtain the GT.

In the LT the relative time is measured on an absolute time scale. Its physical sense is completely defined by formulas (6). As evident from the formulas, in the LT the temporal coordinates (\( t \) and \( t' \)) of an event are time moments of influence of the \( B \)-signal caused by the event upon material particles constituting configuration \( KK' \). This peculiarity of the temporal coordinates determines the relative time as an attribute of the \( B \)-signal. It is valid to say that the relative time is an objective parameter characterizing processes of interactions of material particles. The set of such processes or, in other words, the set of
configurations \( K K' \), having a place in Nature, determines the set of natural times. The absolute time is introduced as a general measure for this set.

3. Maxwell-Galileo Equations

Now it is rather simple to obtain electrodynamics in terms of absolute space-time. For this purpose it is enough to replace operators in Eqs. (10) using formulas (9) and (18). Thus we go into system of readout \( K' \) and replace the relative time by the absolute one. After simple computations the Maxwell equations are converted into the Maxwell-Galileo Equations (M-GEqs):

\[
\frac{\partial G'}{\partial \tau'} + i \nabla' \times G' = 0, \quad \nabla' \cdot G' = 0, \quad \nabla' = i_x' \left( \frac{\partial}{\partial \tau'} + \frac{1}{b} \frac{\partial}{\partial x'} \right) + i_y' \frac{\partial}{\partial y'} + i_z' \frac{\partial}{\partial z'}
\]  

The non-primed and primed components of \( G \) and \( G' \) are connected with each other by relationships (16). The M-GEqs differ from Maxwell’s equations by the operator acting along spatial axes.

Mathematically, Eqs. (19) are isomorphic to Eqs. (10), i.e., any solutions of Eqs. (19) are also the solutions of Eqs. (10). It is possible to say that Eqs. (19) represent electromagnetism in \((\tau, x)\)-form and Eqs. (10) are so in \((t, x)\)-form.

The M-GEqs must be invariant concerning the GT. Indeed, replacing primed operators of Eqs. (19) by formulas (9) and by the following formulas

\[
\frac{\partial}{\partial \tau'} = \frac{\partial}{\partial \tau} + \tan \alpha \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial x'} = \frac{\partial}{\partial x}
\]  

which follow from the GT ones obtain the same equations. In this case, as well as at invariance under Lorentz, relationships (16) should be used.

The operator \( \nabla' \) of Eqs. (19) shows that the LT and the equations invariant with respect to the LT automatically take into consideration the locality principle as an action of the \( B \)-signal with speed \( b \), i.e., Einstein’s local realism is determined by the super-light \( B \)-signal. In the STR the locality principle is realized implicitly by change of the relative time scale.

Let’s find the partial solution of Eqs. (19) in the form:

\[
G' = (F_x(y, z), F_y(y, z), F_z(y, z)) \exp(i(k_1 x + \nu \tau))
\]  

Substituting this expression in system of Eqs. (19) ones obtain the following system

\[
\nu F_x + \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = 0
\]  

\[
\nu F_y - i \left( k_1 + \frac{\nu}{b} \right) F_z = -\frac{\partial F_x}{\partial z}
\]  

\[
i \left( k_1 + \frac{\nu}{b} \right) F_y + \nu F_z = \frac{\partial F_x}{\partial y}
\]
\[ i \left( k_1 + \frac{\nu}{b} \right) F_x + \frac{\partial F_x}{\partial y} + \frac{\partial F_x}{\partial z} = 0 \]  

(25)

Considering this system of the equations as algebraic one, we find that the matrix rank for the unknown \( F_x, F_y, F_z \) and the augmented matrix rank are equal to 3. To find the solution dependent only on the component \( F_x \) we equate to zero the determinant

\[ M = i(k_1 + \nu/b)[\nu^2 - (k_1 + \nu/b)^2]. \]

The condition \( M = 0 \) gives, firstly, ordinary electromagnetic waves. Their velocities at moving off and approaching a wave source are respectively equal to:

\[ c_1 = -1/(1 + \tan(\alpha/2)), \quad c_2 = 1/(1 - \tan(\alpha/2)) \]  

(26)

These formulas also follow from the wave equation for the light after its representation in terms of absolute space - time.

Secondly, from the same condition we obtain the equality \( \nu/k_1 = -b \) that corresponds to a phase velocity of the \( B \)-signal. In this case from Eqs. (22)-(25) ones obtain the following equation for \( F_x \)

\[ \nu^2 F_x + \frac{\partial^2 F_x}{\partial y^2} + \frac{\partial^2 F_x}{\partial z^2} = 0, \]  

(27)

Components \( F_y \) and \( F_z \) are calculated from relationships

\[ \nu F_y + \frac{\partial F_x}{\partial z} = 0, \quad \nu F_z - \frac{\partial F_x}{\partial y} = 0 \]  

(28)

Periodic solutions of Eq. (27) give for complex components of an electromagnetic field the following expressions (the primes are omitted):

\[ G_x = F_0 \exp \left[ i\nu \left( \frac{\tau}{b} - \frac{x}{\nu} \right. + \frac{k_2 y}{\nu} + \frac{k_3 z}{\nu} \right], \quad G_y = -i \frac{k_3}{\nu} G_x, \quad G_z = i \frac{k_2}{\nu} G_x, \quad k_2^2 + k_3^2 = \nu^2 \]  

(29)

Here \( F_0 \) is an arbitrary constant.

Thus the M-GEqs describe, at least, two stationary waves – the light one and the wave named as \( B \)-signal.

4. On Nature of the \( B \)-signal

The stated theory does not introduce any new axioms. For this reason the \( B \)-signal is assumed to be known physics under other name and our theory makes more exact its parameters and/or a functional role in Nature. The most probable candidate (owing to its super-light speed) is the wave of L. de Broglie who postulated and presented the stationary wave in the form, [2]:

\[ \exp \left[ i\nu_0 \left( t - \frac{vx}{c^2} \right) / \sqrt{1 - \nu^2/c^2} \right] \]  

(30)

Here standard designations are used (in our designations \( v/c = \sin \alpha \) and \( c =1 \)).

L. de Broglie defined the phase speed of the stationary wave as \( x/t = c^2/v \). This definition is expressed through the relative time and absolute spatial coordinate, i.e.,
it mixes objective and conventional parameters. For this reason it does not describe a wave transmission correctly. Let’s find the phase speed of the wave, proceeding from kinematics of Galileo. With this in mind we replace the relative time by the absolute time, using L-GT (17). As a result exponent (30) becomes equal to

$$\exp \left[ i\nu_0 \left( \tau' - \tan \frac{\alpha x'}{2c} \right) \right]$$  \hspace{1cm} (31)

From expression (31) immediately follows that the phase speed of L. de Broglie’s wave coincides with the $B$-signal speed

$$b = \frac{x'}{\tau'} = \frac{c}{\tan \left( \frac{\alpha}{2} \right)}$$  \hspace{1cm} (32)

This fact revels that another name of the $B$-signal is de Broglie’s wave. The wave has an electromagnetic nature and its parameters are described by formulas (29). In contrast to the ordinary electromagnetic wave in vacuum the $B$-signal has three modes (two transversal and one longitudinal ones) of the complex vector $\mathbf{G} = \mathbf{E} + i\mathbf{H}$.

Bell’s inequalities, basing on the SRT postulate about the light speed, ignore the $B$-signal. For this reason their violation disclosed by the EPR experiments has no relation to the physical locality principle. As was shown, within the framework of the SRT the principle is taken into account with help of the known dependence of relative time scale on the motion velocity. Thus Bell’s theorem proceeds from an erroneous interpretation of the SRT.

The EPR experiments can be used for direct confirmation of the $B$-signal existence and for measurement of its speed. Really, after some influence on the first particle of an EPR pair there is an time interval, $\Delta \tau < \frac{L}{b}$, inside which the second particle state does not depend on the new state of the first (here $L$ is the base of measurements).

5. The Velocity of Light and Doppler’s Effect

As it is apparent from formulas (26), the dependence of the light speed on the relative velocity $u$ of its source is not the classical addition of velocities. At $\alpha < < 1$ we obtain that $c_{1,2} \cong 1 \pm u/2$ because $u = \tan \alpha \cong \sin \alpha \cong \alpha$. Therefore, to test formulas (26) when $u < < 1$ it is necessary to increase the measurement accuracy of velocities.

From the physical viewpoint, Maxwell’s equations and the M-GEqs. are in conflict with each another despite of their mathematical isomorphism. It needs to be ascertained what system of equations meets conditions of experiments. With this aim in view Doppler’s optical effect may be used. Really, if formulas (26) are true, this effect should arisen as a consequence of the same physical principles which determine its acoustic analogue because the light speed is determined by the state of a substance in which the light is propagated and it is similar in this respect to an acoustic signal. The checkout shows this is the case.

Indeed, the classical theory of Doppler’s acoustic effect, for a source moving off at velocity $u$, predicts the following relationship between the registered ($\nu$) and intrinsic
frequency ($\nu_0$) of a source (of sound or light):

$$\nu = \frac{\nu_0}{1 + u / |c_1|},$$

(33)

where $c_1$ is the speed of a signal (light or sound). In the optical case the signal speed depends on the motion velocity and this dependence is expressed through parameter $\alpha$. Using formulas (14) and (26) from relationship (33) ones obtain the known formula of Doppler’s optical effect:

$$\nu = \nu_0 \left( \frac{1}{\cos \alpha} - \tan \alpha \right)$$

(34)

This formula was found by Einstein. However, his derivation is rather confusing. Indeed, it is based on the following statement: An electromagnetic field phase in the certain point of the four-dimensional space-time does not depend on a choice of a reference frame. This physical statement is correct within the framework of our theory but it is incompatible with the space concept that is actually considered in the SRT. Within the scope of the SRT the notion “the certain point of space-time” conflicts with the following conclusion of Einstein and Pauli, [7], caused by the SRT: Physical characteristics of space have neither positions nor velocities.

At the consecutive interpretation of the SRT, the continuum of points of every reference frame cannot be considered as a mapping of the continual set of points (events) of the physical space-time (owing to dependence of the time scale on a motion velocity). Consequently, the primary image (the physical space-time) in the SRT appears to be uncertain. And this is noted in the given conclusion of Einstein and Pauli.

Thus, it is possible to say that the derivation of Doppler’s effect in SRT and its new derivation confirm the key concept of the given theory: the light speed is determined by the space state characterized with parameter $\alpha$. This fact proves correctness of formulas (26). From the last, for one’s turn, it can conclude that Maxwell’s electrodynamics should be represented as M-GEgs to describe a real physical situation adequately.

The light propagation should have some feature similar to the transformation of a sonic wave into the shock wave when its source velocity exceeds some critical one ($u = 4/3$; then $c_2 > b$). In this case Doppler’s effect should be broken. This fact can be used for testing the theory.

6. Structure of Space

To investigate the space structure ensuing from our postulates let’s consider the following situation. Let’s assume that from the origin point of all coordinate systems $K_i$ material particles move in arbitrary directions. Then for the relativistic description of their movement it is necessary to introduce $n$ configurations $K_i, K'_i$. Each of these configurations is determined by the pair of systems of axes $\{t_i, x_i, y_i, z_i\}$ and $\{t'_i, x'_i, y'_i, z'_i\}$ so that Eqs. (1) and (2) are carried out at $\alpha = \alpha_i$. The index $i$ has to attach also to the spatial and temporal coordinates. All the reference frames $K_i$ are connected with the
same motionless observer, and the reference frame $K'_i$ is attached with the $i$th material particle, ($i = 1, \ldots, n$).

The velocity of the $i$th particle determines the value of $\alpha_i$ for each configuration. In turn, parameter $\alpha_i$ determines speeds of the light and $B$-signal. On the other hand, the two speeds are completely determined by properties of physical space. It is possible, if and only if physical space is a set of $n$ elements and the $i$th element determines velocities $c(\alpha_i)$ and $b(\alpha_i)$. Such an element is the three-dimensional configuration $K_iK'_i$. Physical laws act within the bounds of configurations.

So, we have a new concept of space. According to the concept, space, as a whole, consists of a set of elements. The space element discovered is the configuration $KK'$ connecting two material particles in their binary relations with each another. A descriptive-geometric form of a configuration if a particle moves in straight line is a space-thread. To meet the conservation laws each of two interacting particles should be a source-sink of some substance from which the space-thread is created.

Thus relativistic mechanics does not deal with material points and unbounded continual physical space. Its objects are three-dimensional configurations $KK'$. The end points of configurations are material particles, and its internal points are some physical substance that has all properties of space but it, in contrast to classical space and the SRT, is not a separate physical object.

In a system from $n$ classical particles a quantity of possible elementary configurations, each of which connects only various particles, theoretically can be equal to $N = n(n - 1)/2$, and their total sum is equal to $2^N$. For ordinary conditions this number is huge. For this reason it is difficult to assume that at each given moment of time each particle cooperates with all others. In all likelihood, in the given domain of coordinate space there are free particles which are not belonged to any configurations.

As it is accepted to account, Neother's theorem and the invariance of Maxwell’s equations concerning Lorentz’s group proves uniformity and isotropy of space. Undoubtedly, this opinion can be considered as a correct physical conclusion if its preconditions have solely the objective nature. However, the L-GT proves that it is not so, namely: 1) the absolute space-time is an abstract concept that has only the mathematical nature and it is determined by the subjective will of a researcher; 2) the concept is the basis of any mathematical simulation of physical laws. Therefore, the nature of physical laws has two components – objective and subjective ones. This feature of physical laws entails artifacts that should be comprehended to understand laws correctly. (Generally speaking, the problem of the subjective component of physical laws is difficult. Its discussion goes beyond the scope of the present work.)

Let’s notice that at $u > b$ (when $\alpha > \pi/3$) the correlation of states of material particles cannot be provided. Apparently, such configurations cannot exist and they are disintegrated. If it is so, it is possible the two scripts. First, the particle after destruction of the configuration can remain free and, hence, absolutely non-observable. The collision of such a particle with an observable configuration will look as an absolutely unpredictable event. Second, it can remain observably if there is a way connecting the given particle
with the researcher via other configurations. Let $K_1K'_1, ..., K_nK'_n$ be a sequence of configurations that determines the light way to an observer. In this case ones obtain the following obvious generalization of Doppler’s effect

$$\nu = \prod_{i=1}^{n} \frac{\nu_0}{\cos \alpha_i - \tan \alpha_i}$$  \hspace{1cm} (35)

As it follows from formula (35), the knowledge of the initial and final frequency is not enough for a conclusion about the motion velocity of a light source. There is “a horizon of visibility” (an object becomes non-observable one without instrumentality of other observable objects). The conclusion is of practical significance for cosmology and it can be used to test the given script.

Concerning the space model generated by Maxwell’s equations let’s notice the following. On a functional role (to provide interaction of material bodies) and by origin (material particles are a source-sink of the substance making space) the space concept is indistinguishably from the concept of a field.

7. On a Nature of Shrödinger’s Equation

The thread-like structure of space will have an essential influence on a particle motion if the particle has few threads connecting it with other particles. This supposition is reasonable for micro-particles. If these representations are true, quantum mechanics cannot do without taking into account the thread-like structure of space and finite speeds of interactions, which determines relativistic properties of space.

From this point of view, the greatest interest represents Shrödinger’s equation, for the equation is considered to be not relativistic. Besides, the equation being not invariant with respect to the LT and the GT calls into question the relativity principle. Therefore, checking the offered theory, it is necessary to find out whether Shrödinger’s equation takes into account relativistic properties of the reality and the relativity principle.

It is known that de Broglie’s wave plays an important role in quantum mechanics. We have obtained the wave from electrodynamics. For the reasons we make use of the solution of the M-GEqs that is submitted by formulas (29). The solution determines three components of a complex vector of an electromagnetic field while Shrödinger’s equation is written down for a scalar complex wave function. Therefore, the wave function we introduce as $\psi = \nabla \cdot G$ where the divergence of complex vector $G$ is defined in absolute space-time.

Then from solution (29) ones find

$$\nabla^2 \psi = -i k_1 \left( k_1^2 + k_2^2 + k_3^2 \right) G_x, \quad \frac{\partial \psi}{\partial \tau} = \nu k_1 G_x$$  \hspace{1cm} (36)

Excluding $G_x$ from Eqs. (36), we obtain the equation with structure of Shrödinger’s equation for a free particle, otherwise, for configuration $KK'$ that is free from interaction with other objects

$$i \frac{\partial \psi}{\partial \tau} + \frac{\nu}{k_1^2 + k_2^2 + k_3^2} \nabla^2 \psi = 0$$  \hspace{1cm} (37)
For calculations of the Laplacian factor ones take a component of the wave vector from solution (29), \( k_1 = \nu \tan (\alpha/2) \), \( k_2^2 + k_3^2 = \nu^2 \), de Broglie’s definition of wave-length, \( \lambda = h/m \tan \alpha \), and the kinematic relationship between wave-length and its frequency and speed, \( \lambda \nu = 1/\tan (\alpha/2) \). In result we obtain the equation:

\[
i \frac{\partial \psi}{\partial \tau} + \frac{h \cos \alpha}{2m} \nabla^2 \psi = 0
\]  

(38)

This equation can be named as the relativistic equation of Shrödinger. It becomes approximately relativistic one and turns into Shrödinger’s equation at \( \alpha \ll 1 \), when \( \cos \alpha \approx 1 \).

Thus we have confirmed the theory validity by means of the given derivation of the relativistic equation of Shrödinger. Besides, we have obtained the basis for physical interpretation of wave function and some features of quantum phenomena.

So, in terms of quantum mechanics it is possible to tell that solution (29) and Eq. (38) describe so-called “a pure state”. For the description of “a mixed state” of a quantum object connected to other objects by means of \( n \) threads it is necessary to consider \( n \) systems of the M-GEqs or Maxwell’s equations taking into consideration sources and currents. Such a physical sense gets the superposition principle of quantum mechanics if to consider that objects of its description are configurations \( KK' \).

The thread-like structure of space discovered within the framework of the classical analysis, allows us to explain also such features of the quantum phenomena as a reduction of wave function and wave properties of particles that can be understood as breaking threads and their contact interactions.

The stated representations for the non-observable substance making physical space can have a practical sense only if the substance can be modeling with the help of the ordinary material environment. The derivation of optical effect of Doppler on the basis of its acoustic analogue allows us to hope that such opportunities exist.

**Conclusion**

The physical problem that is necessary to analyze “Einstein’s local realism” was to bring to light the physical space properties contained in Maxwell’s equations and the Lorentz transformation. As is evident from the given work, the problem was rather simple. It transmuted into an extremely involved problem due to the incompatibility tenet of the absolute time with Maxwell’s electromagnetism. The formal faultlessness of SRT and its mathematical beauty, by P. Dirac, ensured a reliable protection of the tenet that turned out to be an insuperable hindrance for the adequate understanding of relativistic properties of reality. In turn, the tenet reliably hid artifacts of the SRT.

The SRT effects having only the mathematical nature and being required for completeness of the theory are: 1) dependences of length, time, and mass upon a motion velocity; 2) the light velocity as the superior limit for velocities of motions and transfers of energy; 3) independence of the light velocity on motion of its source.

The most interesting and practically important result of the stated theory is a discovery of a super-light wave with longitudinal and transversal modes of a complex vector
\( G = E + iH \) of an electromagnetic field. If we are able to generate and detect this wave, we receive, at least, new means of radio communication.

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Noncommutative Geometry constraints and the Standard Renormalization Group approach: Two Doublets Higgs Model as an Example

N. Mebarki and M. Harrat

Laboratoire de Physique Mathématique et Subatomique, Mentouri University, Constantine, Algeria

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Abstract: The Chamssedine-Fröhlich Approach to Noncommutative Geometry (NCG) is extended and applied to the reformulation of the two doublets Higgs model. The Fuzzy mass, coupling and unitarity relations are derived. It is shown that the latter are no more preserved under the renormalization group equations obtained from the standard quantization method. This suggests to look for an appropriate NCG quantization procedure.

Keywords: Noncommutative Geometry; Renormalization Group; Higgs Model

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1. Introduction

In the last decade many interest has been devoted to understand the unsolved problems of the standard model and to find some origins and mechanisms of the various parameters. Many approaches have been developed namely the Connes Noncommutative geometry [1] – [20] formalism. The latter is based on a unitary algebra, where gauge bosons and Higgs are treated in an equal footing. The drawback of this approach, is that we cannot go beyond the standard model[1] – [3]. To overcome this situation, Chamessedine and Fröhlich proposed an extension of the Dirac operator which allows to consider other unified models based on unitary and orthogonal groups $SU(5), SO(10), ..etc.$[21] – [23].
However, the spontaneous breakdown of the symmetry is not well controlled to allow getting reasonable mass and mixing angles relations. Moreover, the elimination of the auxiliary unphysical fields undergo through field equations which is in general a complicated procedure[21] – [23]. The goal of this paper is two fold: First to extend the Chamesseddine - Frohlich approach (through the two doublet Higgs model) by including the strong interaction sector in the new mathematical formalism and generalizing the Dirac operator as well as the scalar product. The elimination of the junk forms is done by applying the orthogonality condition before the construction of the action. Second to derive the tree level Fuzzy mass, coupling and unitarity relations and to show that we cannot preserve these constraints under the ordinary renormalization group flow (running masses, couplings etc.).

In section 2, we give the formalism and derive some of the NCG constraints. We present the renormalization group (RG) analysis and show that the corresponding evolution differential equations, once solved will not in general preserve the tree level NCG relations. Finally, in section 3, we draw our conclusions.

2. Formalism

Consider a model consisting of the spectral triplet ( \( \mathcal{A}, \mathcal{H}, \mathcal{D} \) ) where \( \mathcal{A} \) and \( \mathcal{D} \) are an involutive algebra of operators and unbounded self-adjoint operators on a Hilbert space \( \mathcal{H} \) respectively. Let \( X \) be a compact Riemannian spin-manifold, and (\( \mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1, \chi \)) the Dirac K- cycle where \( \mathcal{H}_1 \equiv L^2(X, \sqrt{g} d^4 x) \) (\( g \) is the metric) acts on the \( \mathcal{A}_1 \) algebra of functions on the \( X \) manifold. Let (\( \mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2 \)) another triplet where \( \mathcal{A}_2 \) is given by [1], [2], [3], [7]:

\[
\mathcal{A}_2 = M_2(\mathbb{C}) \oplus \mathbb{C} \oplus M_3(\mathbb{C})
\]

where \( M_2(\mathbb{C}) \) (resp.\( M_3(\mathbb{C}) \)) are \( 2 \times 2 \) (resp.\( 3 \times 3 \)) matrices algebra and \( \mathcal{H}_2 = h_{2,1} \oplus h_{2,2} \oplus h_{2,3} \) with \( h_{2,1}, h_{2,2} \) and \( h_{2,3} \) are Hilbert spaces on \( \mathbb{C}^2, \mathbb{C} \) and \( \mathbb{C}^3 \) respectively. Then, \( \mathcal{A} \) and \( \mathcal{D} \) are taken to be:

\[
\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2
\]

\[
\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2
\]

and
\[ \mathcal{D} = \mathcal{D}_1 \otimes 1 \oplus \chi \otimes \mathcal{D}_2 \]  

(4)

Where \( \chi \) is chirality operator \( \chi \) acting on the Hilbert space \( \mathcal{H} \) such that:

\[ \chi = \chi^*, \quad \chi^2 = 1, \quad \chi \mathcal{D} = -\mathcal{D} \chi \]  

(5)

In this case, the Hilbert space \( \mathcal{H} \) can be splitted into two orthogonal subspaces \( \mathcal{H}_L \) and \( \mathcal{H}_R \) such that:

\[ \mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R \]  

(6)

and

\[ \mathcal{H}_{L,R} = \frac{1}{2}(1 \mp \chi)\mathcal{H} \]  

(7)

or in the formal form:

\[ \mathcal{H}_L = (\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}) \]  

(8)

and

\[ \mathcal{H}_R = ((\mathbb{C} + \mathbb{C}) \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C} \otimes \mathbb{C}^N \otimes \mathbb{C}) \]  

(9)

Therefore, the \( \mathcal{A} \) algebra which represents the gauge group \( U(2) \otimes U(1) \otimes U(3) \) can be written in the following form:

\[ \mathcal{A} = \mathcal{C}^\infty (X) \otimes (M_2 (\mathbb{C}) \oplus \mathbb{C} \oplus M_3 (\mathbb{C})) \]  

(10)

where here, \( \mathcal{C}^\infty (X) \) is the algebra of differentiable functions. It is to be noted that, an involutive representation \( \pi \) of this algebra is provided by the map:

\[ \pi : \mathcal{A} \rightarrow \mathcal{B} (\mathcal{H}) \]  

(11)

\[ f \rightarrow 1_{4 \times 4} \otimes \text{diag}(f_1, f_2, f_3, f_4, f_5, f_6) \]  

(12)

where \( \mathcal{B} (\mathcal{H}) \) is the algebra of bounded operators in the Hilbert space \( \mathcal{H} \) and to every block diagonal \( f \in \mathcal{A} \), we associate a sixtet \((f_1, f_2, f_3, f_4, f_5, f_6)\) of matrix-valued functions on \( X \) such that:
\[ f_i \in M_2(\mathbb{C}); \ i = 1, 2 \] (13)
\[ f_i \in \mathbb{C}; \ i = 3, 4 \] (14)
\[ f_i \in M_3(\mathbb{C}); \ i = 5, 6 \] (15)

Now, regarding the explicit form of the Hermitian Dirac operator \( D \), it is given by:

\[
D = \begin{pmatrix}
\partial \otimes 1 \otimes 1_N & \gamma_5 \otimes M_{12} \otimes K_{12} & \ldots & \gamma_5 \otimes M_{16} \otimes K_{16} \\
\gamma_5 \otimes M_{21} \otimes K_{21} & \partial \otimes 1 \otimes 1_N & \ldots & \gamma_5 \otimes M_{26} \otimes K_{26} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_5 \otimes M_{61} \otimes K_{61} & \gamma_5 \otimes M_{62} \otimes K_{62} & \ldots & \partial \otimes 1 \otimes 1_N
\end{pmatrix}
\] (16)

where \( \partial = \gamma^\mu \partial_\mu \). The reality condition of the operator \( D \) implies that:

\[ M_{mn}^* = M_{nm}, \ K_{mn}^* = K_{nm}, \ m \neq n = 1, 2, \ldots, 6 \] (17)

The \( M_{mn} \) matrices determine the tree level vacuum expectation values of the higgs fields and control the corresponding symmetry breaking scheme. The \( N \times N, K_{ij} (i, j = 1, \ldots, 6) \) matrices where \( N \) is the fermionic family number; are introduced to include informations about the mixing between the various generations of quarks and leptons.

Now, in order to get the standard model with two Higgs doublets, we have to choose:

\[ K_{13} = K_{24} = K, \] (18)
\[ K_{31} = K_{42} = K^* \]

and take:

\[ M_{13} = \begin{pmatrix} \tilde{M}_u \\ \tilde{M}_d \end{pmatrix} = S' \otimes \tilde{M}_u \otimes 1_N + S \otimes \tilde{M}_d \otimes 1_N \] (19)

and

\[ M_{24} = \begin{pmatrix} 0 \\ \tilde{M}_e \end{pmatrix} = S \otimes \tilde{M}_e \otimes 1_N \] (20)
where

\[ S = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]  \hspace{1cm} (21)

and

\[ S' = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]  \hspace{1cm} (22)

The remaining \( K_{mn} \) and \( M_{mn} \) matrices vanish. The \( \tilde{M}_u \), \( \tilde{M}_d \) and \( \tilde{M}_e \) are the Up, Down quarks-like and leptons mass matrices respectively such that:

\[ \tilde{M}_u = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} \]  \hspace{1cm} (23)

\[ \tilde{M}_d = C_{CKM} \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \]  \hspace{1cm} (24)

and

\[ \tilde{M}_e = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \]  \hspace{1cm} (25)

where

\[ C_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \]  \hspace{1cm} (26)
is the Cabbibo-Kobayashi-Maskawa matrix which is supposed non degenerate and the elements \(m_l, (l = u, c, t, d, s, b, e, \mu, \tau)\) are positive fermions masses).

If we denote by \(\Omega^p(A)\) the algebra of \(p\)-forms where \(\Omega^0(A) = A\), the elements of the canonical universal algebra \(\Omega_D(A)\) can be obtained from those of \(A\) as follows:

\[
\Omega_D(A) := \bigoplus_{p\geq 0} \Omega^p_D(A) = \frac{\pi(\Omega(A))}{\pi(Aux)}
\]  

where:

\[
\Omega^p_D(A) = \left( \frac{\pi(\Omega^p(A))}{\pi(Aux^p)} \right)
\]

represents the \(p\)-form canonical differential algebra and \(Aux^p\) are the \(p\) junks forms (non dynamical auxiliary fields). It is worth to mention that the scalar product between the elements of this algebra is defined as:

\[
\Omega^p_D(A) \otimes \Omega^q_D(A) \rightarrow IR
\]

\[
\langle \omega, \eta \rangle = \delta_{pq} Tr(\omega^* \eta Z)
\]

Here \(Tr\) stands for trace and the positive definite operator \(Z\) has as expression:

\[
Z = \begin{pmatrix}
x & 0 & 0 & 0 & 0 \\
0 & x & 0 & 0 & 0 \\
0 & 0 & y & 0 & 0 \\
0 & 0 & 0 & y & 0 \\
0 & 0 & 0 & 0 & z \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

with

\[
x \rightarrow 1_4 \otimes 1_2 \otimes x \otimes 1_N \\
x \rightarrow 1_4 \otimes 1_1 \otimes y \otimes 1_N \\
z \rightarrow 1_4 \otimes 1_3 \otimes z \otimes 1_N
\]

(\(x, y\) and \(z \in M_3(\mathbb{C})\) and \(1_D\) are \(DxD\) identity matrices). Notice that \(Z\) verifies the following commutation relations:
\( D, Z \) = 0 (32)

and

\[ Z, \pi (A) ] = 0 (33) \]

which corresponds to the case:

\[ x_0 = x, y_0 = y, z_0 = z (34) \]

with \( x, y, z \in \mathbb{R} \).

Now, the 1-form differential algebra \( \Omega^1 D (A) \) reads:

\[ \Omega^1 D (A) \wedge \pi \begin{cases} \Omega^1 (A) \end{cases} = \pi \begin{cases} \sum \alpha_i \delta \beta_i \end{cases} = \sum \pi \begin{cases} \alpha_i \end{cases} \left[ D, \pi \begin{cases} \beta_i \end{cases} \right] (35) \]

Then, the non-vanishing elements are given by:

\[ \pi (\omega) = \pi \begin{cases} \sum \alpha_i \delta \beta_i \end{cases} \]

where \( \alpha_i, \beta_i \in A \).

If we denote by \( \omega_m \mu \) and \( \Phi_{mn} \) the vector and scalar fields respectively and \( N_f \) the fermionic family number (in our case \( N_f = 3 \)) such that:

\[ (\omega_m \mu) = - (\omega_m \mu) (36) \]

and \( \Phi_{mn} = \Phi_{mn} \) (37)

Then, the non-vanishing elements are given by:

\[ \pi (\omega)_{ii} = \gamma_i \otimes \omega_i \mu \otimes \gamma_3 \otimes \gamma_{N_f, i} = \pi \begin{cases} \gamma_i \otimes \Phi_{31} \otimes K_{31} \end{cases} (38) \]

\[ \pi (\omega)_{31} = \gamma_5 \otimes \Phi_{31} \otimes K_{31} \]}

\[ \pi (\omega)_{24} = \gamma_5 \otimes \Phi_{24} \otimes K_{24} \]

\[ \pi (\omega)_{42} = \gamma_5 \otimes \Phi_{42} \otimes K_{42} \] (39)

with

\[ \left\{ [(\omega) \mu, \mathcal{A}] (\nu, \omega) \begin{array}{l} \mu \end{array} \begin{array}{l} \omega \end{array} = \pi \begin{cases} (\omega) \mu \end{cases} \right\} = \pi \begin{cases} (\omega) \mu \end{cases} = (\nu) \begin{cases} \Phi \end{cases} \]

which corresponds to the case:

\[ 0 = [Z, \mathcal{A}] \] (33)
\[ \Phi_{13} = \left( \begin{array}{c} \varphi_1 \\ \varphi_2 - 1 \end{array} \right) \otimes M_d \otimes \mathbf{1}_3 + \left( \begin{array}{c} \chi_1 \\ \chi_2 - 1 \end{array} \right) \otimes M_u \otimes \mathbf{1}_3 \] (40)

and

\[ \Phi_{24} = \left( \begin{array}{c} \varphi_1 \\ \varphi_2 - 1 \end{array} \right) \otimes M_e \otimes \mathbf{1}_1 \] (41)

Regarding the 2-form canonical differential algebra \( \Omega^2_D (\mathcal{A}) \), it is defined as:

\[ \Omega^2_D (\mathcal{A}) = \frac{\pi (\Omega^2 (\mathcal{A}))}{\text{Aux}^2} \] (42)

Where, the 2-form nonphysical (junks) differential algebra \( \text{Aux}^2 \) can be obtained as:

\[ \text{Aux}^2 = \{ \pi (\delta \omega) / \pi (\omega) = 0 \} \] (43)

which implies that:

\[ \omega^m_\mu = 0, \ m = 1, 2, \ldots 6 \] (44)

and

\[ \Phi_{mn} = 0, \ m \neq n = 1, 2, \ldots 6 \] (45)

Straightforward calculations using the fact that:

\[ \pi (\delta \omega) = \left\{ \pi (\delta \alpha_i \delta \beta^j) = \sum_i [\mathcal{D}, \pi (\alpha^i)] [\mathcal{D}, \pi (\beta^j)] \right\} \] (46)

as well as the relations:

\[ | M_{31} |^2 = 1_2 \otimes (\tilde{\mu}_u + \tilde{\mu}_d) \otimes \mathbf{1}_3 \]

\[ | M_{13} |^2 = 1_2 \otimes (\mu_u + \mu_d) 1_3 + \tau_3 \otimes \frac{1}{2} (\mu_u - \mu_d) \otimes 1_3 + \tau_1 \otimes \mu_{ud} \otimes 1_3 \]

\[ | M_{42} |^2 = 1_2 \otimes \tilde{\mu}_e \otimes \mathbf{1}_1 \] (47)
and

\[ | M_{24} |^2 = (1_2 - \tau_3) \otimes \frac{1}{2} \mu_e \otimes 1_1 \]

where

\[ \mu_a = \tilde{M}_a \tilde{M}_a^* \]

\[ \tilde{\mu}_a = \tilde{M}_a^* \tilde{M}_a \]

\[ \mu_{ab} = \tilde{M}_a \tilde{M}_b^* \]  \hspace{1cm} (48)

and

\[ \tilde{\mu}_{ab} = \tilde{M}_b^* \tilde{M}_a \]

(\( \tau_3 \) and \( \tau_1 \) are the Pauli matrices) show that the elements of this algebra have as expressions:

\[
\begin{align*}
(Aux^2)_{11} &= \gamma^\mu \gamma^\nu \otimes 1_3 \otimes 1_N + 1_4 \otimes i \varpi_2 \otimes \frac{1}{2} (\mu_d - \mu_u) \otimes 1_3 \otimes KK^* \\
& \quad + 1_4 \otimes i \varpi_3 \otimes \mu_{ud} \otimes 1_3 \otimes KK^*
\end{align*}
\]

\[
\begin{align*}
(Aux^2)_{22} &= \gamma^\mu \gamma^\nu \otimes \varpi_{1\mu\nu} \otimes 1_3 \otimes 1_N + 1_4 \otimes i \varpi_2 \otimes \mu_e \otimes 1_3 \otimes KK^* \\
(Aux^2)_{33} &= (Aux^2)_{44} = \gamma^\mu \gamma^\nu \otimes \varpi_{\mu\nu} \otimes 1_3 \otimes 1_N \\
(Aux^2)_{55} &= (Aux^2)_{66} = \gamma^\mu \gamma^\nu \otimes y_{\mu\nu} \otimes 1_3 \otimes 1_N
\end{align*}
\]  \hspace{1cm} (49)

and

\[ (Aux^2)_{mn} = 0, \quad m \neq n = 1, 6 \]

with

\[ \varpi_{1\mu\nu} = - \sum_i \alpha_i^i \partial_\mu \partial_\nu \beta_i^i \]

\[ \varpi_{\mu\nu} = - \sum_i \alpha_i^3 \partial_\mu \partial_\nu \beta_i^3 \]

\[ \varpi_2 = - \sum_i \alpha_i^1 [i \tau_3, \beta_i^1] \]  \hspace{1cm} (50)
\[ \mathcal{A}_3 = \sum_i \alpha_i^1 \left[ i \tau_1, \beta_i^1 \right] \]

and

\[ y_{\mu\nu} = -\sum_i \alpha_i^5 \partial_{\mu} \partial_{\nu} \beta_i^5 \] (51)

Now, taking into account the fact that the elements of \( \pi(\Omega^2(A)) \) are given by:

\[ \pi(\Omega^2(A)) = \{ \pi(C) = \pi(\delta\omega) + \pi(\omega^2), \omega \in \Omega^1(A) \} \] (52)

the orthogonality condition leads to the following canonical differential algebra \( \Omega^2_D(A) \) elements:

\[
\begin{align*}
\pi(C)_{11} &= \frac{1}{2} \gamma^{\mu\nu} \otimes F_{\mu\nu}^1 \otimes 1_3 \otimes 1_N \\
\pi(C)_{22} &= \frac{1}{2} \gamma^{\mu\nu} \otimes F_{\mu\nu}^2 \otimes 1_3 \otimes 1_N \\
\pi(C)_{33} &= \frac{1}{2} \gamma^{\mu\nu} \otimes F_{\mu\nu}^3 \otimes 1_3 \otimes 1_N \\
&\quad + 1_4 \otimes (\varphi^* \varphi - 1) \otimes \tilde{\mu}_d \otimes 1_3 \otimes K^* K + 1_4 \otimes (\mathcal{A}^* \mathcal{A} - 1) \otimes \tilde{\mu}_u \otimes 1_3 \otimes K^* K \\
&\quad + 1_4 \otimes \varphi^* \mathcal{A} \otimes \mu_{du} \otimes 1_3 \otimes K^* K + 1_4 \otimes \mathcal{A}^* \varphi \otimes \tilde{\mu}_{du} \otimes 1_3 \otimes K^* K \\
&\quad + \gamma^{\mu\nu} \otimes \frac{TrK^*K}{6N} \left\{ \begin{array}{l} tr(3\mu_d + \mu_e)(\varphi^* \varphi - 1)_{\mu\nu} + tr(3\mu_u)(\mathcal{A}^* \mathcal{A} - 1)_{\mu\nu} \\
+ tr(3\mu_{du})(\varphi^* \mathcal{A})_{\mu\nu} + tr(3\tilde{\mu}_{du})(\mathcal{A}^* \varphi)_{\mu\nu} \end{array} \right\} \quad (53) \\
\pi(C)_{44} &= \frac{1}{2} \gamma^{\mu\nu} \otimes F_{\mu\nu}^4 \otimes 1_3 \otimes 1_N + 1_4 \otimes (\varphi^* \varphi - 1) \otimes \tilde{\mu}_e \otimes 1_3 \otimes K^* K \\
&\quad + \gamma^{\mu\nu} \otimes \frac{TrK^*K}{6N} \left\{ \begin{array}{l} tr(3\mu_d + \mu_e)(\varphi^* \varphi - 1)_{\mu\nu} + tr(3\mu_u)(\mathcal{A}^* \mathcal{A} - 1)_{\mu\nu} \\
+ tr(3\mu_{du})(\varphi^* \mathcal{A})_{\mu\nu} + tr(3\tilde{\mu}_{du})(\mathcal{A}^* \varphi)_{\mu\nu} \end{array} \right\} \\
\pi(C)_{55} &= \pi(e)_{66} = \frac{1}{2} \gamma^{\mu\nu} \otimes F_{\mu\nu}^5 \otimes 1_3 \otimes 1_N \\
\pi(C)_{13} &= -\gamma^5 \gamma^\mu \otimes D_\mu \varphi \otimes M_d \otimes 1_3 \otimes K - \gamma^5 \gamma^\mu \otimes D_\mu \mathcal{A} \otimes M_a \otimes 1_3 \otimes K \\
\pi(C)_{31} &= -\gamma^5 \gamma^\mu \otimes (D_\mu \varphi)^* \otimes M_d^* \otimes 1_3 \otimes K^* - \gamma^5 \gamma^\mu \otimes (D_\mu \varphi)^* \otimes M_a^* \otimes 1_3 \otimes K^* \\
\pi(C)_{24} &= -\gamma^5 \gamma^\mu \otimes D_\mu \varphi \otimes M_e \otimes 1_1 \otimes K \\
\end{align*}
\]

and
\[ \pi (C)_{42} = -\gamma^5 \gamma^\mu \otimes (D^\mu \varphi)^* \otimes M^*_e \otimes 1 \otimes K^* \]  \hfill (54)

where

\[ \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \]  \hfill (55)

\[ \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \]  \hfill (56)

and

\[ D^\mu. = \partial^\mu. + \omega^1_\mu. - \omega^3_\mu \]  \hfill (57)

The \( F^a_{\mu\nu} \)'s are the gauge fields strength tensor.

Before we proceed with further details of the calculation, it is worth to mention that our approach is considered to be an extension of Chamssedine-Frohlich formalism and not that of Conne’s in the sense that like in ours, the former introduce the \( M_{ij} \) and \( K_{ij} \) matrices where in the later are taken to be proportional to the identity matrix (special case). Moreover, the difference between our approach and that of Chamssedine-Frohlich [20] – [22] lies on:

a) The generalization of the scalar product

b) The elimination of the junk forms (or the non dynamical auxiliary fields) by using the orthogonality condition instead of the field equations which is in general a very complicated procedure.

c) The modification of the algebra \( A \) (see Eq.(10)) for the gauge group \( U(2) \otimes U(1) \otimes U(3) \) instead of \( C^\infty (X) \otimes (M_2 (\mathbb{C}) \oplus \mathbb{C}) \) for \( U(2) \otimes U(1) \) in the order to include the strong interactions.

d) The modification and extension of the Dirac operator to have the general form given by Eq.(16). In the case of the standard model and to account for the strong interactions, quark masses and mixings the Dirac operator the following simplified expression:
Notice that, in order to insure that the gauge group $SU(3)$ remains unbroken, the mass matrix along the $M_3 (C)$ direction is taken to be zero, forcing the vanishing of the scalar field expectation value along the same direction. In the Chamssedine-Frohlich approach, the general form of the Dirac operator has the following general form:

$$D = \begin{pmatrix} \partial \otimes 1 \otimes 1_N & \gamma_5 \otimes M_{13} \otimes K_{13} & 0 & 0 & 0 \\ 0 & \partial \otimes 1 \otimes 1_N & \gamma_5 \otimes M_{24} \otimes K_{24} & 0 & 0 \\ \gamma_5 \otimes M_{31} \otimes K_{31} & 0 & \partial \otimes 1 \otimes 1_N & 0 & 0 \\ 0 & \gamma_5 \otimes M_{42} \otimes K_{42} & 0 & \partial \otimes 1 \otimes 1_N & 0 \\ 0 & 0 & 0 & 0 & \partial \otimes 1 \otimes 1_N \end{pmatrix}$$ (58)

and in the case of the standard model it becomes ($M_{13} = M_{23} = 0$):

$$D = \begin{pmatrix} \partial \otimes 1 \otimes 1_N & \gamma_5 \otimes M_{12} \otimes K_{12} & 0 \\ \gamma_5 \otimes M_{21} \otimes K_{21} & \partial \otimes 1 \otimes 1_N & \gamma_5 \otimes M_{23} \otimes K_{23} \\ \gamma_5 \otimes M_{31} \otimes K_{31} & \gamma_5 \otimes M_{32} \otimes K_{32} & \partial \otimes 1 \otimes 1_N \end{pmatrix}$$ (59)

and in the extended approach, the matrices $M_{13}$ and $M_{24}$ contain the quarks and leptons mass respectively. Similarly for the matrices $K_{13}$ and $K_{24}$ where they include the quarks and leptons mixings. However, for the Chamssedine-Frohlich approach, we have just the matrices $M_{12}$ and $K_{12}$ and therefore, we cannot include in a natural way the quarks and their mixings. Furthermore, even if we try to include in their approach the $SU(3)$ gauge group in a commutative way and decoupling it from the rest, we will face the quark mass problem.

In what follows, we will concentrate on the scalar fields sector. Starting from the definition of the bosonic action $\tilde{\Im}_B$:

$$\tilde{\Im}_B = \int d^4x \, Tr (trC^* C)$$ (61)
and after diagonalization, by introducing a mixing angle $\vartheta$ such that:

$$\tan \vartheta = \frac{2tr3\mu_{du}}{tr(3(\mu_d - \mu_u) + \mu_e)} \approx \frac{2m_t m_b}{m_t^2 - m_b^2}$$

and spontaneous breaking of the gauge symmetry as:

$$U(2)_L \otimes U(1) \otimes U(3) \langle \varphi, \kappa \rangle \rightarrow U(1) \otimes U(1)' \otimes U(3)$$

where $\varphi$ and $\kappa$ are two complex doublets where:

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \equiv \begin{pmatrix} \Phi_1^+ \\ \eta_1 + i \kappa_1 \end{pmatrix}$$

and

$$\kappa = \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix} \equiv \begin{pmatrix} \Phi_2^+ \\ \eta_2 + i \kappa_2 \end{pmatrix}$$

The corresponding vacuum expectation values $\langle \varphi \rangle$ and $\langle \kappa \rangle$ are chosen as:

$$\langle \varphi \rangle = \begin{pmatrix} 0 \\ \langle \varphi_0 \rangle \end{pmatrix}, \langle \kappa \rangle = \begin{pmatrix} 0 \\ \langle \kappa_0 \rangle \end{pmatrix}$$

By redefining the scalar fields $\varphi$ and $\kappa$ as:

$$\varphi \rightarrow \alpha_1 \varphi$$

and

$$\kappa \rightarrow \alpha_2 \kappa$$

with

$$\alpha_{1,2} = \frac{\tilde{L}}{m_t \cos \vartheta \pm m_b \sin \vartheta}$$

where
\[ \tilde{L} = \frac{1}{[3 (x + y) Tr K^* K]^{1/2}} \]  

A lengthy calculation leads to the Higgs sector, to the following expression of the potential \( V(\varphi, \chi) \):

\[
V(\varphi, \chi) = \xi_1 (\varphi^* \varphi - \langle \varphi_0 \rangle^2)^2 + \xi_2 (\chi^* \chi - \langle \chi_0 \rangle^2)^2 + \xi_3 (\varphi^* \varphi) + \xi_4 (\chi^* \chi) \\
+ \xi_5 (\varphi^* \varphi) (\chi^* \chi) + \xi_6 (\varphi^* \varphi) (\varphi^* \chi + \chi^* \varphi) \\
+ \xi_7 (\chi^* \chi) (\varphi^* \chi + \chi^* \varphi) + \xi_8 (\varphi^* \chi + \chi^* \varphi)^2 \\
+ \xi_9 (\varphi^* \chi + \chi^* \varphi)
\]  

where

\[
\begin{align*}
\xi_1 &= 3y\Sigma \left( m_1^4 \cos^4 \theta + m_b^4 \sin^4 \theta + 2m_1^2 m_b^2 \cos^2 \theta \sin^2 \theta + 2m_1^3 m_b \sin 2\theta + 2m_1^3 m_t \sin^2 2\theta \right) (\alpha_1)^4 \\
\xi_2 &= 3y\Sigma \left( m_1^4 \sin^4 \theta + m_b^4 \cos^4 \theta + 2m_1^2 m_b^2 \cos^2 \theta \sin^2 \theta - 2m_1^3 m_b \sin 2\theta + 2m_1^3 m_t \sin^2 2\theta \right) (\alpha_2)^4 \\
\xi_3 &= 6y\Sigma \left( m_1^4 \cos^4 \theta + m_b^4 \sin^4 \theta + m_1^2 m_b^2 - 2m_1^3 m_b \sin 2\theta \right) (\alpha_1)^2 \\
\xi_4 &= 6y\Sigma \left( m_1^4 \sin^4 \theta + m_b^4 \cos^4 \theta + m_1^2 m_b^2 - 2m_1^3 m_b \sin 2\theta \right) (\alpha_2)^2 \\
\xi_5 &= 6y\Sigma \left( (m_1^4 + m_b^4) \cos^2 \theta \sin^2 \theta + 2m_1^2 m_b^2 \left( \cos^4 \theta + \sin^4 \theta \right) - 4m_1^3 m_t \sin^2 2\theta \right) (\alpha_1)^2 (\alpha_2)^2 \\
\xi_6 &= 3y\Sigma \left( \sin 2\theta \left( -m_1^4 \cos^2 \theta + m_b^4 \sin^2 \theta \right) + \\
&\phantom{=} \frac{2m_1^2 m_b (\sin^2 \theta \cos 2\theta + \frac{1}{2} \sin^2 2\theta)}{4} \right) (\alpha_1)^3 (\alpha_2) \\
\xi_7 &= 3y\Sigma \left( \sin 2\theta \left( -m_1^4 \sin^2 \theta + m_b^4 \cos^2 \theta \right) - m_1^2 m_b^2 \cos 2\theta \sin 2\theta \right) \right) (\alpha_1) (\alpha_2)^3 \\
\xi_8 &= 3y\Sigma \left( \frac{1}{4} \left( m_1^2 + m_b^2 \right)^2 \sin^2 2\theta + 2m_1^3 m_t \cos^2 2\theta \right) (\alpha_1)^2 (\alpha_2)^2 \\
\xi_9 &= 3y\Sigma \left( \left( m_1^2 - m_b^2 \right)^2 \sin 2\theta + 4m_1^3 m_b \cos 2\theta \right) (\alpha_1) (\alpha_2)
\end{align*}
\]

and

\[
\xi_9 = 3y\Sigma \left( \left( m_1^2 - m_b^2 \right)^2 \sin 2\theta + 4m_1^3 m_b \cos 2\theta \right) (\alpha_1) (\alpha_2)
\]

It is easy now, to obtain the following mass matrices for the real neutral, imaginary neutral and charged scalar fields \((\eta_1, \eta_2), (\chi_1, \chi_2)\) and \((\Phi_1^\pm, \Phi_2^\pm)\) respectively:
\[ M^2(\eta_1, \eta_2) = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} \]

\[ M^2(x_1, x_2) = \begin{pmatrix} \tilde{\Delta}_{11} & \tilde{\Delta}_{12} \\ \tilde{\Delta}_{21} & \tilde{\Delta}_{22} \end{pmatrix} \]  \tag{71}

\[ M^2(\Phi_1^\pm, \Phi_2^\mp) = \begin{pmatrix} \hat{\Delta}_{11} & \hat{\Delta}_{12} \\ \hat{\Delta}_{21} & \hat{\Delta}_{22} \end{pmatrix} \]

where

\[ \Delta_{11} = 6\xi_1 \langle \varphi_0 \rangle^2 + (\xi_5 + 4\xi_8) \langle x_0 \rangle^2 + 6\xi_6 \langle \varphi_0 \rangle \langle x_0 \rangle + \xi_3 \]

\[ \Delta_{12} = \Delta_{21} = 6\xi_6 \langle \varphi_0 \rangle^2 + 3\xi_7 \langle x_0 \rangle^2 + 2(\xi_5 + 2\xi_8) \langle \varphi_0 \rangle \langle x_0 \rangle + \xi_9 \]

\[ \Delta_{22} = (\xi_5 + 4\xi_8) \langle \varphi_0 \rangle^2 + 6\xi_2 \langle x_0 \rangle^2 + 6\xi_7 \langle \varphi_0 \rangle \langle x_0 \rangle + \xi_4 \]

\[ \tilde{\Delta}_{11} = 2\xi_1 \langle \varphi_0 \rangle^2 + \xi_5 \langle x_0 \rangle^2 + 2\xi_6 \langle \varphi_0 \rangle \langle x_0 \rangle + \xi_3 \]

\[ \tilde{\Delta}_{12} = \tilde{\Delta}_{21} = \xi_6 \langle \varphi_0 \rangle^2 + \xi_7 \langle x_0 \rangle^2 + 2\xi_8 \langle \varphi_0 \rangle \langle x_0 \rangle + \xi_9 \]

\[ \tilde{\Delta}_{22} = 2\xi_1 \langle \varphi_0 \rangle^2 + \xi_5 \langle x_0 \rangle^2 + 2\xi_7 \langle \varphi_0 \rangle \langle x_0 \rangle + \xi_4 \]

\[ \hat{\Delta}_{11} = \xi_6/2 + \xi_4 \langle \varphi_0 \rangle^2 + \xi_6 \langle x_0 \rangle^2/2 + \xi_6 \langle \varphi_0 \rangle \langle x_0 \rangle \]

\[ \hat{\Delta}_{12} = \hat{\Delta}_{21} = \xi_6/2 + \xi_6 \langle \varphi_0 \rangle^2 + \xi_7 \langle x_0 \rangle^2 + 2\xi_8 \langle \varphi_0 \rangle \langle x_0 \rangle \]

and

\[ \hat{\Delta}_{22} = \xi_5 \langle \varphi_0 \rangle^2/2 + \xi_2 \langle x_0 \rangle^2 + 2\xi_7 \langle \varphi_0 \rangle \langle x_0 \rangle + \xi_4/2 \]

After a diagonalisation of the above symmetric matrices (Eqs.71), and choosing \( \vartheta = \frac{\pi}{4} \), it is easy to show that:
\[
m_t \approx 2.4m_b \tag{72}
\]
\[
g = (12N_f x)^{-1/2} \tag{73}
\]

\[
g' = \left(12N_f \left(\frac{x}{4} + y\right)\right)^{-1/2} \tag{74}
\]

and

\[
g_3 = (12N_f z)^{-1/2}
\]

where \(g, g'\) and \(g_3\) are the coupling constants of the gauge groups \(U(2), U(1), U(3)\) respectively. Similarly,

\[
\xi_1 \approx \frac{0.17\Sigma}{x (Tr KK^*)^2},
\]
\[
\xi_2 \approx \frac{0.30\Sigma}{x (Tr KK^*)^2},
\]
\[
\xi_3 \approx \frac{0.30m_t^2\Sigma}{Tr KK^*},
\]
\[
\xi_4 \approx \frac{1.50m_t^2\Sigma}{Tr KK^*},
\]
\[
\xi_5 \approx \xi_1,
\]
\[
\xi_6 = \xi_7 = 0,
\]
\[
\xi_8 \approx \frac{0.20\Sigma}{x (Tr KK^*)^2},
\]
\[
\xi_9 \approx \frac{2.m_t^2\Sigma}{Tr KK^*}
\]

\[
\Sigma = Tr (KK^*)^2 - (3N)^{-1} (Tr KK^*)^2 \tag{76}
\]

and

\[
\frac{\langle \varphi_0 \rangle}{\langle \varphi_0 \rangle} \approx 70 \tag{77}
\]

which means that \(\langle \varphi_0 \rangle \ll \langle \varphi_0 \rangle\). Straightforward calculations give
\[ \langle \phi_0 \rangle^2 \approx 9x \, TrKK^* m_t^2 \]  
(78)

and if we take \( M_t \approx 2M_W \), we deduce that:

\[ \sqrt{\frac{TrKK^* x}{x}} \approx 12 \]  
(79)

Regarding the Higgs masses, we obtain first from the eigenstates of Eq.(71); the CP-odd pseudoscalar \( A^0 \), CP-even scalars \( H^0 \) and \( h^0 \) and charged scalars \( H^\pm \) related to the state systems \((\eta_1, \eta_2), (\xi_1, \xi_2) \) and \((\Phi_1^\pm, \Phi_2^\pm) \) respectively, then we obtain in terms of the top quark mass the following approximate mass eigenvalues relations:

\[
\begin{align*}
M_{A^0}^2 &= M_{H^\pm}^2 \approx \frac{2.64 \Sigma}{x \frac{1}{4} (TrKK^*)^2} M_t^2, \\
M_{H^0}^2 &\approx \frac{3.15 \Sigma}{x \frac{1}{2} (TrKK^*)^2} M_t^2, \\
M_{h^0}^2 &\approx \frac{0.42 \Sigma}{x \frac{1}{2} (TrKK^*)^2} M_t^2
\end{align*}
\]  
(80)

with:

\[ M_t^2 = c^2 (m_t \langle \phi_0 \rangle + m_b \langle \kappa_0 \rangle) \]

Now, using the Schwartz inequality, one can deduce that:

\[ \Sigma \leq \frac{26}{9} (TrK^* K)^2 \]  
(81)

and therefore, we get for the CP-odd pseudoscalar \( A^0 \) and charged scalars \( H^\pm \) the constraint:

\[ M_{A^0} = M_{H^\pm} \lesssim 9.57 M_t \]  
(82)

Similarly, for the CP-even scalars \( H^0 \) and \( h^0 \) one can show that:

\[ M_{H^0} \lesssim 10.45 M_t \]  
(83)

and

\[ M_{h^0} \lesssim 3.81 M_t \]  
(84)
Notice that for the $M_{H^\pm}$ and $M_{H^0}$, our results are compatible with the actual experimental bounds which allow for a heavy Higgs scenario.

Regarding the tree level NCG unitarity bound, we have to consider the scalar elastic and inelastic processes of the form $B_1 + B_2 \rightarrow B_3 + B_4$ by using the partial wave decomposition technique to the corresponding amplitude $M$:

$$M(s, t, u) = 16\pi \sum_{l=0}^{\infty} (2l + 1) P_l(\cos \theta) A_l(s) \quad (85)$$

To get upper bound limits to the scalar potential parameters of Eq.(75), one has to require that the tree level unitarity has to be preserved in all possible scattering processes. This is equivalent to the fact that the $s$ partial wave amplitude $A_0$ for the scalar-scalar, gauge boson- gauge boson and gauge boson-scalar has to satisfy $|A_0| \leq 1/2$ in the high energy limit. We remind the reader that at a very high energy, the equivalence theorem states that the amplitude of a scattering process involving longitudinal gauge bosons $W_\mu^\pm$ and $Z_\mu$ may be approximated by the scalar amplitude in which gauge bosons are replaced by their corresponding Goldstone bosons and the unitarity constraints can be implemented by only considering the pure scalar scatterings. In what follows, we limit ourselves to pure scalar scattering processes dominated by quartic interactions and follow the technique introduced in ref.[25]. The $S$ matrix expressed in terms of the physical fields can be transformed into an $S$ matrix for the non physical fields $w_j^\pm, h_j$ and $z_j (i=1, 2)$ such that:

$$\varphi = \begin{pmatrix} w_1^+ \\ \langle \varphi_0 \rangle^* + \frac{1}{2}(h_1 + z_1) \end{pmatrix}, \quad \kappa = \begin{pmatrix} w_2^+ \\ \langle \kappa_0 \rangle^* + \frac{1}{2}(h_2 + z_2) \end{pmatrix} \quad (86)$$

and

$$w_j^\pm = \alpha_j^{-1} \Phi_j^\pm; \quad h_j = 2\alpha_j^{-1} \eta_j; \quad z_i = 2\alpha_j^{-1} \zeta_j; \quad \langle \varphi_0 \rangle^* = \alpha_1^{-1} \langle \varphi_0 \rangle; \quad \langle \kappa_0 \rangle^* = \alpha_2^{-1} \langle \kappa_0 \rangle \quad (87)$$

by making a unitary transformation. Thus, the full set of the scalar scattering processes can be expressed as an $S$ matrix $22 \times 22$ composed of 4 submatrices $M_{6 \times 6}^1, M_{6 \times 6}^2, M_{2 \times 2}^3$ and $M_{8 \times 8}^4$ which do not couple with each other and where the entries are the quartic
couplings mediating the scattering processes. In our case, straightforward simplifications lead to:

(a) Basis \((w_1^+ w_2^-, w_2^+ w_1^-, h_1 z_2, h_2 z_1, z_1 z_2, h_1 h_2)\):

\[
M_{6 \times 6}^1 = \begin{bmatrix}
\xi_5 - \frac{1}{4} \xi_8 & \xi_8 & 0 & 0 & \xi_8 & \xi_8 \\
\xi_8 & \xi_5 + \frac{1}{4} \xi_8 & 0 & 0 & \xi_8 & \xi_8 \\
0 & 0 & \xi_5 & \frac{\xi_8}{2} & 0 & 0 \\
0 & 0 & \frac{\xi_8}{2} & \xi_5 & 0 & 0 \\
\frac{\xi_8}{2} & \frac{\xi_8}{2} & 0 & 0 & \xi_5 + \xi_8 & \frac{\xi_8}{2} \\
\frac{\xi_8}{2} & \frac{\xi_8}{2} & 0 & 0 & \frac{\xi_8}{2} & \xi_5 + \xi_8
\end{bmatrix}
\] (88)

(b) Basis \((w_1^+ w_1^-, w_2^+ w_2^-, \frac{z_1 z_2}{\sqrt{2}}, \frac{z_2 z_1}{\sqrt{2}}, h_1 h_1, h_2 h_2)\):

\[
M_{6 \times 6}^2 = \begin{bmatrix}
4 \xi_1 & \xi_5 + \frac{1}{2} \xi_8 & \sqrt{2} \xi_1 & \frac{1}{\sqrt{2}} \xi_5 & \sqrt{2} \xi_1 & \frac{1}{\sqrt{2}} \xi_5 \\
\xi_5 + \frac{1}{2} \xi_8 & 4 \xi_2 & \frac{1}{\sqrt{2}} \xi_5 & \sqrt{2} \xi_2 & \frac{1}{\sqrt{2}} \xi_5 & \sqrt{2} \xi_2 \\
\sqrt{2} \xi_1 & \frac{1}{\sqrt{2}} \xi_5 & 3 \xi_1 & \xi_5 + \frac{1}{2} \xi_8 & \xi_1 & \frac{1}{2} \xi_5 \\
\frac{1}{\sqrt{2}} \xi_5 & \sqrt{2} \xi_2 & \xi_5 + \frac{1}{2} \xi_8 & 3 \xi_2 & \frac{1}{2} \xi_5 & \xi_2 \\
\sqrt{2} \xi_1 & \frac{1}{\sqrt{2}} \xi_5 & \frac{1}{2} \xi_5 & 3 \xi_1 & \xi_5 + \frac{1}{2} \xi_8 \\
\frac{1}{\sqrt{2}} \xi_5 & \sqrt{2} \xi_2 & \frac{1}{2} \xi_5 & \xi_2 & \xi_5 + \frac{1}{2} \xi_8 & 3 \xi_2
\end{bmatrix}
\] (89)

(c) Basis \((h_1 z_1, h_2 z_2)\):

\[
M_{2 \times 2}^3 = \begin{bmatrix}
2 \xi_1 & \frac{1}{2} \xi_8 \\
\frac{1}{2} \xi_8 & 2 \xi_2
\end{bmatrix}
\] (90)

(d) Basis \((h_1 w_1^+, h_2 w_1^+, z_1 w_1^+, z_2 w_1^+, h_1 w_2^+, h_2 w_2^+, z_1 w_2^+, z_2 w_2^+)\):
\[
M_{8\times8}^4 = \begin{bmatrix}
2\xi_1 & 0 & 0 & 0 & 0 & \frac{1}{2}\xi_8 & 0 & 0 \\
0 & \xi_5 & 0 & 0 & \frac{1}{2}\xi_8 & 0 & 0 & 0 \\
0 & 0 & 2\xi_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \xi_5 & 0 & 0 & \frac{1}{2}\xi_8 & 0 \\
0 & \frac{1}{2}\xi_8 & 0 & 0 & \xi_5 & 0 & 0 & 0 \\
\frac{1}{2}\xi_8 & 0 & 0 & 0 & 2\xi_2 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2}\xi_8 & 0 & 0 & \xi_5 & 0 \\
0 & 0 & \frac{1}{2}\xi_8 & 0 & 0 & 0 & 0 & 2\xi_2
\end{bmatrix}
\] (91)

Now, using Mathematica one can diagonalize the \(M^i\) \((i = 1, 4)\) matrices to get the following eigensvalues:

\[
\Omega_1^1 = \Omega_2^1 = \xi_5 - \frac{1}{2}\xi_8, \quad \Omega_3^1 = \xi_5 + \frac{5}{2}\xi_8, \quad \Omega_4^1 = \Omega_5^1 = \xi_5 + \frac{1}{2}\xi_8
\] (92)

\[
\Omega_{1,2}^2 = 3(\xi_1 + \xi_2) \pm \sqrt{9(\xi_1 - \xi_2)^2 + (2\xi_5 + \xi_8)^2}, \quad \Omega_{3,4}^2 = \Omega_{5,6}^2 = \xi_1 + \xi_2 \pm \sqrt{(\xi_1 - \xi_2)^2 + \frac{1}{4}\xi_8^2}
\] (93)

\[
\Omega_{1,2}^3 = \Omega_{3,4}^2
\] (94)

and

\[
\Omega_1^4 = \Omega_4^1, \quad \Omega_2^4 = \Omega_2^1, \quad \Omega_3^4 = \Omega_5^1, \quad \Omega_4^4 = \Omega_5^1, \quad \Omega_5^4 = \Omega_{5,6}^2, \quad \Omega_6^4 = \Omega_{4,6}^2, \quad \Omega_7^4 = \Omega_1^1, \quad \Omega_8^4 = \Omega_2^1
\] (95)

where \(\Omega_i^j\) stands for the \(i^{th}\) eigenvalue of the matrices \(M^j\). Now, imposing the unitarity condition

\[
|\Omega_i^j| \leq 8\pi
\] (96)

and replacing the \(\xi_i\)'s by their expressions (Eqs.75), we end up with the following solution:

\[
\frac{\Sigma}{(TrKK^*)^2} \lesssim \frac{8\pi x}{14,6}
\] (97)
which can be considered as a new unitarity constraints between the various NCG parameters. Notice that in order that Eq.(97) is compatible with the Schwartz inequality (Eq.(81)), one has to have $x \gtrsim 1.7$.

Regarding the quantization, in the framework of NCG, there is no satisfactory procedure which has been developed yet treating the gauge and higgs bosons in an equal footing. Therefore, we expect that the quantum fluctuations may badly violate the tree level NCG constraints and relations. In principle, the change in the quantization rules is needed around certain energy scale and we have to assume that just below such a scale, the standard quantization method makes a good approximation. In this case, we can start from the classical lagrangian and use the conventional quantization.

In what follow, we apply this approach to our NCG approach of the two doublets Higgs model through simple examples and test whether it is possible to preserve the tree level mass, coupling and unitarity relations. Let us take for example the ratio $\frac{\xi}{g}$. It is easy to see from the relations Eqs.(73)-(74) and Eq.(75) that:

$$\frac{\xi_8}{g} \approx \frac{1.2\Sigma}{x^{1/2} (TrKK^*)^2}$$

where

$$g = \frac{g_1}{\sin \theta_w}$$

Now, if eq.(98) will hold at any scale $\mu$ for given independent values of the NCG parameters $x$, $\Sigma$ and $TrKK^*$, the corresponding $\beta$-functions $\beta_{\xi_8}$ and $\beta_g$ have to verify the same relation of Eq.(98). However, the one loop $\beta$-functions with complicated expressions in terms of the various couplings (see Eq.(99)) do not seem to satisfy this relation for any values of $x$, $\Sigma$ and $TrKK^*$. Therefore, at the one loop order, the constraint Eq.(98) is not preserved under the renormalization flow. To see the complexity of the $\beta$-functions expressions, we have derived (based on refs.[26], [27] and [28]) some of the coupling constants renormalization group equations (R.G.) for the two doublets Higgs model:

$$16\pi^2 \frac{d\beta_{\xi_8}}{dt} = b_1 \xi_8^3; \; (b_1,b_2,b_3)=(\frac{11}{6},-3,-7)$$

$$16\pi^2 \frac{d\beta_g}{dt} = (\Lambda_1+\Lambda_2 g^2) \eta_8; \; \Lambda_1=-\frac{17}{20}g_1^2-\frac{9}{4}g_2^2-8g_3^2; \; \Lambda_2=-\frac{2}{4}$$

$$16\pi^2 \frac{d\beta_{\xi_1}}{dt} = (\Lambda_3+\Lambda_4 g^2) \eta_1; \; \Lambda_3=-\frac{3}{4}g_1^2-\frac{9}{4}g_2^2-8g_3^2; \; \Lambda_4=\frac{1}{4}$$

$$16\pi^2 \frac{d\xi_1}{dt} = 24\xi_1^3+2\xi_1 \xi_8+2\xi_8^2-(9g_2^2+g_1^2)\xi_1+(\frac{2}{4}g_2^2+\frac{27}{40}g_1^4)+\frac{9}{4}g_3^2g_1^2+12g_3^2\xi_1-6g_3^4$$

$$16\pi^2 \frac{d\xi_8}{dt} = 24\xi_8^3+2\xi_1 \xi_8+2\xi_8^2-(9g_2^2+g_1^2)\xi_8+(\frac{2}{4}g_2^2+\frac{27}{40}g_1^4)+\frac{9}{4}g_3^2g_1^2+12g_3^2\xi_8-6g_3^4$$

$$16\pi^2 \frac{d\xi_1}{dt} = 12\xi_1^3+24\xi_1 \xi_8+24\xi_8^2-2(9g_2^2+g_1^2)\xi_1+(\frac{2}{4}g_2^2+\frac{27}{40}g_1^4)-\frac{9}{4}g_3^2g_1^2+3g_3^2\xi_8-6g_3^4$$
and

$$16\pi^2 \frac{d\xi_8}{dt} = 32\xi_8^2 + 4\xi_5\xi_8 + 2\xi_5^2 - (9g_2^2 + \frac{9}{5}g_1^2)\xi_8 + 6(g_t^2 + g_b^2)\xi_8 - 6g_b^2g_t^2$$

Here $g_1, g_2, g_3, g_t$ and $g_b$ denote the electromagnetic, weak, strong, top Yukawa and bottom Yukawa couplings respectively. Moreover, an $SU(5)$ normalization is used for which the Weinberg $\theta_w$ angle is defined through the relation

$$g_2^2\sin^2\theta_w = \frac{3}{5}g_1^2\cos^2\theta_w$$

(100)

and the evolution parameter $t$ is defined as:

$$t = \ln\left(\frac{\mu}{\mu_0}\right)$$

(101)

($\mu_0$ is some reference scale e.g. $\mu_0 = M_Z$). We have also neglected all fermionic Yukawa couplings except those of the top and bottom quarks.

We notice that if the NCG parameters $x, \Sigma$ and $TrKK^*$ are scale dependent, the NCG relations, Schwartz and unitarity bounds (Eqs.(75),(81) and (97)) cannot be simultaneously verified. To be more explicit, let us assume that the relation in Eq.(73) holds for any scale $\mu$. Plugging the latter into the NCG relations Eq.(75), using the fact that:

$$TrKK^* = \frac{4}{9xg_t^2}$$

(102)

and the solutions of the renormalization group equations of $g_1, g_2, g_t$ and $\xi_1$ as are given by Eqs.(99), one can fix exactly the $\mu$-dependence of the $\Sigma$ parameter. However, the latter will not be compatible with the unitarity constraint(Eq.(97)), Schwartz inequality and the other NCG relations (Eqs.(81) and Eqs.(75)) under the renormalization flow of the other couplings $\xi_2, \xi_3...$etc. Thus, the NCG constraints are not renormalization group invariant and we cannot have one scale $\mu$ where all the relations are satisfied simultaneously.

**Conclusion**

As a conclusion, we cannot satisfy simultaneously all the tree level NCG mass, couplings and unitarity relations at the same energy scale and therefore we will face a sort of incompatibility with the renormalization group approach if the NCG constraints holds. Thus, the two doublets higgs model can be nicely constructed classically within this extended Chamesseddine-Frohlich approach to NCG but we have to look for an appropriate quantization procedure in the context of Noncommutative geometry.
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Hamilton-Jacobi Formulation of A Non-Abelian Yang-Mills Theories

W. I. Eshraim* and N. I. Farahat†

Department of Physics
Islamic University of Gaza
P.O.Box 108, Gaza, Palestine

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Abstract: A non-Abelian theory of fermions interacting with gauge bosons is treated as a constrained system using the Hamilton-Jacobi approach. The equations of motion are obtained as total differential equations in many variables. The integrability conditions are satisfied, and the set of equations of motion is integrable. A comparison with Dirac’s method is done.

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1. Introduction

The most common method for investigating the Hamiltonian treatment of constrained systems was initiated by Dirac[1]. The main feature of his method is to consider primary constraints first. All constraints are obtained using consistency conditions. Besides, he showed that the number of degrees of freedom of the dynamical system can be reduced. Hence, the equations of motion of constrained system are obtained in terms of arbitrary parameters.

The canonical method (or Güler’s method) developed Hamilton-Jacobi formulation to investigate constrained systems [2-3]. The Hamilton-Jacobi treatment of constrained systems leads us to obtain the equations of motion as total differential equations in many variables. These equations are integrable if the corresponding system of partial differential equations is a Jacobi system [3,4,5]. Since there are few physical examples were discussed by using Hamilton-Jacobi approach [6-9], it is still necessary to study more

* wibrahim_7@hotmail.com
† nfarahat@iugaza.edu.ps
of them and compare the results that can be obtained by Dirac’s method. In this paper, a non-abelian theory of fermions interacting with gauge bosons will be studied by using both Hamilton-Jacobi formulation and Dirac’s method.

A review of the Hamilton-Jacobi approach can be introduced as follows: If the rank of the Hess matrix

$$A_{ij} = \frac{\partial^2 L(q_i, \dot{q}_i, \tau)}{\partial \dot{q}_i \partial \dot{q}_j}, \quad i, j = 1, 2, \ldots, n,$$

is \((n - r), r < n\), then the standard definition of a linear momenta

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \quad a = 1, 2, \ldots, n - r,$$

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \quad \mu = n - r + 1, \ldots, n,$$

enables us to solve eq.(2) for \(\dot{q}_a\) as

$$\dot{q}_a = \dot{q}_a(q_i, \dot{q}_\mu, p_b) \equiv \omega_a.$$  (4)

Substituting eq.(4), into eq.(3), we obtain the constraints as

$$H'_\mu \equiv p_\mu + H_\mu(\tau, q_i, p_a) = 0,$$  (5)

where

$$H_\mu = -\left. \frac{\partial L}{\partial \dot{q}_\mu} \right|_{\dot{q}_a = \omega_a}.$$  (6)

The usual Hamiltonian \(H_0\) is defined as

$$H_0 = -L + p_\alpha \omega_\alpha - \dot{q}_\mu H_\mu.$$  (7)

Like functions \(H_\mu\), the function \(H_0\) is not an explicit function of the velocities \(\dot{q}_\nu\). Therefore, the Hamilton-Jacobi function \(S(\tau, q_i)\) should satisfy the following set of Hamilton-Jacobi partial differential equations (HJPDE) simultaneously for an extremum of the function:

$$H'_\alpha \left( t_\beta, q_\alpha, P_i = \frac{\partial S}{\partial q_i}, P_0 = \frac{\partial S}{\partial t_0} \right) = 0,$$  (8)

where

$$\alpha, \beta = 0, n - r + 1, \ldots, n; \quad a = 1, 2, \ldots, n - r,$$

and

$$H'_\alpha = p_\alpha + H_\alpha.$$  (9)

The canonical equations of motion are given as total differential equations in variables \(t_\beta,\)

$$dq_p = \frac{\partial H'_\alpha}{\partial p_\alpha} dt_\alpha, \quad p = 0, 1, \ldots, n; \quad \alpha = 0, n - r + 1, \ldots, n,$$  (10)

$$dp_a = -\frac{\partial H'_\alpha}{\partial q_a} dt_\alpha, \quad a = 1, \ldots, n - r,$$  (11)

$$dp_\mu = -\frac{\partial H'_\alpha}{\partial q_\mu} dt_\alpha, \quad \alpha = 0, n - r + 1, \ldots, n.$$  (12)
\[ dZ = \left( -H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_\alpha} \, dt_\alpha \right), \]  

where

\[ Z \equiv S(t_\alpha, q_\alpha), \]

being the action. Thus, the analysis of a constrained system is reduced to solve equations (10-12) with constraints

\[ H'_\alpha(t_\beta, q_\alpha, P_i) = 0, \quad \alpha, \beta = 0, n - r + 1, \ldots, n. \]  

Since the equations above are total differential equations, integrability conditions should be checked. These equations of motion are integrable \[3,4,5\] if and only if the variations of \( H'_\alpha \) vanish identically, that is

\[ dH'_\alpha = 0. \]

If they do not vanish identically, then we consider them as new constraints. This procedure is repeated until a complete system is obtained.

This paper is arranged as follows: Dirac’s method is used in sect.2 and Güler’s method in sect.3. The paper closes with a conclusion in sect.4.

1.1 Dirac’s method

Consider the Lagrangian density for a non-Abelian theory of fermions interacting with
gauge bosons as

\[ L = -\frac{1}{4}(F_{\mu\nu}^a)^2 + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi + \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2, \]

where \( \xi \) can be any finite constant.

In Eq.(17) \( F_{\mu\nu}^a \) is given by the formula

\[ F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c, \]

where \( f^{abc} \) are the structure constants of the Lie algebra and \( g \) represents the coupling constant.

The generalized momenta (2) and (3) read as

\[ \pi^i_a = \frac{\partial L}{\partial \dot{A}_i^a} = -F_{\alpha i}^a, \]

\[ \pi^0_a = \frac{\partial L}{\partial \dot{A}_0^a} = \frac{1}{\xi} \partial_\mu A_\mu^a, \]

\[ p_\psi = \frac{\partial L}{\partial \dot{\psi}} = i\bar{\psi}\gamma^0 = -H_\psi, \]

\[ p_\bar{\psi} = \frac{\partial L}{\partial \dot{\bar{\psi}}} = 0 = -H_{\bar{\psi}}, \]
\begin{equation}
p_{\mu} = \frac{\partial L}{\partial \dot{A}_{\mu}} = 0 = -H_{\mu},
\end{equation}

where we must call attention to necessity of being careful with the spinor indexes. Considering, as usual \( \psi \) as a column vector and \( \overline{\psi} \) as a row vector implies that \( p_\psi \) will be a row vector while \( p_{\overline{\psi}} \) will be a column vector.

Equations (19) and (20), respectively leads us to express the velocities \( \dot{A}_a^i \) and \( \dot{A}_a^0 \) as

\begin{equation}
\dot{A}_a^i = \pi_a^i - \partial_i A_a^0 + g f_{abc} \pi_a^b A^c_i,
\end{equation}

\begin{equation}
\dot{A}_a^0 = \xi \pi_a^0 - \partial_i A_i^a.
\end{equation}

The Hamiltonian density is given by

\begin{equation*}
H_0 = \frac{1}{2} \pi_i^a \pi_i^a - \pi_i^a \partial_i A_0^a - g f_{abc} \pi_a^b A_0^c + \frac{1}{2} \xi \pi_0^a \pi_0^a - \pi_0^a \partial_i A_i^a \\
+ \frac{1}{4} F_{ij}^a F_{ij}^a - \overline{\psi} (i \gamma^i \partial_i + e \gamma^\mu A_\mu - m) \psi.
\end{equation*}

The total Hamiltonian density is constructed as

\begin{equation}
H_T = \frac{1}{2} \pi_i^a \pi_i^a - \pi_i^a \partial_i A_0^a - g f_{abc} \pi_a^b A_0^c + \frac{1}{2} \xi \pi_0^a \pi_0^a - \pi_0^a \partial_i A_i^a \\
+ \frac{1}{4} F_{ij}^a F_{ij}^a - \overline{\psi} (i \gamma^i \partial_i + e \gamma^\mu A_\mu - m) \psi \\
+ \lambda_\psi (p_\psi - i \gamma^0 \overline{\psi}) + \lambda_{\overline{\psi}} p_{\overline{\psi}} + \lambda_\mu p_\mu,
\end{equation}

where \( \lambda_\psi, \lambda_{\overline{\psi}} \) and \( \lambda_\mu \) are Lagrange multipliers to be determined. From the consistency conditions, the time derivative of the primary constraints should be zero, that is

\begin{equation}
\dot{H}_\psi = \{ H'_\psi, H_T \} = - \overline{\psi} (i \overline{\gamma}^i \partial_i - e \gamma^\mu A_\mu + m) - i \lambda_{\overline{\psi}} \gamma^0 \approx 0,
\end{equation}

\begin{equation}
\dot{H}'_\psi = \{ H''_\psi, H_T \} = (i \gamma^i \partial_i + e \gamma^\mu A_\mu - m) \psi + i \gamma^0 \lambda_\psi \approx 0,
\end{equation}

\begin{equation}
\dot{H}_\mu = \{ H'_\mu, H_T \} = \overline{\psi} e \gamma^\mu \psi \approx 0.
\end{equation}

Relations (28) and (29) fix the multipliers \( \lambda_{\overline{\psi}} \) and \( \lambda_\psi \) respectively as

\begin{equation}
\lambda_{\overline{\psi}} = i \overline{\psi} (i \overline{\gamma}^i \partial_i - e \gamma^\mu A_\mu + m) \gamma^0,
\end{equation}

\begin{equation}
\lambda_\psi = i \gamma^0 (i \gamma^i \partial_i + e \gamma^\mu A_\mu - m) \psi.
\end{equation}

Eq.(27) lead to the secondary constraints

\begin{equation}
H''_\mu = \overline{\psi} e \gamma^\mu \psi \approx 0.
\end{equation}

There are no tertiary constraints, since

\begin{equation}
\dot{H}_\mu = \{ H''_\mu, H_T \} = 0.
\end{equation}
By taking suitable linear combinations of constraints, one has to find the first-class, that is
\[ \Phi_1 = H'_\mu = p_\mu, \] (35)
whereas the constraints
\[ \Phi_2 = H'_\psi = p_\psi - i \gamma^0 \overline{\psi}, \] (36)
\[ \Phi_3 = H'_\psi = p_\psi, \] (37)
\[ \Phi_4 = H''_\mu = \overline{\psi} e^{\gamma^\mu} \psi = 0, \] (38)
are second-class.

The equations of motion read as
\[ \dot{A}_a^0 = \{A_a^0, H_T\} = \xi \pi_a^0 - \partial_i A_i^a, \] (39)
\[ \dot{A}_a^i = \{A_a^i, H_T\} = \pi_a^i - \partial_i A_i^a + g f^{abc} A_0^b A_i^c, \] (40)
\[ \dot{\psi} = \{\psi, H_T\} = \lambda_\psi, \] (41)
\[ \dot{\overline{\psi}} = \{\overline{\psi}, H_T\} = \lambda_{\overline{\psi}}, \] (42)
\[ \dot{A}_\mu = \{A_\mu, H_T\} = \lambda_\mu, \] (43)
\[ \dot{\pi}_a^0 = \{\pi_a^0, H_T\} = \partial_i \pi_i^a + g f^{abc} \pi_b^a A_i^c, \] (44)
\[ \dot{\pi}_a^i = \{\pi_a^i, H_T\} = g f^{abc} \pi_b^i A_0^c - \partial_i (F_a^i + \pi_a^0) - F_a^i g f^{abc} A_b^c, \] (45)
\[ \dot{p}_\psi = \{p_\psi, H_T\} = -\overline{\psi} (i \partial_i \gamma^i - e \gamma^\mu A_\mu + m), \] (46)
\[ \dot{p}_{\overline{\psi}} = \{p_{\overline{\psi}}, H_T\} = (i \gamma^i \partial_i + e \gamma^\mu A_\mu - m) \psi + i \gamma^0 \lambda_\psi, \] (47)
\[ \dot{p}_\mu = \{p_\mu, H_T\} = \overline{\psi} e^{\gamma^\mu} \psi. \] (48)

Substituting from Eq. (32) into Eqs. (41) and (47), we get
\[ (i \gamma^\mu \partial_\mu + e \gamma^\mu A_\mu - m) \psi = 0, \] (49)
\[ \dot{p}_{\overline{\psi}} = 0, \] (50)
and from Eq.(31) into (42), we have
\[ \overline{\psi} (i \partial_\mu \gamma^\mu - e \gamma^\mu A_\mu + m) = 0. \] (51)

We will contact ourselves with a partial gauge fixing by introducing gauge constraints for the first-class primary constraints only, just to fix the multiplier \( \lambda_\mu \) in Eq.(27). Since \( p_\mu \) is vanishing weakly, a gauge choice near at hand would be
\[ \phi'_1 = A_\mu = 0. \] (52)

But for this forbids dynamics at all, since the requirement \( \dot{A}_\mu = 0 \) implies \( \lambda_\mu = 0. \)

In the following section the same system will be discussed using Hamilton-Jacobi approach.
1.2 Hamilton-Jacobi method

The set of Hamilton-Jacobi Partial Differential Equations (HJPDE) (8) read as

\[ H'_0 = \pi'^a_i + H_0 = 0, \]  
\[ H'_\psi = p_\psi + H_\psi = p_\psi - i\bar{\psi}\gamma^0 = 0, \]  
\[ H'_\bar{\psi} = p_{\bar{\psi}} + H_{\bar{\psi}} = p_{\bar{\psi}} = 0, \]  
\[ H'_\mu = p_\mu + H_\mu = p_\mu = 0. \]

The equations of motion are obtained as total differential equations as follows:

\[ dA^i_a = \frac{\partial H'_0}{\partial \pi'^a_i} dt + \frac{\partial H'_\psi}{\partial \pi'^a_i} d\psi + \frac{\partial H'_{\bar{\psi}}}{\partial \pi'^a_i} d\bar{\psi} + \frac{\partial H'_\mu}{\partial \pi'^a_i} dA_\mu, \]
\[ = [\pi'^a_i - \partial_i A^0_a + gf^{abc}A^b_0 A^c_i] dt, \]  
\[ dA^0_a = \frac{\partial H'_0}{\partial \pi'^a_0} dt + \frac{\partial H'_\psi}{\partial \pi'^a_0} d\psi + \frac{\partial H'_{\bar{\psi}}}{\partial \pi'^a_0} d\bar{\psi} + \frac{\partial H'_\mu}{\partial \pi'^a_0} dA_\mu, \]
\[ = [\xi \pi'^a_0 - \partial^0 A^0_i] dt, \]  
\[ dp_\psi = -\frac{\partial H'_0}{\partial \psi} dt - \frac{\partial H'_\psi}{\partial \psi} d\psi - \frac{\partial H'_{\bar{\psi}}}{\partial \psi} d\bar{\psi} - \frac{\partial H'_\mu}{\partial \psi} dA_\mu, \]
\[ = [-\bar{\psi} (i\partial_i \gamma^i - e\gamma^\mu A_\mu + m)] dt, \]  
\[ dp_{\bar{\psi}} = -\frac{\partial H'_0}{\partial \bar{\psi}} dt - \frac{\partial H'_\psi}{\partial \bar{\psi}} d\psi - \frac{\partial H'_{\bar{\psi}}}{\partial \bar{\psi}} d\bar{\psi} - \frac{\partial H'_\mu}{\partial \bar{\psi}} dA_\mu, \]
\[ = [(i\gamma^i \partial_i + e\gamma^\mu A_\mu - m)\psi] dt + i\gamma^0 d\psi, \]  
\[ dp_\mu = -\frac{\partial H'_0}{\partial A^\mu_\mu} dt - \frac{\partial H'_\psi}{\partial A^\mu_\mu} d\psi - \frac{\partial H'_{\bar{\psi}}}{\partial A^\mu_\mu} d\bar{\psi} - \frac{\partial H'_\mu}{\partial A^\mu_\mu} dA_\mu, \]
\[ = (\bar{\psi} e\gamma^\mu \psi) dt, \]  
\[ = (\bar{\psi} e\gamma^\mu \psi) dt, \]
\[ d\pi^a_i = \frac{\partial H'_0}{\partial \dot{t}} dt - \frac{\partial H'_0}{\partial \psi} d\psi - \frac{\partial H'_\psi}{\partial \bar{\psi}} d\bar{\psi} - \frac{\partial H'_\mu}{\partial \bar{\mu}} d\bar{\mu}. \] (64)

The integrability conditions imply that the variation of the constraints \( H'_\psi, H'_\bar{\psi} \) and \( H'_\mu \) should be identically zero; that is
\[ dH'_\psi = dp_\psi - i d\bar{\psi} \gamma^0 = 0, \] (65)
\[ dH'_\bar{\psi} = dp_\bar{\psi} = 0, \] (66)
\[ dH'_\mu = dp_\mu = 0. \] (67)

The vanishing of total differential of \( H'_\mu \) leads to a new constraint
\[ H''_\mu = \bar{\psi} e^{\gamma^\mu} \psi. \] (68)

When we taking a gain the total differential of \( H''_\mu \), we notice that it vanishes identically,
\[ dH''_\mu = 0. \] (69)

From Eqs.(57) and (58), respectively we obtain
\[ \dot{A}^i_a = \pi^a_i - \partial_i A^a_0 + gf^{abc} A^b_0 A^c_i, \] (70)
and
\[ \dot{A}^0_i = \xi \pi^0_i - \partial^i A^a_0. \] (71)

Substituting from Eqs. (61) and (62) into Eqs. (65), and (66), respectively we get
\[ \bar{\psi} (i \partial_\mu \gamma^\mu - e \gamma^\mu A_\mu + m) = 0, \] (72)
\[ (i \gamma^\mu \partial_\mu + e \gamma^\mu A_\mu - m) \psi = 0. \] (73)

Also from Eqs. (59-61, 63), we get the following equations of motion:
\[ \dot{\pi}^i_a = gf^{abc} \pi^i_c A^b_0 - \partial_i (F^i_a + \pi^0_a) - F^i_d gf^{abc} A^b_c, \] (74)
\[ \dot{\pi}^0_a = \partial_i \pi^i_a + gf^{abc} \pi^i_c A^0_c, \] (75)
\[ \dot{\psi} = -\bar{\psi} (i \partial_i \gamma^i - e \gamma^\mu A_\mu + m), \] (76)
\[ \dot{\bar{\psi}} = \bar{\psi} e^{\gamma^\mu} \psi. \] (77)

Substituting from Eq. (73) into Eq.(62), we have
\[ \dot{p_\psi} = 0. \] (78)

As a comparison between the above two methods, we get that the Hamilton-Jacobi method and Dirac’s method give the same equations of motion.
2. Conclusion

A non-Abelian theory of fermions interacting with gauge bosons is discussed as constrained system by using both Dirac’s and Hamilton-Jacobi methods. In Dirac’s method the total Hamiltonian composed by adding the constraints multiplied by Lagrange multipliers to the canonical Hamiltonian. In order to derive the equations of motion, one needs to redefine these unknown multipliers in an arbitrary way. However, in the Hamilton-Jacobi approach (or Güler’s method)[2-8], there is no need to introduce Lagrange multipliers to the canonical Hamiltonian. In Hamilton-Jacobi approach it is not necessary to distinguish between first-class and second-class constraints, there is no need to introduce any gauge fixing conditions as in Dirac’s approach. Both the consistency conditions and the integrability conditions lead to the same constraints.

References

Physical Form of the Clustering Parameter and Gravitational Galaxy Clustering

Sajad Masood\textsuperscript{1*}, Manzoor A Malik\textsuperscript{1}, Shakeel Ahmad\textsuperscript{1} and N. A. Rather\textsuperscript{2}

\textsuperscript{1}Department of Physics, University of Kashmir, Hazratbal, Srinagar, 190006, Kashmir, India.
\textsuperscript{2}Department of Mathematics, University of Kashmir, Hazratbal, Srinagar, 190006, Kashmir, India

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Abstract: A theory for a system clustering under gravity is developed for the clustering parameter $b(n, T)$, in terms of a partial differential equation using thermodynamic technique. Various solutions of the differential equation relate $b(n, T)$ with density $n$ and temperature $T$ of the gravitating system. The physical validity of various solutions of $b(n, T)$ on the basis of certain boundary conditions and probability density distribution function is discussed. Results indicate that the clustering parameter depends on the specific combination $nT^{-3}$. The theory also provides a new insight into gravitational clustering.

Keywords: Cosmology-galaxies; Clustering hydrodynamics Method; Gravitational Clustering; Gravitational Thermodynamics; Quasi-equilibrium; Geometry

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1. Introduction

The gravitational clustering of galaxies has played an important role in evolution of the observed Universe. The presently observed clustering of galaxies suggests that their motions have been dominated by mutual gravitational dynamics. The non-linear clustering phenomenon is determined by physical processes involving a lengthy and complex sequence of events. One approach for understanding such galaxy clustering in the Universe deals with the evaluation of n-particle correlation functions between galaxies. This can be done by solving system of Liouville’s equations or BBGKY-hierarchy equations and have been discussed by many workers like Saslaw (1972), Inagaki (1976), Fall and Saslaw (1976)

\textsuperscript{*} smmasoodi@gmail.com
and Peebles (1980), etc. But BBGKY-hierarchy equations are too complicated to handle for higher order correlation functions. However, the lowest order, two-point correlation function can also be pursued for discussing the phenomenon of galaxy clustering which contains information on all the higher n-particle correlations in full BBGKY-hierarchy (Peebles 1980, Zhan 1989, Zhan & Dyer 1989, Hamilton 1993). An alternative simple and more effective statistical approach to two-point correlation functions for non-linear galaxy clustering has been developed by Saslaw and Hamilton (1984) with the help of gravitational thermodynamic results. The physical validity in applying thermodynamic theory of the gravitational clustering of galaxies has been discussed on the basis of computer N-body simulations by Zhan (1989), Itoh et al (1988, 1990, 1993), Saslaw & Sheth (1993). On the basis of the pairwise gravitational interaction of point masses, statistical homogeneity and quasi-equilibrium evolution, an analytical expression for the probability distribution function \( f(N) \) of finding \( N \) galaxies in a randomly positioned volume \( V \) in terms of \( b \), is given by (Saslaw & Hamilton 1984):

\[
f(N, V) = \frac{\bar{N}(1 - b)}{N!} \left[ \bar{N}(1 - b) + Nb \right]^{N-1} e^{-\bar{N}(1 - b) - Nb}
\]

where \( \bar{N} = \bar{n}V \), \( \bar{n} \) being average number density of the particles and the clustering parameter \( b \) is the ratio of gravitational correlation energy \( W_c \) to twice the kinetic energy \( K \) of the peculiar motions:

\[
b = -\frac{W_c}{2K}
\]

The theory can be extended to obtain a non-linear velocity distribution function for quasi-equilibrium gravitational clustering in an expanding Universe (Saslaw et al 1990). The value of \( b \) measures the influence of two-particle correlation function for clustering in a non-linear regime. On the basis of certain physical arguments Saslaw and Hamilton (1984) assumed a simple mathematical expression for \( b \) with expected physical limits \( b = 0 \) (for the perfect gas) and \( b = 1 \) (for complete clustering). The extra-ordinary agreement of \( f(N) \) with the computer N-body simulations and observed galaxy distribution leads to the expectation that it might be based on more fundamental considerations (Saslaw et al 1990). The importance of the clustering parameter, \( b \), on time is revealed by its simple relation with the expansion scale, \( a \), of the Universe (Saslaw 1992).

\[
a(t) = a_* \left( \frac{b^{1/8}}{(1 - b)^{7/8}} \right)^8
\]

Where \( a_* \) is a constant, given by initial state. Saslaw and Sheth (1993) studied in detail the non-linear and time evolution of gravitational galaxy clustering.

The value of \( b \) depends on the two-point correlation function which in turn is a function of density \( n \) and temperature \( T \) of the system in a grand canonical ensemble. Thus it is valuable to understand the functional form of \( b(n, T) \) and derive its expression in terms of \( n \) and \( T \) from first principle. It is the aim of the present paper to develop a theory for \( b(n, T) \) in terms of \( n \) and \( T \) of the system by using gravitational thermodynamic relations and then to see as to how it compares with the Saslaw-Hamilton (1984) guess. Firstly, a
A partial differential equation is developed, from the energy equation of thermodynamics in combination with the equation of state, taking gravitational interaction between particles into consideration. The solutions of the differential equation give the required analytical dependence of \( b \) on \( n \) and \( T \). We are interested in such a solution which satisfies physically viable boundary conditions of the system clustering under gravity. Here, the theory is based on the gravitational quasi-equilibrium hypothesis of thermodynamics. The N-body computer simulation results (Itoh et al 1993) explicitly verify the basic assumption that thermodynamic theory applies to gravitational clustering of galaxies.

This paper is organized as follows: In Section 2, a partial differential equation is developed by using basic thermodynamic results and the equations of state taking gravitational interaction between galaxies into consideration. The differential equation when solved gives a number of explicit solutions in terms of \( n \) and \( T \). In Section 3, the physical validity of various solutions is established on the basis of certain boundary conditions and the probability density distribution function. In Section 4, the results so obtained are compared with other constraints like: adiabatic hypothesis, negative binomial distribution and Riemannien Geometric approach. In Section 5, we briefly discuss the relevance of statistical mechanical approach and finally, in Section 6, the results are discussed.

2. Basic Results of Gravitational Thermodynamics

The applicability of quasi-equilibrium thermodynamics to galaxy clustering was hypothesized previously (Saslaw and Hamilton 1984, Saslaw 1985), and computer N-body experiments confirm the applicability of this approximation (Itoh et al 1988, 1990, 1993). It is further supported by theoretical arguments (Saslaw 1992, Zhan 1989, Zhan and Dyer 1989, Saslaw et al 1990, Saslaw and Sheth 1995, Saslaw and Fang 1996). Since local clustering in the expanding Universe is generally faster than the expansion time scale, quasi-equilibrium evolution seems to be a good approximation. The evolution can proceed approximately adiabatically through a sequence of equilibrium states. This has been tested by N-body simulation results (Itoh et al 1988), which show that relaxed evolution occurs through equilibrium states and applies to a wide range of conditions.

We consider an infinite system of \( N \) single-component particles (galaxies) each of mass \( m \) distributed homogeneously in a volume \( V \) having internal energy \( U \) and pressure \( P \). Assuming galaxies to be point masses which are interacting pairwise gravitationally, we define a quantity \( b \), which measures the influence of gravitational Correlation energy \( W_c \), and is related to two-point Correlation function \( \xi(n, T, r) \) by (Saslaw and Hamilton 1984):

\[
b(n, T) \equiv -\frac{W_c}{2K} = \frac{2\pi G m^2 \bar{n}}{3T} \int_V \xi(n, T, r) \frac{dv}{4\pi r^3} \]

Here \( \bar{n} = \bar{N}/V \) is the average number density and the kinetic energy \( K \) of peculiar motion is related to temperature \( T \) by

\[
K = \frac{3}{2} NT = \frac{1}{2} \sum_{i=1}^{N} mV_i^2
\]
The value of $b(n, T)$ depends on the form of two-particle correlation function $\xi(n, T)$ and it was shown that $\xi \approx r^{-2}$ over a finite range maximizes the gravitational entropy of clustering (Saslaw 1980). Thus specifying the correct form of $b$ as a function of $n$ and $T$ is essential in understanding the gravitational thermodynamics. Saslaw and Hamilton (1984) assumed a simple mathematical form of $b(n, T)$ on the basis of three arguments (i) scale invariance of $b(n, T)$ , (ii) the entropy as a total differential, and (iii) mathematical simplicity.

Without using such approximations here, our main concern is to develop a differential equation for $b$ in terms of $n$ and $T$ from basic principles. This can be done by combining thermodynamic relation .

$$\left( \frac{\partial b}{\partial V} \right)_{T,N} = T \left( \frac{\partial P}{\partial T} \right)_{T,N} - P$$

(5)

With equations of state

$$U = \frac{3}{2} NT(1 - 2b)$$

and

$$P = \frac{NT}{V}(1 - b)$$

The combination then leads to first order partial differential equation

$$3n \left( \frac{\partial b}{\partial n} \right)_T + T \left( \frac{\partial b}{\partial T} \right)_n = 0$$

(6)

The solutions of this partial differential equation can relate $b(n, T)$ to the density and temperature for the gravitational galaxy clustering. The partial differential equation can be solved using standard method (Sneddon 1985) and admits

$$b(n, T) = f(nt^{-3})$$

(7)

as its general solution. The choice of $f(nt^{-3})$ gives different particular solutions for $b(n, T)$ in terms of $nt^{-3}$. We are interested in a solution which satisfies the physically viable boundary conditions of the system for clustering under gravity. One solution of the differential equation (6) is:

$$b = \frac{(\beta nT^{-3})^\alpha}{1 + (\beta nT^{-3})^\alpha}$$

(8)

Where $\beta$ is a positive constant. The value of $\alpha$ can be either $\leq 1$ or $> 1$ and for $\alpha = 1$, equation (12) leads exactly to the assumed expression of $b(n, T)$ by Saslaw and Hamilton (1984). Fig. (1) shows the behaviour of $b$ as a function of $(\beta nt^{-3})$ for various values of $\alpha$. For higher values of $\alpha$, $b$ approaches its maximum value a little faster, but for very high values of $(\beta nt^{-3})$, the distinction becomes less and less obvious.

Various thermodynamic quantities like entropy, chemical potential, specific heat and fluctuations for various solutions of $b(n, T)$ associated with equation (8) can be derived from the standard thermodynamic relations:

$$\left( \frac{S}{N} \right) = \ln(n^{-1}T^{3/2}) - 3b - \frac{1}{\alpha} \ln(1 - b) + S_0$$

(9)
\[ C_V = \frac{3}{2} [1 + 2(3\alpha - 1)b - 6\alpha b^2] \]  

(10)

Fig. (2) shows the behaviour of specific heat as a function of \( b \), for various values of \( \alpha \). For \( b = 0 \), i.e. when the galaxies are uncorrelated, the system behaves as a perfect monoatomic gas \( (C_V = 3/2) \), irrespective of the values of \( \alpha \). Similarly, the value of \( C_V = -3/2 \) as \( b \to 1 \) results from the system becoming virialized on all scales, again independent of the value of \( \alpha \). The higher values of \( \alpha \) suggest that the specific heat peaks around the intermediate value of \( b \).

Further

\[
\left( \frac{\mu}{T} \right) = \ln(nT^{-3/2}) - b + \frac{1}{\alpha} \ln(1 - b)
\]

(11)

The critical value of \( b \) at which specific heat becomes zero is given by

\[ b_c = \frac{(3\alpha - 1) + \sqrt{9\alpha^2 + 1}}{6\alpha} \]

(12)

Fig. (3) shows the critical values of \( b \) for various values of \( \alpha \). For \( \alpha = 1 \), \( b_c = 0.8604 \), for \( \alpha = 2 \), \( b_c = 0.9236 \) and so on. For still higher values of \( \alpha \), the critical value begins to saturate till it reaches a value of unity.

Similarly mean square fluctuations in number of particles \( N \) and internal energy \( U \) is given by:

\[ \langle \Delta N^2 \rangle = \frac{N}{(1 - b)(1 - \alpha b)} \]

(13)

And

\[ \langle \Delta U^2 \rangle \geq 3NT^2 \left[ 5 - 2(9 + \alpha)b + 2(8 + 9\alpha)b^2 - 2\alpha(5 + 3\alpha)b^3 \right] \]

(14)

The solution of equation (6) clearly shows that \( b(n, T) = b(x) \), where \( x = nT^{-3} \). This also agrees explicitly with the result of Sheth (1995). Now the question arises: What is the correct form of \( b(x) \)? Saslaw Hamilton guess, generalization proposed here, Negative binomial, Ruppeiner’s formula; will be discussed each in detail and we will argue the correct form of \( b(x) \) based on various constraints and physical conditions.

3. Constraining the Functional Form of \( b(x) \)

The partial differential equation (6) has number of solutions for \( b(n, T) \) in terms of \( n \) and \( T \). We can investigate the physical validity of these solutions on the basis of: (a) Boundary conditions to be satisfied by the two-point correlation function of gravitational galaxy clustering. (b) Probability density distribution function.

3.1 Boundary Conditions

Two-point correlation function of gravitational galaxy clustering can be assigned certain boundary conditions based on various physically viable constraints. A physically valid
solution of $b(n, T)$ should satisfy following five boundary conditions:

I) The gravitational clustering of galaxies in a homogeneous universe requires $b(n, T)$ to be scale invariant. One can express this boundary condition on the hierarchial scale as $n \rightarrow \lambda^{-3}n$ and $T \rightarrow \lambda^{-1}T$, where $\lambda$ is a scalar multiplier. All the solutions comply with this scale invariance condition. The scaling of temperature indicates that clusters at a higher level of the hierarchy have lower random velocities relative to each other than smaller clusters have relative to each other (Saslaw and Hamilton 1984).

II) When two-particle correlation function increases or $b \rightarrow 1$, the galaxies become more and more clustered. This is because of virial equilibrium, which suggests that at low temperatures and high densities more strongly bound clusters are formed. This boundary condition requires that $b \rightarrow 1$, when $nT^{-3} \rightarrow 1$, and is satisfied by all solutions.

III) As $b \rightarrow 0$, the entropy of the ensemble must tend to be that for the 'Maxwellian gas' distribution i.e. $S \approx \ln(VT^{3/2}/N) + S_0$. This boundary condition is satisfied by $b(n, T)$ for all values of $\alpha$.

Two more boundary conditions which $b(n, T)$ must satisfy are based on the general description of $b$ (Saslaw et al 1990), in more general form of:

$$n(1 - b) = \frac{\phi(b)}{\beta T^{-3}}$$

(15)

Where $\phi(b)$ can be chosen as an arbitrary function of $b$ which satisfies two conditions: Firstly, $\phi(b) = 0$ for $b = 0$ and secondly, the Poisson limit $b\phi'(b) - \phi(b) = 0$. The condition of 'Poisson limit' is determined on the basis of standard two-point correlation function (Landau and Lifshitz 1981)

$$\frac{1}{V^2} \iint \xi_{12}dV_1dV_2 = \frac{1}{N} \left[ \frac{(\Delta N)^2}{N} - 1 \right]$$

(16)

The term on the right hand side is due to mean square number fluctuation given by equation (13).

IV) If $b = 0$, then $\phi(b) = 0$, which implies that the ensemble represents a perfect gas. This condition is a modified form of boundary condition (III). One can see that this boundary condition is satisfied for all values of $\alpha$.

V) The last boundary condition demands that for the Poisson limit $b\phi'(b) - \phi(b) = 0$. It is interesting to note that no solution of $b(n, T)$ except $\alpha = 1$ satisfies this condition, which corresponds to the assumed expression for $b(n, T)$ of Saslaw and Hamilton (1984).
3.2 Galaxy Distribution Function

It is established from above that the first four boundary conditions are satisfied by the solution of partial differential equation given by equation (7) for any value of $\alpha$. The last boundary condition (5) is satisfied only for $\alpha = 1$, which corresponds to Poisson limit. For $\alpha > 1$, the mean square number fluctuation expression $\langle N^2 \rangle$ is negative which is an unphysical result. This would give a Poisson limit equation (16) for $\alpha > 1$, which requires that the integral of the two point correlation function be negative instead of positive or zero which is again unphysical. Now we would like to investigate, what Poisson limit means for $\alpha < 1$ for the gravitational galaxy clustering, on the basis of density distribution function. Saslaw and Hamilton (1984) derived an elegant and simple form for the probability distribution which is given in equation (1). The probability distribution function for finding $N$-galaxies in a volume $V$ of arbitrary shape can be computed following the method used by Saslaw and Hamilton (1984):

$$f(N) = e^{-\psi} \frac{e^{\mu/TN}(e^\psi)^{(N)}_0}{N!}$$  \hspace{1cm} (17)

We evaluate $(e^\psi)^{(N)}_0$ for few values of $N$ at $b = 0$

$$(e^\psi)^{(1)}_0 = \frac{V}{\beta'} \beta'^3 T^{3/2}$$  \hspace{1cm} For all values of $\alpha$  \hspace{1cm} (18)

$$(e^\psi)^{(2)} = \frac{V}{\beta'} T^3 \beta'^2 \left( \frac{V}{\beta'} + 2 \right)$$  \hspace{1cm} for $\alpha = 1$  

$$= \infty$$  \hspace{1cm} for $\alpha < 1$  \hspace{1cm} (19)

Similarly, we have

$$(e^\psi)^{(3)}_0 = \frac{V}{\beta'} T^{9/2} \beta'^3 \left( \frac{V}{\beta'} + 3 \right)^2$$  \hspace{1cm} for $\alpha = 1$  \hspace{1cm} (20)

$$= \infty$$  \hspace{1cm} for $\alpha < 1$

Thus the distribution function is not converging for values other than $\alpha = 1$. The function should be differentiable at all points and possess derivatives for all orders. For $\alpha < 1$, the derivatives at $b = 0$ become infinite at higher orders, and if $\alpha = 1$, the situation is saved. Hence, it can be concluded that the partial differential equation has only one solution which satisfies all the physically viable boundary conditions and gives a distribution function which converges for $b = 0$. A wide range of class for $b(x)$ in terms of functions like trigonometrical, hyperbolic, Polynomial etc do not satisfy all the constraints and other physical boundary conditions including analyticity of distribution function.

4. Comparison with Other Constraints

Although the various boundary conditions define completely the specific functional form of $b(n,T)$, but it is also interesting to test results on the basis of few more constraints like: (a) Adiabatic hypothesis (b) Negative binomial distribution (c) Riemannian Geometric approach.
4.1 Adiabatic Hypothesis

The adiabatic hypothesis earlier used by Saslaw (1992) implies that clustering in an adiabatically expanding universe satisfies cosmic energy equation (Layzer 1966) and can be generalized for any functional form of \( b(x) \):

\[
\frac{1}{a} \frac{da}{db} = \frac{(7 - 6b)}{8(1 - b)^2} + \frac{(1 - 2b)}{8(1 - b)} \frac{1}{x} \frac{dx}{db} \tag{21}
\]

Where \( a(t) \) is the scale factor of the universe. Further, the assumption of adiabatic expansion requires that \( b \) increases monotonically as the universe expands or which means that \( 1/a(da/db) \) must be always positive. Surprisingly \( 1/a(da/db) \) for both the functions like, \( b = x + 1 \) and \( b = (x - 1)/(x + 1) \) remain positive for \( 0 < b < 1 \). A similar situation can also be seen for entropy which increases monotonically with \( b \). Of course, these two functions for \( b(x) \) do not satisfy the physical conditions discussed earlier in section 3. Similarly, we calculate \( 1/a(da/db) \) for the functional form given by equation (8) and is given by

\[
\frac{1}{a} \left( \frac{da}{db} \right) = -\frac{6\alpha b^2 + (7\alpha - 2)b + 1}{8\alpha b(1 - b)^2} \tag{22}
\]

It is always positive for \( \alpha = 1 \) and \( 0 < b < 1 \), and the condition of irreversible growth of clustering is valid as universe expands. The adiabatic hypothesis fails for \( b(x) \) with \( \alpha < 1 \), because at this value equation (22) becomes negative (fig. 4). However, when \( \alpha > 1 \), equation (22) becomes positive for \( 0 < b < 1 \), and hence this distribution fulfills the condition of irreversible growth of clustering. It is interesting to note that adiabatic hypothesis is also satisfied by other functional forms of \( b(x) \), which fail to satisfy other assigned physical boundary conditions.

4.2 Negative Binomial Distribution

The negative binomial distribution fits the observed galaxy distribution and results of Sheth (1995) show that one possible choice for \( b(x) \) yields negative binomial distribution given by:

\[
b = 1 - \frac{\ln(1 + x)}{x} \tag{23}
\]

This distribution fails to satisfy the condition of adiabatic hypothesis of gravitational clustering in the expanding universe given by equation (21). It also violates increasing entropy condition and minimum clustering hypothesis for grand canonical ensemble (Saslaw and Fang 1996). We test the physical validity of above distribution on the basis of various boundary conditions proposed in the previous section. The negative binomial distribution law given by equation (23) satisfies the conditions 1 to 4 (section 3.1) e.g homogeneity test, Maxwellian gas distribution for \( b \to 0 \), and ensemble represents the perfect gas. But it fails the crucial test of Poisson limit based on fluctuation theory given by equation (16). Thus, can we argue at this stage that Poisson limit is essential test for physical viability of distribution function to be applied in the gravitational galaxy clustering?
4.3 Riemannian Geometric Approach

An independent method based on Riemannian geometric theory of thermodynamic fluctuation was used by Ruppeiner (1995, 1996) to determine $b(x)$ explicitly. He has shown that by solving second order differential equation around $b = 0$ (ideal gas limit), $b(x)$ is given by the series:

$$b(x) = \sum_{i=1}^{\infty} a_i (b_1 x)^i$$  \hspace{1cm} (24)

Where $a_1 = 1, a_2 = 29/14, a_3 = 1640/259$ etc and $b_1$ can be set freely in equation (24) to suit the situation. Further Ruppeiner (1996) calculates $b(x)$ by solving differential equation around singular points numerically. At critical points (singular points), usually fluctuations are very large and equilibrium thermodynamics is not suitable approach. It can be shown that Saslaw and Hamilton (1984) type of expression for $b(x)$ is the solution around $b \to 0$ (ideal gas), which corresponds to small thermodynamic fluctuations.

Now, let us test the functional form of $b(x)$ given by equation (24) on the basis of various constraints discussed earlier (section 3). This expression satisfies first condition of homogeneity. According to second constraint if $x \to \infty$ then $b \to 1$. This condition is satisfied by equation (24) only if free parameter $b_1$ is chosen properly. At large values of $x$, the series diverges and again $b_1$ must be chosen in such a way that $b(x)$ converges. When $b \to 0$ then entropy must correspond to ideal gas situation. This condition is also satisfied by equation (24). Another physically viable constraint $\Phi(b)$ defined by equation (15) is satisfied by equation (24). However the Poisson limit condition $b \Phi'(b) = \Phi(b)$ given by equation (16) is not satisfied by Ruppeiner (1996) expression for $b(x)$. While comparing Ruppeiner’s(1996) geometric results (for $b_1 = 1$) of thermodynamics with Saslaw and Hamilton (1984) result for $b(x)$, we observe that both the results match nearly for small values of $x$. But when $x$ increases, there is departure of two results. However, two result can match closely if $b_1$ in geometric analysis is chosen properly. The expressions for temperature and density fluctuations for different functional forms of $b(x)$ normalized at ideal gas (I.g) situation are given by:

$$\langle (\Delta T)^2 \rangle_g = \frac{1}{[-6ab^2 + 2(3\alpha - 1)b + 1]}$$ \hspace{1cm} (25)

And

$$\langle (\Delta N)^2 \rangle_g = \frac{1}{[\alpha b^2 - (\alpha + 1)b + 1]}$$ \hspace{1cm} (26)

The above equations (25 and 26) represent metric elements of temperature and density fluctuations (with gravity and without gravity) in the notation of Ruppeiner (1996). One can observe that the temperature fluctuation expression, equation (24), is nothing but inverse of specific heat at constant volume for the system of galaxies clustering under gravitational interactions. There is a critical value of $b$ given by equation (12), which characterizes the growth of clustering at which thermodynamic instability sets in. The instability arises when specific heat changes from positive value to negative values. When
specific heat at constant volume becomes zero, then temperature fluctuation is infinite. Table 1 gives critical values of $b_c$ for different functional forms of $b$ and are compared with Riemannian geometrical results and negative binomial distribution. There is a very good agreement for critical values of $b$ between Riemannian geometric and our results (for $\alpha = 1$) at which temperature and density fluctuations become non-analytic.

5. Statistical Mechanical Approach

The statistical mechanical description of the cosmological many-body problem developed, for the extended mass system, by Ahmad et al (2002), supports our previous analysis further. The statistical mechanics of the N-body systems is based on the N-body Hamiltonian. From this, the partition function is formed as a function of an N-dimensional integral that incorporates the effects of interaction among all the particles. With some reasonable approximations, the partition function for gravitational clustering of galaxies, incorporating their extended structures such as spherical halos, takes the form:

$$Z_N(T, V) = \frac{V^N}{N!} \left( \frac{2\pi m T}{\Lambda^2} \right)^{3N/2} \left[ 1 + \beta \tilde{n} T^{-3} \gamma(\epsilon/R_1) \right]^{N-1}$$

where

$$\beta = \frac{3}{2}(Gm)^3$$

and

$$\gamma(\epsilon/R_1) = \sqrt{1 + (\epsilon/R_1)^2 + (\epsilon/R_1)^2 \ln \frac{(\epsilon/R_1)}{1 + \sqrt{1 + (\epsilon/R_1)^2}}}$$

Evaluation of the thermodynamical functions is not difficult, once the partition function is known. This is because these functions are first or second order derivatives of the free energy, which follows directly from the partition function. The internal energy $U$ and the Pressure equations of state $P$ obtained via the statistical mechanical approach are:

$$U = \frac{3}{2} N T (1 - 2b_e)$$

$$P = \frac{N T}{V} (1 - b_e)$$

where $b_e$ called the Extended-mass Clustering Parameter, is defined by

$$b_e = \frac{\beta \tilde{n} T^{-3} \gamma(\epsilon/R_1)}{1 + \beta \tilde{n} T^{-3} \gamma(\epsilon/R_1)}$$

since $\beta \tilde{n} T^{-3} = b/(1 - b)$; $b_e$ is related to $b$ by

$$b_e = \frac{b \gamma(\epsilon/R_1)}{1 + b \{\gamma(\epsilon/R_1) - 1\}}$$

$b_e$ takes into account the extended nature of galaxies and has the same physical form as $b$. The softening parameter $\epsilon$ had been introduced to remove the divergences in the
hamiltonian when two galaxies approach each other closely and to account for the extended nature of galaxies. It is expected to have far reaching consequences but is not of any interest here. It may be noted that in the point-mass limit \((\epsilon = 0, \gamma = 1)\), \(b_\epsilon\) reduces to its original point-mass form (Saslaw and Hamilton 1984).

It may be recalled that the quantity \(b_\epsilon\) emerges directly from the calculations and satisfies the same boundary conditions as \(b\). It is obvious that \(b_\epsilon\) tends to zero as \(b \to 0\) and it tends to unity as \(b \to 1\). The fact that \(b_\epsilon\) which is a direct consequence of the statistical mechanical theory of the gravitational many-body problem and has a functional form that corresponds to \(\alpha = 1\) (which is the only value that satisfies all the boundary conditions) is a direct verification of the correctness of the Saslaw-Hamilton (1984) guess.

6. Discussion

On the basis of quasi-equilibrium approach it is possible to use gravitational thermodynamics for deriving a simple partial differential equation which must be satisfied by physically viable constraints and the distribution function of galaxy clustering. The solution of this partial differential equation determines \(b(n, T)\) with density and temperature having specific combination of \(nT^{-3}\). This combination gives scale invariance character of \(b(n, T)\) and satisfies the condition of statistical homogeneity in expanding universe. All the solutions do satisfy this scale invariance condition. It can be argued that scale invariance is the characteristic signature of thermodynamic description of gravitational clustering. However, the functional form of \(b\) with \(nT^{-3}\) is strongly constrained by various physical and boundary conditions. These leads us to an expression close to galaxy clustering description in quasi-equilibrium thermodynamic situation. Different functional forms of \(b(nT^{-3})\) are tested on various physical arguments, but most of them fail at different stages of testing. However, results indicate that if the functional form of \(b(nT^{-3})\) crosses over all the stages of constraints, then it must satisfy the condition of Poisson limit. This corresponds to the condition for the system to behave like an ordinary gas ensemble (un-correlated). The functional form of \(b(nT^{-3})\) given by equation (8) for \(\alpha = 1\) satisfies all the conditions including Poisson limit. Thus we can assign the Poisson Limit \((b \to 0)\) as an essential boundary condition for the distribution function to be satisfied for gravitational clustering. A wide range of class for solution of \(b = f(nT^{-3})\) in terms of functions like trigonometrical, hyperbolic and polynomials etc. do not satisfy all the physical boundary conditions and analyticity of distribution function.

Although the various physical boundary conditions define completely the specific functional form of \(b(nT^{-3})\), but the results can be compared further on the basis of other constraints like: adiabatic hypothesis, negative binomial distribution and Riemannian geometric approach. The assumption of adiabatic hypothesis (Saslaw 1992) requires that \(b(nT^{-3})\) increases monotonically as the universe expands. A comparison of various solutions of \(b(nT^{-3})\) with N-body experiment results Itoh et al (1988) clearly indicates that only one functional form of \(b\) given by equation (8) for \(\alpha = 1\) fits well for \(\Omega_0 = 1\) and 0.1. A comparison of our results with adiabatic hypothesis clearly shows that only
functional form of $b(nT^{-3})$ given by equation (8) with $\alpha = 1$ satisfies this hypothesis. Of course adiabatic hypothesis is satisfied by some functional forms of $b(nT^{-3})$. A detailed study is needed to know why other functional forms of $b(nT^{-3})$ also satisfy the adiabatic hypothesis, when they violate other physical constraints.

The negative binomial distribution (Sheth 1995) satisfies all the physical constraints (Table 1), but fails the crucial test of Poisson limit. It also violates the irresistibility condition of gravitational clustering in an adiabatic expanding universe. Saslaw and Fang (1996) argue that negative binomial distribution has non-physical properties and is unlikely to be dynamically consistent solution of cosmological many body problem. Further, we test the Riemannian geometric result (Ruppeiner 1996) on the basis of various physical conditions. All the assigned boundary conditions, except the Poisson limit condition, are satisfied by Ruppeiner’s (1996) functional form for $b(nT^{-3})$. The results of Riemannian geometric approach of thermodynamics with $\alpha = 1$ (Saslaw & Hamilton 1984) for $b(x)$ can match closely if $b_1$ (a free parameter in geometric analysis) is chosen properly. Again, there exists a close agreement for critical values of $b$ between geometric and our results (for $\alpha = 1$) at which temperature and density fluctuations become non-analytic.

Besides, the functional form of $b_\epsilon$ derived directly from the statistical mechanical description, which reduces to the Saslaw Hamilton (1984) form in the point-mass limit, suggests the value of $\alpha$ to be unity ($\alpha = 1$ satisfies all the boundary conditions) and hence reaffirms the correctness of the physical form of $b$ as assumed by Saslaw and Hamilton (1984).

Finally, we can conclude that various physical boundary conditions and Poisson limit, constraint the functional form of $b(nT^{-3})$. This provides a deeper understanding of the gravitational galaxy clustering on the basis of quasi equilibrium thermodynamics.

**Acknowledgments**

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**References**


Fig. 1 $b$ as a function of $\beta n t^{-3}$ for various values of $\alpha$
Fig. 2 Specific heat as a function of $b$ for fixed values of $\alpha$
Fig. 3 Critical values of $b_c$ as a function of $\alpha$.

Fig. 4 $\frac{1}{\alpha}(\frac{da}{db})$ versus $b$ for fixed values of $\alpha$. 
<table>
<thead>
<tr>
<th>Model</th>
<th>Temperature Fluctuation</th>
<th>Density Fluctuation</th>
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<tr>
<td>Negative Binomial</td>
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<td>0.02</td>
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Table 1. Comparison of critical values of $b$ ($b_c$) for different models with present analysis at which temperature and density fluctuations become non-analytic.
Penrose Model Potential, Compared with Coleman-Weinberg Potential for Early Universe Scalar Evolution

A.W. Beckwith*

Department of Physics and Texas Center for Superconductivity, University of Houston Houston, Texas 77204-5005, USA

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Abstract: We present evidence in terms of a D’Alembertian operator acting on a scalar field minus the first derivative of a potential system, with respect to an inflaton scalar field, that the Penrose model as outlined as an alternative to cosmological big crunch models gives us emergent behavior for an inflaton scalar field in early universe cosmological models. This is in contrast to the Coleman-Weinberg potential which in low temperature conditions is always presenting almost non existent emergent scalar fields. This permits us to state that Penrose’s cyclic universe model in its initial conditions gives us scalar field dynamics consistent with emergent scalar fields which die out quickly as temperature drops after the onset of inflation. We make no attempt to find the particulars of the conformal mapping which allows the alternative to the big crunch Penrose (2007) lectured upon in the inaugural meeting of the IGC at Penn State.

Keywords: Inflaton Scalar Field; Penrose Cyclic Model; Coleman-Weinberg Potential

PACS (2006): 98.80.-k; 98.80.Cq; 98.80.Bp

1. Introduction

We begin first with describing the relevant equation of motion involving the D’Alembertian operation upon a scalar field minus the derivative of a potential system for the scalar potentials with respect to the inflaton. With regards to the Penrose potential, we find largely due to the existence of a scalar mass term added which disappears due to high temperature conditions a procedural addition to permit the inflaton field in high temperatures to nucleate and then to subsequently fade out with dropping temperature conditions as occurs in the aftermath of inflation. The subsequent alternative to the big crunch which

* abeckwith@UH.edu
Penrose (2007) lectured upon in the IGC inaugural meeting involves a conformal mapping which the author has not had time to work on, yet. We can say this though; for a nucleation of an emergent scalar potential, we would first need a low temperature regime which heated up. The author after establishing the validity of Penrose’s (2007) procedure for emergent scalar space time in initial phases refers to a scheme involving worm hole transfer of thermal input into the initial conditions of inflation. The author then closes with some observations as to necessary and sufficient conditions for the formation of the conformal mapping collision of gravitational radiation energy which Penrose (2007) alluded to in his IGC talk, involving the ripples in the pond effect, where gravitational radiation from Black holes, collected in the aftermath of Hawking radiation is collated into a new big bang. Penrose (2007) in conversations with the author at Penn state alluded to the Hawking radiation effect as a causal effect which permits a necessary condition for the formation of necessary conditions for the conformal mapping of many black hole Hawking radiation releases into a singular big bang. The author, without trying to ascertain the particulars of this conformal mapping discusses what he thinks constitutes necessary conditions for the formation of such a transformation.

2. Preliminaries: The D’Albembertain Operation in an Equation of Motion for Emergent Scalar Fields

We begin with the D’Albertain operator as part of an equation of motion for an emergent scalar field. We then refer to the Penrose potential which the author gleaned from the presentation given by Penrose (2007) at the inauguration of the IGC center, at Penn States, August 2007. This potential is then shown, with an initial assumption of Euclidian flat space for computational simplicity to account for, in a high temperature regime an emergent non zero value for the scalar field $\phi$ due to a zero effective mass, at high temperatures. When the mass approaches far lower values, it, a non zero scalar field re appears. Leading to $\phi \xrightarrow{T \to 2.7^0Kelvin} \varepsilon + \approx 0^+$ as a vanishingly small contribution to cosmological evolution, as we have non zero effective mass for low temperatures. This dynamic will be, in the next section, compared to what happens in the Coleman-Weinberg effective potential which is almost always mandating $\phi \xrightarrow{T \to 2.7^0Kelvin} \varepsilon + \approx 0^+$. The author is aware that the Euclidian flat space hypothesis, at the initial phases of nucleation, and indeed a flat space metric is likely not accurate at the initial phases of nucleation. Needless to say, this simple model identifies the initial emergent scalar conditions for what the author views as a quintessence contribution to cosmological evolution. This section also speculates as to what is needed for thermal input to create circumstances for emergent scalar potentials. One model is the discussion by Penrose (2007) as to a non standard solution of innumerable black holes contributing to a future big bang. This is done in terms of a perpetually expanding universe whose matter, as it cools is collated into innumerable black holes. As the thermal temperature drops below the 2.7 K
limit, Penrose (2007) suggests that a conformal transformation makes a bridge between these innumerable black holes releasing material via a Hawking radiation procedure to be collated into a future big bang. The author frankly has conceptual difficulties with this picture, but will bring up what he views as necessary conditions for such a mapping to take place. The other picture, brought up in the next section which the author favors is one of a wormhole transferal of thermal radiation from a prior contracting universe into this present universe, via a pseudo time dependent Wheeler De Witt equation, as given by Crowell (2005). The solution so obtained has symmetries which will be discussed in passing, but in certain respects has a partial WKB character which will be remarked upon by the author.

Let us now begin to initiate how to model the Penrose quintessence scalar field evolution equation. To begin, look at the flat space version of the evolution equation, which Penrose proposed at the IGC inaugural conference. It reads as

$$\ddot{\phi} - \nabla^2 \phi + \frac{\partial V}{\partial \phi} = 0 \quad (1)$$

This is, in the Friedman – Walker metric using the following as a potential system to work with, namely:

$$V(\phi) \sim -\left\{\frac{1}{2} \cdot \left( M(T) + \frac{\kappa}{6} \right) \phi^2 + \frac{\ddot{a}}{4} \phi^4 \right\} \equiv -\left[\frac{1}{2} \cdot \left( M(T) + \frac{\kappa}{6a^2(t)} \right) \phi^2 + \frac{\ddot{a}}{4} \phi^4 \right] \quad (2)$$

This is presupposing $\kappa \equiv \pm 1, 0$, that one is picking a curvature signature which is compatible with an open universe. That means $\kappa = -1, 0$ as possibilities. Picking the closed universe is not compatible with the assumptions of Penrose’s concept which is for an open universe, where Black holes collate matter far into the future and have the material so collected released into a new big bang. While not necessarily endorsing the last part of Penrose’s supposition, we will look at the $\kappa = -1, 0$ values so as to determine what a good emergent scalar field match is. When we do so we find that we need to have an anzatz for the scalar field to work with. We begin with.

$$\ddot{\phi} - \nabla^2 \phi + \frac{\partial V}{\partial \phi} = 0 \Rightarrow \phi^2 = \frac{1}{\ddot{a}} \left\{ c_1^2 - \left[ \alpha^2 + \frac{\kappa}{6a^2(t)} + M(T) \right] \right\} \Leftrightarrow \phi \equiv e^{-\alpha r} \exp(c_1 t) \quad (3)$$

We find the following as far as basic phenomenology, namely

$$\phi^2 = \frac{1}{\ddot{a}} \left\{ c_1^2 - \left[ \alpha^2 + \frac{\kappa}{6a^2(t)} + M(T) \right] \right\} \xrightarrow{M(T \sim \text{high}) \rightarrow 0} \phi^2 \neq 0 \quad (4)$$

$$\phi^2 = \frac{1}{\ddot{a}} \left\{ c_1^2 - \left[ \alpha^2 + \frac{\kappa}{6a^2(t)} + M(T) \right] \right\} \xrightarrow{M(T \sim \text{high}) \rightarrow 0} \phi^2 \approx 0 \quad (5)$$

Our next task will be to come up with a suggestion as to how there could be a switch from Eqn. (4) to Eqn. (5). One suggestion as brought up independently of the Penrose model itself has to do with work the author performed in a different model which if super imposed upon how Eqn. (4) and Eqn. (5) inter relate give us the very real possibility if
we initially have a pre inflationary state of low temperature that the wormhole model
as mentioned in the next section could give us emergent quintessence fields which damp
out quickly. After we establish this in the next section, will be a comparison with the
Coleman Weinberg potential which almost always gives quintessence behavior as given
by Eqn.(5).

3. Emergent Space Time. As generated via a Wormhole

Lorentzian wormholes have been modeled quite thoroughly, and Visser (1995) states that
the wormhole solution does not have an event horizon hiding a singularity, i.e. there is no
singularity in the wormhole held open by dark energy. So being the case, the only case a
wormhole could form would be as a bridge between a prior to a present universe, which
is what Crowell (2005) refers to in his reference on quantum fluctuations of space time
tome which uses a pseudo time like space coordinate to a modified Wheeler – De Witt
equation for a bridge between two universes. We add in another caveat, that the worm
hole solution is dominated by a huge vacuum energy value. This leads to the following
situation, which we present here:

To model this, we use results from Crowell (2005) on quantum fluctuations in space
time which gives a model from a pseudo time component version of the Wheeler De Witt
equation, with a use of the Reissner-Nordstrom metric to help us obtain a solution
which passes through a thin shell separating two space times. The radius of the shell,
r_0(t) separating the two space times is of length l_P in approximate magnitude, leading to
a domination of the time component for the Reissner – Nordstrom metric

\[ dS^2 = -F(r) \cdot dt^2 + \frac{dr^2}{F(r)} + d\Omega^2 \]  \hspace{1cm} (6)

This has:

\[ F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} \cdot r^2 \left. \frac{1}{T \rightarrow 10^{32} \text{Kelvin} \sim \infty} \right\} - \frac{\Lambda}{3} \cdot (r = l_P)^2 \] \hspace{1cm} (7)

This assume that the cosmological vacuum energy parameter has a temperature dependence
as outlined by Park (2003) leading to if

\[ \frac{\partial F}{\partial r} \sim -2 \cdot \frac{\Lambda}{3} \cdot (r \approx l_P) \equiv \eta(T) \cdot (r \approx l_P) \] \hspace{1cm} (8)

As a wave functional solution to a Wheeler De Witt equation bridging two space times.
This solution bridging two space times is similar to that being made between these two
space times with ‘instantaneous’ transfer of thermal heat ,as given by Crowell (2005)

\[ \Psi(T) \propto -\Lambda \cdot \left\{ \eta^2 \cdot C_1 \right\} + A \cdot \eta \cdot \omega^2 \cdot C_2 \] \hspace{1cm} (9)

This has \( C_1 = C_1(\omega, t, r) \) as a pseudo cyclic and evolving function in terms of frequency,
time, and spatial function, with the same thing describable about \( C_2 = C_2(\omega, t, r) \) with
\( C_1 = C_1(\omega, t, r) \neq C_2(\omega, t, r) \). The upshot of this is that a thermal bridge between a shrinking prior universe, collapsing to a singularity, and an expanding universe expanding from a singularity exists, with an almost instantaneous transfer of heat with terms dominated by \( \eta(T) \) exits, and is forming a necessary and sufficient condition for the thermal heat flux. We get that from this is in part due to the identification which we will explicitly state, namely that by assuming that the absolute value of the five dimensional ‘vacuum state’ parameter varies with temperature \( T \), as Beckwith (2007) writes

\[
|\Lambda_{5-\text{dim}}| \approx c_1 \cdot (1/T^\alpha)
\]  

(10)

in contrast with the more traditional four dimensional version of the same, minus the minus sign of the brane world theory version The five dimensional version is actually connected with Brane theory, and higher dimensions, whereas the four dimensional is linked to more traditional De Sitter space time geometry, as given by Park(2003)

\[
\Lambda_{4-\text{dim}} \approx c_2 \cdot T^\beta
\]  

(11)

This is such that If one looks at the range of allowed upper bounds of the cosmological constant, we have that the difference between what Barvinsky (2006) recently predicted, and Park (2003) specifying an upper limit as of 2003, based upon thermal input is a give away that a phase transition is occurring at or before Planck’s time. This allows for a brief interlude of quintessence We should note that this is assuming that a release in gravitons occurs which leads to a removal of graviton energy stored contributions to this cosmological parameter, with \( m_P \) as the Planck mass, i.e. the mass of a black hole of ‘radius’ on the order of magnitude of Planck length \( l_P \sim 10^{-35} \) m. This leads to Planck’s mass \( m_P \approx 2.17645 \times 10^{-8} \) kilograms, as alluded to by Barvinsky (2006)

\[
\Lambda_{4-\text{dim}} \propto c_2 \cdot T \rightarrow \text{graviton production} \rightarrow 360 \cdot m_P^2 << c_2 \cdot [T \approx 10^{32} K]
\]  

(12)

Needless to say, right after the gravitons are released one still is seeing a drop off of temperature contributions to the cosmological constant. Then we can write, for small time values \( t \approx \delta^1 \cdot t_P, 0 < \delta^1 \leq 1 \) and for temperatures sharply lower than \( T \approx 10^{12} \) Kelvin, Beckwith (2007), where for a positive integer \( n \)

\[
\frac{|\Lambda_{4-\text{dim}}|}{|\Lambda_{5-\text{dim}}|} - 1 \approx \frac{1}{n}
\]  

(13)

The transition outlined in Eqn. (12) above has a starting point with extremely high temperatures given by a vacuum energy transferal between a prior universe and our present universe, as outlined by Eqn. (7) and Eqn. (8) above, whereas the regime where we look at an upper bound to vacuum energy in four dimensions is outlined in Eqn. (13) above, so that eventually we can model the behavior of scalar fields as being transformed from cyclic behavior with an imaginary component to a purely real valued scalar equation as given by the next sections argument. We then conclude with a proof of the short term behavior of this quintessence scalar field, in a manner which makes reference to both Eqn. (12) and Eqn (13) above.
4. Why Quintessence Scalar Fields Damp Out

The origins of this methodology lie in looking at first the initial phases of how the Einstein equations evolve, with the assumption made for the sake of simplicity that we can model part of the contributions to the Einstein stress tensor via a Casimir plate treatment made as an approximation to initial ‘domain wall’ treatment of stored energy in the initial phases of nucleation of a vacuum energy. Let us first review the typical implications as to how we get the Einstein field equations. We will then proceed to consider how the Einstein equation, with a bit of emphasis as to the proposal as to the evolution of the vacuum energy actually gives credence to the necessity of short term existence of the quintessence scalar field. The proposal so outlined heavily depends upon a huge vacuum energy being dominant in the derived Einstein field equations, with the combined stress energy tensor contributions set equal to zero. The evolution of the scale factor would be in tandem with adding a new term to the metric \( g_{u,v} \), due to adding in a scale factor contribution to actually read as given by Moffat (2002):

\[
\tilde{g}_{uv} \equiv g_{u,v} - \vartheta \cdot (\partial_u \phi \partial_v \phi)
\]

Let us now look at the Stress tensor in General Relativity. We get, as a take off from Birrell (1984)

\[
T_{u,v} = \frac{2}{\sqrt{-g}} \cdot \frac{\delta S}{\delta g_{u,v}} \equiv 0, \\
S \equiv S_g + S_M,
\]

\[
T_{u,v}|_{S \rightarrow S_M} = \frac{4\pi AG}{a} |_{a \rightarrow \epsilon^+}
\Rightarrow R_{u,v} - \frac{g_{uv} R}{2} + \Lambda_{4-dim} g_{u,v} = -8\pi G T_{u,v}
\Rightarrow \Lambda_{4-dim}|_{initial} \cdot [g_{0,0} - \vartheta \cdot [\partial_0 \phi]^2] = \frac{4\pi AG}{\epsilon^+}
\]

The last segment on the bottom of Eqn. (15) captures the dynamics of the scalar field interaction. The statement made above in particular relies upon the following dynamic made in a short time interval. Namely, assume that the right hand side of the Casimir plate separation, as written up by De Witt (1979) would be relatively constant as the separation of the domain walls of an initial vacuum state configuration became miniscule in size

\[
\Lambda_{4-dim}|_{initial} \cdot [g_{0,0} - \vartheta \cdot [\partial_0 \phi]^2] \xrightarrow{t \rightarrow \epsilon < \text{Planck time}} \text{constant}
\]

In such a short time interval, we would have as the initial cosmological vacuum energy went up, a corresponding drop in the \((\partial_0 \phi) \neq 0\) \xrightarrow{t \sim \text{Planck time}} 0^+. Either the scalar quintessence field would be a constant, or it would go to zero. Considering the non spherical geometry of early brane world geometry, the easiest way to get uniformly consistent criteria would be to have the quintessence scalar field itself rapidly damp out even if we write having the distance of the separation between early vacuum state geometry, designated as \(a\) go nearly to zero. But having the left hand side of \((\partial_0 \phi) \neq 0\) \xrightarrow{t < \text{Planck time}} 0^+
limit initially not zero would be, in tandem with Eqn (16) above a strong lead as to non zero, time changing quintessence fields being a measure of quantum entanglement.

5. Comparison with the Coleman-Weinberg Low Temperature Potential for Scalar Fields

We consider a model of a potential, as given by Coleman-Weinberg which is primarily configured for low temperature regimes. This comes out to be as follows, Kurioukidis (2004), as given by:

\[ V(\phi) \sim V(0) + B\phi^4 \cdot \left[ \ln \left( \frac{\phi^2}{\sigma^2} \right) - \frac{1}{2} \right] \] (17)

If the scalar potential is in itself tending to be small, we find that we obtain a logarithmic expansion which, if we use the other assumptions used in filling out Eqn. (1) above:

\[ \phi^2 \simeq \frac{c_1^2 - \alpha^2}{4B \cdot \left[ 1 + \sum_{j=2}^{N} \frac{1}{j} \right]} \rightarrow 0^+ \] (18)

The more terms are taken in looking at \( \ln \left( \frac{\phi^2}{\sigma^2} \right) \sim \sum_{j=1}^{\infty} (-1)^{j+1} \cdot \frac{[(\phi/\sigma)-1]^j}{j} \), the quicker Eqn (18) converges to zero. I.e. we have an almost instantaneous collapse of the scalar field \( \phi \) to zero, which is in effect saying that the Coleman-Weinberg potential is good for duplicating the results of Eqn. (5) only.

What can be said about a conformal mapping of Hawking radiation leakage from black holes being collated into material for a new ‘big bang’?

First what do we mean in this situation by the Hawking radiation effect? In this we are referring to what happens to the life time of such entities even if the mass drops below a typical threshold. As an example:

\[ \tau = \frac{1}{M_*} \left( \frac{M_{BH}}{M_*} \right)^{\frac{(n+3)}{n+1}} \] (19)

Here, the value of \( M_* \) can be as low as a few TeV, and the formula can give time scales on the order of ”new Planck time” ~10^{-26} s, while \( n \) can be of any dimension needed. The problems come up with supposed Black holes with life times greater than the typical values assumed for the lifetime of the universe. i.e. Penrose stated that in his model that Black holes would be expected to collect matter far into the future, and that Hawking radiation would leak the collected “material” in a fashion which would collectively be re constituted for a new ‘big bang’. Penrose dubbed his collection of Hawking radiation for innumerable black holes a so called “ripple in the pond” effect which would in its own way able to collate material for a new big bang. Ida, Oda, and Park (2003) give a Hawking temperature which is strongly dependent upon dimensionality, but which
affirms roughly that black hole production and evaporation in extra dimension scenarios with TeV scale gravity will still keep much of the qualitative features presented in Eqn (19) above. This then leads us to the following question. What conformal mapping exists which collates material from a Hawking temperature? Ida, Oda, and Park in their article’s Eqn. (27) have a dimensionally dependent expression for Hawking temperature, and the Hawking temperature, $T$, is such that in general as Klaus Kiefer notes (2002) there is a relationship for an interplay between entropy and a Hawking’s temperature $T$ along the lines of, if the surface gravity $K_{\text{surface}} \sim GM/R_0^2$, where the denominator is roughly the radii of a black hole, and, as given by Kiefer (2002)

$$T_{\text{BH}} \cdot dS_{\text{BH}} \sim [K_{\text{surface}}/8\pi G] \cdot dA \tag{20}$$

That the question of collation of thermal radiation is best expressed as to how entropy can be collated into the nucleus of a big bang from innumerable places in space time, as the universe continually expands

Frankly, the author does not see how this is possible. Penrose (2007) in a 20 minute discussion with the author at the inaugural conference at the IGC Penn State university center, in 2007 stated that the necessary conditions for such a conformal mapping collation are contained within individual Black Hole Hawking temperature values. The author points to the dimensionality of the Black holes, and asks if a working conformal mapping is possible.

Penrose may well be right. The authors deliberations as to the inter relationship between entropy and Hawking temperature, as given by Eqn. (20) may reflect upon the author’s limitations, not Penrose. If so, and if Penrose is right, the universe as we know it is far stranger than imagined by current cosmologists. In closing, the author wishes to point to how entropy is linked to IT models of information for the universe, as given by Lloyd (2002)

$$[\# \text{operations}] \leq S(\text{entropy})/(k_B \cdot \ln 2) \sim 10^{120} \tag{21}$$

Parsing the connections between innumerable Black hole contributions to individual entropy into a grand value of entropy as given by Eqn. (21) below is currently beyond the Author’s conceptual understanding at this juncture, but the concept is very intriguing and deserves serious study. In the conclusion below, the Author wishes to also high light what Dr. Smoot at the Challongue school talked about which is the growth of ‘information’ in a way parallel to Eqn. (21) above. Bosonification as mentioned below, where information is initially suppressed but released into cosmology evolution models at the onset of the Big bang favors the worm hole model as mentioned in this document.

**Conclusion**

Eqn. (16) above outlines the current state of the art as to entanglement and its connections to the existence of a worm hole solution for transfer of vacuum energy from a prior universe to our own. In this document, we have alluded to the inescapable conclusion
that entanglement in early universe geometry is intrinsically linked to the existence of short term quintessence in scalar fields, as seen in Beckwith (2007), and Moffat (2002).

This is important, for many reasons. One of them being that it explains the reason for why \( w = -1 \) so early in cosmological evolution. In addition the existence of \((\partial_0 \phi) \neq 0 \) at Planck time limit initially not zero is in tandem with the existence of causal discontinuity written up in another document where the existence of the time dependent Wheeler De Witt worm hole actually leads to causal discontinuity, which if quantum entanglement exists when \((\partial_0 \phi) \neq 0^+\) is in sync with the existence of faster than light transferal of thermal heat from a prior universe to our own.

In a colloquium presentation done by Dr. Smoot (2007); he alluded to the following information theory constructions which bear consideration as to how much is transferred between a prior to the present universe in terms of information ‘bits’. The following are Dr. Smoot’s preliminary analysis of information content in the observable universe:

1. Physically observable bits of information possibly generated in the Universe - \(10^{180}\)
2. Holographic principle allowed states in the evolution / development of the Universe - \(10^{120}\)
3. Initially available states given to us to work with at the onset of the inflationary era - \(10^{10}\)
4. Observable bits of information present due to quantum / statistical fluctuations -\(10^8\)

Our guess is as follows. That the thermal flux so implied by the existence of a worm hole accounts for perhaps \(10^{10}\)bits of information. These could be transferred via a worm hole solution from a prior universe to our present, as alluded to by Eqn. (9) above, and that there could be, perhaps \(10^{120}\) minus \(10^{10}\)bytes of information temporarily suppressed during the initial bozonification phase of matter right at the onset of the big bang itself.

Then after the degrees of freedom dramatically drops during the beginning of the descent of temperature from about \(T \approx 10^{32} \text{Kelvin}\) to at least three orders of magnitude less, as we move out from an initial red shift \(z \approx 10^{25}\) to \(T \approx \sqrt{\epsilon V} \times 10^{28}\text{Kelvin} \sim T_{Hawking} \equiv \frac{\hbar H_{initial}}{2\pi c k_B}\), as outlined by N. Sanchez.

The author wishes to state that he is convinced that the initial starting point of the Penrose model as far as inflation is correct.

What should we look forward to in the future? We should delineate more detail as to what would be transferred, possibly by entanglement information transfer from a prior universe, to our own, as well as understand how additional bytes of information came to be in the present Universe. All this would tie in with an accurate physical understanding of the points raised in the above section. And, in addition, the author is not ruling out in any fashion Dr. Penrose’s (2007) very unorthodox suggestion as to the ‘ripples’ in the pond effect. The picture presented of worm hole transfer favors, implicitly, the big crunch models of cyclic universes. However, the author would be pleasantly surprised and intrigued if entropy collection of innumerable black holes, as related to by a grand value of entropy, as given by Eqn. (21), as mentioned by Lloyd (2002) and that Penrose (2007) is, indeed, verified as to his cyclic universe model.

It is worth mentioning that the author has done work in implementing Dowker’s
view of a causal discontinuity between prior and present universes, Beckwith, (2007), implementing work done by Dowker (2003, 2007). It remains to be seen if such work is necessary in any way to compliment or round out Penrose’s (2007) vision of a cyclic universe, minus the big crunch.

References


Increasing Effective Gravitational Constant in Fractional ADD Brane Cosmology

EL-NABULSI Ahmad Rami*

Department of Nuclear and Energy Engineering, Cheju National University,
Ara-dong 1, Jeju 690-756, South Korea

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Abstract: Arkani–Hamed–Dimopoulos–Dvali brane model with time–increasing scaling gravitational constant is constructed within the framework of fractional action–like variational approach with one positive parameter ‘α’.

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Recently, we formulated a fractional action-like variational approach (FALVA) based on the following fractional action

\[ S[q] \triangleq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} L(\dot{q}, q, \tau) (t - \tau)^{\alpha - 1} d\tau = \int_{t_0=0}^{t} L(q, \dot{q}, \tau) dg_t(\tau) \] (1)

to model correctly nonconservative and dissipative dynamical systems [1,2,3]. Our formalism is inspired in reality from fractional analysis, in particular the Riemann-Liouville fractional integrals and derivatives, playing an important and significant role in the understanding of complex classical and quantum dynamical systems. \( L(\dot{q}, q, \tau) \) is the Lagrangian weighted with \( (t - \tau)^{\alpha - 1} / \Gamma(\alpha) \) and \( \Gamma(1 + \alpha) g_t(\tau) = (t^\alpha - (t - \tau)^\alpha) \) with the scaling properties \( g_{kt}(k\tau) = k^\alpha g_t(\tau), k > 0 \) and \( \alpha \in [-\infty, +\infty] \). For fixed "t", integral (1) becomes a Stieltjes integral. The applications of the fractional calculus to the constrained dynamical systems and the extension of the fractional variational problem of Lagrange were discussed and some interesting cosmological, astrophysical and quantum field features are explored in discussed [4-11]. The extension of Noether’s symmetry theorem to FALVA was also recently discussed [12]. Recently, we have shown that the fractional variational approach when applied to Riemann geometry and Einstein General

* nabulsiahmadrami@yahoo.fr
Relativity, will generate perturbed gravity and friction that decay weakly with the time, revealing some important cosmological features by modifying the gravity [9,11]. Encouraged by these results, we would like in this letter to extend our approach and study a simple cosmological application of FALVA on brane dissipative cosmology. Recently, the cosmological implications of the brane world scenario are investigated when the gravitational coupling $G$ is not constant but rather vary with time [13,14]. We will show that FALVA will lead to a brane world with time-increasing $G$ which may be important to describe a late universe. For this we reconsider the Arkani-Hamed-Dimopoulos-Dvali (ADD) brane model of zero tension with space-time topology of type $M = M^4 \otimes \mathbb{R}^n$, where $M^4$ is the familiar four-dimensional manifold and $\mathbb{R}^n$ is the finite $L$ dimensional extra-dimension. We define the Lagrangian of the theory as $L \rightarrow Lf(\tau)$ where $f(\tau)$ is a function of $\tau$ introduced to take into account the Riemann-Liouville fractional integral of order $\alpha$ of $f$ and rewrite the following fractional action-like as [15-18]:

$$S_{4+n,\alpha} = \frac{c^3}{16\pi G_{(3+n,\tau),\alpha}} \int_{M^4 \otimes \mathbb{R}^n} d^{3+n}x \sqrt{-\tilde{g}} \bar{R}_t \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau) (t - \tau)^{\alpha-1} d\tau,$$

$$\sim \frac{c^3}{16\pi G_{(3+n,\tau),\alpha}} \int_M d^3x \sqrt{-\tilde{g}} \bar{R}_t L^n \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau) (t - \tau)^{\alpha-1} d\tau,$$

$$\equiv \frac{c^3}{16\pi G_{(3+n,\tau),\alpha}} \int_M d^3x \sqrt{-\tilde{g}} \bar{R}_t L^n,$$

(2)

where $\bar{R}_t$ (the Ricci scalar tensor), $\tilde{g}$ are the projections of the corresponding quantities on $M^4$ and

$$D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau) (t - \tau)^{\alpha-1} d\tau,$$

(3)

is the Riemann-Liouville fractional integral of order $\alpha$. The volume of the extra-dimensions is $L^n$. When compared with the standard ADD one corresponding to $\alpha = 1$:

$$S_{4+n,\alpha=1} = \frac{c^3}{16\pi G_{(3+n,\tau),\alpha=1}} \int_M d^3x \sqrt{-\tilde{g}} \bar{R}_t L^n d\tau,$$

(4)

we get

$$G_{(3+n,\tau),\alpha} = G_{(3+n,\tau),\alpha=1} D_t^{-\alpha} f(t)$$

$$= \frac{G_{(3+n,\tau),\alpha=1}}{\Gamma(\alpha)} \int_0^t f(\tau) (t - \tau)^{\alpha-1} d\tau,$$

(5)

while when compared with the 4D action, we easily find:

$$G_{(3+n,\tau),\alpha} \equiv G_{(4+n),\alpha} \propto G_N L^n D_t^{-\alpha} f(t) \equiv \bar{G}_N L^n,$$

(6)

where $\bar{G}_N \equiv G_N D_t^{-\alpha} f (t) \rightarrow G_N$ for $\alpha = 1$ (to be differentiated from $\Delta G$). As a result, we fall into the ADD brane model with time dependent scaling gravitational constant. One can also write:

$$G_{(3+n,\tau),\alpha} \equiv G_{(4+n),\alpha} \propto G_N L^n D_t^{-\alpha} f(t) \equiv G_N \bar{L}^n,$$

(7)
where $\bar{L}_N \equiv L_N D_t^{-\alpha}f(t) \to L_N$ for $\alpha = 1$. In other words, we have an ADD brane model with time-dependent scaling volume of the extra-dimensions [19,20,21]. If $f(t) = K$, $K$ is a constant, $D_t^{-\alpha}K = K t^\alpha / \Gamma (1 + \alpha)[22,23]$ and as a result the effective $\bar{L}_N$ increases with times for $\alpha > 0$. This mechanism could have important consequences on large extra-compactified dimensions [24]. In order to have $G_{(4+n)\alpha} = G_N L^n$, we need to write a fractional 4D action in the sense of (5), that is:

$$S_{4,\alpha} = \frac{c^3}{16\pi G_{(3,\tau)\alpha} \int_{M^4} d^3x \sqrt{-\tilde{g}} R_{\alpha} \frac{1}{\Gamma (\alpha) \int_0^t f(\tau) (t - \tau)^{\alpha - 1} d\tau}. \quad (8)$$

If again $f(\tau) = K$, $K$ is a constant, the effective gravitational constant increases with time [25,26,27]. In fact, higher–dimensional theories imply that the gravitational constant is not fundamental constant. Instead it is related to the sizes of the extra–dimensional space, which are moduli fields in the four-dimensional effective theory [28,29]. It will be of interest to explore in details in the future a FALVA weak dissipative brane cosmology where the gravitational constant varies as $G = G_0 T^\alpha$, $G_0$ being a constant.

References


A Two-dimensional Discrete Mapping with $C^\infty$ Multifold Chaotic Attractors

Zeraoulia Elhadj$^a$*, J. C. Sprott$^b$†

$^a$Department of Mathematics, University of Tébessa, (12000), Algeria.
$^b$Department of Physics, University of Wisconsin, Madison, WI 53706, USA.

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Abstract: This paper introduces a two-dimensional, $C^\infty$ discrete bounded map capable of generating "multifold" strange attractors via period-doubling bifurcation routes to chaos. © Electronic Journal of Theoretical Physics. All rights reserved.

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1. Introduction

Discrete mathematical models are usually derived from theory or experimental observation, or as an approximation to the Poincaré section for some continuous-time models. Many papers have described chaotic systems, one of the most famous being a two-dimensional discrete map suggested by Hénon [3] and studied in detail by others [3,4,11,12]. It is possible to change the form of this map to obtain other chaotic attractors [5,6,7,8,14] or to make some $C^1$-modifications to obtain "multifold" strange chaotic attractors [6] with possible applications in secure communications because of their chaotic properties [9,10].

The Hénon map is a prototypical two-dimensional invertible iterated map with a chaotic attractor and is a simplified model of the Poincaré map for the Lorenz equation proposed by Hénon in 1976 and given by:

$$H(x_n, y_n) = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 - ax_n^2 + by_n \\ x_n \end{pmatrix}$$

* zeraoulia@mail.univ-tebessa.dz and zelhadj12@yahoo.fr
† sprott@physics.wisc.edu
For $b = 0$, the Hénon map reduces to the quadratic map [1], which is conjugate to the logistic map. Bounded solutions exist for the Hénon map over a range of $a$ and $b$ values, and a portion of this range yields chaotic attractors. The Hénon procedure permits the construction of a family of attractors dependent on the two parameters $a$ and $b$, but it does not have attractors with "multifolds." However, appropriate $C^1$-modifications can result in such attractors [6].

In this paper we study a modified Hénon map given by:

$$f(x_n, y_n) = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 - a \sin x_n + by_n \\ x_n \end{pmatrix}, \tag{2}$$

or equivalently:

$$x_{n+1} = 1 - a \sin x_n + bx_{n-1}. \tag{3}$$

where the quadratic term $x^2$ in the Hénon map is replaced by the nonlinear term $\sin x$, and we study this model for all values of $a$ and $b$. The essential motivation for this work is to develop a $C^\infty$ mapping that is capable of generating chaotic attractors with "multifolds" via a period-doubling bifurcation route to chaos which has not been studied before in the literature. The fact that this map is $C^\infty$ in some ways simplifies the study of the map and avoids some problems related to the lack of continuity or differentiability of the map. The choice of the term $\sin x$ has an important role in that it makes the solutions bounded for values of $b$ such that $|b| \leq 1$, and all values of $a$, while they are unbounded for $|b| > 1$. On the other hand, this is not the only possible choice, for example one can use the term $\cos x$.

2. Analytical results

In all proofs given here, we use the following standard results:

**Theorem 1.** Let $(x_n)_n$, and $(z_n)_n$ be two real sequences, if $|x_n| \leq |z_n|$ and $\lim_{n \to +\infty} |z_n| = A < +\infty$, then $\lim_{n \to +\infty} |x_n| \leq A$, or if $|z_n| \leq |x_n|$, and $\lim_{n \to +\infty} |z_n| = +\infty$ then, $\lim_{n \to +\infty} |x_n| = +\infty$.

The proof of this result is available in standard mathematics books and will not be given here.

We use this result to construct a sequence $(z_n)_n$ that satisfies the above conditions for determining whether the difference equation (3) has bounded or unbounded orbits.

**Theorem 2.** For all values of $a$ and $b$ the sequence $(x_n)_n$ given in (3) satisfies the following inequality:

$$|1 - x_n + bx_{n-2}| \leq |a|, \tag{4}$$
Proof. We have for every \( n > 1 \): \( x_n = 1 - a \sin x_{n-1} + bx_{n-2} \), then, one has:

\[
| -x_n + 1 + bx_{n-2} | = | a \sin x_{n-1} | \leq |a|,
\]

since \( \sup_{x \in \mathbb{R}} | \sin x | = 1 \).

Theorem 3. For every \( n > 1 \), and all values of \( a \) and \( b \), and for all values of the initial conditions \((x_0, x_1) \in \mathbb{R}^2\), the sequence \((x_n)\) satisfies the following equalities:

(a) If \( b \neq 1 \), then:

\[
x_n = \begin{cases} 
\frac{n-1}{b-1} + b\frac{n-1}{2} x_1 - a \sum_{p=1}^{\frac{n-1}{2}} b^{p-1} \sin x_{n-(2p-1)}, & \text{if } n \text{ is odd}, \\
\frac{n-1}{b-1} + b\frac{n}{2} x_0 - a \sum_{p=1}^{\frac{n}{2}} b^{p-1} \sin x_{n-(2p-1)}, & \text{if } n \text{ is even},
\end{cases}
\]

(b) If \( b = 1 \), then:

\[
x_n = \begin{cases} 
\frac{n-1}{2} + x_1 - a \sum_{p=1}^{\frac{n-1}{2}} \sin x_{n-(2p-1)}, & \text{if } n \text{ is odd}, \\
\frac{n}{2} + x_0 - a \sum_{p=1}^{\frac{n}{2}} \sin x_{n-(2p-1)}, & \text{if } n \text{ is even}
\end{cases}
\]

Proof. Assume that \( n \) is odd, then we have for every \( n > 1 \), the following equalities:

\[
x_n = 1 - a \sin x_{n-1} + bx_{n-2}, \quad (8)
\]

\[
x_{n-2} = 1 - a \sin x_{n-3} + bx_{n-4}, \quad (9)
\]

\[
x_{n-4} = 1 - a \sin x_{n-5} + bx_{n-6} \ldots \quad (10)
\]

Then the results in (6) and (7) are obtained by successive substitutions of (9), (10),... into (8) for all \( k = n - 2, n - 4, ..., 2 \). The other cases can be obtained using the same logic.

Theorem 4. The fixed points \((l, l)\) of the map (3) exist if one of the following conditions holds:

(i) If \( a \neq 0 \), and \( b \neq 1 \), then \( l \) satisfies the following conditions:

\[
\begin{align*}
1 - a \sin l + (b - 1) l &= 0, \quad \text{and } l \leq \frac{1 + |a|}{1-b}, \quad \text{if } b > 1, \\
\frac{1 + |a|}{1-b} &\leq l, \quad \text{if } b < 1,
\end{align*}
\]

(ii) If \( b = 1 \), and \( |a| \geq 1 \), then, \( l \) is given by : \( l = \arcsin \left( \frac{1}{2a} \right) \).

(iii) If \( b \neq 1 \), and \( a = 0 \), then, \( l \) is given by \( l = \frac{1}{1-b} \).

(iv) If \( a = 0 \), and \( b = 1 \), there are no fixed points for the map (3).

Proof. The proof is direct except for the case (i) where we apply Theorem 2, and
therefore one concludes that all fixed points of the map (3) are confined to the interval 
$(-\infty, \frac{1+|a|}{1-b}]$ if $b > 1$, and to $[\frac{1+|a|}{1-b}, +\infty)$ if $b < 1$. On the other hand, case (iii) gives a simple linear second-order difference equation $x_n = 1 + bx_{n-2}$, for which the situation is standard.

Since the location of the fixed points for map (3) cannot be calculated analytically, their stability will be studied numerically.

2.1 Existence of bounded orbits

In this subsection, we determine sufficient conditions for the map (3) to have bounded solutions. This is the interesting case since it includes the periodic, quasi-periodic, and chaotic orbits. First we prove the following theorem:

**Theorem 5.** The orbits of the map (3) are bounded for all $a \in \mathbb{R}$, and $|b| < 1$, and all initial conditions $(x_0, x_1) \in \mathbb{R}^2$.

**Proof.** From equation (3) and the fact that $\sin x$ is a bounded function for all $x \in \mathbb{R}$, one has the following inequalities for all $n > 1$:

$$|x_n| \leq 1 + |a| + |bx_{n-2}|,$$  \hspace{1cm} (12)

$$|x_{n-2}| \leq 1 + |a| + |bx_{n-4}|,$$  \hspace{1cm} (13)

$$... |x_{n-4}| \leq 1 + |a| + |bx_{n-2}|.$$  \hspace{1cm} (14)

This implies from (12), (13), (14), ... that:

$$|x_n| \leq (1 + |a|) + |b| (1 + |a| + |bx_{n-4}|),$$  \hspace{1cm} (15)

$$|x_n| \leq (1 + |a|) + (1 + |a|) |b| + |b|^2 |x_{n-4}|,$$  \hspace{1cm} (16)

$$|x_n| \leq (1 + |a|) + (1 + |a|) |b| + |b|^2 |x_{n-4}|,$$  \hspace{1cm} (17)

Hence, from (13) and (17) one has:

$$|x_n| \leq (1 + |a|) + (1 + |a|) |b| + |b|^2 (1 + |a|) + |b|^3 |x_{n-6}|,$$  \hspace{1cm} (18)

Since $|b| < 1$, then the use of (18) and induction about some integer $k$ using the sum of a geometric growth formula permits us to obtain the following inequalities for every $n > 1, k \geq 0$:

$$|x_n| \leq (1 + |a|) \left( 1 + \frac{|b|^k}{1 - |b|} \right) + |b|^k |x_{n-2k}|.$$  \hspace{1cm} (19)

where $k$ is the biggest integer $j$ such that $j \leq \frac{n}{2}$. Thus one has the following two cases:

1) if $n$ is odd, i.e. $\exists m \in \mathbb{N}$, such that $n = 2m + 1$, then the biggest integer $k \leq \frac{n}{2}$ is $k = \frac{n-1}{2}$, for which $(x_n)_n$ satisfies the following inequalities:

$$|x_{2m+1}| \leq (1 + |a|) \left( 1 + \frac{|b|^m}{1 - |b|} \right) + |b|^m |x_1| = z_m.$$  \hspace{1cm} (20)
(2) if \( n \) is even, i.e. \( \exists m \in \mathbb{N} \), such that \( n = 2m \), then, the biggest integer \( k \leq \frac{n}{2} \) is \( k = \frac{n}{2} \), for which \( x_n \) satisfies the following inequalities:

\[
|x_{2m}| \leq (1 + |a|) \left( \frac{1 - |b|^{m}}{1 - |b|} \right) + |b|^m |x_0| = u_m,
\]

(21)

Thus, since \( |b| < 1 \), the sequences \((z_m)_m\) and \((u_m)_m\) are bounded, and one has:

\[
\begin{align*}
\forall m \in \mathbb{N}, \quad & z_m \leq \frac{(1+|a|)}{1-|b|} + \left| x_0 - \frac{(1+|a|)}{1-|b|} \right|, \\
\forall m \in \mathbb{N}, \quad & u_m \leq \frac{(1+|a|)}{1-|b|} + \left| x_0 - \frac{(1+|a|)}{1-|b|} \right|,
\end{align*}
\]

(22)

Thus Formulas (20), (21), and inequalities (22) give the following bounds for the sequence \((x_n)_n\):

\[
|x_n| \leq \max \left( \frac{(1+|a|)}{1-|b|} + \left| x_0 - \frac{(1+|a|)}{1-|b|} \right|, \frac{(1+|a|)}{1-|b|} + \left| x_0 - \frac{(1+|a|)}{1-|b|} \right| \right).
\]

(23)

Finally, for all values of \( a \) and all values of \( b \) satisfying \( |b| < 1 \) and all initial conditions \( (x_0, x_1) \in \mathbb{R}^2 \), one concludes that all orbits of the map (3) are bounded, i.e. in the subregion of \( \mathbb{R}^4 \):

\[
\Omega_1 = \{ (a, b, x_0, x_1) \in \mathbb{R}^4 / |b| < 1 \}.
\]

(24)

Hence the proof is completed.

2.2 Existence of unbounded orbits

In this subsection, we determine sufficient conditions for which the orbits of the map (3) are unbounded. We first prove the following theorem:

**Theorem 6.** The map (3) possesses unbounded orbits in the following subregions of \( \mathbb{R}^4 \):

\[
\Omega_2 = \left\{ (a, b, x_0, x_1) \in \mathbb{R}^4 / |b| > 1, \text{ and both } |x_0|, |x_1| > \frac{|a| + 1}{|b| - 1} \right\},
\]

(25)

and

\[
\Omega_3 = \left\{ (a, b, x_0, x_1) \in \mathbb{R}^4 / |b| = 1, \text{ and } |a| < 1 \right\}.
\]

(26)

**Proof.** (a) For every \( n > 1 \), we have: \( x_n = 1 - a \sin x_{n-1} + bx_{n-2} \). Then \( |bx_{n-2} - a \sin x_{n-1}| = |x_n - 1| \) and \( |bx_{n-2}| - |a \sin x_{n-1}| \leq |x_n - 1| \). (We use the inequalities: \( |x| - |y| \leq ||x| - |y|| \leq |x - y| \)). This implies that

\[
|bx_{n-2}| - |a \sin x_{n-1}| \leq |x_n| + 1.
\]

(27)

Since \( |a \sin x_{n-1}| \leq 1 \), this implies \( -|a \sin x_{n-1}| \geq -|a| \), and \( |bx_{n-2}| - |a \sin x_{n-1}| \geq |bx_{n-2}| - |a| \). Finally, one has from (27) that:

\[
|bx_{n-2}| - (|a| + 1) \leq |x_n|.
\]

(28)
Then, by induction as in the previous section, one has:

\[ |x_n| \geq \begin{cases} 
   \left( \frac{|a|+1}{|b|-1} + |x_1| \right) |b|^\frac{n-1}{2} + \frac{|a|+1}{|b|-1}, & \text{if } n \text{ is odd,} \\
   \left( \frac{|a|+1}{|b|-1} + |x_0| \right) |b|^\frac{n}{2} + \frac{|a|+1}{|b|-1}, & \text{if } n \text{ is even.} 
\end{cases} \]  

(29)

Thus, if \( |b| > 1 \), and both \( |x_0|, |x_1| > \frac{|a|+1}{|b|-1} \), one has: \( \lim_{n \to +\infty} |x_n| = +\infty \).

(b) For \( b = 1 \), one has:

\[ |x_n| \geq \begin{cases} 
   (1 - |a|) \left( \frac{n-1}{2} \right) + x_1, & \text{if } n \text{ is odd,} \\
   (1 - |a|) \left( \frac{n}{2} \right) + x_0, & \text{if } n \text{ is even.} 
\end{cases} \]  

(30)

Hence, if \( |a| < 1 \), then one has: \( \lim_{n \to +\infty} x_n = +\infty \).

For \( b = -1 \), one has from Theorem 3 the inequalities:

\[ x_n \leq \begin{cases} 
   - \left( \frac{n-1}{2} \right) + x_1 + \sum_{p=1}^{\lfloor \frac{n-1}{2} \rfloor} a (-1)^{p-1} \sin x_{n-(2p-1)} , & \text{if } n \text{ is odd,} \\
   - \left( \frac{n}{2} \right) + x_0 + \sum_{p=1}^{\lfloor \frac{n}{2} \rfloor} a (-1)^{p-1} \sin x_{n-(2p-1)} , & \text{if } n \text{ is even.} 
\end{cases} \]  

(31)

Because \( a (-1)^{p-1} \sin x_{n-(2p-1)} \leq |a| \), then one has:

\[ x_n \leq \begin{cases} 
   (|a| - 1) \left( \frac{n-1}{2} \right) + x_1, & \text{if } n \text{ is odd,} \\
   (|a| - 1) \left( \frac{n}{2} \right) + x_0, & \text{if } n \text{ is even,} 
\end{cases} \]  

(32)

Thus, if \( |a| < 1 \), then one has: \( \lim_{n \to +\infty} x_n = -\infty \).

Hence, the proof is completed.

It can be easily seen from the above results that the Hénon-like map of the form (3) with a sine function may exhibit with respect to the parameter \( b \) the following behaviors:

(i) If \( |b| < 1 \), then the map (3) is bounded (See Theorem 5).

(ii) If \( |b| \geq 1 \), then the map (3) is unbounded (See Theorem 6).
3. Numerical simulations

3.1 Some observed multifold attractors

In this section we present some observed multifold chaotic attractors obtained by an appropriate choice of the parameters $a$ and $b$. All the phase portraits presented in this paper are done in the $xy$-plane. We observe that the chaotic attractors evolve around a large number of fixed points, and it appears that the number of these points increase with increasing $a$ when $b$ is fixed.

There are several possible ways for a discrete dynamical system to transition from regular behavior to chaos. Bifurcation diagrams display these routes and allow one to identify the chaotic regions in $ab$-space from which the chaotic attractors can be determined. In this subsection we will illustrate some observed chaotic attractors, along with some other dynamical phenomena.

![Chaotic multifold attractors of the map (3) obtained for (a) $a = 2.4, b = -0.5$. (b) $a = 2, b = 0.2$. (c) $a = 2.8, b = 0.3$. (d) $a = 2.7, b = 0.6$.](image)

**Fig. 1** Chaotic multifold attractors of the map (3) obtained for (a) $a = 2.4, b = -0.5$. (b) $a = 2, b = 0.2$. (c) $a = 2.8, b = 0.3$. (d) $a = 2.7, b = 0.6$.

3.2 Route to Chaos

It is well known that the Hénon map [3] typically undergoes a period-doubling route to chaos as the parameters are varied. By contrast, the Lozi map [8] has no period-doubling route, but rather it goes directly from a border-collision bifurcation developed from a stable periodic orbit [5]. Similarly, the chaotic attractor given in [14] is obtained from a border-collision period-doubling bifurcation scenario. This scenario involves a sequence of pairs of bifurcations, whereby each pair consists of a border-collision bifurcation and a pitchfork bifurcation. The other map given in [13] is obtained from a quasi-periodic route to chaos. Thus, the four chaotic systems go via different and distinguishable routes to chaos. Furthermore, the *multifold* chaotic attractors presented in Fig. 1 are obtained...
from the map (3) via a period-doubling bifurcation route to chaos as shown in Fig. 5 (a).

3.3 Dynamical behaviors with parameter variation

In this subsection, the dynamical behaviors of the map (3) are investigated numerically.

![Figure 2](image)

**Fig. 2** Regions of dynamical behaviors in $ab$-space for the map (3).

Figure 2 shows regions of unbounded (white), fixed point (gray), periodic (blue), and chaotic (red) solutions in the $ab$-plane for the map (3), where we use $10^6$ iterations for each point. On the other hand, if we fix parameter $b = 0.3$ and vary $-1 \leq a \leq 4$, the map (3) exhibits the dynamical behaviors as shown in Fig. 5.

![Figure 3](image)

**Fig. 3** Chaotic multifold attractors of the map (3) obtained for (a) $a = 3.4, b = -0.8$. (b) $a = 3.6, b = -0.8$. (c) $a = 4, b = 0.5$. (d) $a = 4, b = 0.9$.

In the interval $-1 \leq a \leq 0.76$, the map (3) converges to a fixed point. For $0.76 < a \leq 1.86$, there is a series of period-doubling bifurcations as shown in Fig. 5 (a). In the interval $1.86 < a \leq 2.16$, the orbit converges to a chaotic attractor. For $2.16 < a \leq 2.27$, it converges to a fixed point. For $2.27 < a \leq 2.39$, there are periodic windows. For
Fig. 4 Multifold chaotic attractors of the map (3) obtained for $b = 0.3$ and (a) $a = 3$, (b) $a = 5$, (c) $a = 7$, (d) $a = 10$.

Fig. 5 (a) Bifurcation diagram for the map (3) obtained for $b = 0.3$ and $-1 \leq a \leq 4$. (b) Variation of the Lyapunov exponents of map (3) over the same range of $a$

$2.39 < a \leq 2.92$, it converges to a chaotic attractor. For $a > 2.92$, the map (3) is chaotic. For example, the Lyapunov exponents for $a = 3$ and $b = 0.3$ are $\lambda_1 = 0.56186$ and $\lambda_2 = -1.76583$, giving a Kaplan-Yorke dimension of $D_{KY} = 1.31818$. There are also fixed points and periodic orbits. This map is invertible for all $b \neq 0$, especially for $|b| < 1$,
and there is no hyperchaos since the sum of the Lyapunov exponents $\lambda_1 + \lambda_2 = \ln |b|$ is never positive. Generally, if we fix $b = 0.3$ and $-150 \leq a \leq 200$, map (3) is chaotic over all the range as shown in Fig. 6, except for the small intervals mentioned above and shown in Fig. 5.

![Graph](image)

**Fig. 6** Variation of the Lyapunov exponents of map (3) over the range $-150 \leq a \leq 200$ with $b = 0.3$.

However, if we fix parameter $a = 3$ and vary $b \in \mathbb{R}$, the map (3) exhibits very complicated dynamical behaviors as shown in Fig. 7, which shows some fixed points and some periodic windows. However, a large fraction of the region has chaotic attractors. Finally, for $|b| > 1$, the map (3) does not converge as shown in the previous section analytically.

For the map (3) we have calculated the attractors on a grid in $ab$-space (for $-1 \leq a \leq 4$) where the system is chaotic. There is a very wide variety of possible multifold chaotic attractors with different numbers of folds, only some of which are shown in Figs. 1, 3, and 4. The plots for the attractors do not show basin boundaries because the basins for bounded orbits include the entire $xy$-plane for $|b| < 1$.

There are regions of $ab$-space where two coexisting attractors occur as shown in black in Fig. 8, both in the regular and chaotic regimes. For example, with $a = 2$ and $b = -0.6$, a two-cycle $(1.314326, -0.584114)$ coexists with a period-3 strange attractor. Similarly, for $a = 2.2$ and $b = -0.36$, there is a strange attractor surrounded by a second period-3 strange attractor as shown in black in Fig. 9 with their corresponding basins of attraction shown in yellow and magenta, respectively.

Figure 8 was obtained by using 200 different random initial conditions and looking for cases where the distribution of the average value of $x$ on the attractor is bimodal. Since there is no rigorous test for bimodality, this was done by sorting the 200 values of $<x>$ and then dividing them into two equal groups. The group with the smallest range of $<x>$ was assumed to represent one of the attractors, and a second attractor was assumed to exist if the largest gap in the values of those in the other group was twice the
Fig. 7 a) Bifurcation diagram for the map (3) obtained for \( a = 3 \) and \( -1 \leq b \leq 1 \). (b) Variation of the Lyapunov exponents of map (3) for the same range of \( b \).

Fig. 8 The regions of \( ab \)-space where multiple attractors are found (shown in black).

range of the first group. The coexisting attractors were then confirmed in a separate calculation.

Conclusion

This paper reported the results of a detailed study of a \( C^\infty \) two-dimensional discrete map capable of generating smooth multifold strange attractors via period-doubling bifurcations.
Fig. 9 Two coexisting attractors occur for $a = 2.2$ and $b = -0.36$, where a strange attractor is surrounded by a second period-3 strange attractor with their corresponding basins of attraction shown in yellow and magenta, respectively.

References


Bosons-Parafermions Wess-Zumino Model*

L. Maghlaoui\textsuperscript{1} and N. Belaloui\textsuperscript{2†}

\textsuperscript{1}Université M’hamed Bouguerra, Faculté des Sciences, Département de Physique, Boumerdès, Algeria
\textsuperscript{2}LPMPS, Faculté des Sciences Exactes, Département de Physique, Université Mentouri Constantine, Algeria

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Abstract: A Wess-Zumino model in terms of bosons and parafermions of order $p = 2$ is investigated. We show that the parasupercharges associated to the parasupersymmetric transformations satisfy the $p = 2$ trilinear relations. The closure of the transformations algebra is established with a trilinear product rule for the fermionic elements. Finally, we verify that these parasupercharges are really the generators of the parasupersymmetric transformations.

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1. Introduction

Generalized statistics first was proposed by H. S. Green in 1953 [1]. Known nowadays as paraquantization, it leads to two families of generalized statistics (parabose and parafermi statistics which include the ordinary bose and fermi statistics as particular cases) which are collectively called parastatistics [2].

A fundamental question is faced respect to a spin statistics theorem which assert that integer-spin fields can not be quantized with the help of anticommutators and half-integer-spin fields can not be quantized with the help of commutators [3, 4, 5]. In the literature, A. B. Gorokov [6] published a very nice review article of a generalized statistics which consist on a modern review of different aspects of generalized statistics and first of all the paraboson and parafermion systems. Indeed, as is mentioned in [6], all particles

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† n._belaloui@yahoo.fr
which we consider to be elementary obey either bose or fermi statistics. This concerns both particles which are directly observable in our laboratories and those which can not. The reason expected for this general law of nature is hidden in deep properties of matter itself: identity and repeatability. The main idea of A. B. Govorkov [6] consist to adopt the fact that the invariance of the density matrix with respect to the permutation of all variables of identical particles as the starting definition of indistinguishability of the particles. Previously, some authors, Pauli [7], Dirac [8] and others have always underscored the possibility of describing the statistics of identical particles not only by means of symmetric and antisymmetric representations but also by multidimensional representations of the group of permutation of the coordinates and spin variables of particles, even it was proved [9] that it is impossible to establish commutation relations for operators corresponding to multidimensional representations of permutation groups. Afterwards, a theorem [10] came to reinforce the foregoing about the fact that the statistics of identical particles can be parafermi or parabose statistics. As a result of the starting definition of indistinguishability of the particles, the particles creation and annihilation operators obey to trilinear commutation relations. One principal result of this is the following [6]: Finite order parabose and parafermi statistics allow to put a definite number of particles in an antisymmetric state and a symmetric state, respectively, not exceeding a fixed number \( p \), the so-called order of the paraquantization.

The physics of elementary particles is based on two main symmetries: Poincaré and internal symmetries. A symmetry different from all of these was brought to the attention of the particles physics community by Wess and Zumino, and goes by the name of supersymmetry, the Wess-Zumino [11, 12] model has allowed anticommutation relations of the generators of supersymmetry which transform bosons into fermions and vice versa.

Generalized statistics and supersymmetry may be unified in what is called parasupersymmetry, which is a symmetry between bosons and parafermions. The parasupersymmetry structure will of course depend on the number \( p \) of parafermions that can occupy the same state. The paraextension of the Wess-Zumino model in the formalism of the superspace is developed in [13], the aim of this work is to investigate the most immediate field theoretical realization of the parasuperPoincaré algebra without the context of the parasuperspace. Following the same procedure in [14], we demonstrate by explicit calculation that the spinor parasupercharges of the theory considered as linear operators in the Fock space satisfy the parasuperPoincaré algebra. We study the closure of the transformations algebra and verify that the parasupercharges constructed are the generators of the parasupersymmetry transformations considered.

2. The Model

Introducing parasupersymmetry in the context of a simple four dimensional massive and free field theory: paraquantum version of the Wess-Zumino model, let us assume that the model possess one parafermion \( \Psi^\alpha \) of order \( p = 2 \) which is a Majorana spinor, on shell, that is \((i\partial_\mu\gamma^\mu\Psi - m) = 0\). Let us assume that the two degrees of freedom of
the parafermion $\Psi^\alpha$ impose the introduction of two bosonic degrees of freedom in order to form a realization of parasupersymmetry. We introduce then one ordinary real scalar field $A$ and one ordinary real pseudoscalar field $B$ subject to $(\Box + m^2)A = (\Box + m^2)B = 0$. Here we assume that the mass dimension of the parafermion $\Psi$ is, like in the ordinary case, always 3/2 and of course the ordinary fields $A$ and $B$ have the dimension 1. With an adequate symmetrization of the parafermionic field, the lagrangian density describing this simple system is given by:

\[
\mathcal{L} = \frac{1}{4} [\overline{\Psi}, (i\gamma^\mu \partial_\mu - m)\Psi] + \frac{1}{2} (\partial_\mu A)^2 + \frac{1}{2} (\partial_\mu B)^2 \\
- \frac{1}{2} m^2 A^2 - \frac{1}{2} m^2 B^2
\] (1)

where all the fields have the same mass $m$. and $\overline{\Psi} = \Psi^T \gamma^\alpha$. The basic commutation relations for the $p = 2$ parafermionic fields (at fixed time) are

\[
\langle \Psi_i(x, t) \Psi_j(y, t) \Psi_k(z, t) \rangle = 2\delta(x - y) \delta_{ij} \Psi_k(x, t) \\
+ 2\delta(y - z) \delta_{jk} \Psi_i(x, t)
\] (2)

\[
\langle \Psi_i(x, t) \Psi_j(y, t) \overline{\Psi}_k(z, t) \rangle = 2\delta(y - z) \delta_{jk} \Psi_i(x, t)
\] (3)

\[
\langle \Psi_i(x, t) \overline{\Psi}_j(y, t) \Psi_k(z, t) \rangle = 0
\] (4)

with the notation $\langle abc \rangle = abc + cba$. Those for the ordinary fields $A$ and $B$ are

\[
[A(x, t), \pi(y, t)] = i\delta(x - y)
\] (5)

\[\pi(x, t), \pi(y, t) = [A(x, t), A(y, t)] = 0
\] (6)

\[B(x, t), \pi'(y, t) = i\delta(x - y)
\] (7)

\[\pi'(x, t), \pi'(y, t) = [B(x, t), B(y, t)] = 0
\] (8)

where

\[
\pi(x, t) = \frac{\delta \mathcal{L}}{\delta (\partial_0 A(x, t))}, \pi'(x, t) = \frac{\delta \mathcal{L}}{\delta (\partial_0 B(x, t))}
\]

Furthermore, bosonic and parafermionic fields are taken to commute among themselves

\[ [\Psi_i, A] = [\Psi_i, B] = [A, B] = 0 \]

Now, the later development of this work necessitates the plane wave expansions of the Majorana parafield $\Psi^\alpha$ and the ordinary fields $A$ and $B$ which are given by

\[
\Psi^\alpha(x) = \frac{1}{(2\pi)^{3/2}} \sum_s \int d^3 p \left( \frac{m}{\omega_p} \right)^{1/2} \\
\left( d(\vec{p}, s) u_\alpha(\vec{p}, s) e^{-ipx} \\
+ d^+(\vec{p}, s) v_\alpha(\vec{p}, s) e^{ipx} \right)
\] (9)

\[
A(x) = \frac{1}{(2\pi)^{3/2}} \int d^3 p \left( \frac{1}{2\omega_p} \right)^{1/2} \\
\left\{ a(\vec{p}) e^{-ipx} + a^+(\vec{p}) e^{ipx} \right\}
\] (10)

\[
B(x) = \frac{1}{(2\pi)^{3/2}} \int d^3 p \left( \frac{1}{2\omega_p} \right)^{1/2} \\
\left\{ b(\vec{p}) e^{-ipx} + b^+(\vec{p}) e^{ipx} \right\}
\] (11)
with the usual notations:

\[ \omega_p = (\bar{p}^2 + m^2)^{1/2} \]
\[ (p - m)u(\bar{p}, s) = 0 \]
\[ (p + m)v(\bar{p}, s) = 0 \]

and where, in terms of the modes, the relations (2 – 8) are rewritten in the following forms:

\[ \langle d(\bar{p}, s)d^+(\bar{k}, l)d(p', r) \rangle = 2\delta(\bar{p} - \bar{k})\delta_{ls}d(p', r) \]
\[ + 2\delta(\bar{p}' - \bar{k})\delta_{tr}d(\bar{p}, s) \tag{12} \]
\[ \langle d^+(\bar{p}, s)d^+(\bar{k}, l)d(p', r) \rangle = 2\delta(\bar{p}' - \bar{k})\delta_{ls}d^+(\bar{p}, s) \tag{13} \]
\[ \langle d(\bar{p}, s)d(\bar{k}, l)d(p', r) \rangle = 0 \tag{14} \]

\[ [a(\bar{p}), a^+(\bar{k})] = [b(\bar{p}), b^+(\bar{k})] = \delta(\bar{p} - \bar{k}) \tag{15} \]
\[ [a(\bar{p}), a(\bar{k})] = [b(\bar{p}), b(\bar{k})] = [a(\bar{p}), d^+(\bar{k}, l)] \]
\[ = [a(\bar{p}), b(\bar{k})] = [b(\bar{p}), d^+(\bar{k}, l)] = 0 \tag{16} \]

3. Parasupersymmetry

3.1 Parasupersymmetric Transformations

We now investigate continuous transformations of the fields \(A, B\) and the parafield \(\Psi^\alpha\), which will be the parasupersymmetric transformations of the theory defined by (1). We are then led to verify that the action

\[ S = \int \left\{ \frac{1}{4} \left[ \bar{\Psi}, (i\gamma^\mu \partial_\mu - m)\Psi \right] + \frac{1}{2} (\partial_\mu A)^2 + \frac{1}{2} (\partial_\mu B)^2 - \frac{1}{2} m^2 A^2 - \frac{1}{2} m^2 B^2 \right\} d^4x \tag{17} \]

is left invariant. To do this, it suffices that, under these transformations, the free lagrangian density changes by a total derivative

\[ \delta L = \partial_\mu J^\mu \tag{18} \]

where \(J^\mu\) is a conserved Noether parasupersymmetric current density. As in the ordinary Wess-Zumino model, let us consider the following variations which transform
parafermions and bosons into each other

$$\delta \Psi = (-i\gamma^\mu \partial_\mu + m)(A - i\gamma^5 B)\varepsilon$$

$$\delta \overline{\Psi} = \varepsilon(A - i\gamma^5 B)(i\gamma^\mu \partial_\mu - m)$$

$$\delta A = \frac{1}{2} [\varepsilon, \Psi]$$

$$\delta B = \frac{-i}{2} [\varepsilon, \gamma^5 \Psi]$$

which are rewritten with an appropriate symmetrization of the product $\varepsilon \Psi$ and where $\varepsilon$ is a constant Majorana spinor. It is important to notice here that, by analogy with the ordinary supersymmetric case for which $\varepsilon^\alpha$ is anticommuting just like $\Psi^\alpha$, here, we have to take the components $\varepsilon^\alpha$ as paraGrassmann which obey the algebra (specific to the order $p = 2$)

$$\varepsilon^\alpha \varepsilon^\beta \varepsilon^\gamma + \varepsilon^\gamma \varepsilon^\beta \varepsilon^\alpha = 0$$

Note that this implies $(\varepsilon^\alpha)^3 = 0$. The components $\varepsilon^\alpha$ are then assumed to have non-trivial commutations relations with the parafields $\Psi^\alpha$

$$[[\varepsilon^\alpha, \Psi_i (\overrightarrow{x}, t)], \overline{\Psi}_j (\overrightarrow{y}, t)] = 2\varepsilon^\alpha \delta_{ij} \delta(\overrightarrow{x} - \overrightarrow{y})$$

$$[[\varepsilon^\alpha, \Psi_i (\overrightarrow{x}, t)], \Psi_j (\overrightarrow{y}, t)] = 0$$

$$[[\varepsilon^\alpha, \Psi_i (\overrightarrow{x}, t)], \varepsilon^\beta] = 0$$

and by analogy to the ordinary case, they are assumed to commute with $A$ and $B$. One can rewrite the precedent relations as follows:

$$[[\varepsilon^\alpha, d(\overrightarrow{p}, s)], d^+(\overrightarrow{k}, l)] = 2\varepsilon^\alpha \delta_{s,l} \delta(\overrightarrow{p} - \overrightarrow{k})$$

$$[[\varepsilon^\alpha, d(\overrightarrow{p}, s)], d(\overrightarrow{k}, l)] = 0$$

$$[[\varepsilon^\alpha, d(\overrightarrow{p}, s)], \varepsilon^\beta] = 0$$

$$[a(\overrightarrow{p}), \varepsilon^\alpha] = [b(\overrightarrow{p}), \varepsilon^\alpha] = 0$$

Let us now proceed to the evaluation of the Noether curent $J^\mu$. Computing the variation of the lagrangian by the use of the transformations $(19 - 22)$ and the equations of motion of the fields, one can write

$$\delta \mathcal{L} = \partial_\mu V^\mu$$

where

$$V^\mu = \frac{1}{4} [\varepsilon, \partial^\mu (A - i\gamma^5 B) \Psi]$$

The conserved parasupercurent $J^\mu$ is given by the relation

$$J^\mu = V^\mu - \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta \phi_i$$
where

\[
\frac{\partial L}{\partial \partial_\mu \phi_i} \delta \phi_i = \frac{\partial L}{\partial \partial_\mu A} \delta A + \frac{\partial L}{\partial \partial_\mu B} \delta B + \frac{\partial L}{\partial \partial_\mu \Psi} \delta \Psi = \frac{1}{4} [\bar{\psi}, \partial_\mu (A - i \gamma^5 B)] \Psi
\]

\[
- \frac{i}{4} \left[ \bar{\Psi} (i \gamma^\mu \partial_\mu + m)(A - i \gamma^5 B), \varepsilon \right]
\]

(34)

so that

\[
J^\mu = \frac{i}{4} \left[ \bar{\Psi} (i \gamma^\mu \partial_\mu + m)(A - i \gamma^5 B), \varepsilon \right]
\]

(35)

which can be noted as

\[
J^\mu = \frac{1}{\lambda} [\bar{k}, \varepsilon]
\]

where \( \lambda \) is a real constant which has to be determined.

### 3.2 Parasupersymmetric Algebra

Now, if this \( p = 2 \) extension of the Wess-Zumino model is a field theoretical realization of the parasupersymmetric algebra, the spinor parasupercrace

\[
Q_a = \int d^3 \vec{x} k^a
\]

(36)

must satisfy the commutation relations and the trilinear relations of a parasupersymmetric Poincaré algebra (in the sense of Debergh and Becker)

\[
[P_\mu, P_\nu] = 0
\]

(37)

\[
[M_{\mu\nu}, P_\rho] = 0
\]

(38)

\[
[M_{\mu\nu}, M_{\rho\sigma}] = - i (\eta_{\mu\nu} M_{\rho\sigma} - \eta_{\mu\sigma} M_{\rho\nu})
\]

\[
- i (\eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho})
\]

(39)

\[
[M_{\mu\nu}, Q_a] = - (\sigma^4_{\mu\nu})_{ab} Q_b
\]

(40)

\[
[P_\mu, Q_a] = 0
\]

(41)

\[
\langle Q_a, \overline{Q}_b, Q_c \rangle = 4 P_\mu \gamma^\mu_{ab} Q_c + 4 Q_a P_\mu \gamma^\mu_{cb}
\]

(42)

\[
\langle Q_a, \overline{Q}_b, \overline{Q}_c \rangle = 4 P_\mu \gamma^\mu_{ab} \overline{Q}_c
\]

\[
+ 4 Q_a (c^{-1} \gamma^\mu)_{bc} P_\mu
\]

(43)

\[
\langle \overline{Q}_a, \overline{Q}_b, Q_c \rangle = 4 Q_a (c^{-1} \gamma^\mu)_{bc} P_\mu
\]

\[
+ 4 (c^{-1} \gamma^\mu)_{ab} P_\mu \overline{Q}_c
\]

(44)

where

\[
\sigma^4_{\mu\nu} = \frac{i}{4} [\gamma_\mu, \gamma_\nu]
\]
and $c$ is the charge conjugation matrix. In particular, let us show that the $Q_a$ satisfy the $p = 2$ trilinear relation (42). To do this, we have to give a representation of the para-supercharges (36) as a linear operator acting on a Fock space defined by a fundamental state $|0\rangle$ satisfying

$$a |0\rangle = b |0\rangle = 0$$

and

$$d(\overrightarrow{p}, s) d^+(\overrightarrow{k}, l) |0\rangle = 2 \delta_is \delta(\overrightarrow{p} - k) |0\rangle$$
$$d(\overrightarrow{p}, s) |0\rangle = 0$$

which fixes the order of the paraquantization $p = 2$. With the same calculation steps as in [14] one can obtain these expressions for $Q_a$ and $\overline{Q}_a$

$$Q_a = \frac{i\lambda}{2^{1/2}} m^{1/2} \sum_s \int d^3p (\left\{ C(\overrightarrow{p}) d^+(\overrightarrow{p}, s) v(\overrightarrow{p}, s) \right\}_a$$
$$- \{ D(\overrightarrow{p}) d(\overrightarrow{p}, s) u(\overrightarrow{p}, s) \}_a)$$

$$\overline{Q}_a = -\frac{i\lambda}{2^{1/2}} m^{1/2} \sum_s \int d^3p (\left\{ \overline{v}(\overrightarrow{p}, s) d(\overrightarrow{p}, s) D(\overrightarrow{p}) \right\}_a$$
$$- \{ \overline{u}(\overrightarrow{p}, s) d^+(\overrightarrow{p}, s) C(\overrightarrow{p}) \}_a)$$

where

$$C(\overrightarrow{p}) = (a - i \gamma^5 b)$$
$$D(\overrightarrow{p}) = (a^+ - i \gamma^5 b^+)$$

By the use of the same steps of calculus in [14] in the context of the paraquantization by
the use of the relations, one can show that (12-16):
\[
\langle Q_a, Q_b, Q_c \rangle \\
= \int d^3p \frac{\lambda^2}{2} \left\{ \{a^+(\overline{p})a(\overline{p}) + b^+(\overline{p})b(\overline{p}) \} + \left\{ 1 + \frac{1}{2} \sum_s [d^+(\overline{p}, s), d(\overline{p}, s)] \right\} p_\mu \gamma_\mu^{ab} \right. \\
\left. \left( i\lambda \left( \frac{m}{2} \right)^{1/2} \sum_r \int d^3p \gamma C_{\gamma}(p') d^+(p', r) u_\gamma(p', r) \right) - \left( i\lambda \left( \frac{m}{2} \right)^{1/2} \sum_r \int d^3p \gamma D_{\gamma}(p') d(p', r) u_\gamma(p', r) \right) \right. \\
\left. + i\lambda \left( \frac{m}{2} \right)^{1/2} \sum_s \int d^3p (C_{\alpha\alpha}(p') d^+(p', s) u_\alpha(p', s) \right. \\
\left. - D_{\alpha\alpha}(p') d(p', s) u_\alpha(p', s)) \right) \\
\left\{ \int \frac{\lambda^2}{2} d^3p [a^+(\overline{p})a(\overline{p}) + b^+(\overline{p})b(\overline{p}) + 1] \right\} \\
+ \left\{ \frac{1}{2} \sum_r [d^+(p', r), d(p', r)] p_\mu \gamma_\mu^{cb} \right\} + R_1 \right) \quad (50)
\]

where, again by the use of the symmetry properties, the contribution of the terms proportional to \((\gamma^\mu \gamma^5)_{ab}\) in \(R_1\) vanishes so that:

\[
R_1 = \sum_{ls} \frac{i}{16} \frac{\lambda^3}{2} \int d^3p \int d^3k p\mu \gamma\mu^{ac} C_{\beta\beta} \gamma_{\beta}(k', l) \gamma_{\beta}(k', l) \\
\left[ d^+(\overline{p}, s) C^+(\overline{k}, l) d(\overline{p}, s) - d(\overline{p}, s) C^+(\overline{k}, l) \right] \\
\left[ d^+(\overline{p}, s) - \sum_{ls} \frac{i}{16} \frac{\lambda^3}{2} \int d^3p \int d^3k p\mu \gamma\mu^{ac} \right. \\
D_{\beta\beta} \gamma_{\beta}(k', l) d(\overline{p}, s) \gamma_{\beta}(k', l) d^+(\overline{p}, s) \\
\left. - d^+(\overline{p}, s) d(\overline{p}, s) \right]
\]

In the other hand, one can prove that the energy momentum takes the form

\[
P_\mu = \int d^3p \left\{ a^+(\overline{p})a(\overline{p}) + b^+(\overline{p})b(\overline{p}) + 1 \right\} \\
+ \left\{ \frac{1}{2} \sum_s [d^+(\overline{p}, s), d(\overline{p}, s)] \right\} p_\mu 
\]

so that, the relation (50) can be rewritten in the form

\[
\langle Q_a, Q_b, Q_c \rangle = \frac{1}{2} \lambda^2 P_\mu \gamma_\mu^{ab} Q_c + \frac{1}{2} \lambda^2 Q_a P_\mu \gamma_\mu^{cb} + R_1
\]
Thus if we choose $\lambda = (2)^{3/2}$, we obtain
\begin{equation}
\langle Q_a, \bar{Q}_b, Q_c \rangle = 4P_\mu \gamma^{\mu}_{ab}Q_c + 4Q_a P_\mu \gamma^{\mu}_{cb} + R_1 \tag{51}
\end{equation}

a straightforward calculation permits to verify that
\begin{equation}
[Q_a, P_\mu] = 0 \tag{52}
\end{equation}

Clearly, from the relations (51), (52), $R_1$ must also be conserved, in fact, one can verify that
\begin{equation}
[R_1, P_\mu] = 0
\end{equation}

Now, and like in the work of [15], since in general $R_1$ is non zero, the parasupersymmetric algebra (51) is complicated. A choice of a vacuum state analogous to (42) would reduce it to
\begin{equation}
\langle Q_a, \bar{Q}_b, Q_c \rangle = 4P_\mu \gamma^{\mu}_{ab}Q_c + 4Q_a P_\mu \gamma^{\mu}_{cb}
\end{equation}

since in the corresponding Fock space, $R_1$ is represented by the null operator.

Now, in the same way, one obtain
\begin{align*}
\langle Q_a, \bar{Q}_b, Q_c \rangle &= 4P_\mu \gamma^{\mu}_{ab}Q_c + 4Q_a (c^{-1}\gamma^{\mu})_{bc} P_\mu + R_2 \\
\langle \bar{Q}_a, Q_b, Q_c \rangle &= 4\bar{Q}_a (c^{-1}\gamma^{\mu})_{bc} P_\mu + 4 (c^{-1}\gamma^{\mu})_{ab} P_\mu \bar{Q}_c + R_3
\end{align*}

Where
\begin{align*}
R_2 &= \sum_{ls} \frac{i}{16} \lambda^3 \int d^3p \int d^3k p_\mu (\gamma^{\mu})_{ac} C_{\beta\bar{\beta}} (\bar{k}, l) \\
&\quad \left[ d(\bar{p}, s) d^+(\bar{k}, l) d^+(\bar{p}, s) - d^+(\bar{p}, s) d^+(\bar{k}, l) \right] \\
&\quad - \sum_{ls} \frac{i}{16} \lambda^3 \int d^3p \int d^3k p_\mu (\gamma^{\mu})_{ac} D_{\beta\bar{\beta}}(\bar{k}, l) \left[ d^+(\bar{p}, s) d(\bar{k}, l) d(\bar{p}, s) - d(\bar{p}, s) d(\bar{k}, l) d^+(\bar{p}, s) \right]
\end{align*}

and
\begin{align*}
R_3 &= -\sum_{ls} \frac{i}{16} \lambda^3 \int d^3p \int d^3k p_\mu (\gamma^{\mu} c^{-1})_{ac} C_{\beta\bar{\beta}} (\bar{k}, l) \\
&\quad \left[ d(\bar{p}, s) d^+(\bar{k}, l) d^+(\bar{p}, s) - d^+(\bar{p}, s) d^+(\bar{k}, l) \right] \\
&\quad + \sum_{ls} \frac{i}{16} \lambda^3 \int d^3p \int d^3k p_\mu (\gamma^{\mu} c^{-1})_{ac} D_{\beta\bar{\beta}}(\bar{k}, l) \left[ d^+(\bar{p}, s) d(\bar{k}, l) d(\bar{p}, s) - d(\bar{p}, s) d(\bar{k}, l) d^+(\bar{p}, s) \right]
\end{align*}

Which, as in the case of $R_1$, are again represented by the null operators in the Fock space.
4. Closure

Notice that, to the trilinear nature of the parasupercalgebras (42 − 44) correspond the fact that the infinitesimal transformations (19 − 22) must close onto an algebra that involves trilinear relations. Indeed, let us calculate the term

$$
\delta_3 \delta_2 \partial_1 A = -\frac{i}{4} \gamma_{\mu}^{\alpha} \left[ \varepsilon_{\alpha}, \varepsilon_{\beta} \right] \left[ \varepsilon_{\gamma}, \partial_{\mu} \Psi_{\alpha} \right]
$$

using the properties of the paragGrassmann $\varepsilon_{\alpha}$ one finds:

$$
\delta_3 \delta_2 \partial_1 A = \frac{1}{2} \left( -\frac{i}{2} \left[ \varepsilon_{\alpha}, \varepsilon_{\beta} \right] \varepsilon_{\gamma} \right) \varepsilon_{\mu} \Psi_{\alpha}
$$

which can be rewritten as

$$
\delta_3 \delta_2 \delta_1 A = \partial_{\mu} \delta_\mu A
$$

where the subscript $\bar{\theta}$ of the transformation $\delta$ is given by

$$
\bar{\theta} = -\frac{i}{2} \left[ \varepsilon_{\alpha}, \varepsilon_{\beta} \right] \varepsilon_{\gamma} \varepsilon_{\mu} + \frac{i}{2} \left[ \varepsilon_{\alpha}, \varepsilon_{\beta} \right] \varepsilon_{\gamma} \varepsilon_{\mu} - \frac{i}{2} \left[ \varepsilon_{\alpha}, \varepsilon_{\beta} \right] \varepsilon_{\gamma} \varepsilon_{\mu}
$$

Finally, the closure of the transformations algebra takes the form

$$
(\delta_3 \delta_2 \delta_1 + \delta_1 \delta_3 \delta_2 + \delta_2 \delta_3 \delta_1) A = \partial_{\mu} \delta_{\bar{\theta}} A = \partial_{\mu} \delta_{\bar{\theta}} A
$$

(53)

where

$$
(\xi_{\mu})^{\alpha} = -\frac{i}{2} \left[ \varepsilon_{\alpha}, \varepsilon_{\beta} \right] \varepsilon_{\gamma} \varepsilon_{\mu} + \frac{i}{2} \left[ \varepsilon_{\alpha}, \varepsilon_{\beta} \right] \varepsilon_{\gamma} \varepsilon_{\mu} - \frac{i}{2} \left[ \varepsilon_{\alpha}, \varepsilon_{\beta} \right] \varepsilon_{\gamma} \varepsilon_{\mu}
$$

Analogous relations for $B$ and $\Psi_{\alpha}$ with the same parameter $\xi$ are derived. One can see that the algebra (42 − 44) is nothing other that the Hamiltonian rewriting of (53). Their distinctive feature is the occurrence of a trilinear product rule for the fermionic elements which is the translating of a $p = 2$ parasuperalgebra.
5. Generators of the Parasupersymmetric Transformations

Now that the required transformations are derived, one is tempted to seek whether the spinor parasupercharge $Q_a$ defined by (48) gives the correct parasupersymmetric transformations (19 – 22) of the fields $A(x), B(x)$ and $\Psi^a(x)$. Indeed, it is easy to check that $Q_a$ generates (19 – 22) through the relation:

$$\delta \phi = i[[\bar{\epsilon}, Q_a], \phi]$$

where $\phi$ stands for the fields $A(x), B(x)$ or the parafield $\Psi^a(x)$.

**a-Calculation of $-i[[\bar{\epsilon}a, Q_a], A]$**

By inserting into the trilinear commutator the Fourier expansions obtained earlier for the relevant quantities, one can write:

$$-i[[\bar{\epsilon}a, Q_a], A] = -i\frac{1}{2} \sum_s \int d^3p \left\{ C_{ab}(\overrightarrow{p})d^+(\overrightarrow{p}, s)u_b(\overrightarrow{p}, s) \right\}$$

$$- \left\{ D_{ab}(\overrightarrow{p})d(\overrightarrow{p}, s)u_b(\overrightarrow{p}, s) \right\}$$

$$\frac{1}{(2\pi)^{3/2}} \int \frac{1}{(2\omega_k)^{1/2}} d^3k \left\{ a(\overrightarrow{k})e^{ikx} + a^+(\overrightarrow{k})e^{-ikx} \right\}$$

which is simplified as follows

$$-i[[\bar{\epsilon}a, Q_a], A] = \frac{1}{2} \sum_s \int d^3p \int \left( \frac{1}{2\omega_k} \right)^{1/2} d^3k$$

$$([\bar{\epsilon}_a, d^+(\overrightarrow{p}, s)u_b(\overrightarrow{p}, s)][C_{ab}(\overrightarrow{p}), a^+(\overrightarrow{k})]e^{-ikx}$$

$$- [\bar{\epsilon}_a, d(\overrightarrow{p}, s)u_b(\overrightarrow{p}, s)][D_{ab}(\overrightarrow{p}), a(\overrightarrow{k})]e^{ikx})$$

by the use of

$$[C_{ab}(\overrightarrow{p}), a^+(\overrightarrow{k})] = \delta(\overrightarrow{p} - \overrightarrow{k})\delta_{ab}$$

$$[D_{ab}(\overrightarrow{p}), a(\overrightarrow{k})] = -\delta(\overrightarrow{p} - \overrightarrow{k})\delta_{ab}$$

and, working out the integral over $k$ one finds

$$-i[[\bar{\epsilon}_a, Q_a], A] = \frac{1}{2} [\bar{\epsilon}_a, \Psi_a] = \delta A$$

in the same way, the use of the relations

$$[C_{ab}(\overrightarrow{p}), b^+(\overrightarrow{k})] = -i\gamma^5_{ab}\delta(\overrightarrow{p} - \overrightarrow{k})$$

$$[D_{ab}(\overrightarrow{p}), b(\overrightarrow{k})] = i\gamma^5_{ab}\delta(\overrightarrow{p} - \overrightarrow{k})$$

leads to:

$$-i[[\bar{\epsilon}_a, Q_a], B] = \delta B$$
b-Calculation of $-i[[\varepsilon_a, Q_a], \Psi_b]$

One can write:

$$-i[[\varepsilon_a, Q_a], \Psi_b] = \frac{1}{2} \frac{1}{(2\pi)^{3/2}} \sum_{s,l} \int d^3p \int d^3k \left( \frac{m}{2\omega_k} \right)^{1/2} \{[[\varepsilon_a, d^+ p], d(\vec{k}, l)] C_{ac}(\vec{p}) v_c(\vec{p}, s)$$

$$u_b(\vec{k}, l)e^{-ikx} + [[\varepsilon_a, d^+ (\vec{p}, s)], d^+ (\vec{k}, l)] C_{ac}(\vec{p})$$

$$v_c(\vec{p}, s)v_b(\vec{k}, l)e^{ikx} - [[\varepsilon_a, d(\vec{p}, s)], d(\vec{k}, l)]$$

$$D_{ac}(\vec{p}) u_c(\vec{p}, s) u_b(\vec{k}, l)e^{-ikx}$$

$$- \left[[\varepsilon_a, d(\vec{p}, s)], d^+ (\vec{k}, l) \right] D_{ac}(\vec{p})$$

$$u_c(\vec{p}, s)v_b(\vec{k}, l)e^{ikx} \}$$

Using (27) and (28), one finds

$$-i[[\varepsilon_a, Q_a], \Psi_b] = \frac{1}{2} \frac{1}{(2\pi)^{3/2}} \sum_{s,l} \int d^3p \int d^3k \left( \frac{m}{2\omega_k} \right)^{1/2}$$

$$\{2\delta_{ls}\delta(\vec{p} - \vec{k}) \varepsilon_a C_{ac}(\vec{p}) v_c(\vec{p}, s)$$

$$u_b(\vec{k}, l)e^{-ikx} - 2\delta_{ls}\delta(\vec{p} - \vec{k}) \varepsilon_a$$

$$D_{ac}(\vec{p}) u_c(\vec{p}, s) v_b(\vec{k}, l)e^{ikx} \}$$

the relations

$$\sum_s u_c(\vec{p}, s)v_b(\vec{p}, s) = (\frac{p + m}{2m}c)_{cb}$$

$$\sum_s v_c(\vec{p}, s)u_b(\vec{p}, s) = (\frac{p - m}{2m}c)_{cb}$$

lead to

$$-i[[\varepsilon_a, Q_a], \Psi_b] = \frac{1}{(2\pi)^{3/2}} \sum_s \int d^3p (\varepsilon_a C_{ac}(\vec{p})) (\frac{p + m}{2m}c)_{cb}$$

$$e^{-ipx} - \varepsilon_a D_{ac}(\vec{p}) (\frac{p - m}{2m}c)_{cb} e^{ipx} \}$$

which can be rewritten as

$$-i[[\varepsilon_a, Q_a], \Psi_b] = -i\gamma^\mu \varepsilon_a \partial_\mu A + (\gamma^\mu \gamma^5)_{ba} \varepsilon_a \partial_\mu B$$

$$= \delta \Psi_b$$

Conclusion

In this work, we have investigated the most simple paraextension of the Wess-Zumino model outside the parasuperspace formalism. This model forms a field theoretical realization of the parasuperPoincaré algebra, where, as a bosons-parafermions system, the
parasupercharges of this model satisfy the trilinear commutations relations dictated by these types of systems.

The distinctive feature of the closure is the occurrence of a trilinear product rule for the fermionic elements which is the translating of the $p = 2$ parasuperalgebra. Unlike the ordinary bilinear case, this result imply that the three times repeated applications of the transformations on a field gives rise to a translation of the transformed field. This closure is reinforced by the verification that these parasupercharges are the effective generators of these transformations.

References

Geometrodynamics of Information on Curved Statistical Manifolds and its Applications to Chaos

C. Cafaro* and S. A. Ali†

Department of Physics, State University of New York at Albany-SUNY, 1400 Washington Avenue, Albany, NY 12222, USA

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Abstract: A novel information-geometrodynamical approach to chaotic dynamics (IGAC) on curved statistical manifolds based on Entropic Dynamics (ED) is presented and a new definition of information geometrodynamical entropy (IGE) as a measure of chaoticity is proposed. The general classical formalism is illustrated in a relatively simple example. It is shown that the hyperbolicity of a non-maximally symmetric 6N-dimensional statistical manifold $M_s$ underlying an ED Gaussian model describing an arbitrary system of $3N$ degrees of freedom leads to linear information-geometric entropy growth and to exponential divergence of the Jacobi vector field intensity, quantum and classical features of chaos respectively. An information-geometric analogue of the Zurek-Paz quantum chaos criterion in the classical reversible limit is proposed. This analogy is illustrated applying the IGAC to a set of $n$-uncoupled three-dimensional anisotropic inverted harmonic oscillators characterized by an Ohmic distributed frequency spectrum.

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1. Introduction

The lack of a unified characterization of chaos in classical and quantum dynamics is well-known. In the Riemannian [1] and Finslerian [2] (a Finsler metric is obtained from a Riemannian metric by relaxing the requirement that the metric be quadratic on each tangent space) geometrodynamical approach to chaos in classical Hamiltonian systems, an active field of research concerns the possibility of finding a rigorous relation among

* carlocafaro2000@yahoo.it
† alis@alum.rpi.edu
the sectional curvature, the Lyapunov exponents, and the Kolmogorov-Sinai dynamical entropy (i.e. the sum of positive Lyapunov exponents) [3]. The largest Lyapunov exponent characterizes the degree of chaoticity of a dynamical system and, if positive, it measures the mean instability rate of nearby trajectories averaged along a sufficiently long reference trajectory. Moreover, it is known that classical chaotic systems are distinguished by their exponential sensitivity to initial conditions and that the absence of this property in quantum systems has led to a number of different criteria being proposed for quantum chaos. Exponential decay of fidelity, hypersensitivity to perturbation, and the Zurek-Paz quantum chaos criterion of linear von Neumann’s entropy growth [4] are some examples [5]. These criteria accurately predict chaos in the classical limit, but it is not clear that they behave the same far from the classical realm.

The present work makes use of the so-called Entropic Dynamics (ED) [6]. ED is a theoretical framework that arises from the combination of inductive inference (Maximum relative Entropy Methods, [7]) and Information Geometry (Riemannian geometry applied to probability theory) (IG) [8]. As such, ED is constructed on statistical manifolds. It is developed to investigate the possibility that laws of physics - either classical or quantum - might reflect laws of inference rather than laws of nature.

This article is a follow up of a series of the authors works [9, 10, 11]. In this paper, the ED theoretical framework is used to explore the possibility of constructing a unified characterization of classical and quantum chaos. We investigate a system with $3N$ degrees of freedom (microstates), each one described by two pieces of relevant information, its mean expected value and its variance (Gaussian statistical macrostates). This leads to consider an ED model on a non-maximally symmetric $6N$-dimensional statistical manifold $\mathcal{M}_s$. It is shown that $\mathcal{M}_s$ possesses a constant negative Ricci curvature that is proportional to the number of degrees of freedom of the system, $R_{\mathcal{M}_s} = -3N$. It is shown that the system explores statistical volume elements on $\mathcal{M}_s$ at an exponential rate. We define a dynamical information-geometric entropy $S_{\mathcal{M}_s}$ of the system and we show it increases linearly in time (statistical evolution parameter) and is moreover, proportional to the number of degrees of freedom of the system. The geodesics on $\mathcal{M}_s$ are hyperbolic trajectories. Using the Jacobi-Levi-Civita (JLC) equation for geodesic spread, it is shown that the Jacobi vector field intensity $J_{\mathcal{M}_s}$ diverges exponentially and is proportional to the number of degrees of freedom of the system. Thus, $R_{\mathcal{M}_s}$, $S_{\mathcal{M}_s}$ and $J_{\mathcal{M}_s}$ are proportional to the number of Gaussian-distributed microstates of the system. This proportionality leads to conclude there is a substantial link among these information-geometric indicators of chaoticity.

Finally, an information-geometric analog of the Zurek-Paz quantum chaos criterion is suggested. We illustrate this point by use of an $n$-set of inverted harmonic oscillators (IHO). In the ED formalism, the IHO system is described by a curved $n$-dimensional statistical manifold that is conformally related to an Euclidean one.
2. Specification of the Gaussian ED-model

Maximum relative Entropy (ME) methods are used to construct an ED model that follows from an assumption about what information is relevant to predict the evolution of the system. Given a known initial macrostate (probability distribution) and that the system evolves to a final known macrostate, the possible trajectories of the system are examined. A notion of distance between two probability distributions is provided by IG. As shown in [12, 13] this distance is quantified by the Fisher-Rao information metric tensor.

We consider an ED model whose microstates span a $3N$-dimensional space labelled by the variables $\{\vec{X}\} = \{\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(N)}\}$ with $\vec{x}^{(\alpha)} \equiv (x_1^{(\alpha)}, x_2^{(\alpha)}, x_3^{(\alpha)})$, $\alpha = 1, \ldots, N$ and $x_a^{(\alpha)} \in \mathbb{R}$ with $a = 1, 2, 3$. We assume the only testable information pertaining to the quantities $x_a^{(\alpha)}$ consists of the expectation values $\langle x_a^{(\alpha)} \rangle$ and variance $\Delta x_a^{(\alpha)} = \sqrt{\langle (x_a^{(\alpha)} - \langle x_a^{(\alpha)} \rangle)^2 \rangle}$. The set of these expectation values define the $6N$-dimensional space of macrostates of the system. A measure of distinguishability among the states of the ED model is obtained by assigning a probability distribution $P(\vec{X} | \vec{\Theta})$ to each macrostate $\vec{\Theta}$ where $\{\vec{\Theta}\} = \{(1)\theta_a^{(\alpha)}, (2)\theta_a^{(\alpha)}\}$ with $\alpha = 1, 2, \ldots, N$ and $a = 1, 2, 3$. The process of assigning a probability distribution to each state endows $\mathcal{M}$ with a metric structure. Specifically, the Fisher-Rao information metric defined in (12) is a measure of distinguishability among macrostates. It assigns an IG to the space of states.

2.1 The Gaussian Statistical Manifold $\mathcal{M}_S$

We consider an arbitrary system evolving over a $3N$-dimensional space. The variables $\{\vec{X}\} = \{\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(N)}\}$ label the $3N$-dimensional space of microstates of the system. All information relevant to the dynamical evolution of the system is assumed to be contained in the probability distributions. For this reason, no other information is required. Each macrostate may be viewed as a point of a $6N$-dimensional statistical manifold with coordinates given by the numerical values of the expectations $\langle x_a^{(\alpha)} \rangle$ and variance $\Delta x_a^{(\alpha)} = \sqrt{\langle (x_a^{(\alpha)} - \langle x_a^{(\alpha)} \rangle)^2 \rangle}$. The available information is contained in the following $6N$ information constraint equations,

$$\langle x_a^{(\alpha)} \rangle = \int_{-\infty}^{+\infty} dx_a^{(\alpha)} x_a^{(\alpha)} P_a^{(\alpha)}(x_a^{(\alpha)} | (1)\theta_a^{(\alpha)}, (2)\theta_a^{(\alpha)})$$

and

$$\Delta x_a^{(\alpha)} = \left[\int_{-\infty}^{+\infty} dx_a^{(\alpha)} \left(x_a^{(\alpha)} - \langle x_a^{(\alpha)} \rangle\right)^2 P_a^{(\alpha)}(x_a^{(\alpha)} | (1)\theta_a^{(\alpha)}, (2)\theta_a^{(\alpha)})\right]^{1/2}.$$
where \((1)\theta_a^{(\alpha)} = \langle x_a^{(\alpha)} \rangle\) and \((2)\theta_a^{(\alpha)} = \Delta x_a^{(\alpha)}\) with \(\alpha = 1, 2, \ldots, N\) and \(a = 1, 2, 3\). The probability distributions \(P_a^{(\alpha)}\) are constrained by the conditions of normalization,
\[
\int_{-\infty}^{+\infty} dx_a^{(\alpha)} P_a^{(\alpha)} (x_a^{(\alpha)} \mid (1)\theta_a^{(\alpha)}, (2)\theta_a^{(\alpha)}) = 1.
\] (2)

The Gaussian distribution is identified by information theory as the maximum entropy distribution if only the expectation value and the variance are known. ME methods allows to associate a probability distribution \(P_{\vec{X}} \mid \vec{\Theta}\) to each point in the space of states \(\vec{\Theta}\). The distribution that best reflects the information contained in the prior distribution \(m_{\vec{X}}\) updated by the information \((\langle x_a^{(\alpha)} \rangle, \Delta x_a^{(\alpha)})\) is obtained by maximizing the relative entropy
\[
S(\vec{\Theta}) = -\int_{\{\vec{x}\}} d^{3N} \vec{X} P(\vec{X} \mid \vec{\Theta}) \log \left( \frac{P(\vec{X} \mid \vec{\Theta})}{m(\vec{X})} \right). \tag{3}
\]

As a working hypothesis, the prior \(m_{\vec{X}}\) is set to be uniform since we assume the lack of prior available information about the system (postulate of equal a priori probabilities). Upon maximizing (3), given the constraints (1) and (2), we obtain
\[
P(\vec{X} \mid \vec{\Theta}) = \prod_{\alpha=1}^{N} \prod_{a=1}^{3} P_a^{(\alpha)} (x_a^{(\alpha)} \mid \mu_a^{(\alpha)}, \sigma_a^{(\alpha)}) \tag{4}
\]
where
\[
P_a^{(\alpha)} (x_a^{(\alpha)} \mid \mu_a^{(\alpha)}, \sigma_a^{(\alpha)}) = \left( \frac{2\pi \left[ \sigma_a^{(\alpha)} \right]^2 \right)^{-\frac{1}{2}} \exp \left[ -\frac{(x_a^{(\alpha)} - \mu_a^{(\alpha)})^2}{2 \left( \sigma_a^{(\alpha)} \right)^2} \right] \tag{5}
\]
and \((1)\theta_a^{(\alpha)} = \mu_a^{(\alpha)}, (2)\theta_a^{(\alpha)} = \sigma_a^{(\alpha)}\). For the rest of the paper, unless stated otherwise, the statistical manifold \(\mathcal{M}_S\) will be defined by the following expression,
\[
\mathcal{M}_S = \left\{ P(\vec{X} \mid \vec{\Theta}) \text{ in (4)} : \vec{X} \in \mathbb{R}^{3N}, \vec{\Theta} \in D_\Theta = \left[ (-\infty, +\infty)_\mu \times (0, +\infty)_\sigma \right]^{3N} \right\}. \tag{6}
\]

The probability distribution (4) encodes the available information concerning the system. Note we assumed uncoupled constraints among microvariables \(x_a^{(\alpha)}\). In other words, we assumed that information about correlations between the microvariables need not to be tracked. This assumption leads to the simplified product rule (4). However, coupled constraints would lead to a generalized product rule in (4) and to a metric tensor (12) with non-trivial off-diagonal elements (covariance terms). For instance, the total probability distribution \(P(x, y \mid \mu_x, \sigma_x, \mu_y, \sigma_y)\) of two dependent Gaussian distributed microvariables
\(x\) and \(y\) reads

\[
P(x, y|\mu_x, \sigma_x, \mu_y, \sigma_y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}} \times 
\exp\left\{-\frac{1}{2(1-r^2)} \left[ \frac{(x - \mu_x)^2}{\sigma_x^2} - 2r \frac{(x - \mu_x)(y - \mu_y)}{\sigma_x\sigma_y} + \frac{(y - \mu_y)^2}{\sigma_y^2} \right] \right\},
\]

where \(r \in (-1, +1)\) is the correlation coefficient given by

\[
r = \frac{\langle (x - \langle x \rangle)(y - \langle y \rangle) \rangle}{\sqrt{\langle x - \langle x \rangle \rangle} \sqrt{\langle y - \langle y \rangle \rangle}} = \frac{\langle xy \rangle - \langle x \rangle \langle y \rangle}{\sigma_x\sigma_y}.
\]

The metric induced by (7) is obtained by use of (12), the result being

\[
g_{ij} = \begin{bmatrix}
\frac{-1}{\sigma_x^2(r^2-1)} & 0 & \frac{r}{\sigma_x\sigma_y(r^2-1)} & 0 \\
0 & \frac{-2r^2-2}{\sigma_x^2(r^2-1)} & 0 & \frac{r^2}{\sigma_x\sigma_y(r^2-1)} \\
\frac{r}{\sigma_x\sigma_y(r^2-1)} & 0 & \frac{-1}{\sigma_y^2(r^2-1)} & 0 \\
0 & \frac{r^2}{\sigma_x\sigma_y(r^2-1)} & 0 & \frac{-2r^2}{\sigma_y^2(r^2-1)}
\end{bmatrix},
\]

where \(i, j = 1, 2, 3, 4\). The Ricci curvature scalar associated with manifold characterized by (9) is given by

\[
R = g^{ij}R_{ij} = -\frac{8(r^2-2) + 2r^2(3r^2-2)}{8(r^2-1)}.
\]

It is clear that in the limit \(r \to 0\), the off-diagonal elements of \(g_{ij}\) vanish and the Scalar \(R\) reduces to the result obtained in [10], namely \(R = -2 < 0\). Correlation terms may be fictitious. They may arise for instance from coordinate transformations. On the other hand, correlations may arise from external fields in which the system is immersed. In such situations, correlations among \(x^{(a)}_\alpha\) effectively describe interaction between the microvariables and the external fields. Such generalizations would require more delicate analysis. Before proceeding, a comment is in order. Most probability distributions arise from the maximum entropy formalism as a result of simple statements concerning averages (Gaussians, exponential, binomial, etc.). Not all distribution are generated in this manner however. Some distributions are generated by combining the results of simple cases (multinomial from a binomial) while others are found as a result of a change of variables (Cauchy distribution). For instance, the Weibull and Wigner-Dyson distributions can be obtained from an exponential distribution as a result of a power law transformation [15].

2.1.1 Metric Structure of \(\mathcal{M}_S\)

We cannot determine the evolution of microstates of the system since the available information is insufficient. Not only is the information available insufficient but we also do not know the equation of motion. In fact there is no standard "equation of motion". Instead we can ask: how close are the two total distributions with parameters \((\mu^{(a)}_\alpha, \sigma^{(a)}_\alpha)\)
and \((\mu^a + d\mu^a, \sigma^a + d\sigma^a)\)? Once the states of the system have been defined, the
next step concerns the problem of quantifying the notion of change from the state \(\vec{\Theta}\) to the state \(\vec{\Theta} + d\vec{\Theta}\). A convenient measure of change is distance. The measure we seek is
given by the dimensionless distance \(ds\) between \(P\left(\vec{X} \mid \vec{\Theta}\right)\) and \(P\left(\vec{X} \mid \vec{\Theta} + d\vec{\Theta}\right)\),
\[
ds^2 = g_{\mu\nu} d\Theta^\mu d\Theta^\nu \quad \text{with} \quad \mu, \nu = 1, 2, \ldots, 6N, \tag{11}
\]
where
\[
g_{\mu\nu} = \int d\vec{X} P\left(\vec{X} \mid \vec{\Theta}\right) \frac{\partial \log P\left(\vec{X} \mid \vec{\Theta}\right)}{\partial \Theta^\mu} \frac{\partial \log P\left(\vec{X} \mid \vec{\Theta}\right)}{\partial \Theta^\nu} \tag{12}
\]
is the Fisher-Rao information metric. Substituting (4) into (12), the metric \(g_{\mu\nu}\) on \(\mathcal{M}_s\) becomes a \(6N \times 6N\) matrix \(M\) made up of \(3N\) blocks \(M_{2\times2}\) with dimension \(2 \times 2\) given by,
\[
M_{2\times2} = \begin{pmatrix}
    \left(\sigma^a\right)^{-2} & 0 \\
    0 & 2 \times \left(\sigma^a\right)^{-2}
\end{pmatrix} \tag{13}
\]
with \(\alpha = 1, 2, \ldots, N\) and \(a = 1, 2, 3\). From (12), the “length” element (11) reads,
\[
ds^2 = \sum_{\alpha=1}^{N} \sum_{a=1}^{3} \left[ \frac{1}{\left(\sigma^a\right)^2} d(\mu^a)^2 + \frac{2}{\left(\sigma^a\right)^2} d(\sigma^a)^2 \right]. \tag{14}
\]
We bring attention to the fact that the metric structure of \(\mathcal{M}_s\) is an emergent (not fundamental) structure. It arises only after assigning a probability distribution \(P\left(\vec{X} \mid \vec{\Theta}\right)\) to each state \(\vec{\Theta}\).

2.1.2 Curvature of \(\mathcal{M}_s\)

Given the Fisher-Rao information metric, we use standard differential geometry methods applied to the space of probability distributions to characterize the geometric properties of \(\mathcal{M}_s\). Recall that the Ricci scalar curvature \(R\) is given by,
\[
R = g^{\mu\nu} R_{\mu\nu}, \tag{15}
\]
where \(g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho\) so that \(g^{\mu\nu} = (g_{\mu\nu})^{-1}\). The Ricci tensor \(R_{\mu\nu}\) is given by,
\[
R_{\mu\nu} = \partial_\gamma \Gamma^\gamma_{\mu\nu} - \partial_\nu \Gamma^\gamma_{\mu\lambda} + \Gamma^\gamma_{\mu\nu} \Gamma^\xi_{\xi\xi} - \Gamma^\gamma_{\mu\xi} \Gamma^\xi_{\nu\xi}. \tag{16}
\]
The Christoffel symbols \(\Gamma^\rho_{\mu\nu}\) appearing in the Ricci tensor are defined in the standard manner as,
\[
\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left( \partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right). \tag{17}
\]
Using (13) and the definitions given above, we can show that the Ricci scalar curvature becomes
\[
R_{\mathcal{M}_s} = R^\alpha_{\alpha} = \sum_{\rho \neq \sigma} K(\epsilon_\rho, e_\sigma) = -3N < 0. \tag{18}
\]
The scalar curvature is the sum of all sectional curvatures of planes spanned by pairs of orthonormal basis elements \( \{ e_\rho = \partial_{\Theta_\rho(p)} \} \) of the tangent space \( T_pM_s \) with \( p \in M_s \),

\[
K(a, b) = \frac{R_{\mu\nu\rho\sigma} a^\mu b^\nu a^\rho b^\sigma}{(g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma}) a^\mu b^\nu a^\rho b^\sigma},
\]

where \( \langle e_\rho, h_\sigma \rangle = \delta_\rho^\sigma \). Notice that the sectional curvatures completely determine the curvature tensor. From (18) we conclude that \( M_s \) is a \( 6N \)-dimensional statistical manifold of constant negative Ricci scalar curvature. A detailed analysis on the calculation of Christoffel connection coefficients using the ED formalism for a four-dimensional manifold of Gaussians can be found in [10].

2.1.3 Anisotropy and Compactness

It can be shown that \( M_s \) is not a pseudosphere (maximally symmetric manifold). The first way this can be understood is from the fact that the Weyl Projective curvature tensor [16] (or the anisotropy tensor) \( W_{\mu\nu\rho\sigma} \) defined by

\[
W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{R_{M_s}}{n(n-1)} (g_{\nu\sigma} g_{\mu\rho} - g_{\mu\rho} g_{\nu\sigma}),
\]

with \( n = 6N \) in the present case, is non-vanishing. In (20), the quantity \( R_{\mu\nu\rho\sigma} \) is the Riemann curvature tensor defined in the usual manner by

\[
R^\alpha_{\beta\rho\sigma} = \partial_\rho \Gamma^\alpha_{\beta\sigma} - \partial_\sigma \Gamma^\alpha_{\beta\rho} + \Gamma^\alpha_{\lambda\rho} \Gamma^\lambda_{\beta\sigma} - \Gamma^\alpha_{\lambda\sigma} \Gamma^\lambda_{\beta\rho}. \tag{21}
\]

Considerations regarding the negativity of the Ricci curvature as a strong criterion of dynamical instability and the necessity of compactness of \( M_s \) in ”true” chaotic dynamical systems is under investigation [14].

The issue of symmetry of \( M_s \) can alternatively be understood from consideration of the sectional curvature. In view of (19), the negativity of the Ricci scalar implies the existence of expanding directions in the configuration space manifold \( M_s \). Indeed, from (18) one may conclude that negative principal curvatures (extrema of sectional curvatures) dominate over positive ones. Thus, the negativity of the Ricci scalar is only a sufficient (not necessary) condition for local instability of geodesic flow. For this reason, the negativity of the scalar provides a strong criterion of local instability. Scenarios may arise where negative sectional curvatures are present, but the positive ones could prevail in the sum so that the Ricci scalar is non-negative despite the instability in the flow in those directions. Consequently, the signs of the sectional curvatures are of primary significance for the proper characterization of chaos.

Yet another useful way to understand the anisotropy of the \( M_s \) is the following. It is known that in \( n \) dimensions, there are at most \( \frac{n(n+1)}{2} \) independent Killing vectors (directions of symmetry of the manifold). Since \( M_s \) is not a pseudosphere, the information metric tensor does not admit the maximum number of Killing vectors \( K_\nu \) defined as

\[
\mathcal{L}_K g_{\mu\nu} = D_\mu K_\nu + D_\nu K_\mu = 0, \tag{22}
\]
where $D_\mu$, given by

$$D_\mu K_\nu = \partial_\mu K_\nu - \Gamma^\rho_{\mu\nu} K_\rho,$$

is the covariant derivative operator with respect to the connection $\Gamma$ defined in (17). The Lie derivative $\mathcal{L}_K g_{\mu\nu}$ of the tensor field $g_{\mu\nu}$ along a given direction $K$ measures the intrinsic variation of the field along that direction (that is, the metric tensor is Lie transported along the Killing vector) [17]. Locally, a maximally symmetric space of Euclidean signature is either a plane, a sphere, or a hyperboloid, depending on the sign of $R$. In our case, none of these scenarios occur. As will be seen in what follows, this fact has a significant impact on the integration of the geodesic deviation equation on $\mathcal{M}_s$. At this juncture, we emphasize it is known that the anisotropy of the manifold underlying system dynamics plays a crucial role in the mechanism of instability. In particular, fluctuating sectional curvatures require also that the manifold be anisotropic. However, the connection between curvature variations along geodesics and anisotropy is far from clear and is currently under investigation.

Krylov was the first to emphasize [18] the use of $R < 0$ as an instability criterion in the context of an $N$-body system (a gas) interacting via Van der Waals forces, with the ultimate hope to understand the relaxation process in a gas. However, Krylov neglected the problem of compactness of the configuration space manifold which is important for making inferences about exponential mixing of geodesic flows [19]. Why is compactness so significant in the characterization of chaos? True chaos should be identified by the occurrence of two crucial features: 1) strong dependence on initial conditions and exponential divergence of the Jacobi vector field intensity, i.e., stretching of dynamical trajectories; 2) compactness of the configuration space manifold, i.e., folding of dynamical trajectories. Compactness [2, 20] is required in order to discard trivial exponential growths due to the unboundedness of the “volume” available to the dynamical system. In other words, the folding is necessary to have a dynamics actually able to mix the trajectories, making practically impossible, after a finite interval of time, to discriminate between trajectories which were very nearby each other at the initial time. When the space is not compact, even in presence of strong dependence on initial conditions, it could be possible in some instances (though not always), to distinguish among different trajectories originating within a small distance and then evolved subject to exponential instability.

The statistical manifold defined in (6) is compact. This can be seen as follows. It is known form IG that there is a one-to-one relation between elements of the statistical manifold and the parameter space. More precisely, the statistical manifold $\mathcal{M}_s$ is homeomorphic to the parameter space $\mathcal{D}_\Theta$. This implies the existence of a continuous, bijective map $h_{\mathcal{M}_s, \mathcal{D}_\Theta}$:

$$h_{\mathcal{M}_s, \mathcal{D}_\Theta} : \mathcal{M}_S \ni P \left( \vec{X} \big| \vec{\Theta} \right) \rightarrow \vec{\Theta} \in \mathcal{D}_\Theta$$

where $h_{\mathcal{M}_s, \mathcal{D}_\Theta}^{-1} \left( \vec{\Theta} \right) = P \left( \vec{X} \big| \vec{\Theta} \right)$. The inverse image $h_{\mathcal{M}_s, \mathcal{D}_\Theta}^{-1}$ is the so-called homeomorphism map. In addition, since homeomorphisms preserve compactness, it is sufficient to restrict ourselves to a compact subspace of the parameter space $\mathcal{D}_\Theta$ in order to ensure
that $\mathcal{M}_S$ is itself compact.

3. Canonical Formalism for the Gaussian ED-model

The geometrization of a Hamiltonian system by transforming it to a geodesic flow is a well-known technique of classical mechanics associated with the name of Jacobi [21]. Transformation to geodesic motion is obtained in two steps: 1) conformal transformation of the metric; 2) rescaling of the time parameter [22]. The reformulation of dynamics in terms of a geodesic problem allows the application of a wide range of well-known geometrical techniques in the investigation of the solution space and properties of equations of motions. The power of the Jacobi reformulation is that all of the dynamical information is collected into a single geometric object - the manifold on which geodesic flow is induced - in which all the available manifest symmetries are retained. For instance, integrability of the system is connected with the existence of Killing vectors and tensors on this manifold [23, 24].

In this section we study the trajectories of the system on $\mathcal{M}_S$. We emphasize ED can be derived from a standard principle of least action (of Maupertuis-Euler-Lagrange-Jacobi type) [6, 25]. The main differences are that the dynamics being considered here, namely ED, is defined on a space of probability distributions $\mathcal{M}_S$, not on an ordinary linear space $V$ and the standard coordinates $q_{\mu}$ of the system are replaced by statistical macrovariables $\Theta^\mu$. The geodesic equations for the macrovariables of the Gaussian ED model are given by,

$$\frac{d^2\Theta^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{d\Theta^\nu}{d\tau} \frac{d\Theta^\rho}{d\tau} = 0 \quad (25)$$

with $\mu = 1, 2, ..., 6N$. Observe the geodesic equations are nonlinear second order coupled ordinary differential equations. They describe a reversible dynamics whose solution is the trajectory between an initial and a final macrostate. The trajectory can be equally well traversed in both directions.

3.1 Geodesics on $\mathcal{M}_S$

We determine the explicit form of (25) for the pairs of statistical coordinates $(\mu_a^{(\alpha)}, \sigma_a^{(\alpha)})$. Substituting the expression of the Christoffel connection coefficients into (25), the geodesic equations for the macrovariables $\mu_a^{(\alpha)}$ and $\sigma_a^{(\alpha)}$ associated to the microstate $x_a^{(\alpha)}$ become,

$$\frac{d^2\mu_a^{(\alpha)}}{d\tau^2} - \frac{2}{\sigma_a^{(\alpha)}} \frac{d\mu_a^{(\alpha)}}{d\tau} \frac{d\sigma_a^{(\alpha)}}{d\tau} = 0, \quad \frac{d^2\sigma_a^{(\alpha)}}{d\tau^2} - \frac{1}{\sigma_a^{(\alpha)}} \left( \frac{d\sigma_a^{(\alpha)}}{d\tau} \right)^2 + \frac{1}{2\sigma_a^{(\alpha)}} \left( \frac{d\mu_a^{(\alpha)}}{d\tau} \right)^2 = 0, \quad (26)$$
with $\alpha = 1, 2, \ldots, N$ and $a = 1, 2, 3$. This is a set of coupled ordinary differential equations, whose solutions are

$$
\mu^{(\alpha)}(\tau) = \left(\frac{B^{(\alpha)}}{2\beta^{(\alpha)}}\right)^2 \frac{1}{\cosh(2\beta^{(\alpha)}\tau) - \sinh(2\beta^{(\alpha)}\tau)} + C^{(\alpha)}_a,
$$

$$
\sigma^{(\alpha)}(\tau) = B^{(\alpha)} \frac{\cosh(\beta^{(\alpha)}\tau) - \sinh(\beta^{(\alpha)}\tau)}{\cosh(2\beta^{(\alpha)}\tau) - \sinh(2\beta^{(\alpha)}\tau)} + \frac{B^{(\alpha)}}{s^{(\alpha)}}.
$$

(27)

The quantities $B^{(\alpha)}_a, C^{(\alpha)}_a, \beta^{(\alpha)}_a$ are real integration constants that can be evaluated upon specification of boundary conditions. We are interested in the stability of the trajectories on $M_s$. It is known [25] that the Riemannian curvature of a manifold is intimately related to the behavior of geodesics on it. If the Riemannian curvature of a manifold is negative, geodesics (initially parallel) rapidly diverge from one another. For the sake of simplicity, we assume very special initial conditions: $B^{(\alpha)}_a \equiv \Lambda, \beta^{(\alpha)}_a \equiv \lambda \in \mathbb{R}^+, C^{(\alpha)}_a = 0, \forall \alpha = 1, 2, \ldots, N$ and $a = 1, 2, 3$. However, the conclusions drawn can be generalized to more arbitrary initial conditions. We observe that since every maximal geodesic is well-defined for all temporal parameters $\tau$, $M_s$ constitute a geodesically complete manifold [26]. It is therefore a natural setting within which one may consider global questions and search for a weak criterion of chaos [2].

4. Exponential Divergence of the Jacobi Vector Field Intensity

The actual interest of the Riemannian formulation of the dynamics stems from the possibility of studying the instability of natural motions through the instability of geodesics of a suitable manifold, a circumstance that has several advantages. First of all a powerful mathematical tool exists to investigate the stability or instability of a geodesic flow: the Jacobi-Levi-Civita equation for geodesic spread [27]. The JLC-equation describes covariantly how nearby geodesics locally scatter. It is a familiar object both in Riemannian geometry and theoretical physics (it is of fundamental interest in experimental General Relativity). Moreover the JLC-equation relates the stability or instability of a geodesic flow with curvature properties of the ambient manifold, thus opening a wide and largely unexplored field of investigation of the connections among geometry, topology and geodesic instability, hence chaos.

Consider the behavior of the one-parameter family of neighboring geodesics $F_{G_M}^{(\lambda)}(\lambda) \equiv \{ \Theta^{(\lambda)}_{\lambda}(\tau; \lambda) \}_{\lambda \in \mathbb{R}^+}$ where

$$
\mu^{(\alpha)}_a(\tau; \lambda) = \frac{\Lambda^2}{2\lambda \cosh(2\lambda\tau) - \sinh(2\lambda\tau)} \frac{1}{\cosh(2\lambda\tau) - \sinh(2\lambda\tau)} + \frac{\Lambda^2}{8\lambda},
$$

$$
\sigma^{(\alpha)}_a(\tau; \lambda) = \frac{\cosh(\lambda\tau) - \sinh(\lambda\tau)}{\cosh(2\lambda\tau) - \sinh(2\lambda\tau)} + \frac{\Lambda^2}{8\lambda}.
$$

(28)
with $\alpha = 1, 2, \ldots, N$ and $a = 1, 2, 3$. The relative geodesic spread on a (non-maximally symmetric) curved manifold as $M_s$ is characterized by the Jacobi-Levi-Civita equation, the natural tool to tackle dynamical chaos [17, 27],

$$\frac{D^2 \delta \Theta^\mu}{D\tau^2} + R^\mu_{\nu \rho \sigma} \frac{\partial \Theta^\nu}{\partial \tau} \frac{\partial \Theta^\rho}{\partial \tau} = 0$$

(29)

where the Jacobi vector field $J^\mu$ is defined as,

$$J^\mu \equiv \delta \Theta^\mu \overset{\text{def}}{=} \delta \lambda \Theta^\mu = \left( \frac{\partial \Theta^\mu(\tau; \lambda)}{\partial \lambda} \right) \bigg|_{\tau=\text{const}} \delta \lambda.$$

(30)

Notice that the JLC-equation appears intractable already at rather small $N$. For isotropic manifolds, the JLC-equation can be reduced to the simple form,

$$\frac{D^2 J^\mu}{D\tau^2} + K J^\mu = 0, \; \mu = 1, \ldots, 6 N$$

(31)

where $K$ is the constant value assumed throughout the manifold by the sectional curvature. The sectional curvature of manifold $M_s$ is the $6N$-dimensional generalization of the Gaussian curvature of two-dimensional surfaces of $\mathbb{R}^3$. If $K < 0$, unstable solutions of equation (31) assumes the form

$$J(\tau) = \frac{1}{\sqrt{-K}} \omega(0) \sinh \left( \sqrt{-K} \tau \right)$$

(32)

once the initial conditions are assigned as $J(0) = 0$, $\frac{dJ(0)}{d\tau} = \omega(0)$ and $K < 0$. Equation (29) forms a system of $6N$ coupled ordinary differential equations linear in the components of the deviation vector field (30) but nonlinear in derivatives of the metric (12). It describes the linearized geodesic flow: the linearization ignores the relative velocity of the geodesics. When the geodesics are neighboring but their relative velocity is arbitrary, the corresponding geodesic deviation equation is the so-called generalized Jacobi equation [28, 29]. The nonlinearity is due to the existence of velocity-dependent terms in the system. Neighboring geodesics accelerate relative to each other with a rate directly measured by the curvature tensor $R_{\alpha\beta\gamma\delta}$. Substituting (28) in (29) and neglecting the exponentially decaying terms in $\delta \Theta^\mu$ and its derivatives, integration of (29) leads to the following asymptotic expression of the Jacobi vector field intensity,

$$J_{M_s} = \|J\| = (g_{\mu\nu} J^\mu J^\nu)^{\frac{1}{2}} \approx 3 N e^{\lambda_J \tau}.$$

(33)

We conclude that the geodesic spread on $M_s$ is described by means of an exponentially divergent Jacobi vector field intensity $J_{M_s}$, a classical feature of chaos. In our approach the quantity $\lambda_J$,

$$\lambda_J \overset{\text{def}}{=} \lim_{\tau \to \infty} \frac{1}{\tau} \ln \left[ \frac{\|J_{M_s}(\tau)\|}{\|J_{M_s}(0)\|} \right]$$

(34)

would play the role of the conventional Lyapunov exponents.
5. Linearity of the Information Geometrodynamic Entropy

We investigate the stability of the trajectories of the ED model considered on $\mathcal{M}_s$. It is known [25] that the Riemannian curvature of a manifold is closely connected with the behavior of the geodesics on it. If the Riemannian curvature of a manifold is negative, geodesics (initially parallel) rapidly diverge from one another. For the sake of simplicity, we assume very special initial conditions: $B_\alpha^{(a)} \equiv \Lambda$, $\beta_\alpha^{(a)} \equiv \lambda \in \mathbb{R}^+$, $C_\alpha^{(a)} = 0$, $\forall \alpha = 1, 2, ..., N$ and $a = 1, 2, 3$. However, the conclusion we reach can be generalized to more arbitrary initial conditions. Recall $\mathcal{M}_s$ is the space of probability distributions $\{P(\bar{X}|\bar{\Theta})\}$ labeled by $6N$ statistical parameters $\bar{\Theta}$. These parameters are the coordinates for the point $P$, and in these coordinates a volume element $dV_{\mathcal{M}_s}$ reads,

$$dV_{\mathcal{M}_s} = \sqrt{g}d^{6N}\bar{\Theta} = \prod_{\alpha=1}^{N} \prod_{a=1}^{3} \frac{\sqrt{2}}{(\sigma_\alpha^{(a)})^2} \mu_\alpha^{(a)}(0) \sigma_\alpha^{(a)}(0) d\mu_\alpha^{(a)} d\sigma_\alpha^{(a)}. \quad (35)$$

The volume of an extended region $\Delta V_{\mathcal{M}_s}(\tau; \lambda)$ of $\mathcal{M}_s$ is defined by,

$$\Delta V_{\mathcal{M}_s}(\tau; \lambda) \equiv \prod_{\alpha=1}^{N} \prod_{a=1}^{3} \int_{\mu_\alpha^{(a)}(0)}^{\mu_\alpha^{(a)}(\tau)} \int_{\sigma_\alpha^{(a)}(0)}^{\sigma_\alpha^{(a)}(\tau)} \frac{\sqrt{2}}{(\sigma_\alpha^{(a)})^2} d\mu_\alpha^{(a)} d\sigma_\alpha^{(a)}. \quad (36)$$

where $\mu_\alpha^{(a)}(\tau)$ and $\sigma_\alpha^{(a)}(\tau)$ are given in (28). The quantity that encodes relevant information about the stability of neighboring volume elements is the average volume $\bar{V}_{\mathcal{M}_s}(\tau; \lambda)$,

$$\bar{V}_{\mathcal{M}_s}(\tau; \lambda) \equiv \langle \Delta V_{\mathcal{M}_s}(\tau; \lambda) \rangle_{\tau} \equiv \frac{1}{\tau} \int_{0}^{\tau} \Delta V_{\mathcal{M}_s}(\tau'; \lambda) d\tau' \leftarrow \tau \approx e^{3N\lambda\tau}. \quad (37)$$

This asymptotic regime of diffusive evolution in (37) describes the exponential increase of average volume elements on $\mathcal{M}_s$. The exponential instability characteristic of chaos forces the system to rapidly explore large areas (volumes) of the statistical manifold. It is interesting to note that this asymptotic behavior appears also in the conventional description of quantum chaos where the entropy increases linearly at a rate determined by the Lyapunov exponents [30]. The linear increase of entropy as a quantum chaos criterion was introduced by Zurek and Paz [4]. In our information-geometric approach a relevant quantity that can be useful to study the degree of instability characterizing the ED model is the information-geometric entropy defined as,

$$S_{\mathcal{M}_s} \equiv \lim_{\tau \to \infty} \log \bar{V}_{\mathcal{M}_s}(\tau; \lambda). \quad (38)$$

Substituting (37) in (38), we obtain

$$S_{\mathcal{M}_s} = \lim_{\tau \to \infty} \left\{ \frac{1}{\tau} \int_{0}^{\tau} \prod_{\alpha=1}^{N} \prod_{a=1}^{3} \int_{\mu_\alpha^{(a)}(0)}^{\mu_\alpha^{(a)}(\tau')} \int_{\sigma_\alpha^{(a)}(0)}^{\sigma_\alpha^{(a)}(\tau')} \frac{\sqrt{2}}{(\sigma_\alpha^{(a)})^2} d\mu_\alpha^{(a)} d\sigma_\alpha^{(a)} d\tau' \right\} \leftarrow \tau \approx 3N\lambda\tau. \quad (39)$$
The entropy $S_{M_s}$ in (39) is the asymptotic limit of the natural logarithm of the statistical weight $\langle \Delta V_{M_s} \rangle_\tau$ defined on $M_s$. Its linear growth in time is reminiscent of the aforementioned quantum chaos criterion. Indeed, equation (39) may be considered the information-geometric analog of the Zurek-Paz chaos criterion.

In conclusion, we have shown,

$$R_{M_s} = -3N, \quad S_{M_s} \xrightarrow{\tau \to \infty} 3N\lambda \tau, \quad J_{M_s} \xrightarrow{\tau \to \infty} 3Ne^{\lambda \tau}. \quad (40)$$

The Ricci scalar curvature $R_{M_s}$, the information-geometric entropy $S_{M_s}$ and the Jacobi vector field intensity $J_{M_s}$ are proportional to the number of Gaussian-distributed microstates of the system. This proportionality leads to the conclusion that there exists a substantial link among these information-geometric measures of chaoticity, namely

$$R_{M_s} \sim S_{M_s} \sim J_{M_s}. \quad (41)$$

Equation (41), together with the information-geometric analog of the Zurek-Paz quantum chaos criterion, equation (39), represent the fundamental results of this work. We believe our theoretical modelling scheme may be used to describe actual systems where transitions from quantum to classical chaos scenario occur, but this requires additional analysis. In the following section, we briefly consider some similarities among the von Neumann, Kolmogorov-Sinai and Information-Geometrodynamical entropies.

6. On the Von Neumann, Kolmogorov-Sinai and Information Geometrodynamical Entropies

In conventional approaches to chaos, the notion of entropy is introduced, in both classical and quantum physics, as the missing information about the systems fine-grained state [5, 31]. For a classical system, suppose that the phase space is partitioned into very fine-grained cells of uniform volume $\Delta v$, labelled by an index $j$. If one does not know which cell the system occupies, one assigns probabilities $p_j$ to the various cells; equivalently, in the limit of infinitesimal cells, one can use a phase-space density $\rho(X_j) = \frac{p_j}{\Delta v}$. Then, in a classical chaotic evolution, the asymptotic expression of the information needed to characterize a particular coarse-grained trajectory out to time $\tau$ is given by the Shannon information entropy (measured in bits),

$$S_{\text{classical}}^{(\text{chaotic})} = -\int dX \rho(X) \log_2 (\rho(X) \Delta v) = -\sum_j p_j \log_2 p_j \sim K \tau. \quad (42)$$

where $\rho(X)$ is the phase-space density and $p_j = \frac{v_j}{\Delta v}$ is the probability for the corresponding coarse-grained trajectory. $S_{\text{classical}}^{(\text{chaotic})}$ is the missing information about which fine-grained cell the system occupies. The quantity $K$ represents the linear rate of information increase and it is called the Kolmogorov-Sinai entropy (or metric entropy) ($K$ is the sum of positive Lyapunov exponents, $K = \sum_j \lambda_j$). $K$ quantifies the degree of classical chaos. It is worthwhile emphasizing that the quantity that grows asymptotically as $K \tau$ is really
the average of the information on the left side of equation (42). This distinction can be ignored however, if we assume that the chaotic system has roughly constant Lyapunov exponents over the accessible region of phase space. In quantum mechanics the fine-grained alternatives are normalized state vectors in Hilbert space. From a set of probabilities for various state vectors, one can construct a density operator

\[ \hat{\rho} = \sum_j \lambda_j |\psi_j\rangle \langle \psi_j| \, , \, \hat{\rho} |\psi_j\rangle = \lambda_j |\psi_j\rangle \]  

(43)

The normalization of the density operator, \( \text{tr} (\hat{\rho}) = 1 \), implies that the eigenvalues make up a normalized probability distribution. The von Neumann entropy of the density operator \( \hat{\rho} \) (measured in bits) [32],

\[ S_{\text{quantum}}^{(\text{chaotic})} = - \text{tr} (\hat{\rho} \log_2 \hat{\rho}) = - \sum_j \lambda_j \log_2 \lambda_j \]  

(44)

can be thought of as the missing information about which eigenvector the system is in. Entropy quantifies the degree of unpredictability about the system's fine-grained state.

Recall that decoherence is the loss of phase coherence between the set of preferred quantum states in the Hilbert space of the system due to the interaction with the environment. Moreover, decoherence induces transitions from quantum to classical systems. Therefore, classicality is an emergent property of an open quantum system. Motivated by such considerations, Zurek and Paz investigated implications of the process of decoherence for quantum chaos.

They considered a chaotic system, a single unstable harmonic oscillator characterized by a potential \( V(x) = -\frac{\lambda x^2}{2} \) (\( \lambda \) is the Lyapunov exponent), coupled to an external environment. In the reversible classical limit [33], the von Neumann entropy of such a system increases linearly at a rate determined by the Lyapunov exponent,

\[ S_{\text{quantum}}^{(\text{Zurek-Paz})} \sim \frac{\lambda}{\tau} \]  

(45)

Notice that the consideration of \( 3N \) uncoupled identical unstable harmonic oscillators characterized by potentials \( V_i(x) = -\frac{\lambda_i x^2}{2} \) (\( \lambda_i = \lambda_j \); \( i, j = 1, 2, ..., 3N \)) would simply lead to

\[ S_{\text{quantum}}^{(\text{Zurek-Paz})} \sim 3N \lambda \tau \]  

(46)

The resemblance of equations (39) and (46) is remarkable. In what follows, we apply our information geometrical method to an \( n \)-set (for \( n = 2 \)) of uncoupled inverted harmonic oscillators, each with different frequency, and show we obtain asymptotic linear IGE growth. The case for arbitrary \( n \)-set in three dimensions is presented in the Appendix.

7. The Information Geometry of a 2-set of Inverted Harmonic Oscillators (IHO)

In this section, our objective is to characterize chaotic properties of a 2-set of one-dimensional inverted harmonic oscillators, each with different frequency \( \omega_1 \neq \omega_2 \) using
the formalism presented in this paper. We will study the asymptotic behavior of the geometrodynamical entropy and the functional dependence of the Ricci scalar curvature of the 2-dimensional manifold $\mathcal{M}_{IHO}^{(2)}$ underlying the ED model of the IHOs on the frequencies $\omega_i, \ i = 1, 2$. Recent investigation explore the possibility of using well established principles of inference to derive Newtonian dynamics from relevant prior information codified into an appropriate statistical manifold [34]. In that work the basic assumption is that there is an irreducible uncertainty in the location of particles so that the state of a particle is defined by a probability distribution. The corresponding configuration space is a statistical manifold the geometry of which is defined by the information metric. The trajectory follows from a principle of inference, the method of Maximum Entropy. There is no need for additional "physical" postulates such as an action principle or equation of motion, nor for the concept of mass, momentum and of phase space, not even the notion of time. The resulting "entropic" dynamics reproduces Newton’s mechanics for any number of particles interacting among themselves and with external fields. Both the mass of the particles and their interactions are explained as a consequence of the underlying statistical manifold.

In what follows, we introduce the basics of the general formalism for an $n$-set of IHOs. This approach is similar (mathematically but not conceptually) to the geometrization of Newtonian dynamics used in the Riemannian geometrodynamical to chaos [1, 35].

7.1 Informational Geometrization of Newtonian Dynamics

Newtonian dynamics can be recast in the language of Riemannian geometry applied to probability theory, namely, Information Geometry. In our case, the system under investigation has $n$ degrees of freedom and a point on the $n$ dimensional configuration space manifold $\mathcal{M}_{IHO}^{(n)}$ is parametrized by the $n$ Lagrangian coordinates $(\theta_1, \ldots, \theta_n)$. Moreover, the system under investigation is described by the Lagrangian $L$,

$$L = T \left( \dot{\theta}_1, \ldots, \dot{\theta}_n \right) - \Phi (\theta_1, \ldots, \theta_n) = \frac{1}{2} \delta_{ij} \dot{\theta}_i \dot{\theta}_j + \frac{1}{2} \sum_{j=1}^{n} \omega_j^2 \theta_j^2$$

(47)

so that the Hamiltonian function $\mathcal{H} = T + \Phi \equiv E$ is a constant of motion. For the sake of simplicity, let us set $E = 1$. According to the principle of stationary action - in the form of Maupertuis - among all the possible isoenergetic paths $\gamma(t)$ with fixed end points, the paths that make vanish the first variation of the action functional

$$\mathcal{I} = \int_{\gamma(t)} \frac{\partial L}{\partial \dot{\theta_i}} \dot{\theta}_i d\tau$$

(48)

are natural motions. As the kinetic energy $T$ is a homogeneous function of degree two, we have $2T = \dot{\theta}_i \frac{\partial \mathcal{L}}{\partial \theta_i}$, and Maupertuis’ principle reads

$$\delta \mathcal{I} = \delta \int_{\gamma(t)} 2T d\tau = 0.$$  

(49)
The manifold $M_{IHO}^{(n)}$ is naturally given a proper Riemannian structure. In fact, let us consider the matrix
\[ g_{ij}(\theta_1, \ldots, \theta_n) = [1 - \Phi(\theta_1, \ldots, \theta_n)] \delta_{ij} \] (50)
so that Maupertuis’ principle becomes
\[ \delta \int_{\gamma(t)} T dt = \delta \int_{\gamma(t)} \left( T^2 \right)^{\frac{1}{2}} d\tau = \delta \int_{\gamma(t)} \left\{ [1 - \Phi(\theta_1, \ldots, \theta_n)] \delta_{ij} \dot{\theta}_i \dot{\theta}_j \right\}^{\frac{1}{2}} \]
\[ = \delta \int_{\gamma(t)} \left( g_{ij} \dot{\theta}_i \dot{\theta}_j \right)^{\frac{1}{2}} d\tau = \delta \int_{\gamma(s)} ds = 0, \quad ds^2 = g_{ij} d\theta^i d\theta^j \] (51)
thus natural motions are geodesics of $M_{IHO}^{(n)}$, provided we define $ds$ as its arclength. The metric tensor $g_{ij}(\cdot, \cdot)$ of $M_{IHO}^{(n)}$ is then defined by
\[ g = g_{ij} d\theta^i \otimes d\theta^j \] (52)
where $(d\theta^1, \ldots, d\theta^n)$ is a natural base of $T_{\theta}M_{IHO}^{(n)}$ - the cotangent space at the point $\theta$ - in the local chart $(\theta^1, \ldots, \theta^n)$. This is known as the Jacobi metric (or kinetic energy metric). Denoting by $\nabla$ the canonical Levi-Civita connection, the geodesic equation
\[ \nabla_{\dot{\gamma}} \dot{\gamma} = 0 \] (53)
becomes, in the local chart $(\theta^1, \ldots, \theta^n),$
\[ \frac{d^2 \theta^i}{ds^2} + \Gamma^i_{jk} \frac{d\theta^j}{ds} \frac{d\theta^k}{ds} = 0 \] (54)
where the Christoffel coefficients are the components of $\nabla$ defined by
\[ \Gamma^i_{jk} = \left\langle d\theta^i, \nabla_j e_k \right\rangle = \frac{1}{2} g^{im} \left( \partial_j g_{km} + \partial_k g_{mj} - \partial_m g_{jk} \right), \] (55)
with $\partial_i = \frac{\partial}{\partial \theta^i}$. Since $g_{ij}(\theta_1, \ldots, \theta_n) = [1 - \Phi(\theta_1, \ldots, \theta_n)] \delta_{ij}$, from the geodesic equation we obtain
\[ \frac{d^2 \theta^i}{ds^2} + \frac{1}{2 (1 - \Phi)} \left[ \frac{\partial (1 - \Phi)}{\partial \theta_j} \frac{d\theta^i}{ds} \frac{d\theta^j}{ds} - g^{ij} \frac{\partial (1 - \Phi)}{\partial \theta_j} g_{km} \frac{d\theta^k}{ds} \frac{d\theta^m}{ds} \right] = 0, \] (56)
whereupon using $ds^2 = (1 - \Phi)^2 d\tau^2$, we verify that (56) reduces to
\[ \frac{d^2 \theta^i}{d\tau^2} + \frac{\partial \Phi(\theta_1, \ldots, \theta_n)}{\partial \theta_i} = 0, \quad i = 1, \ldots, n. \] (57)
Equation (57) are Newton’s equations. It is worthwhile emphasizing that the transformation to geodesic motion on a curved statistical manifold is obtained in two key steps: the conformal transformation of the metric, $\delta_{ij} \rightarrow g_{ij} = (1 - \Phi) \delta_{ij}$ and, the rescaling of the temporal evolution parameter, $d\tau^2 \rightarrow ds^2 = 2 (1 - \Phi)^2 d\tau^2$. 
7.2 The 2-set of Inverted Anisotropic One-dimensional Harmonic Oscillators

As a simple physical example, we examine the IG associated with a 2-set configuration of IHOs. In this case, the metric tensor \( g_{ij} \) appearing in (50) takes the form

\[
\begin{equation}
    g_{ij}(\theta_1, \theta_2) = [1 - \Phi(\theta_1, \theta_2)] \cdot \delta_{ij}(\theta_1, \theta_2) \quad \text{with} \quad i, j = 1, 2. \tag{58}
\end{equation}
\]

where the function \( \Phi(\theta_1, \theta_2) \) is given by,

\[
\begin{equation}
    \Phi(\theta_1, \theta_2) = \sum_{j=1}^{2} \Phi_j(\theta_j), \quad \Phi_j(\theta_j) = -\frac{1}{2} \omega_j^2 \theta_j^2. \tag{59}
\end{equation}
\]

Hence the metric tensor \( g_{ij} \) on \( M^{(2)}_{IHO} \) becomes,

\[
\begin{equation}
    g_{ij} = \begin{pmatrix}
        1 + \frac{1}{2} (\omega_1^2 \theta_1^2 + \omega_2^2 \theta_2^2) & 0 \\
        0 & 1 + \frac{1}{2} (\omega_1^2 \theta_1^2 + \omega_2^2 \theta_2^2)
    \end{pmatrix}. \tag{60}
\end{equation}
\]

Using the standard definition of the Ricci scalar (15), we obtain

\[
\begin{equation}
    R_{M^{(2)}_{IHO}}(\omega_1, \omega_2) = \frac{4 (\theta_1^2 \omega_1^4 + \theta_2^2 \omega_2^4) - 4 (\theta_1^2 + \theta_2^2) \omega_1^2 \omega_2^2 - 8 (\omega_1^2 + \omega_2^2)}{(\theta_1^2 \omega_1^2 + \theta_2^2 \omega_2^2 + 2)^3}. \tag{61}
\end{equation}
\]

In the limit of a flat frequency spectrum, \( \omega_1 = \omega_2 = \omega \), the scalar curvature (61) is constantly negative,

\[
\begin{equation}
    R_{M^{(2)}_{IHO}}(\omega) = \frac{-16 \omega^2}{[2 + (\theta_1^2 + \theta_2^2) \omega^2]^3} < 0, \quad \forall \omega \geq 0. \tag{62}
\end{equation}
\]

However, in presence of distinct frequency values, \( \omega_1 \neq \omega_2 \), it is possible to properly choose the \( \omega \)'s so that \( R_{M^{(2)}_{IHO}}(\omega_1, \omega_2) \) becomes either negative or positive. In addition, we notice that the manifold underlying the IHO model is anisotropic since its associated Weyl projective curvature tensor components are non-vanishing. For the special case, \( \omega_1 = \omega_2 \), we obtain

\[
\begin{equation}
    W_{1212}(\omega) = \frac{8 \omega^4 (\theta_1^4 + \theta_2^4) + 2 \omega^6 (\theta_1^4 + \theta_1^2 \theta_2^2) + 4 \omega^6 \theta_1^2 \theta_2^2}{(\theta_1^2 \omega^2 + \theta_2^2 \omega^2 + 2)^3}. \tag{63}
\end{equation}
\]

Clearly, the frequency parameter \( \omega \) drives the degree of anisotropy of the statistical manifold \( M^{(2)}_{IHO} \) and, as expected, in the limit of vanishing \( \omega \), we recover the flat \( (R = 0) \), isotropic \( (W = 0) \) Euclidean manifold characterized by metric \( \delta_{ij} \). This result is a concrete example of the fact that conformal transformations change the degree of anisotropy of the ambient statistical manifold underlying the Newtonian dynamics. Our only remaining task is to compute the information geometrodynamical entropy \( S_{M^{(2)}_{IHO}}(\tau; \omega_1, \omega_2) \), defined as

\[
\begin{equation}
    S_{M^{(2)}_{IHO}}(\tau; \omega_1, \omega_2) \overset{\text{def}}{=} \lim_{\tau \to \infty} \log \left[ \left\langle \Delta V_{M^{(2)}_{IHO}}(\tau; \omega_1, \omega_2) \right\rangle \right]. \tag{64}
\end{equation}
\]
The quantity \( \left\langle \Delta V_{\mathcal{M}_{IHO}^{(2)}} (\tau; \omega_1, \omega_2) \right\rangle_{\tau} \) appearing in (64) is the average volume element, defined by

\[
\left\langle \Delta V_{\mathcal{M}_{IHO}^{(2)}} (\tau; \omega_1, \omega_2) \right\rangle_{\tau} = \frac{1}{\tau} \int_{0}^{\tau} \Delta V_{\mathcal{M}_{IHO}^{(2)}} (\tau; \omega_1, \omega_2) \, d\tau',
\]

with the statistical volume element \( \Delta V_{\mathcal{M}_{IHO}^{(2)}} \) given by

\[
\Delta V_{\mathcal{M}_{IHO}^{(2)}} (\tau; \omega_1, \omega_2) = \int_{\{\theta\}} \left[ 1 + \frac{1}{2} \left( \omega_1^2 \theta_1^2 + \omega_2^2 \theta_2^2 \right) \right] \, d\theta_1 \, d\theta_2.
\]

(65)

\[
\tau \rightarrow \infty \approx \frac{1}{6} \theta_1 \theta_2 \left( \omega_1^2 \theta_1^2 + \omega_2^2 \theta_2^2 \right).
\]

Recall that the two Newtonian equations of motion for each inverted harmonic oscillator are given by,

\[
d^2 \theta_j / d\tau^2 - \omega_j^2 \theta_j = 0, \forall j = 1, 2.
\]

(67)

Hence, the asymptotic behavior of such macrovariables on manifold \( \mathcal{M}_{IHO}^{(2)} \) is given by,

\[
\theta_j (\tau) \sim \xi_j e^{\omega_j \tau}, \xi_j \in \mathbb{R}, \forall j = 1, 2.
\]

(68)

Substituting \( \theta_1 (\tau') = \xi_1 e^{\omega_1 \tau'} \) and \( \theta_2 (\tau') = \xi_2 e^{\omega_2 \tau'} \) into (66), we obtain

\[
\Delta V_{\mathcal{M}_{IHO}^{(2)}} (\tau; \omega_1, \omega_2) \sim \frac{\xi_1 \xi_2}{6} e^{(\omega_1 + \omega_2)\tau} \left( \xi_1^2 e^{2\omega_1 \tau} \omega_1^2 + \xi_2^2 e^{2\omega_2 \tau} \omega_2^2 \right).
\]

(69)

By direct computation, we find the average of (69) is given by,

\[
\left\langle \Delta V_{\mathcal{M}_{IHO}^{(2)}} (\tau; \omega_1, \omega_2) \right\rangle_{\tau} \sim \frac{1}{\tau} \int_{0}^{\tau} \left[ \frac{\xi_1 \xi_2}{6} e^{(\omega_1 + \omega_2)\tau'} \left( \xi_1^2 e^{2\omega_1 \tau'} \omega_1^2 + \xi_2^2 e^{2\omega_2 \tau'} \omega_2^2 \right) \right] d\tau'.
\]

(70)

Assuming as a working hypothesis that \( \xi_1 = \xi_2 = \xi \), we obtain

\[
\frac{1}{\tau} \int_{0}^{\tau} \left[ \frac{\xi_1 \xi_2}{6} e^{(\omega_1 + \omega_2)\tau'} \left( \xi_1^2 e^{2\omega_1 \tau'} \omega_1^2 + \xi_2^2 e^{2\omega_2 \tau'} \omega_2^2 \right) \right] d\tau' = \begin{cases} 
\frac{1}{12} \xi^6 \omega \exp(4\omega \tau) / \tau, & \text{if } \omega_1 = \omega_2, \\
\frac{1}{18} \xi^6 \omega_1 \exp(3\omega_1 \tau) / \tau, & \text{if } \omega_1 \gg \omega_2, \\
\frac{1}{18} \xi^6 \omega_2 \exp(3\omega_2 \tau) / \tau, & \text{if } \omega_2 \gg \omega_1.
\end{cases}
\]

(71)
Finally, substituting (71) in (64), we obtain
\[
S_{\mathcal{M}^{(2)}_{IHO}}(\tau; \omega_1, \omega_2) \overset{\tau \to \infty}{\propto} \begin{cases} 
2\omega \tau, & \text{if } \omega_1 = \omega_2, \\
\omega_1 \tau, & \text{if } \omega_1 \gg \omega_2, \\
\omega_2 \tau, & \text{if } \omega_2 \gg \omega_1. 
\end{cases}
\] (72)

It is clear that the information-geometrodynamical entropy \(S_{\mathcal{M}^{(2)}_{IHO}}(\tau; \omega_1, \omega_2)\) exhibits classical linear behavior in the asymptotic limit, with proportionality coefficient \(\Omega = \omega_1 + \omega_2\),
\[
S_{\mathcal{M}^{(2)}_{IHO}}(\tau; \omega_1, \omega_2) \overset{\tau \to \infty}{\propto} \Omega \tau. 
\] (73)

Equation (73) expresses the asymptotic linear growth of our information geometrodynamical entropy for the IHO system considered. This result (for \(n = 2\)) extends the result of Zurek-Paz (46). This result, together with the authors previous works [10, 11] lend substantial support for the IG approach advocated in the present article.

8. Final Remarks

A Gaussian ED statistical model has been constructed on a \(6N\)-dimensional statistical manifold \(\mathcal{M}_s\). The macro-coordinates on the manifold are represented by the expectation values of microvariables associated with Gaussian distributions. The geometric structure of \(\mathcal{M}_s\) was studied in detail. It was shown that \(\mathcal{M}_s\) is a curved manifold of constant negative Ricci curvature \(-3N\). The geodesics of the ED model are hyperbolic curves on \(\mathcal{M}_s\). A study of the stability of geodesics on \(\mathcal{M}_s\) was presented. The notion of statistical volume elements was introduced to investigate the asymptotic behavior of a one-parameter family of neighboring volumes \(F_{V_{\mathcal{M}_s}}(\lambda) \equiv \{V_{\mathcal{M}_s}(\tau; \lambda)\}_{\lambda \in \mathbb{R}^+}\). An information-geometric analog of the Zurek-Paz chaos criterion was suggested. It was shown that the behavior of geodesics is characterized by exponential instability that leads to chaotic scenarios on the curved statistical manifold. These conclusions are supported by a study based on the geodesic deviation equations and on the asymptotic behavior of the Jacobi vector field intensity \(J_{\mathcal{M}_s}\) on \(\mathcal{M}_s\). A Lyapunov exponent analog similar to that appearing in the Riemannian geometric approach to chaos was suggested as an indicator of chaoticity. On the basis of our analysis a relationship among an entropy-like quantity, chaoticity and curvature is proposed, suggesting to interpret the statistical curvature as a measure of the entropic dynamical chaoticity.

The results obtained in this work are significant, in our opinion, since a rigorous relation among curvature, Lyapunov exponents and Kolmogorov-Sinai entropy is still under investigation [3]. In addition, there does not exist a well defined unifying characterization of chaos in classical and quantum physics [5] due to fundamental differences between the
two theories. In addition, the role of curvature in statistical inference is even less understood. The meaning of statistical curvature for a one-parameter model in inference theory was introduced in [36]. Curvature served as an important tool in the asymptotic theory of statistical estimation. Therefore the implications of this work is twofold. Firstly, it helps understanding possible future use of the statistical curvature in modelling real processes by relating it to conventionally accepted quantities such as entropy and chaos. On the other hand, it serves to cast what is already known in physics regarding curvature in a new light as a consequence of its proposed link with inference.

As a simple physical example, we considered the information-geometry $\mathcal{M}^{(2)}_{IHO}$ associated with a 2-set configuration of inverted harmonic oscillators. It was determined that in the limit of a flat frequency spectrum ($\omega_1 = \omega_2 = \omega$), the scalar curvature $R_{\mathcal{M}^{(2)}_{IHO}}(\omega_1, \omega_2)$ is constantly negative. In the case of distinct frequencies, i.e., $\omega_1 \neq \omega_2$, it is possible - for appropriate choices of $\omega_1$ and $\omega_2$ - to obtain either negative or positive values of $R_{\mathcal{M}^{(2)}_{IHO}}(\omega_1, \omega_2)$. Moreover, it was shown that $\mathcal{M}^{(2)}_{IHO}$ is an anisotropic manifold since the Weyl projective curvature tensor has a non-vanishing component $W_{1212}$. It was found that the information geometrodynamical entropy of the IHO system exhibits asymptotic linear growth. This IHO example is generalized to arbitrary values of $n$ in the Appendix.

The descriptions of a classical chaotic system of arbitrary interacting degrees of freedom, deviations from Gaussianity and chaoticity arising from fluctuations of positively curved statistical manifolds are being investigated [14]. The work here presented is shown to be useful to investigate chaotic quantum spectra arising, for instance, from the Poisson and Wigner-Dyson quantum level spacing distributions [15, 37, 38]. We remark that based on the results obtained from the chosen ED models, it is not unreasonable to think that should the correct variables describing the true degrees of freedom of a physical system be identified, perhaps deeper insights into the foundations of models of physics and reasoning (and their relationship to each other) may be uncovered.

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**Appendix A: The $n$-set of inverted anisotropic three-dimensional harmonic oscillators**

We now generalize the results obtained in this article for the $n$-set of IHOs. The information metric on the $3n$-dimensional statistical manifold $\mathcal{M}^{(3n)}_{IHO}$ is given by

$$g_{ij}(\theta_1, \ldots, \theta_{3n}) = [1 - \Phi(\theta_1, \ldots, \theta_{3n})] \cdot \delta_{ij}(\theta_1, \ldots, \theta_{3n}),$$

(74)

where

$$\Phi(\theta_1, \ldots, \theta_{3n}) = \sum_{j=1}^{3n} \Phi_j(\theta_j), \quad \Phi_j(\theta_j) = -\frac{1}{2} \omega_j^2 \theta_j^2.$$
The information geometrodynamical entropy $S_{M_{IHO}^{(3n)}} (\tau; \omega_1, ..., \omega_{3n})$ is defined as

$$S_{M_{IHO}^{(3n)}} (\tau; \omega_1, ..., \omega_{3n}) \overset{\text{def}}{=} \lim_{\tau \to \infty} \left[ \log \left( \int \Delta V_{M_{IHO}^{(3n)}} (\tau; \omega_1, ..., \omega_{3n}) \right) \right],$$

where the average volume element $\Delta V_{M_{IHO}^{(3n)}}$ is given by

$$\left\langle \Delta V_{M_{IHO}^{(3n)}} (\tau; \omega_1, ..., \omega_{3n}) \right\rangle_{\tau} = \frac{1}{\tau} \int_0^\tau \Delta V_{M_{IHO}^{(3n)}} (\tau'; \omega_1, ..., \omega_{3n}) d\tau',$$

and the statistical volume element $\Delta V_{M_{IHO}^{(3n)}}$ is defined as

$$\Delta V_{M_{IHO}^{(3n)}} (\tau'; \omega_1, ..., \omega_{3n}) = \int d^3n \vec{\theta} \left( 1 + \frac{1}{2} \sum_{j=1}^{3n} \omega_j^2 \theta_j^2 \right)^{\frac{3n}{2}}.$$

Substituting (77) and (78) in (76) we obtain the general expression for $S_{M_{IHO}^{(3n)}} (\tau; \omega_1, ..., \omega_{3n})$,

$$S_{M_{IHO}^{(3n)}} (\tau; \omega_1, ..., \omega_{3n}) \overset{\text{def}}{=} \lim_{\tau \to \infty} \left\{ \frac{1}{\tau} \int_0^\tau \left[ \int d^3n \vec{\theta} \left( 1 + \frac{1}{2} \sum_{j=1}^{3n} \omega_j^2 \theta_j^2 \right)^{\frac{3n}{2}} \right] d\tau' \right\}.$$

To evaluate (79) we observe $\Delta V_{M_{IHO}^{(3n)}}$ can be written as

$$\Delta V_{M_{IHO}^{(3n)}} (\tau'; \omega_1, ..., \omega_{3n}) = \int d^3n \vec{\theta} \left( 1 + \frac{1}{2} \sum_{j=1}^{3n} \omega_j^2 \theta_j^2 \right)^{\frac{3n}{2}}.$$

$$= \int d\theta_1' \int d\theta_2' ... \int d\theta_{3n-1}' \left[ \int \left( 1 + \frac{1}{2} \sum_{j=1}^{3n} \omega_j^2 \theta_j^2 \right)^{\frac{3n}{2}} \right]$$

$$\approx \frac{1}{3n} \frac{1}{2^{\frac{3n}{2}}} \left( \prod_{i=1}^{3n} \theta_i' \right)^{\frac{3n}{2}} \sum_{j=1}^{3n} \omega_j^2 \theta_j^2.$$

(80)

Since the $n$-Newtonian equations of motions for each IHO are given by

$$\frac{d^2 \theta_j}{d\tau^2} - \omega_j^2 \theta_j = 0, \forall j = 1, ..., 3n,$$

the asymptotic behavior of such macrovariables on manifold $M_{IHO}^{(3n)}$ is given by

$$\theta_j (\tau) \overset{\tau \to \infty}{\approx} \Xi_j e^{\omega_j \tau}, \Xi_j \in \mathbb{R}, \forall j = 1, ..., 3n.$$

(82)

We therefore obtain

$$\Delta V_{M_{IHO}^{(3n)}} (\tau; \omega_1, ..., \omega_{3n}) \overset{\tau \to \infty}{\approx} \frac{1}{3n} \frac{1}{2^{\frac{3n}{2}}} \left( \prod_{i=1}^{3n} \Xi_i \right) \cdot \exp \left( \sum_{i=1}^{3n} \omega_i \tau \right) \left[ \sum_{j=1}^{3n} \Xi_j^2 e^{2\omega_j \tau} \right]^{\frac{3n}{2}}.$$

(83)
Upon averaging (83) we find
\[ \langle \Delta V_{\mathcal{IHO}}^{(3n)} (\tau; \omega_1, \ldots, \omega_{3n}) \rangle_{\tau} \approx \frac{1}{\tau} \int_0^\tau \frac{1}{\frac{3n}{2} \tau^2} \left( \prod_{i=1}^{3n} \Xi_i \right) \cdot \exp \left( \frac{\Omega \tau^2}{2} \right) \left( \sum_{j=1}^{3n} \Xi_j^2 e^{2\omega_j \tau^2} \right) \frac{3n}{2} \tau^2 \, d\tau'. \]

where \( \Omega = \sum_{i=1}^{3n} \omega_i \). As a working hypothesis, we assume \( \Xi_i = \Xi_j = \Xi \forall i, j = 1, \ldots, 3n \). Furthermore, assume that \( n \to \infty \) so that the spectrum of frequencies becomes continuum and, as an additional working hypothesis, assume this spectrum is linearly distributed,
\[ \rho (\omega) = \omega \text{ with } \int_0^{\Omega_{\text{cut-off}}} \rho (\omega) \, d\omega = 1, \Omega_{\text{cut-off}} = \xi \Omega, \xi \in \mathbb{R} \] (85)

Therefore, we obtain
\[ \langle \Delta V_{\mathcal{IHO}}^{(3n)} (\tau; \omega_1, \ldots, \omega_{3n}) \rangle_{\tau} \approx \frac{1}{\frac{3n}{2} \tau^2} \frac{\xi^2 \Omega^2}{2} \frac{3n}{2} \exp \left( \frac{3n \xi \Omega \tau}{\tau} \right). \] (86)

Finally, substituting (86) into (76), we obtain the remarkable result
\[ S_{\mathcal{IHO}}^{(3n)} (\tau; \omega_1, \ldots, \omega_{3n}) \approx \frac{\xi^2 \Omega^2}{2} \frac{3n}{2} \exp \left( \frac{3n \xi \Omega \tau}{\tau} \right). \] (87)

Equation (87) displays the asymptotic, linear information geometrodynamical entropy growth of the generalized \( n \)-set of inverted harmonic oscillators and extends the result of Zurek-Paz to an arbitrary \( n \)-set of anisotropic inverted harmonic oscillators [4].

References


Stochastic Measures and Modular Evolution in Non-equilibrium Thermodynamics

Enrique Hernandez-Lemus *, and Jesus K. Estrada-Gil

Computational Genomics Department, Instituto Nacional de Medicina Genomica
Periférico Sur No. 4124, Torre Zafiro 2, Piso 6, Álvaro Obregón 01900, México, D.F., México

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Abstract: We present an application of the theory of stochastic processes to model and categorize non-equilibrium physical phenomena. The concepts of uniformly continuous probability measures and modular evolution lead to a systematic hierarchical structure for (physical) correlation functions and non-equilibrium thermodynamical potentials. It is proposed that macroscopic evolution equations (such as dynamic correlation functions) may be obtained from a non-equilibrium thermodynamical description, by using the fact that extended thermodynamical potentials belong to a certain class of statistical systems whose probability distribution functions are defined by a stationary measure; although a measure which is, in general, different from the equilibrium Gibbs measure. These probability measures obey a certain hierarchy on its stochastic evolution towards the most probable (stationary) measure. This in turns defines a convergence sequence. We propose a formalism which considers the mesoscopic stage (typical of non-local dissipative processes such as the ones described by extended irreversible thermodynamics) as being governed by stochastic dynamics due to the effect of non-equilibrium fluctuations. Some applications of the formalism are described.

Keywords: Stochastic Processes; Non-equilibrium Thermodynamics

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Scope

The paper is outlined as follows: Section 1 is a brief introduction to the problem of applying measure theoretical tools to the study of many-particle physical systems, also some recent developments in the field are mentioned. We sketch how the probability measures approach has been applied to equilibrium systems (states). In section 2 we

* Corresponding author: e hernandez@inmegen.gob.mx
present an extension of such method for the case of non-equilibrium systems (processes) by means of the Choquet-Meyer Theorem on continuous measures. In this section two propositions (2.1 and 2.3) are made in terms of non-equilibrium stochastic measures as how to deal with systems out of local thermodynamic equilibrium within a modeling approach. The formalism dealing with Linear Response with Memory, (physical) Time Correlation Functions. Section 3 includes some results of the successful application of these methods on the aforementioned problems. Section 4 presents some brief concluding remarks.

1. States and Processes in Statistical Physics

One of the great unfinished tasks of non-equilibrium physics, whether statistical or phenomenological is that of finding a general, model-independent way of describing out of equilibrium states, similar to the Gibbs-Boltzmann formalism for the statistical-kinetic description that leads to a complete thermodynamical definition of systems in equilibrium. The Gibbs-Boltzmann formalism also applies to systems close to equilibrium by means of the celebrated local equilibrium hypothesis. This hypothesis lies at the heart of both linear response theory and linear (classic) irreversible thermodynamics.

The main problem in finding such a general theory of non-equilibrium states lies in the fact that, due to their intrinsic dissipative nature, such systems do not have a unique definition of entropy for each state [50, 13, 51], because of the non-volume preserving property of its phase space [25, 11, 27]. Since phase space volume is not preserved it is, in general impossible to define a unique global measure [59], derive from this measure a probability distribution function and characterize from it a non-equilibrium system. Nevertheless some work in that direction has been proved useful in recent years. For example, by extending the works of Sinai [62] and later of Ruelle and Bowen [59, 4] Gallavotti and Cohen [21, 22] wrote down an expression for a non-trivial measure characteristic of a stationary non-equilibrium state obeying this so-called SRB statistics. This measure plus the so-called chaotic hypothesis make possible to formulate, at least at a formal stage a proposal for a general theory of non-equilibrium phenomena. In this way probability measures give a new insight to our mathematical and physical understanding of non-equilibrium.

1.1 Physical Description and Measurability

Let us consider a description of a system in terms of its states. For continuous systems the configurations are countable subsets $X$ of $\mathbb{R}^\nu$ such that $X \cap \Lambda$ is finite for every bounded $\Lambda \subset \mathbb{R}^\nu$. A state should be defined as a probability measure on the set of all such $X$. We must give to each open set $\Lambda \subset \mathbb{R}^\nu$ the probability of finding exactly $n$ particles in
Λ and also the probability distribution of their positions. For every bounded open set Λ and for every integer \( n \geq 0 \), let \( \mu^n_\Lambda \geq 0 \) be a measure on \( \Lambda^n \subset \mathbb{R}^{n\nu} \). We shall say that the \( \mu^n_\Lambda \) form a system of density distributions (SDD) if they satisfy the following conditions [60]:

\[
\mu^0_\emptyset(\mathbb{R}^0) = 1
\]

for every \( \Lambda \subset \Lambda' \)

\[
\mu^n_\Lambda(x_1, x_2, \ldots, x_n) = \sum_{p=0}^{\infty} \frac{(n+p)!}{n!p!} \int_{\Lambda' \setminus \Lambda} dx_{n+1} \ldots \int_{\Lambda' \setminus \Lambda} dx_{n+p+1} \mu^{n+p}_\Lambda(x_1, x_2, \ldots, x_{n+p})
\]

The above conditions imply the normalization condition

\[
\sum_n \int dx_1 \ldots dx_n \mu^n_\Lambda(x_1, x_2, \ldots, x_n) = 1
\]

The (statistical) correlation functions\(^2\) corresponding to the density distribution \( \mu^n_\Lambda \) are defined by:

\[
\rho(x_1, x_2, \ldots, x_n) = \sum_{p=0}^{\infty} \frac{(n+p)!}{p!} \int dx_{n+1} \ldots dx_{n+p+1} \mu^{n+p}_\Lambda(x_1, x_2, \ldots, x_{n+p})
\]

Here \( x_1, x_2, \ldots, x_n \in \Lambda \). Equation 4 could be formally inverted to obtain:

\[
\mu^n_\Lambda(x_1, x_2, \ldots, x_n) = \frac{1}{n!} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \int_{\Lambda} dx_{n+1} \ldots dx_{n+p} \rho(x_1, x_2, \ldots, x_{n+p})
\]

Since the series in the right-hand side of equation 5 may be non-convergent, there could be a case when the statistical correlation functions may not determine the state of the system. Nevertheless, it is usually assumed that the series converge.

It is possible to associate with every SDD \( \mu^n_\Lambda \) a state on a \( B^* \)-algebra \( \mathcal{A} \) that we will construct as follows:

For every bounded open set \( \Lambda \subset \mathbb{R}^\nu \) let \( \mathcal{K}_\Lambda^\nu \) be the space of real continuous functions on \( (\mathbb{R}^\nu)^n \) with support in \( \Lambda^n \). Let us call \( \mathcal{K}_\Lambda \) the space of sequences \( (f^n)_{n \geq 0} \) where \( f^n \in \mathcal{K}_\Lambda^\nu \) and \( f^n = 0 \) for large enough \( n \). Let us now define \( \mathcal{K} = \bigcup_\Lambda \mathcal{K}_\Lambda \).

\(^2\) These statistical correlation functions are different from the physical correlation functions that we will talk about later. The former are the moments of the SDD or Ensemble (as the physicist call it), correlating partitions of the probability measurable space; the latter are correlations between physical quantities which evolve in time. Obviously both kinds of correlation functions are related but their links are non-trivial, and in fact make up for a lot of the actual battlefield of the statistical physicist.
We denote by $T$ the topological sum $\sum_{n \geq 0} (\mathbb{R}^{\nu})^n$ of disjoint copies of the spaces $(\mathbb{R}^{\nu})^n$. If $f = (f^n)_{n \geq 0} \in \mathcal{K}$, a function $Sf$ on $T$ is defined so that its restriction to $(\mathbb{R}^{\nu})^n$ is:

$$Sf(x_1, x_2, \ldots, x_n) = \sum_{p \geq 0} \sum_{i_1}^{n} \ldots \sum_{i_p}^{n} f^p(x_{i_1}, \ldots, x_{i_p})$$  \hspace{1cm} (6)

Let us now consider functions on $T$ of the form $\varphi(Sf_1, \ldots, Sf_q)$ where $f_1, \ldots, f_q \in \mathcal{K}$ and $\varphi$ is a bounded continuous complex function on $\mathbb{R}^q$. With respect to the usual operations on functions and the complex conjugation operation $(\ast)$, the $\varphi(Sf_1, \ldots, Sf_q)$ form a commutative algebra $\tilde{A}$ with an involution. The closure $\mathcal{A}$ of $\tilde{A}$ with respect to the uniform norm is an Abelian $\mathbb{B}^\ast$-algebra [60].

Given an SDD $\mu^n_\Lambda$, a state $\rho$ on $\mathcal{A}$ is defined as follows. For each $\varphi(Sf_1, \ldots, Sf_q)$, let $\Lambda$ be such that $f_1, \ldots, f_q \in \mathcal{K}_\Lambda$. We write

$$\rho(\varphi(Sf_1, \ldots, Sf_q)) = \sum_{n \geq 0} \int dx_1 \ldots dx_n \mu^n_\Lambda(x_1, \ldots, x_n) \times \varphi(Sf_1, \ldots, Sf_q)$$  \hspace{1cm} (7)

$f_i = f_i(x_1, \ldots, x_n) \ \forall i$, this definition (equation 7) is independent of the choice of the volume $\Lambda$ and extends by continuity to a state on $\mathcal{A}$. If we call $\mathcal{F}$ the set of states defined as above (equation 7) from an SDD; the mapping $\mu^n_\Lambda \rightarrow \rho$ is then one-to-one onto $\mathcal{F}$. It could be seen that the translations of $\mathbb{R}^{\nu}$ are automorphisms of $\mathcal{A}$. The algebra here obtained $\mathcal{A}$ is not separable, hence a direct decomposition of invariant states into ergodic states is not possible. However, it can be shown that this could be attained if by an indirect treatment [61].

We are now on position of studying the time evolution of the just defined states (i.e. a physical process) as a particular automorphism of the algebra $\mathcal{A}$. The time evolution of the system occupying the finite region $\Lambda \in \mathbb{R}^{\nu}$, with respect to a given physical interaction $^3$ $\Phi$, is defined by a one-parameter group of automorphisms of $\mathcal{A}_\Lambda$ which associates to the observable $A(A(0))$ at time 0, the following observable at time $t$ ($A(t)$):

$$e^{it H_\Phi(\Lambda)} A e^{-it H_\Phi(\Lambda)} = \sum_{n=0}^{\infty} i^n \frac{t^n}{n!} [H_\Phi(\Lambda), A]^n$$  \hspace{1cm} (8)


It has been proved [57] that in the so-called thermodynamic limit ($\Lambda \rightarrow \infty$), for a general class of physical interactions $\Phi$ with associated hamiltonians $^4$ $H_\Phi$ the following

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3 Whose specific nature it will not affect the following consideration.

4 A Hamiltonian is a function which essentially gives the energy of the system in terms of the positions of the particles and the nature of the interactions between them, in this case it is sufficient to know that is a bounded function of the $x_i$'s.
limit exists:

$$\lim_{\Lambda \to \infty} e^{i\tau H_\Lambda(A)} A e^{-i\tau H_\Lambda(A)} = \Gamma_\tau A$$  \hfill (9)

Hereafter we will call \( \Gamma_\tau \) the modular automorphism operator and \( \Gamma_\tau A \) will be called the \( \tau \)-modular evolution of \( A \) or just the time evolution of \( A \).

1.2 Equilibrium States, Gibbs Measures and KMS Condition

Let us consider a quantum dynamical system described by a von Neumann algebra \( A \) (representing the system’s physical observable quantities) with a one-parameter automorphism group (taking into account the time evolution of the observable quantities). In equilibrium statistical mechanics the entropy \( S \) of a given state \(< \cdot >\) (labelled by an observable) is defined as:

$$S = -\operatorname{Tr} \hat{\rho} \ln \hat{\rho}$$  \hfill (10)

where \( \hat{\rho} = \rho / \operatorname{Tr} \rho \) is the so-called normalized density matrix. We are to consider states lying on a constant observable energy surface \( E = < H > \) with \( H \) the system’s hamiltonian and the trace taken with respect to some measure \( \Xi \) (an ergodic measure). For such systems, a global equilibrium state is characterizable as the state of maximal entropy within this set (of constant energy \( E \)). As one knows the constant energy and the normalization of the probability matrix \( \rho \) \( \operatorname{Tr} \hat{\rho} = 1 \) constraints are introduced in the maximization procedure as Lagrange multipliers [43]. Doing so one gets the usual Gibbs states \( \rho = e^{-\beta H} \).

It turns out that by analyzing the meaning of this entropy (making it coincident with the equilibrium thermodynamic entropy) one gets two main results. The parameter \( \beta \) is proportional to the inverse temperature of the system and the ergodic measure \( \Xi \) is identical to the Gibbs measure \( \rho \) [68]. This equilibrium condition \( \Xi = \rho \) (usually called Gibbs condition) has the restriction of being limited to apply to finite systems [41].

Now let us examine the quantum mechanical description of a system with an arbitrary number of degrees of freedom under a constraint \( \eta \) from the standpoint of a measure theory. The expectation value of an observable quantity \( A \) in a state \( < \cdot >_\eta \) could be characterized by a density matrix \( \rho_\eta \) as follows:

$$\langle A \rangle_\eta = \frac{\operatorname{Tr} \rho_\eta A}{\operatorname{Tr} \rho_\eta}$$  \hfill (11)

It is possible to introduce in connection with this measure a so-called modular Hamiltonian \( H^* \) as:

\( ^5 \) In order to avoid the confusion between thermodynamic entropies and probabilistic or informational entropies we will use term entropy when referring to the former and entropic-measure when referring to the latter.
Here $\beta^*$ is a number introduced for later convenience (e.g. as a parameter of periodicity for the trace of density matrices). The modular evolution of a property $A$ is given by an action-like similarity mapping [68] as:

$$\Gamma_\tau(A) = e^{iH^*\tau} A e^{-i\beta^*H^*}$$  \hspace{1cm} (13)

The cyclical behavior of the trace (i.e. $Tr\rho_\eta(\tau) = Tr\rho_\eta^*(\tau + i\beta^*)$) gives the following condition on the modular evolution between two states $N, M$:

$$\langle \Gamma_\tau(N) M \rangle_\eta = \langle M \Gamma_{\tau+i\beta^*}(N) \rangle_\eta$$  \hspace{1cm} (14)

This condition is called the KMS condition [42]. This Kubo-Martin-Schwinger (KMS) condition (proposed by those authors as an auxiliary boundary condition for Green’s functions) can be adopted as a simple characterizing property of equilibrium states in the algebraic formulation (also called $C^*$-statistical mechanical formulation), which makes sense for an infinite system (in contrast to Gibbs Ansatz) and hence enables one to study an infinite system directly, by-passing the thermodynamic limit. This is a generalization of the theorem concerning Laplace transforms, which says that the Laplace transform of a function of energy $E$, which is zero for $E<0$, is analytic as a function of the conjugate variable (here, time $t$) in the upper half-t-plane. The KMS generalization of this says that, if the energy of a theory is positive then the equilibrium state gives a two-point two-time correlation function that is analytic in time-difference in a strip in the upper-half plane of width $1/K^*T$, where $T$ is the temperature, and is periodic with imaginary period $1/K^*T$, $K^*$ is a measure-specific constant. The periodicity follows from the cyclicity of the trace. The justification of this KMS condition as the characterization of equilibrium states is that KMS states along with ground states and ceiling states (corresponding to $±0$ temperature) are precisely those states which are stable against local perturbations.

If the system is in a global equilibrium state $\langle \cdot \rangle_\eta =< \cdot >, \beta^*$ is just the inverse temperature $\beta = 1/K_BT$ and the modular hamiltonian $H^*$ is the usual mechanical hamiltonian $H$. In this case modular evolution generates time evolution. Stationarity and dynamical stability are minimal requirements for a state to be called an equilibrium state; A passivity theorem [31] can be rephrased by saying that it is precisely the KMS states which are distinguished by the second law of thermodynamics in Kelvin’s formulation: there are no cyclic processes converting heat into mechanical work if the state of the systems obeys the KMS condition [58].

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6 Of course it is possible to replace this modular evolution by the action of a Liouvillian super-operator or propagator. See for example [1], also [52] For a modern introduction to modular operators see [20]

7 For a proof of the theorem in the above context of modular evolution see reference [57].
Local equilibrium could also be defined through a KMS-type condition. We can think of local thermodynamic equilibrium as a state that cannot be distinguished from a global equilibrium state by infinitesimally localized measurements [36]. The quantitative description of the term infinitesimally localized measurements could be given in terms of a one-parameter scaling procedure [36] over the observable. We introduce a parametric diffeomorphism as a scaling procedure as follows.

Definition 1.3

Uniparametric State Space Scaling

A diffeomorphism of the state space that preserves the topological and ergodic character of such space and projects a set of points (called quasi-local points) into equilibrium points by means of a change of scale is called a Uniparametric State Space Scaling (USSS).

Let us consider an observable depending on n-points in real space $A = A(x_1, x_2, \ldots, x_n)$. For the sake of simplicity we will consider a unique field $\phi$ spanned by the various values of the observable and depending on spacial localization as follows: $A = \phi(x_1)\phi(x_2)\ldots\phi(x_n)$. A $\xi$-scaling diffeomorphism (USSS) $\Lambda_\xi$ is given as:

$$\Lambda_\xi A = \Sigma(\xi)^n \phi(\chi_\xi x_1)\ldots\phi(\chi_\xi x_n)$$

(15)

with $\Lambda_\xi A$ the value of $A$ after a scaling by a factor $\xi$, $\Sigma(\xi)$ a scaling function (postulated in order for the scaling to be well-defined) and

$$\chi_\xi(x_i) = x^* + \xi(x_i - x^*)$$

(16)

$x^*$ is the point to which quasi-local points $x_i$ shrink after localization, $x^*$ is called an equilibrium point-cell in the sense of Onsager. It is possible to see that $\chi_{\xi=0}(x_i) = x^* \forall i$ and that $\chi_{\xi=1}(x_i) = x_i$ so that if the system scales with $\xi \rightarrow 0$ we could talk about local equilibrium (i.e. all points shrink to an equilibrium one) and if $\xi \rightarrow 1$ we are in a highly delocalized stage and hence local equilibrium is not attained. Of course repeated application of the scaling $\Lambda_\xi$ would define a coarse graining operator in the sense evoked by Ehrenfest and formalized by Zwanzig [70].

It is worth noticing that this scaling procedure generates, for each application, a sequence of probability distributions $\rho_0, \rho_1, \ldots, \rho_n, \ldots$; here $\rho_0$ is the initial random distribution, generally taken as a uniform distribution, a fact called Stosszahlansatz (by Boltzmann), molecular chaos hypothesis or Initial Random Phase Approximation. Since the USSS procedure is, by construction, convergent to the state of thermodynamic equilibrium, $\rho_\infty$ should be the equilibrium distribution function, i.e. the equilibrium Gibbs distribution, $\rho_\infty = \rho$. A weak convergence condition is generally assumed:

8 We call quasi-local to those points with a value of the property $A$ sufficiently close to its equilibrium value $A^*$ so that after a finite scaling they will indeed take the value $A^*$. Hence quasi-local points are good candidates for representing local equilibrium states.
\[ \lim_{n \to \infty} \rho_n = \rho \] (17)

Even if the state of equilibrium (global or local) is not attained the description of the system under a modular evolution could be done. In this case the measure will depend on the space-time localization described by the scaling \( \Lambda_\xi \). For a variety of conditions a systematization for the scaling effect could be done in terms of a time-dependent distribution function \( \Omega_\tau(\xi) \) for the values of \( \xi \). The time \( \tau \) can be considered as the indicator for the direction of convergence. In some cases the distribution function is given in the form of a stationary absolutely continuous uniform measure (or a family of Cauchy convergent continuous measures) and so the weak convergence argument-equation 17-applies.

Several physically relevant problems could be better understood in terms of uniformly continuous measures of parameter state spaces. The problem of extending the notion of Gibbs states to quasi-local interactions [45], coupled quantum systems [37], the hierarchical structure of physical descriptions in terms of the coarse-graining [30] exemplifies such problems. The recent arise of the so-called non-extensive statistical mechanics [64] and the somehow dubious \( q \)-based entropies [72] are an indirect consequence of a problem that has been called non-canonical averaging [2]. This problem has been related with the issue of aging in physical systems in the form of anomalous phenomena such as the ones surrounding the glass transition, a problem that could also be treated as the effect of a non-local in time measure. Quasi-local interactions, coarse-graining-scale effects and non-canonicity are all in a class of phenomena for which the description in terms of absolutely continuous measures (phase space densities) results enlightening.

As we already noticed a central issue in non-equilibrium physics is the description of irreversible processes in a formalism akin to the Gibbs-Boltzmann Ansatz. Since we have seen that a measure-theoretical description carries on the double duty of characterizing equilibrium (and also local equilibrium) states through a KMS condition, and also providing a tool for calculation of the evolution of equilibrium quantities (i.e. modular evolution via an equilibrium distribution called the Gibbs measure \( \rho \)) it is natural to think of a possible extension of this procedure for the characterization of non-equilibrium processes. Nevertheless, even if this goal has been pursued by some of the most brilliant minds in thermal physics: Boltzmann, Landau, Onsager, Zwanzig, Green and many others; the problem has been proved to be very difficult to tackle. The reason is that since we have to base our measure theoretical description on a non-stationary hamiltonian (a strongly time-dependent quantity), the explicit form of the associated measure is given in terms of the solution (impossible in practice) of the complete many-body problem. Several approaches have been attempted, ranging from projection operator techniques [70, 71] to memory functions [3], variational principles [68], etc.

In the next section we will introduce an alternative based on a kinetic-theory-founded
[15] extended irreversible thermodynamical formalism [13, 7, 16] taking into account the fact that in the typical time and length scales of extended non-equilibrium phenomena (the so-called mesoscopic stage) the fluctuations of the macro-variables will play a very important role.

2. Non-equilibrium Measures

By application of the so called modified moment method for the kinetic theory extraction of macroscopic non-equilibrium information, it has been proved [12] that a probabilistic measure (a “quasi-gibbsian” measure) exists for an arbitrary pair of (non-equilibrium) steady states given in terms of the entropy production (more precisely the uncompensated heat production) at the steady states. This measure gives rise to an extended Gibbs relation that generalizes the Gibbs relation for the entropy change usually employed in equilibrium thermodynamics [7].

The distribution function \( f_{\text{ne}} = f_0 + f_1^{\text{ne}} + f_2^{\text{ne}} + \ldots \), if examined under the so-called functional hypothesis [15, 17] gives rise to an irreversible thermodynamical formalism that, in spite of being of a nonlocal and nonlinear (far from equilibrium) nature obeys a form of canonical thermal averaging; this fact will be very useful later in this study. Here \( f_0 \) is the equilibrium distribution function and \( f_i^{\text{ne}} \), \( i = 1, 2, \ldots \) are non-equilibrium contributions to the distribution function also called higher order moments of the distribution function.

The specific form of the measure \( \mu \) depends on the solution of the kinetic theoretical description of the system under study. Once again, except for a few ideal systems like the classical and quantum diluted gas there is no closed solution for the kinetic equations. However, since the measure depends on the kinetic solution in terms of a distribution function \( \mu = \mu (f_0, f_1^{\text{ne}}, f_2^{\text{ne}}, \ldots) \) (even without having the explicit solution for it), it is possible to see that the passage from the so-called chaotic stage (given by Boltzmann’s stosszahlansatz) to the fully-developed stationary distribution shall induce a dynamic convergence sequence in the measure. Given these facts the main contribution of this work could be summarized in two principles:

Proposition 2.1

The measure \( \mu \), associated with the state space modular evolution \( \Gamma_+ (\vec{X}) \) for a system out of (local or global) thermodynamic equilibrium and described by variables \( \vec{X}(t) \) is given by a sequence of time dependent distribution functions. This means that we are able to recognize the measure as the stationary solution of a stochastic process \( \Theta_+(t) \).

More explicitly we can state that:
\[
\lim_{t \uparrow \tau} \Theta_\tau(t) = \mu_\tau
\]

(18)

\[
\Gamma_\tau A = \int_{\Omega} \Theta_\tau(t) F(A(t), \tau) \, dt = \int_{\tau} \mu_\tau F^\tau(A(t)) \, dt
\]

(19)

\(F\) is called the kernel or modulus of the given modular evolution \(\Gamma\) and is a still-undefined \(^9\) continuous monotonic function of the state space fields \(\vec{X}\). The dynamic variable \(A\) is an arbitrary function of the state space fields \(\vec{X}, A(t) \equiv A(\vec{X}(t))\). \(\Theta_\tau\) is a time-continuous stochastic process. The sequence induced by equation 19 converges according to the distribution \(\Omega_\tau\) already mentioned in connection with the scaling procedure in the previous section. Although the explicit form for the distribution function for the scaling parameter \(\Omega_\tau\) is \textit{a priori} unknown we will see that is homeomorphic to the convergence sequence for the kinetic distribution function \(f^{ne}\).

\textbf{Theorem 2.2}

The stationary state of the Stochastic process \(\Theta_\tau\) corresponds to the non-equilibrium probability measure \(\mu\) at least in the sense of means (i.e. almost-surely) according to equation (18).

\textbf{Proof}

Let \(\Omega_\tau(\xi)\) be a distribution of \(\tau\)-time dependent scaling factors in the sense of USSS. Let us assume that the distribution is stationary and has a compact support (i.e. it is dense in state space). By construction of the USSS, the stationary distribution corresponds to an equilibrium (local or global) distribution of state space points. Now let \(\mu = \mu(f_0, f_1^{ne}, f_2^{ne}, \ldots)\) be an \(\epsilon\)-dependent or kinetic distribution, where \(\epsilon\) is called the uniformity parameter or Knudsen’s parameter. By definition \(\mu\) is stationary and has compact support in phase space. Since the stationary distribution \(\mu(f_0)\) is an equilibrium (local or global) distribution there exists a transformation from one equilibrium distribution to the other (i.e. in the thermodynamic limit both distributions are equivalent).

This condition suffices, to take into account the fact that as \(\xi\) goes to zero the kinetic distribution \(f^{ne}\) should converge to the equilibrium distribution \(f_0\). This is so, since the condition of equilibrium (in the sense of spatial homogeneity of the thermodynamic potentials) gets attained (see equation 16), notwithstanding the difference in the speed of convergence of both distributions.

In some sense, \textit{Theorem 2.2} is a kind of informal version of the Monotone-Convergence Theorem [67]. The second proposition, namely equation (19) follows \textit{Theorem 2.2} plus

\(^9\) System-dependent in the Physicists saying
the assumption of modular evolution. Let us see this in terms of measures:

Let $F$ be a locally convex topological vector space and $K$ a convex, compact subset of $F$. The dual $\mathcal{M}$ of the closure of $K$ consists on the measures on $K$. Denoting by $\mathcal{M}_+$ the convex cone of positive measures and by $\mathcal{M}_1$ the set of positive measures of norm 1 (i.e., the probability measures). If $\mu \in \mathcal{M}_1$ there exists $\rho \in K$, such that:

$$ f(\rho) = \int f(\sigma) d\mu(\sigma) \quad (20) $$

$\rho$ is usually called the resultant of $\mu$ [5]. If $\mu \in \mathcal{M}_1$ has a resultant $\rho$ then $\mu$ can be approximated weakly by measures $\mu' \in \mathcal{M}_1$ with resultant $\rho$ and finite support $^{10}$

If we take this result in terms of equation 18, equation 19 follows directly. In the second expression at the right hand side of equation 19 has been used the so-called tower property of conditional probabilities [67].

We have still retained the notation of discrete evolution in order to show up the analogy with usual (i.e. finite volume) Gibbs measures, however we must note that $\mu_\tau F^\tau$ represents the operation of a time continuous stochastic process $\Theta_\tau(t)$ over a field through the action of a $\tau$-continuous semigroup $F$. $^{11}$

Once we have defined this measure we are able to make explicit assumptions to what the time evolution of a macro-variable will be in terms of correlation functions under the context of non-local irreversible thermodynamics.

**Proposition 2.3**

The time evolution of a (non-local) irreversible thermodynamic field $Z$ given in terms of a modular evolution operator $\Gamma_\tau$ generates a form for the 2-time correlation function given by the $\tau$-continuous stochastic map:

$$ \langle Z(t), Z(t') \rangle_{EIT} = \int_{-\infty}^t Z(t) \mu_{(t-t')} Z(t') dt' \quad (21) $$

Similar expressions could be write down for higher order and crossed correlations.

As it could have been already noticed Propositions 2.1 and 2.3, are an extension of linear response theory for the (non-linear) case of a time dependent measure. At the moment and having recognized the impossibility to derive this measure from microscopic

$^{10}$ A formal proof of this Choquet-Meyer Theorem (too lengthy to be included here) is given in reference [8]

$^{11}$ Since $F^\tau$ represents the time evolution of a mesoscopic and hence possibly dissipative system the inverse operation may not satisfy existence and/or unicity so we will refer to the effect of operating $F$ as the action of a semigroup.
models we have assumed it as an stochastic variable. Of course if this stochastic measure converges (at least in the sense of means) -see equation 17- to the Gibbs measure $\rho$ we recover the usual linear response theory.

2.1 Linear Response with Memory

Non-equilibrium thermodynamics often addresses the problem of transport through material media. Hyperbolic transport equations (such as the MCV [6, 65] equations and the telegrapher’s equation [26]) taking into account the lag on the response due to the finite velocity of perturbations have been derived from several points of view. Ranging from phenomenological arguments to purely microscopic transport [26] and also probabilistic methods such as the persistent random walk [28]. On the other hand linear response theory represented a powerful theoretical tool to cope with transport phenomena from a dynamic and thermodynamic standpoint. In the last years linear response functions incorporating memory (i.e. the effect of the lag in the response to an applied field) appeared in such problems as delayed transport in electronic devices, molecular hydrodynamics and rheology of structured fluids, and also solid state phenomena such as the dynamics of Abrikosov vortices.

Let us first consider two related dynamic variables say the magnetic field $\vec{H}(t)$, and the magnetization $\vec{M}$. At constant temperature in the low-field limit there is a linear relation between the magnetic field and the magnetization:

$$\vec{M} = \chi^0_T \vec{H}(t)$$

(22)

Here $\chi^0_T$ is a tensor response coefficient called the susceptibility tensor. However, it is known that it takes some time $\tau_M$ for a material media to achieve magnetization under exposure to a magnetic field. In other words, there is a lag in the response to the field. This effect is due to the fact that magnetization is a non-equilibrium process. If we look at the effect of mesoscopic fluctuations of the magnetic field due to non-equilibrium evolution in the sense already evoked we will see a possible explanation. Since the magnetic field is in disequilibrium we can assume it undergoes a stochastic time evolution. It is possible to associate an automorphism $\Gamma_\eta$ on the magnetic field with this stochastic evolution:

$$\vec{H}(t)_{sto} = \Gamma_\eta \vec{H}(t)$$

(23)

If we consider linear response between the magnetization $\vec{M}$ and the field $\vec{H}(t)_{sto}$ we get:

$$\vec{M} = \chi^0_T \vec{H}(t)_{sto} = \chi^0_T \Gamma_\eta \vec{H}(t)$$

(24)

But since the modular automorphism is induced by a time-continuous measure $\mu_\tau$ we have:
\[ \vec{\mathcal{M}} = \chi^0_T \int_{-\infty}^{t} \mu_T(\vec{H}(t')) \, dt' \] (25)

By proposing a form for the stochastic process associated with the measure \( \mu_T \) of the automorphism \( \Gamma \), we can model this non-equilibrium process. If we look for the behavior in a weak stochastic limit we are able to consider a first order stochastic process. Let us consider \( \Gamma \) as a modular automorphism whose associated stochastic process is a simple exponential decay in time since this is a prototypic first order stochastic process \[39\]. In this case equation 25 reads:

\[ \vec{\mathcal{M}} = \chi^0_T \int_{-\infty}^{t} \lambda_P \exp\left(\frac{t - t'}{\tau_P}\right) \vec{H}(t') \, dt' \] (26)

Here \( \lambda_P \) is the amplitude of the associated distribution and \( \tau_P \) is a characteristic time (akin to Poisson’s time). If we define a non-equilibrium susceptibility \( \chi^\text{ne}_T \) (up to first order) as \( \chi^\text{ne}_T = \chi^0_T \lambda_P \), and recognize Poisson’s time as the relaxation time associated with the non-equilibrium process of magnetization we get:

\[ \vec{\mathcal{M}} = \int_{-\infty}^{t} \chi^\text{ne}_T \exp\left(\frac{t - t'}{\tau_M}\right) \vec{H}(t') \, dt' \] (27)

Equation 27 is the usual expression of a linear response with exponential memory (also called fading memory). If we look at the differential representation of equation 27 we will find the hyperbolic MCV-type transport equation. It is worth noticing that even if we start with a linear response relation and the measure is induced by a first order stochastic process, the resulting equation possess a non-markovian character, i.e. memory in the response. If the time scale of the system is much more slower than its stochastic time (\( t_{\text{char}} >> \tau_P \)) then we can consider the limit \( \tau_P \rightarrow 0 \). In this case the exponential decay distribution is just a \( \delta \)-distribution and the usual linear response without memory is recovered. By means of the analysis of the associated measure (and its related stochastic process) one is enable to perform a selection of the effects that we will take into account in a very transparent way. This will be a very important issue when considering a collection of irreversible processes taking place in a non-equilibrium system, since in this case the presence of irreversible couplings (a question related to the relative relaxation times of the various processes) could be cope-with in a systematic way.

2.2 Time Correlation Functions

Time correlation functions are usually considered to represent the mean behavior of a large set of microscopic dynamic variables under certain averaging assumptions (canonicity, number density conservation, energy density conservation, etc.) usually taken into account by some kind of Lagrange multiplier formalism (in the case of extended thermodynamics these multipliers are introduced, for example, by Liu’s Method) \[19, 38, 44\]. Nevertheless by applying a novel irreversible thermodynamics formalism \[14\] it has been stated that they also represent Gibbsian ensemble averages of a collection of macroscopic
field variables, when the latter are considered (as is the case in extended irreversible ther-
modynamics [13]) as coming from stochastic processes (in general non-markovian noises) 
in a mesoscopic length-scale [18, 12]. This statement has only been proved formally for 
dilute gaseous systems by means of quantum kinetic theory [13, 18, 12], but there is 
strong evidence supporting its validity for several other physically significant systems. 
We will use this statement as an ansatz.

In order for the field averages to be of a more general character, we will use the equiv-
ality between 2-time correlation functions and propagators [70, 71], or time evolution 
operators acting on a dynamical variable. The dynamic variables are the set of extended 
thermodynamical potentials over its gibbsian set. Time evolution operators are obtained 
as inner products with dual vectors of the aforementioned thermodynamical potentials. 
From the theory of stochastic processes [39, 48] we obtain the statistical properties of 
such inner products by introducing a probability measure \( \mu \) (or weighting function) in 
the last stages of study of the problem.

The 2-time auto-correlation function for a dynamical variable \( A \) will then be given as 
(see eq. 21):

\[
\langle A(t)A(t') \rangle \tau = \int A(t) \mu_\tau A(t') \, dt'
\]

(28)

As we already stated we are in a position to see the measure \( \mu_\tau \) as a dual complement 
of \( A \), i.e. \( \mu_\tau A = A^\dagger \) with \( (A(t), A(t)^\dagger) = \| A(t) \|^2 \) [63]. We could also look in a very 
similar way to 2-time crossed-correlation functions and also to higher order correlations. 
Let us look at an example coming from extended irreversible thermodynamics [34].

According with the formalism of Extended Irreversible Thermodynamics (EIT) [38] 
the time evolution of the (entropy-like) compensation function \( \Psi \) is given by:

\[
Td_\tau \Psi = d_\tau U + Pd_\tau V - \sum_i \Upsilon_i d_\tau C_i + \sum_j X_j \odot d_\tau \Phi_j
\]

(29)

We see that equation (29) is nothing but the formal extension of the celebrated Gibbs 
equation of equilibrium thermodynamics for the case of a multi-component non-equilibrium 
system. This extended Gibbs relation is brought about in the theory by imposition of 
some consistency conditions on the non-equilibrium part of the distribution function 
[14, 18, 15]. The quantities appearing therein are the standard ones, \( T \) is the local tem-
perature, \( P \) and \( V \) the pressure and volume, \( \Upsilon_i \) is the chemical potential for the species 
"i", etc. \( X_j \) and \( \Phi_j \) are extended thermodynamical fluxes and forces.

We will look up for the effect that a non-equilibrium process, say mass flux will have 
on the thermodynamic description of the system. In the case of a binary fluid mixture we 
will take our set of relevant variables \( G = S \cup \mathcal{F} \) to consist in the temperature \( T(\vec{r},t) \) and 
concentration of one of the species \( C_2(\vec{r},t) \) fields as the slow varying (classical) parameters
set $S$ and the mass flux of the same species $\vec{J}_2(\vec{r}, t)$ as a fast variable on the extended set $\mathcal{F}$. For the fast dynamic variables (such as the mass flux $\vec{J}_2$ and its conjugated thermodynamic force $\vec{X}$) characteristic times are much smaller than for the so-called conserved fields (mass, energy and momentum densities, etc.) so, the effect of fluctuations in these variables will be greater. We will take this fact into account by associating to these fields a memory in the response akin to the memory described in the last section.

We propose linear constitutive equations with exponential memory kernel (see section 2.1) for the following reasons: a) The associated transport equations are hyperbolic (of the Maxwell-Cattaneo-Vernotte type) \cite{6, 65} so causality is taken into account, b) These hyperbolic transport equations are compatible with the postulates of EIT \cite{38, 44, 53, 54, 55}, c) Similar equations can be derived for coupled non-markovian stochastic processes \cite{66, 47} and since stochasticity has been associated with the major role of fluctuations in the mesoscopic description level of EIT the outcome will improve our understanding of phenomena occurring in such mesoscopic stages, d) The stochastic process associated with this set of two coupled constitutive equations called a semimartingale has been extensively studied and its known to accept absolutely continuous measures a fact that will become very useful \cite{10, 67, 33}.

The constitutive equations are therefore chosen to be,

$$\vec{J}_2(\vec{r}, t) = \int_{-\infty}^{t} \lambda_1 \vec{u} e^{(t-t')/\tau_1} \vec{Y}(\vec{r}, t')dt'$$

(30)

$$\vec{X}(\vec{r}, t) = \int_{-\infty}^{t} \lambda_2 e^{(t-t')/\tau_2} \vec{J}_2(\vec{r}, t'')dt''$$

(31)

The $\lambda_i$’s are time-independent, but possibly anisotropic amplitudes, $\vec{u}$ is a unit vector in the direction of mass flow and $\tau_i$’s are the associated relaxation times considered path-independent scalars. Since we have a linear (thought non-markovian) relation between thermodynamic fluxes and forces some features of the Onsager-Casimir formalism will still hold. The resulting thermodynamic potentials and its derivatives are assumed to belong to the quasi-gibbsian set of time dependent macroscopic functions mentioned in reference \cite{12}, over which we will perform a thermodynamic average (i.e. a phase field average as measured by a quasi-gibbsian measure characteristic of the non-equilibrium thermodynamic state space) in order to obtain the 2-time dependent correlation function for, say the concentration-concentration correlation function $\langle C_1(\vec{r}, t), C_1(\vec{r}, t') \rangle$. If the thermodynamic average is performed under isotropic canonical conditions \cite{68, 69, 32} the resulting correlation function is given by:

$$\langle C_1(\vec{r}, t), C_1(\vec{r}, t') \rangle = \int_{\Phi} C_1(\vec{r}, t) \mu(t-t')C_1(\vec{r}, t')dt'$$

(32)

For the case being treated it was showed \cite{34} that equation 32 is given as follows:
\[ \langle C_1(\vec{r}, t), C_1(\vec{r}, t') \rangle = \int_\Phi \Xi(\vec{r}) e^{-n(t+t')} W(t-t') dt' \]  

(33)

with

\[ \Xi(\vec{r}) = AB + B^2 \]  

(34)

\[ A = \int_0^T C_p(T) \frac{1}{H(\vec{r})} \]  

(35)

\[ B = \left[ \frac{\lambda_1 \lambda_2 \tau_1 H(\vec{r}) T}{n(1-n\tau_1)^2} \right] \left[ \frac{\lambda_1 \tau_1^2}{(1-n\tau_1)} + \frac{\lambda_1 (1-n\tau_1) - 1}{(1-n\tau_2)} \right] \]  

(36)

\[ W(t-t') \] is a stochastic process representative of the measure \( \mu(t-t') \); hence the presence of non-equilibrium correlations is taken into account by means of this stochastic evolution of the measure. Equation (33) is the non-regular part of the composition field time correlation function (i.e. the value of the correlation function near the critical point after scaling of the thermal and concentration fields). \( C_p \) is the heat capacity, \( T(\vec{r}) \) and \( H(\vec{r}) \) are known continuous functions for the amplitudes of the temperature field and the chemical potential field.

For a broad family of stochastic measures equation (33) asymptotically converges to a limit given by [29]:

\[ \langle C_1(\vec{r}, t), C_1(\vec{r}, t') \rangle = \kappa \Xi(\vec{r}) e^{-2nt} \]  

(37)

where \( \kappa \) is a constant depending on the explicit stochastic measure under consideration. For an unitary Ornstein-Uhlenbeck measure \( W(t-t') = e^{lt-l't'} \) convergence implies \( \kappa = \frac{1}{1-n} \). As we shall see this distribution resulted very appropriate to model this kind of non-equilibrium critical system [34]. In the case of an unitary Gaussian measure \( W(t-t') = e^{l(t-t')^2} \) for example, an asymptotic solution could not be given in terms of an exponential decay, a typical feature of the critical decay of fluctuations. In this Gaussian case the solution consist of two contributions: an exponential decay plus an error function type mode. This difference eliminates the short-time plateau (known as critical slowing down of fluctuations) present in correlation functions obtained under colored noise measures.

Equation 37 is a form of the van Hove expression for the density-density correlation function, also called a dynamic structure factor.

One of the advantages of this formalism with respect to others, such as mode-coupling theory is that we can test different kinds of stochastic relaxational couplings (delta-correlated, quasi-markovian, gaussian, lorentzian, Ornstein-Uhlenbeck type, non-markovian, Ornstein-Uhlenbeck character for linear non-markovian statistics except in the case where stochastic resonance was present.

\[ 12 \text{In the case studied then (reference [34]) the convergence sequence was Cauchy, with stationary Ornstein-Uhlenbeck character for linear non-markovian statistics except in the case where stochastic resonance was present.} \]
simultaneous non-linear multi-modal coupling, etc.) just by changing the weighting functions \( W(t - t') \) (that is the family of probability measures \( \mu_\tau \)), within a unifying thermodynamic scheme.

2.3 Hysteresis in Non-equilibrium Systems

It is possible to describe how the phenomena of hysteresis could be well understood in the context of an extended irreversible thermodynamic formalism by pointing out that hysteretic phenomena is a consequence of the existence of dissipative internal processes in the system as was showed elsewhere [35]. The different contributions to the uncompensated heat production can be explicitly incorporated in the description by characterizing each one by its relaxation time. In order to do so it is important to take into account the effect of dynamical coupling. One way of taking this effects into account is by examining the coupling of the associated stochastic measures, then looking for processes whose measures have stochastic times of the same order, since in this case a dynamic coupling is possible. By enslaving those stochastic processes to the faster one it is possible to develop a hierarchy of measures. In this case the analysis of relaxation times is also physically clearer than by using the standard hysteresis operators [56, 49]

3. Some Successful Examples of the Results of the Application of the Formalism

Since we introduced the non-equilibrium stochastic measure \( \mu_\tau \) in Propositions 1 and 2 as a tool for modeling systems out of local or global equilibrium its physical validity will be ultimately conditioned by the results given by its use against experimental data. We will give here some brief results of some cases where its application has led to satisfactory results. Let us first consider the extended thermodynamical description of criticality as given in ref.[34]. If we compare the results for the temperature dependence of the diffusion as obtained from the calculated 2-time concentration autocorrelation function in the neighborhood of the critical consolute point, as given by equation 33 under a suitable measure with the experimental light scattering spectra as reported and by looking at the results there, it is clear that the stochastic measure was a reliable tool for the description of a highly complex non-equilibrium property.

For the case of non-equilibrium couplings and its relation to the observed phenomena of hysteresis (cf. section 3.3) it has been possible to show [35] that if the relaxation times of the phenomena are compared to the stochastic time given by the associated measure, then a criteria is given as to when will we experimentally observe hysteretic phenomena.

In brief, by applying simple and reasonable modeling approximations within the formalism outlined in the preceding sections, we were able to reconstruct the hyperbolic
transport MCV equations (cf eq. 27) of linear response theory with memory. Also a well known result of condensed matter theory and phase transitions was obtained, namely the Van Hove equation for the density autocorrelation function (eq.37). Finally some previous results on the phenomena of hysteresis in terms of relaxation times resulted firmly grounded within the framework presented in this paper.

Concluding Remarks

In equilibrium statistical mechanics a measure theoretical representation (Gibbs formalism) has provided a great insight and also powerful tools for the characterization of equilibrium states (via Gibbs or KMS conditions) and the calculation of equilibrium quantities (via thermodynamic averages, i.e. averages given by equilibrium measures). Similar tools has been derived for systems close to equilibrium (in local equilibrium) by means of linear irreversible thermodynamics and linear response theory. Although this task has been impossible to take further away from equilibrium, by using a similar approach we have been able to model expressions representing some non-equilibrium situations in a non-local regime. The application of this formalism (based on the assumption of stochastic evolution at the mesoscopic level of description) permits a systematic study of such dissipative systems and in the cases outlined give rise to good agreement with available experimental data. Much work has to be done in order to fully understand the consequences of this and related formalisms, such as the ergodic dynamic systems approach of Gallavotti and Cohen [21] and the (much debated) fractional calculus approach of Tsallis [64].

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References


Beltrami Flow of an Unsteady Dusty Fluid Between Parallel Plates in Anholonomic Co-ordinate System

B. J. Gireesha, C. S. Bagewadi* and C. S. Vishalakshi

Department of Mathematics, Kuvempu University, Shankaraghatta-577 451, Shimoga, Karnataka, India

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Abstract: An analytical study of Beltrami flow of viscous dusty fluid between two parallel plates has been studied. The flow is due to influence of movement of plates. Flow analysis is carried out using differential geometry techniques and exact solutions of the problem are obtained using Laplace Transform technique also which are discussed with the help of graphs drawn for different values of Reynolds number. Further the expressions for skin-friction are obtained at the boundaries.

Keywords: Frenet Frame Field System; Beltrami Flow, Dusty Fluid; Velocity of Dust Phase and Fluid Phase, Unsteady Flow, Vorticity, Reynolds Number

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1. Introduction

A dusty fluid is a mixture of fluid and fine dust particles. Its study is important in areas like environmental pollution, smoke emission from vehicles, emission of effluents from industries, cooling effects of air conditioners, flying ash produced from thermal reactors and formation of raindrops, etc. Also it is useful in the study of lunar ash flow which explains many features of lunar soil.


* prof_bagewadi@yahoo.co.in and bjgireesu@rediffmail.com

Some researchers like Kanwal [10], Truesdell [19], Indrasena [9], Purushotham [16], Bagewadi, Shantharajappa and Gireesha [1, 2, 3] have applied differential geometry techniques to investigate the kinematical properties of fluid flows in the field of fluid mechanics. Further, the authors [2, 3] have studied two-dimensional dusty fluid flow in Frenet frame field system. Recently the authors [7, 8] have studied the flow of unsteady dusty fluid under varying different pressure gradients like constant, periodic and exponential. In this paper we study the Beltrami flow of a dusty fluid between two infinite parallel plates in anholonomic co-ordinate system. Further by considering the fluid and dust particles are at rest initially, the analytical expressions are obtained for velocities of fluid and dust particles using Laplace transform technique in different three cases. The changes in the velocity profiles for different Reynold’s number are shown graphically.

2. Equations of Motion

The governing equations of motion of unsteady viscous incompressible fluid with uniform distribution of dust particles are [18]:

For fluid phase

\[ \nabla \cdot \overrightarrow{u} = 0 \quad \text{(Continuity)} \]  
\[ \frac{\partial \overrightarrow{u}}{\partial t} + (\overrightarrow{u} \cdot \nabla) \overrightarrow{u} = -\rho^{-1} \nabla p + \overrightarrow{g} + \nu \nabla^2 \overrightarrow{u} + \frac{kN}{\rho} (\overrightarrow{v} - \overrightarrow{u}) \quad \text{(Linear Momentum)} \]  

For dust phase

\[ \nabla \cdot \overrightarrow{v} = 0 \quad \text{(Continuity)} \]  
\[ \frac{\partial \overrightarrow{v}}{\partial t} + (\overrightarrow{v} \cdot \nabla) \overrightarrow{v} = \overrightarrow{g} + \frac{k}{m} (\overrightarrow{u} - \overrightarrow{v}) \quad \text{(Linear Momentum)} \]  

We have following nomenclature:

\( \overrightarrow{u} \)—velocity of the fluid phase, \( \overrightarrow{v} \)—velocity of dust phase, \( \rho \)—density of the gas, \( \overrightarrow{g} = -\nabla \varphi \), \( \varphi \)—gravitational potential, \( p \)—pressure of the fluid, \( N \)—number of density of dust particles, \( \nu \)—kinematic viscosity, \( k = 6\pi a\mu \)—Stoke’s resistance (drag coefficient), \( a \)—spherical radius of dust particle, \( m \)—mass of the dust particle, \( \mu \)—the co-efficient of viscosity of fluid particles, \( t \)—time.
Let \( \overrightarrow{s}, \overrightarrow{n}, \overrightarrow{b} \) be triply orthogonal unit vectors tangent, principal normal, binormal respectively to the spatial curves of congruences formed by fluid phase velocity and dusty phase velocity lines respectively as shown in the figure-1.

![Figure 1 Frenet Frame Field System](image)

Geometrical relations are given by Frenet formulae [4]

\[
\begin{align*}
\text{i) } \frac{\partial \overrightarrow{s}}{\partial s} &= k_s \overrightarrow{n}, \quad \frac{\partial \overrightarrow{n}}{\partial s} = \tau_s \overrightarrow{b} - k_s \overrightarrow{s}, \quad \frac{\partial \overrightarrow{b}}{\partial s} = -\tau_s \overrightarrow{n} \\
\text{ii) } \frac{\partial \overrightarrow{n}}{\partial n} &= k'_n \overrightarrow{s}, \quad \frac{\partial \overrightarrow{b}}{\partial n} = -\sigma'_n \overrightarrow{s}, \quad \frac{\partial \overrightarrow{s}}{\partial n} = \sigma'_n \overrightarrow{b} - k'_n \overrightarrow{n} \quad (5) \\
\text{iii) } \frac{\partial \overrightarrow{b}}{\partial b} &= k''_b \overrightarrow{s}, \quad \frac{\partial \overrightarrow{n}}{\partial b} = -\sigma''_b \overrightarrow{s}, \quad \frac{\partial \overrightarrow{s}}{\partial b} = \sigma''_b \overrightarrow{n} - k''_b \overrightarrow{b} \\
\text{iv) } \nabla \cdot \overrightarrow{s} &= \theta_{ns} + \theta_{bs}; \quad \nabla \cdot \overrightarrow{n} = \theta_{bn} - k_s; \quad \nabla \cdot \overrightarrow{b} = \theta_{nb} 
\end{align*}
\]

where \( \partial/\partial s, \partial/\partial n \) and \( \partial/\partial b \) are the intrinsic differential operators along fluid phase velocity (or dust phase velocity ) lines, principal normal and binormal. The functions \( (k_s, k'_n, k''_b) \) and \( (\tau_s, \sigma'_n, \sigma''_b) \) are the curvatures and torsion of the above curves and \( \theta_{ns} \) and \( \theta_{bs} \) are normal deformations of these spatial curves along their principal normal and binormal respectively.

### 3. Formulation and Solution of the Problem

Let us consider an unsteady flow of an incompressible viscous fluid with uniform distribution of dust particles between two infinite parallel plates separated by a distance \( h \) under conservative body forces as shown in the figure-2.

The flow is due to the influence of movement of the plates. Both the fluid and the dust particle clouds are supposed to be static at the beginning. The dust particles are assumed to be spherical in shape and uniform in size. The number density of the dust particles is taken as a constant throughout the flow. Under these assumptions the flow will be a parallel flow in which the streamlines are along the tangential direction and the velocities are varies along binormal direction and with time \( t \), since we extended the fluid to infinity in the principal normal direction.
Fig. 2 Geometry of the flow.

For the above described flow the velocities of fluid and dust are of the form

\[ \vec{u} = u_s \vec{s}, \quad \vec{v} = v_s \vec{s} \]

where \((u_s, u_n, u_b)\) and \((v_s, v_n, v_b)\) are velocity components of fluid and dust particles respectively.

By virtue of system of equations (5) the intrinsic decomposition of equations (2) and (4) give the following forms;

From the vector identities we know that

\[ (\vec{u} \cdot \nabla) \vec{u} = \nabla \left( \frac{u^2}{2} \right) - (\vec{u} \times \vec{w}) \quad \text{(6)} \]

where \(\vec{w} = \nabla \times \vec{u}\) i.e., the vorticity vector for fluid phase, similarly the identity (6) is true for dust phase also with vorticity vector \(\vec{w}_p = \nabla \times \vec{v}\).

Since the forces are conservative in nature, we have \(\vec{g} = -\nabla \varphi\), here \(\varphi\) is the gravitational potential. Hence using equation (6) in (2) and (4) one can obtain

\[ \frac{\partial \vec{u}}{\partial t} + \nabla \left( \frac{u^2}{2} \right) - (\vec{u} \times \vec{w}) = -\rho^{-1} \nabla p - \nabla \varphi + \nu \nabla^2 \vec{u} + \frac{kN}{\rho} (\vec{v} - \vec{u}) \quad \text{(7)} \]

\[ \frac{\partial \vec{v}}{\partial t} + \nabla \left( \frac{v^2}{2} \right) - (\vec{v} \times \vec{w}_p) = -\nabla \varphi + \frac{k}{m} (\vec{u} - \vec{v}) \quad \text{(8)} \]

For a Beltrami flow, we know that \(\vec{w} = \nabla \times \vec{u} = \alpha \vec{u}\) and \(\vec{w}_p = \nabla \times \vec{v} = \alpha \vec{v}\) for fluid and dust phase respectively, where \(\alpha\) is a scalar quantity. So we get

\[ \vec{u} \times \vec{w} = 0 \quad \text{and} \quad \vec{v} \times \vec{w}_p = 0. \quad \text{(9)} \]

By taking curl on both sides of equation (7) and (8) and using (9), we see that

\[ \frac{\partial \vec{w}}{\partial t} = \nu \nabla^2 \vec{w} + \frac{kN}{\rho} (\vec{w}_p - \vec{w}) \quad \text{(10)} \]

\[ \frac{\partial \vec{w}_p}{\partial t} = \frac{k}{m} (\vec{w} - \vec{w}_p) \quad \text{(11)} \]
By virtue of system of equations (5) the intrinsic decomposition of equations (10) and (11) and using the fact that $\overrightarrow{w} = \alpha \overrightarrow{u}$ and $\overrightarrow{w}_p = \alpha \overrightarrow{v}$ one can obtain the following forms:

$$\frac{\partial u_s}{\partial t} = \nu \left[ \frac{\partial^2 u_s}{\partial b^2} - C_r u_s \right] + \frac{kN}{\rho} (v_s - u_s)$$  \hspace{1cm} (12)

$$2u^2_s k_s = \nu \left[ 2\sigma''_b \frac{\partial u_s}{\partial b} - u_s k_s^2 \right]$$ \hspace{1cm} (13)

$$0 = \nu \left[ u_s k_s \tau_s - 2k''_b \frac{\partial u_s}{\partial b} \right]$$ \hspace{1cm} (14)

$$\frac{\partial v_s}{\partial t} = k \left( u_s - v_s \right)$$ \hspace{1cm} (15)

$$v^2_s k_s = 0.$$ \hspace{1cm} (16)

where $C_r = (\sigma''_n + k''_n + k''_b + \sigma''_b)$ is called curvature number [3].

From equation (16) we see that $v^2_s k_s = 0$ which implies either $v_s = 0$ or $k_s = 0$. The choice $v_s = 0$ is impossible, since if it happens then $u_s = 0$, which shows that the flow doesn’t exist. Hence $k_s = 0$, it suggests that the curvature of the streamline along tangential direction is zero. Thus no radial flow exists.

**CASE-1:** In this case, the equations (12) and (15) are to be solved when subjected to the following initial and boundary conditions;

Initial condition; at $t = 0$; $u_s = 0$, $v_s = 0$

Boundary condition; for $t > 0$; $u_s = 0$, at $b = 0$ and $u_s = u_0$, at $b = h$

where $u_0$ is a constant.

Let us consider the following non-dimensional quantities,

$$u_s^* = \frac{u_s h}{U}, \quad v_s^* = \frac{v_s h}{U}, \quad b^* = \frac{b}{h}, \quad t^* = \frac{tU}{h^2};$$

where $U$ is the characteristic velocity.

Using the above non-dimensional quantities we get the non-dimensionalized form of the equations (12), (15) and the boundary conditions as follows;

$$\frac{\partial u_s}{\partial t} = \frac{h}{Re} \frac{\partial^2 u_s}{\partial b^2} - \frac{h^3 C_r}{Re} u_s + \frac{kNh^2}{\rho U} (v_s - u_s)$$ \hspace{1cm} (17)

$$\frac{\partial v_s}{\partial t} = \frac{k h^2}{m U} (u_s - v_s)$$ \hspace{1cm} (18)

$$u_s = 0 \quad \text{at} \quad b = 0 \quad \text{and} \quad u_s = \frac{hu_0}{U} \quad \text{at} \quad b = 1.$$ \hspace{1cm} (19)

where $Re =Uh/\nu$ is the Reynold’s number.

We define Laplace transformations of $u_s$ and $v_s$ as

$$U_s = \int_0^\infty e^{-zt}u_s dt \quad \text{and} \quad V_s = \int_0^\infty e^{-zt}v_s dt$$ \hspace{1cm} (20)
Applying the Laplace transform to equations (17), (18) and to (19), then by using initial conditions one obtains

\[ xU_s = \frac{h}{Re} \frac{d^2 U_s}{db^2} - \frac{h^3 C_r}{Re} U_s + \frac{h^2 l}{U_T} (V_s - U_s) \]  
\[ (21) \]

\[ xV_s = \frac{h^2}{U_T} (U_s - V_s) \]  
\[ (22) \]

\[ U_s = 0 \text{ at } b = 0 \text{ and } U_s = \frac{hu_0}{Ux} \text{ at } b = 1 \]  
\[ (23) \]

where \( l = \frac{mN}{\rho} \) and \( \tau = \frac{m}{T} \). Equation (22) implies

\[ V_s = \frac{h^2}{(h^2 + xU_T)} U_s \]  
\[ (24) \]

Eliminating \( V_s \) from (21) and (24) we obtain the following equation

\[ \frac{d^2 U_s}{db^2} - Q^2 U_s = 0 \]  
\[ (25) \]

where \( Q^2 = h^2 C_r + \frac{xRe}{h} \left( 1 + \frac{lh^2}{(xU_T + h^2)} \right) \).

The velocities of fluid and dust particle are obtained by solving the equation (25) subjected to the boundary conditions (23) as follows

\[ U_s = \frac{hu_0}{U} \left\{ \frac{\sinh(Qb)}{x \sinh(Q)} \right\} \]

Using \( U_s \) in (24) we obtain \( V_s \) as

\[ V_s = \frac{u_0 h^3}{U(xU_T + h^2)} \left\{ \frac{\sinh(Qb)}{x \sinh(Q)} \right\} \]

By taking inverse Laplace transform to \( U_s \) and \( V_s \), one can obtain \( u_s \) and \( v_s \), as;

\[ u_s = \frac{hu_0}{U} \left( \frac{\sinh(Xb)}{\sinh(X)} \right) - \frac{2\pi h^2 u_0}{URe} \sum_{r=0}^{\infty} (-1)^r \sin(r\pi b) \times \left[ \frac{e^{x_1t}(h^2 + x_1U_T)^2}{x_1[(h^2 + x_1U_T)^2 + lh^2]} + \frac{e^{x_2t}(h^2 + x_2U_T)^2}{x_2[(h^2 + x_2U_T)^2 + lh^2]} \right] \]

\[ v_s = \frac{hu_0}{U} \left( \frac{\sinh(Xb)}{\sinh(X)} \right) - \frac{2\pi h^4 u_0}{URe} \sum_{r=0}^{\infty} (-1)^r \sin(r\pi b) \times \left[ \frac{e^{x_1t}(h^2 + x_1U_T)}{x_1[(h^2 + x_1U_T)^2 + lh^2]} + \frac{e^{x_2t}(h^2 + x_2U_T)}{x_2[(h^2 + x_2U_T)^2 + lh^2]} \right] \]

Shearing Stress (Skin Friction): The Shear stress at the boundaries \( b = 0 \) and \( b = 1 \)
are given by

\[
D_0 = \frac{u_0 \mu X}{U \sinh(X)} - \frac{2\pi^2 \mu h^2 u_0}{U \text{Re}} \sum_{r=0}^{\infty} (-1)^r r^2 \\
\times \left[ \frac{e^{x^2t}(h^2 + x_1 U^2)^2}{x_1[(h^2 + x_1 U^2)^2 + lh^4]} + \frac{e^{x^2t}(h^2 + x_2 U^2)^2}{x_2[(h^2 + x_2 U^2)^2 + lh^4]} \right]
\]

\[
D_1 = \frac{u_0 \mu X}{U \cosh(X)} - \frac{2\pi^2 \mu h^2 u_0}{U \text{Re}} \sum_{r=0}^{\infty} r^2 \\
\times \left[ \frac{e^{x^2t}(h^2 + x_1 U^2)^2}{x_1[(h^2 + x_1 U^2)^2 + lh^4]} + \frac{e^{x^2t}(h^2 + x_2 U^2)^2}{x_2[(h^2 + x_2 U^2)^2 + lh^4]} \right]
\]

**CASE-2:** In this case, solve the equations (12) and (15) when subjected to the following initial and boundary conditions i.e., both lower and upper plates are moving with uniform velocity as:

Initial condition; at \( t = 0 \); \( u_s = 0, v_s = 0 \)

Boundary condition; for \( t > 0 \); \( u_s = u_0, \) at \( b = 0 \) and \( u_s = u_1, \) at \( b = h \)

where \( u_0 \) and \( u_1 \) are constants.

By applying the same procedure as in case-1, one can obtain \( u_s \) and \( v_s \) as follows;

\[
u_s = \frac{h}{U} \left[ \frac{u_1 \sinh(Xb) - u_0 \sinh(X(b-1))}{\sinh(X)} \right] - \frac{2\pi h^2}{U \text{Re}} \sum_{r=0}^{\infty} r[u_1(-1)^r - u_0] \sin(r\pi b) \\
\times \left[ \frac{e^{x^2t}(h^2 + x_1 U^2)^2}{x_1[(h^2 + x_1 U^2)^2 + lh^4]} + \frac{e^{x^2t}(h^2 + x_2 U^2)^2}{x_2[(h^2 + x_2 U^2)^2 + lh^4]} \right]
\]

\[
v_s = \frac{h}{U} \left[ \frac{u_1 \sinh(Xb) - u_0 \sinh(X(b-1))}{\sinh(X)} \right] - \frac{2h^4\pi^2}{U \text{Re}} \sum_{r=0}^{\infty} r[u_1(-1)^r - u_0] \sin(r\pi b) \\
\times \left[ \frac{e^{x^2t}(h^2 + x_1 U^2)}{x_1[(h^2 + x_1 U^2)^2 + lh^4]} + \frac{e^{x^2t}(h^2 + x_2 U^2)}{x_2[(h^2 + x_2 U^2)^2 + lh^4]} \right]
\]

**Shearing Stress (Skin Friction):** The Shear stress at the boundaries \( b = 0 \) and \( b = 1 \) are given by

\[
D_0 = \frac{\mu X}{U} \left[ \frac{u_1 - u_0 \cosh(X)}{\sinh(X)} \right] - \frac{2\pi^2 h^2 \mu}{U \text{Re}} \sum_{r=0}^{\infty} r^2[u_1(-1)^r - u_0] \\
\times \left[ \frac{e^{x^2t}(h^2 + x_1 U^2)^2}{x_1[(h^2 + x_1 U^2)^2 + lh^4]} + \frac{e^{x^2t}(h^2 + x_2 U^2)^2}{x_2[(h^2 + x_2 U^2)^2 + lh^4]} \right]
\]

\[
D_1 = \frac{\mu X}{U} \left[ \frac{u_1 \cosh(X) - u_0}{\sinh(X)} \right] - \frac{2\pi^2 h^2 \mu}{U \text{Re}} \sum_{r=0}^{\infty} r^2[u_1 - u_0(-1)^r] \\
\times \left[ \frac{e^{x^2t}(h^2 + x_1 U^2)^2}{x_1[(h^2 + x_1 U^2)^2 + lh^4]} + \frac{e^{x^2t}(h^2 + x_2 U^2)^2}{x_2[(h^2 + x_2 U^2)^2 + lh^4]} \right]
\]
CASE-3: For this case, consider an initial and boundary conditions as;

Initial condition; at \( t = 0 \); \( u_s = 0, v_s = 0 \)
Boundary condition; for \( t > 0 \); \( u_s = u_0 + u_1 t \), at \( b = 0 \) and \( u_s = u_2 + u_3 t \), at \( b = h \).

where \( u_0, u_1, u_2 \) and \( u_3 \) are constants. Now we obtain \( u_s \) and \( v_s \) as follows;

\[
\begin{align*}
\frac{u_s}{U} &= \frac{h}{U} \left[ \frac{u_2 \sinh(Xb) - u_0 \sinh(X(b - 1))}{\sinh(X)} \right] - \frac{2\pi h^2}{U Re} \sum_{r=0}^{\infty} r[u_2(-1)^r - u_0] \sin(r\pi b) \\
&\times \left[ \frac{e^{x_1t}(h^2 + x_1 U \tau)^2}{x_1[(h^2 + x_1 U \tau)^2 + lh^4]} + \frac{e^{x_2t}(h^2 + x_2 U \tau)^2}{x_2[(h^2 + x_2 U \tau)^2 + lh^4]} \right] \\
&+ \left[ \frac{t h^3}{U^2} - \frac{h^2 Re(1 + l)}{2XU^2} \coth(X) \right] \\
&\times \left[ \frac{u_3 \sinh(Xb) - u_1 \sinh(X(b - 1))}{\sinh(X)} \right] \\
&+ \left[ \frac{h^2 Re(1 + l)}{2XU^2} \right] \left[ \frac{u_3 b \cosh(Xb) - u_1 (b - 1) \cosh(X(b - 1))}{\sinh(X)} \right] \\
&- \frac{2h^4\pi}{U^2 Re} \sum_{r=0}^{\infty} r[u_2(-1)^r - u_0] \sin(r\pi b) \\
&\times \left[ \frac{e^{x_1t}(h^2 + x_1 U \tau)}{x_1[(h^2 + x_1 U \tau)^2 + lh^4]} + \frac{e^{x_2t}(h^2 + x_2 U \tau)}{x_2[(h^2 + x_2 U \tau)^2 + lh^4]} \right] \\
&- \frac{2h^4\pi}{U^2 Re} \sum_{r=0}^{\infty} r[u_2(-1)^r - u_0] \sin(r\pi b) \\
&\times \left[ \frac{e^{x_1t}(h^2 + x_1 U \tau)}{x_1[(h^2 + x_1 U \tau)^2 + lh^4]} + \frac{e^{x_2t}(h^2 + x_2 U \tau)}{x_2[(h^2 + x_2 U \tau)^2 + lh^4]} \right].
\end{align*}
\]

\[
\begin{align*}
\frac{v_s}{U} &= \frac{h}{U} \left[ \frac{u_2 \sinh(Xb) - u_0 \sinh(X(b - 1))}{\sinh(X)} \right] \\
&- \frac{2h^4\pi}{U^2 Re} \sum_{r=0}^{\infty} r[u_2(-1)^r - u_0] \sin(r\pi b) \\
&\times \left[ \frac{e^{x_1t}(h^2 + x_1 U \tau)}{x_1[(h^2 + x_1 U \tau)^2 + lh^4]} + \frac{e^{x_2t}(h^2 + x_2 U \tau)}{x_2[(h^2 + x_2 U \tau)^2 + lh^4]} \right] \\
&+ \left[ \frac{h^2 Re(1 + l)}{2XU^2} \right] \left[ \frac{u_3 b \cosh(Xb) - u_1 (b - 1) \cosh(X(b - 1))}{\sinh(X)} \right] \\
&- \frac{2h^4\pi}{Re} \sum_{r=0}^{\infty} r[u_3(-1)^r - u_1] \sin(r\pi b) \\
&\times \left[ \frac{e^{x_1t}(h^2 + x_1 U \tau)}{x_1[(h^2 + x_1 U \tau)^2 + lh^4]} + \frac{e^{x_2t}(h^2 + x_2 U \tau)}{x_2[(h^2 + x_2 U \tau)^2 + lh^4]} \right].
\end{align*}
\]

Shearing Stress (Skin Friction): The Shear stress at the boundaries \( b = 0 \) and \( b = 1 \) are given by
particle decreases and ultimately as $\tau$ fine i.e., mass of the dust particles is negligibly small then the relaxation time of dust much steeper than that of dust particles. Further one can observe that if the dust is very respectively, which are parabolic in nature. It is observed that the path of fluid particles be the same. Also we see that the fluid particles will reach the steady state earlier than

$$D_0 = \frac{\mu h X}{U} \left[ \frac{u_2 - u_0 \cosh(X)}{\sinh(X)} \right] - \frac{2\pi^2 h^2}{UR\Re} \sum_{r=0}^{\infty} r^2 [u_2(-1)^r - u_0]$$

$$\times \left[ \frac{e^{x_1[(h^2 + x_1 U\tau)^2 + lh^4]}}{x_1[(h^2 + x_1 U\tau)^2 + lh^4]} + \frac{e^{x_2[(h^2 + x_2 U\tau)^2 + lh^4]}}{x_2[(h^2 + x_2 U\tau)^2 + lh^4]} \right]$$

$$+ \mu \left[ \frac{th^4 X}{U^2} - \frac{h^2 \Re(1 + l)}{2U^2} \coth(X) \right] \left[ \frac{u_3 - u_1 \cosh(X)}{\sinh(X)} \right]$$

$$+ \frac{\mu h^2 \Re(1 + l)u_1}{2U^2} - \frac{2h^4\pi^2}{UR\Re} \sum_{r=0}^{\infty} r^2 [u_3(-1)^r - u_1]$$

$$\times \left[ \frac{e^{x_1[(h^2 + x_1 U\tau)^2 + lh^4]}}{x_1[(h^2 + x_1 U\tau)^2 + lh^4]} + \frac{e^{x_2[(h^2 + x_2 U\tau)^2 + lh^4]}}{x_2[(h^2 + x_2 U\tau)^2 + lh^4]} \right]$$

$$+ \frac{\mu h^2 \Re(1 + l)u_3}{2U^2} - \frac{2h^4\pi^2}{UR\Re} \sum_{r=0}^{\infty} r^2 [u_3 - u_1(-1)^r]$$

$$\times \left[ \frac{e^{x_1[(h^2 + x_1 U\tau)^2 + lh^4]}}{x_1[(h^2 + x_1 U\tau)^2 + lh^4]} + \frac{e^{x_2[(h^2 + x_2 U\tau)^2 + lh^4]}}{x_2[(h^2 + x_2 U\tau)^2 + lh^4]} \right]$$

where

$$x_1 = -\frac{1}{2\Re U\tau} \left( (h^2 C_r + r^2\pi^2) hU\tau + \Re h^2(1 + l) \right)$$

$$+ \frac{1}{2\Re U\tau} \sqrt{((h^2 C_r + r^2\pi^2) hU\tau + \Re h^2(1 + l))^2 - 4\Re U\tau (h^2 C_r + r^2\pi^2) h^3}$$

$$x_2 = -\frac{1}{2\Re U\tau} \left( (h^2 C_r + r^2\pi^2) hU\tau + \Re h^2(1 + l) \right)$$

$$- \frac{1}{2\Re U\tau} \sqrt{((h^2 C_r + r^2\pi^2) hU\tau + \Re h^2(1 + l))^2 - 4\Re U\tau (h^2 C_r + r^2\pi^2) h^3}$$

$$X = \sqrt{h^2 C_r}$$

**Conclusion**

The figures 3, 4 and 5 represents the velocity profiles for the fluid and dust particles respectively, which are parabolic in nature. It is observed that the path of fluid particles much steeper than that of dust particles. Further one can observe that if the dust is very fine i.e., mass of the dust particles is negligibly small then the relaxation time of dust particle decreases and ultimately as $\tau \to 0$ the velocities of fluid and dust particles will be the same. Also we see that the fluid particles will reach the steady state earlier than
the dust particles. The Reynolds number ($Re$) which means the inertial motion over the viscous resistance. One can observed that the impressive effect of Reynolds number on the velocity field. It is seen that the Reynolds number is favorable to the velocity fields i.e., for a constant value $t$, the velocity profiles for both fluid and dust particles increases as Reynolds number increases.

The graphs are drawn for the following values $h = 1, U = 0.8, r = 2, \tau = 0.5, C_r = 1, u_0 = 0.5, u_1 = 1, u_2 = 1.5, u_3 = 2, l = 1, t = 0.2$.

Fig. 3 Variation of fluid and dust phase velocity with $b$

Fig. 4 Variation of fluid and dust phase velocity with $b$
Fig. 5 Variation of fluid and dust phase velocity with $b$

References


Exact Solution of The Non-Central Modified Kratzer Potential Plus a Ring-Shaped Like Potential By The Factorization Method

J. Sadeghi\textsuperscript{a,b}\textsuperscript{*} and B. Pourhassan \textsuperscript{a}\textsuperscript{†}

\textsuperscript{a} Sciences Faculty, Department of Physics, Mazandaran University, P.O. Box 47415-416, Babolsar, Iran
\textsuperscript{b} Institute for Studies in Theoretical Physics and Mathematics (IPM), P.O. Box 19395-5531, Tehran, Iran

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Abstract: In this paper, we study the Schrödinger equation with a non-central modified Kratzer potential plus a ring-shaped like potential, which is not spherically symmetric. Thus, the standard methods for separation of variables do not quite apply. However, we are able to separate variables using a simple extension of the standard method, which leads to solutions in the associated Laguerre function for the radial part and Jacobi polynomials for the polar angle part. We also introduce an interesting pair of first-order ladder operators, which allow us to generate the energy eigenvalues for all states of the system. The obtained results show that the lack of spherical symmetry removes the degeneracy of second quantum number $m$ which completely expected.

Keywords: Modified Kratzer Potential; Schrödinger Equation; Factorization Method; Raising and Lowering Operators


1. Introduction

One of the important work in theoretical physics is to obtain exact solution of the Schrödinger equation for special potentials. It is well known that exact solution of Schrödinger equation

\textsuperscript{*} pouriya@ipm.ir
\textsuperscript{†} b.pourhassan@umz.ac.ir
are only possible for certain cases. The exact solution of Schrödinger equation for a class of non-central potential already studied in quantum chemistry [1, 2, 3]. In most of these studies, the eigenvalues and eigenfunctions are obtained by separation of variables. The path integral for particles moving in non-central potentials is evaluated to find the energy spectrum of the system [4]. In recent years, many work yields to obtain the exact solution of the ring-shaped potential [5]. This potential usually add to certain potential, for example Kratzer potential. The Kratzer potential [6, 7] have played an important role in the history of the molecular structure and interactions [8]. This potential offered one of the most important exactly models of atomic and molecular physics and quantum chemistry. It may be apply to energy spectrum for the CO diatomic molecule with different quantum numbers. Also the analytical solution of the radial Schrödinger equation is of high important in non-relativistic quantum mechanics since the wave function contains all the necessary information to describe a quantum system fully. As we know the quasi-exactly solution for the radial Schrödinger equation within a given potential is given by Ref [9]. Recently a new potential which is called the modified Kratzer’s type of molecular potential proposed. Very recently Chen and Dong [10] found a new ring-shaped potential and now Chen and Dai [11] try to solve the Schrödinger equation for the modified Kratzer potential plus this new ring-shaped potential. In this paper we consider non-central modified Kratzer potential plus a ring-shaped like potential which related to the angle dependent part of Schrödinger equation. In the spherical coordinates, this potential is defined as

$$V(r, \theta) = D(r - a)^2 + \frac{\beta' r^2}{r^2 \sin^2 \theta} + \frac{\gamma r^2}{r^2 \sin^2 \theta},$$

(1)

where $\beta'$ and $\gamma$ are strictly positive constants, $D$ is the dissociation energy and $a$ is the equilibrium internuclear separation. It is clear that the limiting case of $\beta' = 0$ and $\gamma = 0$ reduce to the modified Kratzer potential [12]. In other word the first term of this potential is the modified Kratzer potential and other terms are the angle dependent parts. There are different methods used to obtain the exact solutions of the Schrödinger equation for the ring-shaped like potentials. The most important of this methods, for example, are the standard methods [2], the path integral approach [4], the supersymmetric quantum mechanics and shape invariance method [13] and the Nikiforov-Uvarov (NU) method [15]. The NU method is based on solving the second-order linear differential equation by reducing to a generalized equation of hypergeometric type. But we use another way. In this paper the factorization method is used to solve the Schrödinger equation for potential introduced in equation (1). This work is organized as follows: in section 2 we introduce the factorization method. Radial part of Schrödinger equation considered in section 3. in section 4 we study the polar angle part of Schrödinger equation. Finally in section 5 we have conclusion for obtained result.
2. Factorization Method

In this section we show our way for solve Schrödinger equation. In the spherical coordinates, the Schrödinger equation is,

$$-\frac{\hbar^2}{2\mu} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] \psi + V(r, \theta) \psi = E \psi, \quad (2)$$

where $V(r, \theta)$ is non-central modified Kratzer potential plus a ring-shaped like potential as equation (1). In order to find exact solution of equation (2) we give spherical total wave function as

$$\psi(r, \theta, \varphi) = R(r) \Theta(\theta) \Phi(\varphi). \quad (3)$$

Then put wave function (3) into Schrödinger equation (2). The wave equation for the non-central modified Kratzer potential plus a ring-shaped like potential separated into independent variable and the following equations are obtained,

$$\frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} + \frac{2\mu}{\hbar^2} \left[ E - \frac{D(r - a)^2}{r} \right] R - \frac{\lambda}{r^2} R = 0, \quad (4)$$

$$\frac{d^2 \Theta(\theta)}{d\theta^2} + \cot \theta \frac{d\Theta(\theta)}{d\theta} + \left[ \lambda - \frac{m^2}{\sin^2 \theta} - \frac{2m}{\hbar^2} \left( \frac{(\beta' + \gamma \cos \theta)}{\sin^2 \theta} \right) \right] \Theta(\theta) = 0, \quad (5)$$

and

$$\frac{d^2 \Phi(\varphi)}{d\varphi^2} + m^2 \Phi(\varphi) = 0. \quad (6)$$

Now we can solve above equations separately. equations (4) and (5) are more complicated but solution of equation (6) is well known. therefore we focus on the equations (4) and (5). In the next sections we discuss about the radial part and polar angle part of Schrödinger equation corresponding to the modified Kratzer potential plus a ring-shaped like potential. In that case we choose suitable variable for obtain exact solutions. It lead to the associated Laguerre and Jacobi equations. For obtain exact solutions of the Schrödinger equation the factorization information from associated function were employed to solve the corresponding equations. It lead us to have some rising and lowering operators which are first order equations. These operators help us to obtain all quantum states and energy spectrum for different quantum number $n$ and $m$.

3. The Solutions of the Radial Part of the Schrödinger Equation

In equation (4) we obtained the radial part of Schrödinger equation for non-central modified Kratzer potential plus a ring-shaped like potential which can be rewritten as,

$$r \frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} + \frac{2\mu}{\hbar^2 r} \left[ (E - D)r^2 + 2Da - (Da^2 + \frac{\lambda \hbar^2}{2\mu}) \right] R = 0. \quad (7)$$
This equation can be solved by many methods but in our case we choose suitable variable as follows

$$R(r) = U(r)L(r). \quad (8)$$

By using variable (8) in equation (7) we have

$$rL''(r) + \left[ 2r\frac{U'}{U} + 2 \right] L'(r) + \left[ r\frac{U''}{U} + 2\frac{U'}{U} + \frac{2\mu}{\hbar^2}(E - D)r + \frac{4D\mu a}{\hbar^2} - \frac{2\mu}{\hbar^2}(Da^2 + \frac{\lambda\hbar^2}{\mu}) \frac{1}{r} \right] R = 0. \quad (9)$$

In order to obtain the parameters \((D\) and \(\lambda)\), eigenfunction and eigenvalue for the non-central modified Kratzer potential plus a ring-shaped like potential we compare equation (9) with the following associated Laguerre differential equation [17, 18, 19],

$$rL''(\alpha,\beta)_{n,m} + \left( 1 + \alpha - \beta r \right) L'(\alpha,\beta)_{n,m}(r) + \left[ (n - \frac{m}{2})\beta - \frac{m}{2}(\alpha + \frac{1}{2}) \right] L^{(\alpha,\beta)}_{n,m}(r) = 0, \quad (10)$$

so we obtain the \(U(r)\), \(D\) and \(\lambda\),

$$U(r) = e^{-\beta r} r^{\frac{\alpha - 1}{2}}, \quad (11)$$

$$D = \frac{\hbar^2}{4\mu a} \left[ (n - \frac{m}{2})\beta + \frac{(\alpha + \beta)}{2} \right] \quad (12)$$

and

$$\lambda = \frac{1}{2} \left[ \frac{m}{2}(\alpha + \frac{m}{2}) + \frac{\alpha - 1}{2} + \frac{(\alpha - 1)^2}{4} - \frac{1}{2a} (n - \frac{m}{2})\beta - \frac{1}{4a}(\alpha + \beta) \right]. \quad (13)$$

From equations (8) and (11) one can obtain the corresponding eigenfunction,

$$R(r) = e^{-\beta r} r^{\frac{\alpha - 1}{2}} L^{(\alpha,\beta)}_{n,m}(r). \quad (14)$$

In that case the exact energy eigenvalues of the radial part of the Schrödinger equation with the non-central modified Kratzer potential plus a ring-shaped like potential are derived as,

$$E_{n,m} = \frac{\hbar^2}{4\mu a} \left[ (n - \frac{m}{2})\beta + \frac{\alpha + \beta}{2} - \frac{\beta^2}{2} \right]. \quad (15)$$

In here we note that the solution associated Laguerre in the Rodrigues representation are,

$$L^{(\alpha,\beta)}_{n,m}(x) = \frac{a_{n,m}(\alpha,\beta)}{r^{\alpha + \frac{m}{2}} e^{-\beta r}} \frac{d}{dx} \left[ r^{\alpha + \frac{m}{2}} e^{-\beta r} \right]^{n-m}, \quad (16)$$

where \(a_{n,m}(\alpha,\beta)\) is the normalization coefficient, and also obtained by,

$$a_{n,m}(\alpha,\beta) = (-1)^m \sqrt{\frac{\beta^{\alpha + m + 1}}{\Gamma(n - m + 1)\Gamma(n + \alpha + 1)}}. \quad (17)$$
In here we also discuss the raising and lowering operators which is corresponding to the redial part of modified Kratzer potential plus a ring - shaped like potential. So, we can factorize the associated Laguerre differential equation with respect to the parameters \( n \) and \( m \) as follows,

\[
A_{n,m}^+(r)A_{n,m}^-(r)L_{n,m}^{(\alpha,\beta)}(r) = (n - m)(n + \alpha)L_{n,m}^{(\alpha,\beta)}(r)
\]

\[
A_{n,m}^-(r)A_{n,m}^+(r)L_{n-1,m}^{(\alpha,\beta)}(r) = (n - m)(n + \alpha)L_{n-1,m}^{(\alpha,\beta)}(r),
\]

where the differential operators as functions of parameters \( n \) and \( m \) are respectively :

\[
A_{n,m}^+(r) = r \frac{d}{dr} - \beta r + \frac{1}{2}(2n + 2\alpha - m)
\]

\[
A_{n,m}^-(r) = -r \frac{d}{dr} + \frac{1}{2}(2n - m).
\]

Note that the shape invariance equation (18) can also be written as the raising and lowering relation,

\[
A_{n,m}^+(r)L_{n-1,m}^{(\alpha,\beta)}(r) = \sqrt{(n - m)(n + \alpha)}L_{n-1,m}^{(\alpha,\beta)}(r)
\]

\[
A_{n,m}^-(r)L_{n,m}^{(\alpha,\beta)}(r) = \sqrt{(n - m)(n + \alpha)}L_{n,m}^{(\alpha,\beta)}(r).
\]

So, we obtain the raising and lowering operators for the redial part of the modified Kratzer potential plus a ring - shaped like potential. These operators help us to have a bound states for that system. In order to show the effect of the modified Kratzer potential plus ring - shaped like potential on the energy spectrum, the obtained energy spectrum of radial part of Schrödinger equation is illustrated in Fig. 1, as function of quantum number \( n \).

4. The Solutions of the Polar Angle Part of Schrödinger Equation

Now we try to obtain the eigenvalues and eigenfunctions of the polar angle of Schrödinger equation similar to the solution of the radial part. By introducing a new variable \( x = \cos \theta \), we can write equation (5) as follows,

\[
\frac{d^2 \Theta(x)}{dx^2} - \frac{2x}{1-x^2} \frac{d\Theta(x)}{dx} + \left( \frac{\lambda(1-x^2) - m'^2 - \frac{2m'}{\hbar^2}(\beta' + \gamma x)}{(1-x^2)^2} \right) \Theta(x) = 0.
\]

As equation (8) in previous section we choose suitable variable as,

\[
\Theta(x) = U(x)P(x).
\]

Now we put variable (22) to equation (21) and obtain,

\[
(1 - x^2)P''(x) + \left[ \frac{2U''(1-x^2)}{U} - 2x \right] P'(x)
\]

\[
+ \left[ (1-x^2) \frac{U''}{U} - 2x \frac{U'}{U} + \lambda - \frac{m'^2 - \frac{2m'}{\hbar^2}(\beta' + \gamma x)}{1-x^2} \right] P(x) = 0.
\]
Here, in order to obtain the exact solution of the Schrödinger equation for the non-central modified Kratzer potential plus ring-shaped like potential we compare equation (23) with the following associated Jacobi differential equation [17],

\[(1 - x^2)P''_{n,m}(x) - [\alpha - \beta + (\alpha + \beta + 2)x] P'_{n,m}
+ \left[ n(\alpha + \beta + n + 1) - \frac{m(\alpha + \beta + m + (\alpha - \beta))}{1 - x^2} \right] P_{n,m}(x) = 0, \quad (24)\]

where

\[m = \frac{2m'}{\hbar^2} - \left(\frac{\alpha + \beta}{2}\right). \quad (25)\]

So, the energy spectrum is,

\[E_{n,m} = \frac{\hbar^2}{4\mu a} \left[ (n - \frac{m'}{\hbar^2} + \frac{(\alpha + 1)\beta + \alpha + \beta}{2} - \frac{\beta^2}{2} \right], \quad (26)\]

where

\[\alpha + \beta = \beta', \quad \alpha - \beta = \gamma, \quad (27)\]

The corresponding wave function are found as,

\[\Theta_{n,m}(x) = \left(\frac{\beta - \alpha}{2}\right) \int \left(\frac{dx}{1 - x^2}\right) - \left(\frac{\alpha + \beta}{2}\right) \int \frac{x}{(1 - x^2)} dx P_{n,m}^{(\alpha,\beta)}(x). \quad (28)\]

The associated Jacobi functions \(P_{n,m}^{(\alpha,\beta)}(x)\) as the solution of the differential equation have the following Rodrigues representation,

\[P_{n,m}^{(\alpha,\beta)}(x) = \frac{a_{n,m}(\alpha, \beta)}{(1 - x)^{\alpha + \frac{n}{2}}(1 + x)^{\beta + \frac{n}{2}} \left(\frac{d}{dx}\right)^{n-m} \left((1 - x)^{\alpha+n}(1 + x)^{\beta+n}\right). \quad (29)\]

Now we are going to discuss the raising and lowering operators corresponding to the polar angle part of Schrödinger equation for non-central modified Kratzer potential plus ring-shaped like potential. As mentioned in Refs [14, 17, 18], we can write the associated Jacobi differential equation (23) as the following,

\[A_{n,m}^+ A_{n,m}^- P_{n,m}^{(\beta',\gamma)}(x) = B_{n,m} P_{n-1,m}^{(\beta',\gamma)}(x) \]
\[A_{n,m}^- A_{n,m}^+ P_{n,m}^{(\beta',\gamma)}(x) = B_{n,m} P_{n,m}^{(\beta',\gamma)}(x) \quad (30)\]

where

\[B_{n,m} = \frac{(n - m)(\beta' + \gamma + 2n)(\beta' + m + n)}{(\beta' + 2n)^2} \]
\[A_{n,m}^+(x) = (1 - x^2) \frac{d}{dx} - (\beta' + n)x - \gamma \frac{\beta' + n + m}{\beta' + 2n} \]
\[A_{n,m}^-(x) = -(1 - x^2) \frac{d}{dx} - nx + \gamma \frac{n - m}{\beta' + 2n} \quad (31)\]
In the case of shape invariance with respect to \( m \) we have,

\[
A_m^+(x)A_m^-(x)P_{n,m}^{(\beta',\gamma)}(x) = C_{n,m}P_{n,m}^{(\beta',\gamma)}(x)
\]
\[
A_m^-(x)A_m^+(x)P_{n,m-1}^{(\beta',\gamma)}(x) = C_{n,m}P_{n,m-1}^{(\beta',\gamma)}(x),
\]

(32)

where,

\[
C(n,m) = (n - m + 1)(\beta' + n + m)
\]

and lader operators are,

\[
A_m^+ = \sqrt{1-x^2} \frac{d}{dx} + \frac{(m-1)}{\sqrt{1-x^2}} x
\]
\[
A_m^- = -\sqrt{1-x^2} \frac{d}{dx} + \frac{\gamma + (\beta' + m)}{\sqrt{1-x^2}} x
\]

(34)

These operators help us to discuss about the bound states for the corresponding potential. In both part we can see that the energy spectrum obtained in equations (15) and (26) are independent of parameters \( \beta' \) and \( \gamma \). Also we show the effect of the modified Kratzer potential plus a ring - shaped like potential on the energy spectrum obtained from the angular part of Schrödinger equation in Fig. 2, as function of quantum number \( m \).

**Conclusion**

In this paper we find exact solution of the Schrödinger equation for non - central modified Kratzer potential plus a ring - shaped like potential by the factorization method. We have shown that the energy spectrum and the corresponding eigenfunctions of the Schrödinger equation with non - central potentials can be easily obtained by using the factorization method. We saw that the energy spectrum of radial part and polar angle part of Schrödinger equation are independent of parameters \( \beta' \) and \( \gamma \). As we see in figures (1) and (2) the degeneracy of second quantum number is completely removed and also the variation of energy spectrum for the special value of \( n \) the value of \( m \leq n - 1 \) decreases but this process for the figure (2) completely increases. These results may be in future apply to the CO like diatomic molecule for different quantum numbers. We know that it is possible to study the modified Kratzer potential plus another ring - shaped like potential [19] and to solve exactly the Schrödinger equation for this system.

**References**


Fig. 1 The energy spectrum of solving the radial part of Schrödiger equation. $\beta$ and $\alpha$ are arbitrary numbers.

Fig. 2 The energy spectrum of solving the angular part of Schrödiger equation. $\beta$ and $\alpha$ are arbitrary numbers.
Discrete Self-Similarity between RR Lyrae Stars and Singly-Excited Helium Atoms

Robert L. Oldershaw*

Earth Sciences Building
Amherst College
Amherst, MA 01002, USA

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Abstract: Classical variable stars called RR Lyrae stars have pulsating outer envelopes constituted of excited atoms. Here we demonstrate that the qualitative and quantitative properties of RR Lyrae variables and one subclass of their atomic scale constituents: singly-excited helium atoms undergoing transitions between Rydberg states, share a remarkable degree of self-similarity. In terms of masses, radii, oscillation periods, morphologies and kinematics the stellar and atomic analogues obey a simple set of discrete self-similar scaling equations. The concept of stellar/atomic self-similarity may prove useful in the search for a deeper understanding of both stellar and atomic systems.

Keywords: Self-Similarity; Fractals; RR Lyrae Stars; Variable Stars; Rydberg Atoms; Cosmology PACS (2006): 05.45.Df; 97.30.Kn; 05.30.Pr; 97.30.b; 97.90.+j; 98.80.k

1. Introduction

Fractal structures are common to virtually all realms of nature [1]. Galaxy distributions, fluid turbulence, topographic shapes, neuronal interconnections and the clustering of stars or plasma particles represent a small sampling of the broad domains wherein fractal phenomena are ubiquitous. A key property of fractal systems is their self-similarity, in which similar morphological or temporal patterns recur on different size or time scales throughout the hierarchical structure of the system. Here we demonstrate a surprisingly robust self-similarity between the classical pulsating stars called RR Lyrae variables and one subclass of their atomic scale constituents: helium atoms undergoing transitions between adjacent energy levels.

* roldershaw@amherst.edu
2. Preliminaries

At the outset of this investigation we define a heuristic set of discrete self-similar scaling equations that allow us to correlate observed stellar mass, radius and oscillation period values with experimental measurements for helium atoms in Rydberg states. Initially we will treat these scaling equations as axioms, deferring a discussion of their origin until the end of our investigation, except to say that they were derived decades ago without reference to RR Lyrae stars. The scaling equations are:

\[ R \approx \Lambda r \]  \hspace{1cm} (1)

\[ P \approx \Lambda p \]  \hspace{1cm} (2)

\[ M \approx \Lambda^D m. \]  \hspace{1cm} (3)

R, P and M are radii, periods and masses of RR Lyrae stars; r, p and m are the counterpart parameters of helium atoms; and \( \Lambda \) and \( D \) are dimensionless scaling constants equal to 5.2 x 10\(^{17} \) and 3.174, respectively.

RR Lyrae variables [2,3] are blue giant stars with classifications of A or F. They are thought to pulsate primarily in the fundamental radial mode \((l = 0)\), and \((R_{\text{max}} - R_{\text{min}})/R_{\text{min}} \approx 10\%\). It has been “definitely established” that the oscillation takes place in the outer envelope of the star, rather than in its core [2]. RR Lyrae stars tend to oscillate with a single period, but cases of double-mode pulsation with \(P_1/P_0 \approx 0.746\) are not uncommon. Although the periods of RR Lyrae variables range from \(\approx 0.2\) days to \(\approx 1.0\) days, the overwhelming majority have periods between 0.25 days and 0.75 days. Three subtypes have been identified: RRe variables with nearly sinusoidal light curves and periods of roughly 0.3 days, RRA variables with asymmetric light curves and periods of roughly 0.5 days and RRb variables (often combined with the RRA class) with intermediate light curve asymmetry and periods of roughly 0.7 days [3]. The most typical mass for RR Lyrae stars is \(<M> \approx 0.6\ M_\odot\) and their radii range from approximately 3.7 \(R_\odot\) to 7.2 \(R_\odot\) [4].

Given the approximate value of \(<M>\), we find that \(<M> \approx \Lambda^D m_{He},\) in agreement with Eq.3 to within a factor of 0.045. Given the stellar radius range of 3.7 \(R_\odot\) to 7.2 \(R_\odot\) and Eq. 1, we can estimate that the self-similar radius range for the \(^4\text{He}\) atom should be 4.95 \(x 10^{-7}\) cm to 9.64 \(x 10^{-7}\) cm. Then using the general radius versus principal quantum number (n) relation [5], \(r \approx 2n^2a_0,\) for Rydberg atoms with low angular momentum quantum numbers (l), where \(a_0\) is the Bohr radius of \(\approx 0.53 \times 10^{-8}\) cm, we can determine that the relevant range of n values is roughly 6.8 to 9.5, which rounds off to 7 \(\leq n \leq 10.\) Since RR Lyrae stars appear to be mostly radial mode oscillators, we assume that the relevant range of l values for \(^4\text{He}\) is \(0 \leq l \leq 1.\) We also make the tentative assumption that the most likely atomic scale transitions for \(^4\text{He}\) in 1sns or 1snp states with \(7 \leq n \leq 10\) are single-level transitions, i.e., \(\Delta n = 1.\)
3. Quantitative Test

With the above preliminaries completed, we are now ready to put the putative example of discrete cosmological self-similarity to a crucial test: RR Lyrae oscillation periods should match up uniquely with the specified $^4$He transition periods when the latter are scaled in accordance with Eq.2. Table 1 presents the $^4$He data [6] needed for this test, and the predicted RR Lyrae periods derived from that data.

<table>
<thead>
<tr>
<th>$n_1 \rightarrow n_2; , ^3S$</th>
<th>$\Delta E$ (atomic units)</th>
<th>Transition Period $1/\nu$ (sec)</th>
<th>Predicted RR Lyrae Oscillation Period (days)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$8p \rightarrow 7s; , ^3S$</td>
<td>0.00318</td>
<td>$4.7725 \times 10^{-14}$</td>
<td>0.2872</td>
</tr>
<tr>
<td>$8p \rightarrow 7s; , ^1S$</td>
<td>0.00284</td>
<td>$5.3559 \times 10^{-14}$</td>
<td>0.3223</td>
</tr>
<tr>
<td>$8s \rightarrow 7s; , ^3S$</td>
<td>0.00270</td>
<td>$5.6275 \times 10^{-14}$</td>
<td>0.3387</td>
</tr>
<tr>
<td>$8s \rightarrow 7s; , ^1S$</td>
<td>0.00253</td>
<td>$6.0056 \times 10^{-14}$</td>
<td>0.3614</td>
</tr>
<tr>
<td>$9p \rightarrow 8s; , ^3S$</td>
<td>0.00216</td>
<td>$7.0340 \times 10^{-14}$</td>
<td>0.4233</td>
</tr>
<tr>
<td>$9p \rightarrow 8s; , ^1S$</td>
<td>0.00194</td>
<td>$7.8446 \times 10^{-14}$</td>
<td>0.4722</td>
</tr>
<tr>
<td>$9s \rightarrow 8s; , ^3S$</td>
<td>0.00183</td>
<td>$8.3029 \times 10^{-14}$</td>
<td>0.4997</td>
</tr>
<tr>
<td>$9s \rightarrow 8s; , ^1S$</td>
<td>0.00172</td>
<td>$8.8339 \times 10^{-14}$</td>
<td>0.5317</td>
</tr>
<tr>
<td>$10p \rightarrow 9s; , ^3S$</td>
<td>0.00153</td>
<td>$9.9130 \times 10^{-14}$</td>
<td>0.5966</td>
</tr>
<tr>
<td>$10p \rightarrow 9s; , ^1S$</td>
<td>0.00138</td>
<td>$1.0991 \times 10^{-13}$</td>
<td>0.6615</td>
</tr>
<tr>
<td>$10s \rightarrow 9s; , ^3S$</td>
<td>0.00129</td>
<td>$1.1778 \times 10^{-13}$</td>
<td>0.7089</td>
</tr>
<tr>
<td>$10s \rightarrow 9s; , ^1S$</td>
<td>0.00123</td>
<td>$1.2353 \times 10^{-13}$</td>
<td>0.7435</td>
</tr>
</tbody>
</table>

Table 1 Transition Data for $^4$He (1sns and 1snp), Singlet and Triplet States, $7 \leq n \leq 10$, $\Delta n = 1$, and Predicted RR Lyrae Periods

If the concept of discrete self-similarity between RR Lyrae stars and $^4$He atoms has merit, then the 12 predicted oscillation periods listed in Table 1 should be identifiable in RR Lyrae period distributions. As an initial indication of the general agreement between predictions and observations, we note that the 12 periods can be subdivided naturally into 3 subgroups representing the $n = 8 \rightarrow 7$, $n = 9 \rightarrow 8$ and $n = 10 \rightarrow 9$ transitions. The average oscillation periods for these three subgroups is roughly 0.3 days, 0.5 days and 0.7 days, which corresponds rather well with the observed average periods for RRc, RRa and RRb stars, respectively. Also, note that there are potentially diagnostic “gaps” in the distribution of predicted periods at roughly 0.37 – 0.41 days, 0.43 – 0.46 days, 0.54 – 0.58 days and 0.61 – 0.65 days. To achieve a more rigorous quantitative test, we need
a sizeable sample of RR Lyrae variables that is reasonably homogeneous and analyzed with care. Fortunately an excellent test sample has recently become available.

One of the secondary benefits of microlensing searches for stellar-mass dark matter objects has been the discovery of substantial numbers of variable stars. The Optical Gravitational Lensing Experiment (OGLE) team has recently published a catalog [7] of RR Lyrae stars found within the Large Magellanic Cloud galaxy. The period data were subjected to narrow binning and multiple-binning analyses, yielding period distributions that surpass the accuracy of previous period histograms. Figure 1 shows the distribution of periods for 84 RR Lyrae variables found in the rich star cluster NGC 1835.

![Fig. 1 Period Distribution for RR Lyrae Stars in NGC 1835][7]

One can identify 10 structural features in this distribution: 6 peaks, 2 “shoulders”, 1 gap and 1 “valley”. Each of these structural features matches up, to within 1% or better, with one of the periods or gaps that were uniquely predicted solely on the basis of the physical properties of $^4$He ($1sns,p$) transitions with $7 \leq n \leq 10$ and $\Delta n = 1$. In Table 2 we summarize these results.

### 4. Caveats

It is important to acknowledge the physical factors that tend to shift atomic transition frequencies away from their unperturbed values, or add spurious frequencies to the test spectra. In the case of $^4$He ($1sns,p$) transitions with $7 \leq n \leq 10$ and $\Delta n = 1$, there is the expected doubling of lines due to a splitting that permits singlet or triplet systems. Sets of spurious periods will be introduced into the test spectra if systems with $l > 1$ (and therefore $-1 \leq m \leq +1$) “contaminate” the test sample. Likewise, if systems other than helium but in He-like configurations such as Li$^+$ ($1sns,p$) or H$^-$ ($1sns,p$) are present, then additional sets of spurious transition periods will be detected. If $^3$He and $^6$He isotopes
Observed Feature (days) | $^4$He Transition | Predicted Period (days) | Error (days)
---|---|---|---
0.28 [peak] | 1s8p – 1s7s ($^3$S) | 0.2872 | 0.007
0.32 [peak] | 1s8p – 1s7s ($^1$S) | 0.3223 | 0.002
0.36 [peak] | 1s8s – 1s7s ($^1$S) | 0.3614 | 0.001
0.50 [peak/shoulder] | 1s9s – 1s8s ($^3$S) | 0.4997 | 0.0003
0.54 [peak] | 1s9s – 1s8s ($^1$S) | 0.5317 | 0.008
0.59 [shoulder] | 1s10p – 1s9s ($^3$S) | 0.5966 | 0.007
0.66 [peak] | 1s10p – 1s9s ($^1$S) | 0.6615 | 0.002
0.71 [peak] | 1s10s – 1s9s ($^3$S) | 0.7089 | 0.001
0.44 [gap] | | 0.43 – 0.46 | 0
0.62 [valley] | | 0.61 – 0.65 | 0

Table 2: Observed and Predicted Periods (NGC 1835)

are present in the test sample, then still more sets of spurious transition periods will be observed. Lastly with regard to spurious periods, systems undergoing transitions with $\Delta n > 1$ can add further sets of transition periods to the spectra.

Turning now to shifts away from the unperturbed positions of transition frequencies or periods, there are at least four physical factors that can cause significant shifts. Due to the relatively large charge separations of Rydberg configurations, ambient electric fields can cause substantial energy level shifts. Likewise, the energy levels of Rydberg atoms are susceptible to significant shifting due to their sensitivity to ambient magnetic fields. Temperature and pressure, which are often quite high in astrophysical settings, can also broaden and shift peaks in the test spectra.

Given these four sources of spurious periods and the four physical causes of line shifting, which would tend to be present at some level in non-laboratory settings, we should maintain reasonable expectations for the discreteness of non-laboratory period distributions of $^4$He atoms and the self-similar period distributions of RR Lyrae stars. Moreover, given the stochastic nature of quantum phenomena, we would expect different samples of either type of systems to show significant variability in the numbers of specific transition periods present.

Conclusions

Starting from a knowledge of the physical properties of RR Lyrae variable stars and a set of tentative self-similar scaling equations, we have demonstrated that RR Lyrae stars and $^4$He atoms ($1sns.p; 7 \leq n \leq 10$) undergoing energy level transitions with $\Delta n = 1$ are quantitatively self-similar in terms of masses, radii and oscillation periods. The stellar and
atomic analogues share the qualitative properties of being spherical systems undergoing energetic pulsations of limited duration that take place in their outer envelopes. An interesting question for future research is whether this remarkable self-similarity can be extended to include other classes of variable stars such as ZZ Ceti, W Virginis, δ Scuti, Cepheid and Mira variables.

Finally, it should be mentioned that the self-similar scale transformation equations used in this paper were developed in 1985 for a discrete fractal paradigm called the Self-Similar Cosmological Paradigm (SSCP) [8-11]. The SSCP emphasizes nature’s intrinsic and well-stratified hierarchical organization, proposing that the hierarchy is divided into discrete cosmological Scales, of which we can currently observe the Atomic, Stellar and Galactic Scales. The SSCP also proposes that the discrete Scales are rigorously self-similar to one another, such that for each class of fundamental particle, composite system or physical phenomenon on any given Scale there are self-similar analogues on all other Scales. At present the number of Scales cannot be known, but for reasons of natural philosophy it is tentatively proposed that there are a denumerably infinite number of Scales, ordered in terms of their intrinsic ranges of space, time and mass scales. The spatial (R), temporal (T) and mass (M) parameters of discrete self-similar analogues on neighboring Scales Ψ and Ψ-1 are related by the following set of discrete self-similar Scale transformation equations.

\[ R_\Psi = \Lambda R_{\Psi-1}, \]
\[ T_\Psi = \Lambda T_{\Psi-1}, \]
\[ M_\Psi = \Lambda^D M_{\Psi-1}, \]

where Λ and D are empirically determined dimensionless scale factors equal to 5.2 x 10^{17} and 3.174, respectively. The value of \( \Lambda^D \) is 1.70 x 10^{56}. The symbol Ψ is used as an index to distinguish different Scales, such that

\[ \Psi = \{\ldots, -2, -1, 0, +1, +2, \ldots\}, \]

and the Stellar Scale is usually assigned Ψ = 0. Thus, Atomic Scale systems and phenomena are designated by Ψ = -1 and Galactic Scale systems and phenomena are assigned Ψ = +1. The fundamental self-similarity of the SSCP and the recursive character of the discrete scaling equations suggest that nature is an infinite discrete fractal, in terms of its morphology, kinematics and dynamics. Perhaps the most thorough and accessible resource for exploring the SSCP is the author’s website [12].

References


[12] A website devoted to the Self-Similar Cosmological Paradigm can be found at www.amherst.edu/~rloldershaw.
Brownian Dynamics of Nanoparticles Moving Near a Fluctuating Membrane

A. Bendouch, M. Benhamou*, and H. Kaidi

Laboratoire de Physique des Polymères et Phénomènes Critiques, Faculté des Sciences Ben M’sik, B.P. 7955, Casablanca, Morocco

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Abstract: This work deals with Brownian dynamics study of small nanoparticles moving near an attractive penetrable fluid membrane. As consequence, these particles are pushed towards the interface, under a change of a suitable physical parameter, such as temperature, pressure or membrane environment. For simplicity, we assume that the particle size is small enough in comparison to the roughness of the membrane. In addition, the particles are supposed to be of very low density (their mutual interactions can be ignored). Then, the only remaining interaction is a mean-force external potential computed exactly in some recent work. The latter that originates from the strong membrane undulations, is a function of the perpendicular distance $z$. Brownian dynamics are studied through the time particle density, which solves the Smoluchowski equation. This density is determined exactly around the fluid membrane, where the essential of phenomenon takes place. In particular, far from the interface, the beads diffuse as usual. But inside the thermal fluctuations region, the Brownian particles diffuse and effectuate small oscillations, with a frequency $\omega$ scaling as $\omega \sim \kappa^{3/8}$, where $\kappa$ accounts for the bending rigidity constant of the membrane. We emphasize that the present Brownian dynamics study reveals the existence of a characteristic time $\tau \sim \kappa^{-3/4}$, which can be interpreted as the time beyond which the particles reach their final equilibrium state. For early times ($t < \tau$), however, the particles are out equilibrium. After a long time ($t > \tau$), the beads reach their final equilibrium state, and occupy new holes and valleys. Finally, this work must be considered as a natural extension of a recent one that was concerned with the static study of the colloidal organization in contact with a fluctuating fluid membrane.

Keywords: Membranes; Nanoparticles; Thermal Fluctuations; Interactions; Brownian Dynamics; Time Particle Density

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* Corresponding author: m.benhamou@univh2m.ac.ma
1. Introduction

The cell membranes are of crucial importance to life, because they separate the cell from the surrounding environment and act as a selective barrier for the import and export of materials. A detailed description of the structural organization and basic functions of biomembranes can be found in Refs. [1−7]. The cell membranes essentially present as a phospholipid bilayer combined with a variety of proteins and cholesterol (mosaic fluid model). In particular, the latter ensures the bilayer fluidity. A phospholipid is an amphiphile molecule possessing a hydrophilic polar head attached to two hydrophobic (fatty acyl) chains. The phospholipids move freely on the membrane surface. On the other hand, the thickness of a bilayer membrane is of the order of 50 Angstroms. These two properties allow to consider it as a two-dimensional fluid membrane. The fluid membranes, self-assembled from surfactant solutions, may have a variety of shapes and topologies [8], which have been explained in terms of bending energy [9, 10].

Generally, the fluid membranes are not pure, that is they are in the presence of various entities, like proteins, small and macro-ions, or more complex structures [11]. For example, the membrane suspensions used in detergency and cosmetics are usually in contact with numerous additives (macromolecules and colloids), in order to improve their efficiency and to control their viscoelastic properties [12].

To model the organization of these external entities and their influence on the mechanical and topological properties of fluid membranes, the simple way consists in regarding these as small spherical colloids. We believe that this assumption has a good physical meaning, when one is concerned with those phenomena occurring at scales greater than the characteristic size of neighboring entities (diameter of particles, gyration radius of macromolecules, etc.).

Physics of nanoparticles in contact with a fluctuating fluid membrane is the subject of very recent theoretical works [13−15]. These were concerned with quantitative studies of the static organization of these nanoparticles that interact with impenetrable of penetrable interfaces. More precisely, the problem was the computation of the particle density profile and the effective pair-potential. The latter is mediated by membrane undulations. Beside their mutual interactions, the particles experience an external mean-force potential generated by the presence of the fluid membrane. Also, the attention has been paid to the study of the aggregation transition [15], driving colloids from a dispersed phase (gas) to a dense one (liquid).

In this work, the question is addressed to a Brownian dynamics study of nanoparticles of very low density, which are in contact with a penetrable interface. More precisely, we are interested in how the particles are pushed by the external potential towards this interface. This Brownian dynamics will be studied through the time evolution of the particle density, when some suitable parameter (temperature, pressure, environment...) is changed from an initial value to a final one.

The Brownian motion dominates various time-dependent phenomena ranging from suspensions [16−21] to polymer solutions [22]. This motion can be investigated us-
ing two approaches, the Smoluchowski equation and the Langevin equation. Although the two theoretical formulations are different, but they are physically equivalent. The Smoluchowski equation that is a generalization of the usual diffusion equation, has a clear relevance to thermodynamics of irreversible processes. On the other hand, the Langevin equation has no direct relationship to thermodynamics, but it provides a successful tool for the description of wider classes of stochastic processes.

To study the Brownian dynamics of a colloidal suspension near an interacting fluid membrane, we use an approach based on the Smoluchowski equation. The latter describes the time evolution of the particle density, and involves a known mean-force external potential experienced by particles [14]. As simplified assumptions, we suppose that the particles are point-like and of very low density. The first assumption remains valid as long as we are concerned with strong thermal fluctuations of the fluid membrane. While the second means that the mutual interactions between particles can be ignored. Therefore, the only remaining interaction is an external potential originating from the membrane undulations (see below). In the distance-range of interest (around the membrane), we determine the exact form for the time particle density. This depends on the perpendicular distance \( z \) from the projection plane, mean roughness of the membrane \( \xi_\perp \) (transverse fluctuations size), time \( t \), and some characteristic time-scale \( \tau \sim \xi_\perp^3 \sim \kappa^{-3/4} \) (\( \kappa \) being the bending rigidity constant). The latter can be interpreted as the necessary time during which the beads are trapped in holes and valleys. In addition to these variables, the time particle density depends on the coupling constant measuring the interaction strength between colloids and membrane.

The remaining of presentation proceeds as follows. Sec. II is devoted to a succinct recall of the induced interactions between beads, in particular, the mean-force potential we are interested in. Brownian dynamics study is the aim of Sec. III. We draw some concluding remarks in the last section.

2. Induced Interactions

Consider a dilute suspension of \( N \) colloids in contact with a fluid interface. For the sake of simplicity, we assume that the particles are point-like. This assumption makes sense only if the particle size is much smaller than the surface roughness \( \xi_\perp \). Typically, the particles under consideration have diameter of a few tens of nanometers, whereas the surface roughness lies in the micrometer range. In the Monge representation, a point on the surface can be described by a three-dimensional position-vector \( \mathbf{r} = (\rho, z) \in \mathbb{R}^3 \), where \( \rho = (x, y) \in \mathbb{R}^2 \) is the transverse vector and \( z = h(x, y) \) the perpendicular distance from the plane located at \( z = 0 \). Here, the height function \( h(x, y) \) may take either positive or negative values. The total Hamiltonian \( \mathcal{H} \) of the system is given by [14, 15]

\[
\mathcal{H} = \mathcal{H}_m + \mathcal{H}_{cc} + \mathcal{H}_{cm} .
\]
The first contribution $\mathcal{H}_m$ is the standard Canham-Helfrich free energy [9, 23]

$$\mathcal{H}_m[h] = \frac{1}{2} \int d^2 \rho \left[ \kappa (\Delta h)^2 + \sigma (\nabla h)^2 + \mu h^2 \right].$$  \hspace{1cm} (2)

Here, $\kappa$ is the bending rigidity constant, $\sigma$ the surface tension coefficient, and $\mu$ the confinement parameter. For convenience, we shall use the rescaling parameters: $\hat{\kappa} = \kappa/k_B T$, $\hat{\sigma} = \sigma/k_B T$, and $\hat{\mu} = \mu/k_B T$, with $T$ the absolute temperature and $k_B$ the Boltzmann’s constant.

We shall need the expression of the height-height correlation function

$$G(\rho - \rho') = \langle h(\rho) h(\rho') \rangle_0 - \langle h(\rho) \rangle_0 \langle h(\rho') \rangle_0 .$$  \hspace{1cm} (3)

The notation $\langle ... \rangle_0$ means the thermal average, which is performed with the Canham-Helfrich Hamiltonian in the absence of particles. We recall that the general expression of the propagator is [13, 14]

$$G(\rho) = \frac{1}{2\pi \sqrt{\sigma^2 - 4\hat{\mu} \hat{\kappa}}} \left[ K_0 (\rho\xi_-) - K_0 (\rho\xi_+) \right] ,$$  \hspace{1cm} (4)

with the two lengths

$$\xi_- = \left[ \hat{\sigma} - \sqrt{\sigma^2 - 4\hat{\mu} \hat{\kappa}} \right]^{1/2}, \quad \xi_+ = \left[ \hat{\sigma} + \sqrt{\sigma^2 - 4\hat{\mu} \hat{\kappa}} \right]^{1/2} .$$  \hspace{1cm} (5)

There, $K_\nu (z)$ is the second-order modified Bessel function [24]. From the obtained expression of the propagator $G(\rho)$, we extract the mean-squared displacement

$$\xi_+^2 = G(0) = \frac{1}{2\pi \sqrt{\sigma^2 - 4\hat{\mu} \hat{\kappa}}} \ln \left( \frac{\xi_+}{\xi_-} \right) .$$  \hspace{1cm} (6)

For tensionless membranes, the corresponding mean roughness $\xi_+$ becomes [13, 14]

$$\xi_+ = 2^{-3/2} (\hat{\kappa}, \hat{\mu})^{-1/4} .$$  \hspace{1cm} (7)

The other distance-scale to consider is the in-plane correlation length $\xi_\parallel$ measuring the correlations extent, that is $G(\rho) \sim e^{-\rho/\xi_\parallel}$, for $\rho >> \xi_\parallel$. It is given by [13, 14]: $\xi_\parallel = 2^{1/2} (\hat{\kappa}/\hat{\mu})^{1/4}$. These characteristic lengths then scale with the bending rigidity constant $\kappa$ as: $\xi_\perp \sim \kappa^{-1/4}$ and $\xi_\parallel \sim \kappa^{1/4}$. Therefore, the thermal fluctuations are strong only for those membranes of high bending rigidity constant (high $\xi_\parallel$ or small $\xi_\perp$).

In equality (1), $\mathcal{H}_{cc}$ represents the direct colloid–colloid interaction. The latter can be ignored as long as we are concerned with a very dilute gas (no mutual interactions between colloids). The last term in this equality, $\mathcal{H}_{cm}$, is the colloid-interface interaction. It is generally a complicated function of particles positions and configurations of the interface. To simplify the study, we assume that $\mathcal{H}_{cm}$ is a contact interaction, and depends only on
the relative perpendicular distances between particles and surface. We then adopt the following form for this extra interaction

$$\frac{H_{cm}[h]}{k_B T} = -\frac{w}{2} \sum_{i=1}^{N} \delta \left[ z_i - h(\rho_i) \right], \quad (8)$$

with $\delta$ the Dirac distribution. In this definition, the discrete sum is performed over all particles of positions $r_i = (\rho_i, z_i)$. There, $w > 0$ is the (surface) coupling constant. In fact, $w$ plays the role of an extrapolation length as usually encountered in Surface Critical Phenomena [25 – 27]. In this model, the surface is penetrable and the colloids can accommodate on both sides of the interface.

On the other hand, the interaction energy between particles generated by the membrane undulations was found to be [14]

$$U(r_1, ..., r_N) = -k_B T \ln \langle e^{-\beta H_{cm}[h]} \rangle_0 = \sum_{i=1}^{N} U_1(r_i) + \frac{1}{2} \sum_{\{i,j\}} U_2(r_i, r_j) + ... \quad (9)$$

Using the cumulant method used in Statistical Field Theory [28, 29], it has been shown [14] that the $p$-body potential $U_p$ expresses in terms of functions

$$\Phi_N(r_1, ..., r_N) = \left\langle \prod_{i=1}^{N} \delta \left[ z_i - h(\rho_i) \right] \right\rangle_0. \quad (10)$$

These functions have been exactly computed in Ref. [14]. In this paper, we are concerned only by colloidal suspensions of very low density (ideal gases). This means that the mutual interactions between particles can be neglected. Therefore, the only remaining interaction is $U_1$, which describes the direct potential between colloids and interface. Its expression is [14]

$$\frac{U(z)}{k_B T} = -\frac{w}{2} \Phi_1(z) \quad , \quad (11)$$

with the function

$$\Phi_1(z) = \frac{1}{\sqrt{2\pi\xi_\perp}} \exp \left\{ -\frac{z^2}{2\xi_\perp^2} \right\} \quad . \quad (12)$$

(The subscript 1 on $U_1$ is discarded). Explicitly, we have

$$U(z) = U_0 \exp \left\{ -\frac{z^2}{2\xi_\perp^2} \right\} \quad , \quad (13)$$

with the amplitude

$$U_0 = -\frac{w}{2\sqrt{2\pi\xi_\perp}} k_B T < 0 \quad . \quad (14)$$

Then, $|U_0|$ represents the depth of the potential.

Let us discuss the above expression of the external potential experienced by beads.
Firstly, this potential exhibits a minimum at the origin \( z = 0 \). In addition, it is symmetric around this point.

Secondly, remark that the potential depth \(|U_0|\) depends on three kinds of parameters, namely the absolute temperature \( T \), the colloid-membrane coupling constant \( w \), and the bending rigidity constant \( \kappa \) (through \( \xi_\perp \sim \kappa^{-1/4} \)). For example, if \( T \) and \( w \) are fixed, the potential depth behaves as : \(|U_0| \sim \kappa^{1/4} \). Therefore, this external potential felt by colloids has significant values only for those membranes of higher bending rigidity constant (small \( \xi_\perp \)). Then, \(|U_0|\) can be varied by changing the environment of the membrane or by introducing some inclusions. Now, if \( T \) and \( \kappa \) are fixed to some values, the potential depth linearly increases with increasing surface coupling constant \( w \).

Finally, notice that \(|U_0|\) must be small in comparison with the thermal energy \( k_B T \). This implies that the coupling constant \( w \) is bounded from above, i.e. \( w < w^* = 2\sqrt{2\pi} \xi_\perp \sim 5.0133 \times \xi_\perp \).

The (reduced) external potential experienced by colloids, \( U(z)/k_B T \), is depicted in Fig. 1, upon perpendicular distance \( z \) (expressed in \( \xi_\perp \) unit). This curve is drawn with parameter : \( w = 0.9 \times \xi_\perp \). As shown in this figure, the potential takes its higher values around \( z = 0 \), and fails exponentially for distances much greater than \( \xi_\perp \). Remark the symmetric character of \( U(z) \) due to the fact that the membrane is penetrable and the particles may exist in its two sides.

In Fig. 2, we report the (reduced) external potential upon the perpendicular distance, for two values of the coupling constant \( w \) : \( w_1 = 0.5 \times \xi_\perp \) and \( w_2 = 0.9 \times \xi_\perp \). As it should be, the curve drawn with parameter \( w_2 \) is below that associated with \( w_1 < w_2 \).

The above expression of the one-body potential is the principal ingredient for the Brownian dynamics study of very low density particles, which are located near a soft membrane.

### 3. Time Evolution of the Particle Density

Consider an assembly of colloidal particles moving around a fluctuating fluid membrane. Under a sudden change of a suitable parameter, the system is out equilibrium. We are interested in the investigation of the time evolution of the particle density before the colloidal suspension reaches its final equilibrium state. The time particle density, \( \rho(z,t) \), represents the number of colloids per unit volume, at distance \( z \) and at time \( t \). For simplicity, we will neglect mutual interactions between particles. This hypothesis makes sense at least for small particle densities. Therefore, the only interaction felt by the beads is an external potential caused by the membrane undulations.

#### 3.1 Basic Equations

We first recall the expression of the equilibrium particle density

\[
\rho_{\text{eq}}(z) = A \exp \left\{ -\frac{U(z)}{k_B T} \right\},
\]  

(15)
where \( A \) is a normalization constant and \( U(z) \) the (attractive) one-body potential originating from the membrane undulations.

Now, the crucial question is the determination of the particle density \( \rho(z,t) \), as a function of space-variable \( z \) and time \( t \). More precisely, we are interested in how the colloidal system reaches its final equilibrium state under a change of some pertinent parameter, such as temperature, pressure, or membrane environment. The key of this problem is to write a partial differential equation satisfied by the density \( \rho(z,t) \). To this end, we recall that the diffusion phenomenon is correctly described by the Fick’s law.

The latter suggests that, if the density is not uniform, there is a flux, \( j(z,t) \), which is directly proportional to the spatial gradient of the density, that is

\[
\begin{align*}
j &= -D \frac{\partial \rho}{\partial z} - \frac{1}{\zeta} \frac{\partial U}{\partial z},
\end{align*}
\]

with \( D \) the diffusion constant defined later, \( \zeta \) the friction coefficient, of which the inverse \( 1/\zeta \) is the mobility. If we design by \( a \) the radius of the particles and by \( \eta_s \) the solvent viscosity, the friction coefficient \( \zeta \) can be calculated from hydrodynamics [30] : \( \zeta = 6\pi \eta_s a \).

Come back to formula (16) and notice, first, that the additional flux appearing in its right-hand side is the contribution of the driving force, which pushes the particles towards the membrane. Second, emphasize that an important relation can emerges from Eq. (16). Indeed, when the system is at the equilibrium state, the flux \( j \) vanishes, the density \( \rho(z,t) \) tends to its equilibrium value \( \rho_{eq} \), and we have

\[
D \frac{\partial \rho_{eq}}{\partial z} + \frac{1}{\zeta} \frac{\partial U}{\partial z} = 0.
\]

Since \( \rho_{eq} \) is also given by Eq. (15), then, by identification, we obtain the famous Einstein relation

\[
D = \frac{k_B T}{\zeta},
\]

which states that the diffusion constant characterizing the thermal motion is related to the quantity \( \zeta \), which expresses the response to an external field.

On the other hand, the density \( \rho(z,t) \) must satisfy the local conservation law of matter

\[
\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial z} = 0.
\]

Combining the above relation with equality (16) yields

\[
\begin{align*}
\frac{\partial \rho}{\partial t} &= \frac{1}{\zeta} \frac{\partial}{\partial z} \left( k_B T \frac{\partial \rho}{\partial z} + \rho \frac{\partial U}{\partial z} \right).
\end{align*}
\]

This is the so-called Smoluchowski equation satisfied by the particle density \( \rho(z,t) \). This equation must be supplemented by two boundary conditions, that is

\[
\begin{align*}
\rho(z,t = 0) &= \rho_i(z), & \rho(z,t = \infty) &= \rho_f(z).
\end{align*}
\]

If temperature \( T \) and coupling constant \( w \) are fixed, the initial and final equilibrium particle densities, \( \rho_i(z) \) and \( \rho_f(z) \), are completely determined by two characteristic lengths \( \xi_i \) and \( \xi_f \).
and \( \xi^f_{\perp} \), respectively. We will assume that the equilibrium breaking is caused by a change of the membrane environment, such that: \( \xi^i_{\perp} < \xi^f_{\perp} \). This process is then accompanied by an increasing of the membrane undulations.

### 3.2 Time Evolution of the Particle Density

The expected time particle density solves the above differential equation, with the one-body potential \( U(z) \) defined in Eq. (11). As first remark, the solution of the Smoluchowski equation (20) is not trivial. To get an approximative solution, we shall distinguish two interesting distance-regimes: (1) \( z < \xi_{\perp} \) (small distances compared to \( \xi_{\perp} \)), and (2) \( z > \xi_{\perp} \) (higher ones). The second regime corresponds to the situation where particles are located in a region far from the fluid membrane, where the one-body potential is weak enough. In this region, the particles freely diffuse according to the usual diffusion laws [1–7]. For small-distances, however, the diffusion of particles is sensitive to the membrane undulations, and then, they move in holes and valleys. In this regime, the potential is harmonic, that is

\[
U(z) \simeq U_0 + \frac{1}{2} k z^2 , \quad z < \xi_{\perp} , \tag{21a}
\]

with the elastic constant

\[
k = -\frac{U_0}{\xi^2_{\perp}} = \frac{w}{2\sqrt{2\pi\xi^2_{\perp}}} k_B T > 0 . \tag{21b}
\]

Of course, in the absence of the colloid-membrane interaction \( (w = 0) \), the one-body potential vanishes.

For simplicity, we suppose that the colloidal system is of very low density, and the colloid-membrane interactions are strong enough. Hence, the particles are located in a space-region of size \( \xi_{\perp} \), and then, the bulk density is practically zero. With these considerations, the equilibrium particle density can be written as

\[
\rho_{eq}(z) \simeq \rho_0 \exp \left\{ -\frac{W(z)}{k_B T} \right\} , \tag{22a}
\]

with the harmonic potential

\[
W(z) = \frac{1}{2} k z^2 . \tag{22b}
\]

In relation (22), \( \rho_0 \) is nothing else but the (maximum) value of the particle density at the origin \( z = 0 \). Therefore, formula (22) is valid in all \( z \)-range, and in particular, \( \rho(\infty) \rightarrow 0 \).

Within the harmonic approximation, the differential equation (20) becomes

\[
\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial z^2} + k \frac{\partial \rho}{\xi z} + \frac{k}{\xi} \rho . \tag{23}
\]

Some non-straightforward algebra yields the solution

\[
\rho(z,t) = \rho_f(z) + \left[ 2 \pi D \tau_f \left( 1 - e^{-2t/\tau_f} \right) \right]^{-1/2} \int_{-\infty}^{+\infty} dy \exp \left\{ -\frac{(ze^{t/\tau_f} - y)^2}{2D\tau_f \left( e^{2t/\tau_f} - 1 \right)} \right\} \left[ \rho_i(y) - \rho_f(y) \right] . \tag{24}
\]
The initial and final equilibrium particle densities are given by

\[ \rho_i(z) = \rho_0^i \exp \left\{ -\frac{k_i z^2}{2k_B T} \right\} = \rho_0^i \exp \left\{ -\frac{z^2}{2D\tau_i} \right\}, \] (25)

\[ \rho_f(z) = \rho_0^f \exp \left\{ -\frac{k_f z^2}{2k_B T} \right\} = \rho_0^f \exp \left\{ -\frac{z^2}{2D\tau_f} \right\}, \] (26)

with the time-scales \( \tau_i \) and \( \tau_f \)

\[ \tau_i = \frac{\zeta}{k_i}, \quad \tau_f = \frac{\zeta}{k_f}. \] (27)

Explicitly, we have

\[ \tau_i = D^{-1} \frac{\sqrt{2\pi}}{w} \left( \xi_{i\perp}^i \right)^3, \quad \tau_f = D^{-1} \frac{\sqrt{2\pi}}{w} \left( \xi_{f\perp}^f \right)^3 < \tau_i, \] (28)

where \( \xi_{i\perp}^i \) and \( \xi_{f\perp}^f \) are the initial and final mean roughnesses. As it should be, the times \( \tau \)'s depend on the characteristics of the fluid membrane and its interactions with beads, through the mean roughness \( \xi_{\perp} \) and coupling constant \( w \). In particular, \( \tau_f \) can be interpreted as the time beyond which the colloidal system reaches its final equilibrium state. Then, strong membrane undulations (or strong colloid-membrane interaction) necessitate a small time before the colloidal system tends to its final state. In fact, the time-scale \( \tau_f \) may have another physical meaning, and can be regarded as the time during which the particles are trapped in new holes and valleys.

Integration over the variable \( y \) in Eq. (24) yields the closer form for the time particle density

\[ \rho(z,t) = \rho_0^i \left[ 1 + \eta \left( e^{-2t/\tau_f} - 1 \right) \right]^{-1/2} \exp \left\{ -\frac{1}{1 + \eta \left( e^{-2t/\tau_f} - 1 \right)} \frac{z^2}{2D\tau_i} \right\}, \] (29)

with the reduced time-shift

\[ \eta = \frac{\tau_i - \tau_f}{\tau_i} > 0. \] (30)

We have used the matter conservation law

\[ \int_{-\infty}^{+\infty} dz \rho(z,t) = \int_{-\infty}^{+\infty} dz \rho_i(z) = \int_{-\infty}^{+\infty} dz \rho_f(z) \equiv \Gamma. \] (31)

This implies the relationship

\[ \sqrt{2\pi D\tau_f} \rho_0^f = \sqrt{2\pi D\tau_i} \rho_0^i = \Gamma. \] (32)

The notation \( \Gamma \) means the adsorbance. The latter is defined as the total number of colloids per unit area (excess particles) located near interface.

The result (29) calls the following remarks.

Firstly, we note that it is easy to see that the solution (29) satisfies the two boundary conditions (20a), where the initial and final equilibrium states are defined in Eqs. (25) and (26).
Secondly, when it is reduced by $\rho^i_0$, the time particle density is a universal scaling function, independently on the particular details of the characteristics of the membrane and its interactions with colloids. This function depends on three dimensionless factors, which are the renormalized distance $z/\sqrt{2D\tau_i}$, the time-ratio $t/\tau_f$ and the time-shift $\eta = (\tau_i - \tau_f)/\tau_i$. Therefore, all microscopic details are entirely contained in $\tau_i$ and $\tau_f$.

Finally, we note that the time particle density curve exhibits a maximum at $z = 0$, and it is symmetric around this point, whatever be the values of $t/\tau_f$ and $\eta$.

In Fig. 3, we report the reduced time particle density $\rho(z,t)/\rho^i_0$ upon the renormalized distance $z/\sqrt{2D\tau_i}$, choosing three values of the time-ratio $t/\tau_f$: 0, 0.5, and $\infty$. The former and the third one correspond to the initial and final states, respectively. These curves are drawn with parameter $\eta = 0.2$.

Conclusions

This work was addressed to the Brownian dynamics study of small colloidal particles in contact with an attractive penetrable fluid membrane. More precisely, the main question was how these particles move around the interface by changing some suitable physical parameter, which may be temperature, pressure or membrane environment. To apprehend the problem, we started from three hypothesizes: (1) the particle size is small enough when it is compared to the mean roughness of the membrane $\xi_\perp$, (2) they are of very low density (in order to forget their mutual interactions), and (3) strongly interact with the membrane.

The used theoretical formalism for the study was the Smoluchowski equation, solved exactly around the fluid membrane (region of size $\xi_\perp$), where the essential of phenomenon occurs. Far from the interface, the Brownian dynamics is insignificant, since most of particles are condensed near the interface (strong colloid-membrane attraction). Within this distance-region, the mean-force external potential was approximated by an harmonic one. Therefore, we were in the presence of Brownian particles moving in an harmonic potential. In addition to the normal diffusion, the particles experience small oscillations with a frequency $\omega$ scaling as $\omega \sim \xi_\perp^{-3/2} \sim \kappa^{3/8}$, where $\kappa$ is the bending rigidity constant of the membrane. Hence, the harmonic approximation used for the external potential is largely justified for those fluid membranes of smaller bending rigidity constant.

In the description of Brownian dynamics, it appeared a time-scale $\tau \sim \xi_\perp^{3} \sim \kappa^{-3/4}$. The latter was interpreted as the necessary time during which beads reach their final equilibrium state. Also, this same scale-time can be viewed as the time during which particles are trapped in holes and valleys of size $\xi_\perp$.

Finally, questions, such as the computation of the time correlation and response functions, and the investigation of hydrodynamic interactions effects, are under consideration.
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References


Figure captions

**Fig. 1 :** Reduced external potential $U(z)/k_B T$ versus the renormalized distance $z/\xi_{\perp}$. This curve is drawn with parameter $w = 0.5 \times \xi_{\perp}$.

**Fig. 2 :** Reduced external potential $U(z)/k_B T$ versus the renormalized distance $z/\xi_{\perp}$, for two values of the coupling constant $w$: $w_1 = 0.5 \times \xi_{\perp}$ (solid line) and $w_2 = 0.9 \times \xi_{\perp}$ (dashed line).

**Fig. 3 :** Reduced time particle density, $\rho(z,t)/\rho_0$, versus the renormalized distance $z/\sqrt{2D\tau_i}$, with three values of the time-ratio $t/\tau_f$: 0 (dashed line), 0.5 (solid line), $\infty$ (line in dots). These curves are drawn with parameter $\eta = 0.2$. 
Figure 1
Figure 2
Figure 3
Influence of Third Order Perturbation on Heisenberg Hamiltonian of Thick Ferromagnetic Films

P. Samarasekara*

Department of Physics, University of Ruhuna, Matara, Sri Lanka

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Abstract: The effect of third order perturbation on the classical Heisenberg Hamiltonian of thick ferromagnetic has been investigated for the first time. Energy of thick films with layers up to 10000 has been plotted for sc(001) and fcc(001) ferromagnetic compounds. Unlike the second order perturbation, the third order perturbation does not increase the total energy by any considerable amount. For the thicknesses approximately N=45 and 40, the anisotropy energy is small for sc(001) and fcc(001), respectively, indicating that the energy required to rotate from easy to hard direction is really small at these thicknesses. The energy curves of sc (001) and fcc(001) with N=10000 have been flattened by reducing the smooth part of the curve compared with those of second order perturbation. The angle between the easy and hard direction is 97.4° and 32.45° for sc(001) and fcc(001) thick film with N=10000, respectively. The overshooting parts began to appear after introducing second or third order perturbation, and hence the angle between easy and hard directions is not 90° in the overshooting part of curves. The third and second order perturbation vanish at θ=0° and 90° directions.

Keywords: Heisenberg Hamiltonian; third order perturbation; anisotropy; magnetic films

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1. Introduction

Most of the films found in the reality consist of 1000-10000 layers. But it is difficult to find reports related to the theoretical work of thick films. The Heisenberg Hamiltonian of thick ferromagnetic films with third order perturbation has been solved in this report, after taking the stress induced anisotropy and the demagnetization factor into account in addition to the spin exchange spin interaction, dipole interaction, second and fourth order anisotropies. The stress induced anisotropy arises due to the difference between the thermal expansion coefficients of substrate and the film while the sample is cooling.

* pubudas@phy.ruh.ac.lk
down or heating. This anisotropy constant mostly depends on the magnitude of applied stress and the magnetostriction coefficient. These quantities depend on parameters such as thermal expansion coefficients, Young’s modulus and Poisson’s ratio. The in plane and out plane demagnetization factors are 0 and 1 (or $4\pi$) in SI (or CGS) units, respectively.

The properties of ferromagnetic films are extensively studied nowadays, due to their potential applications of magnetic memory devices and microwave devices. Because it is difficult to understand the behavior of exchange anisotropy and its applications in magnetic sensors and media technology, the theoretical researches of exchange anisotropy are widely expanded [2]. Bloch spin wave theory is sometimes applied to study magnetic properties of ferromagnetic thin films [3]. Although the magnetization of some thin films is oriented in plane due to dipole interaction, the perpendicular orientation is preferred at the surface due to the broken symmetry of uniaxial anisotropy energy. Two dimensional Heisenberg model has been earlier used to explain the magnetic anisotropy in the presence of dipole interaction [4]. Magnetic properties of ferromagnetic thin films with alternating super layers were investigated by Ising model [5].

2. The Model

The Heisenberg Hamiltonian of ferromagnetic films can be written as following.

$$H = -\frac{J}{2} \sum_{m,n} \vec{S}_m \cdot \vec{S}_n + \frac{\omega}{2} \sum_{m \neq n} \left( \frac{\vec{S}_m \cdot \vec{S}_n}{r_{mn}^3} - \frac{3(\vec{S}_m \cdot \vec{r}_{mn})(\vec{r}_{mn} \cdot \vec{S}_n)}{r_{mn}^5} \right) - \sum_m D^{(2)}_\lambda (S^z_m)^2 +$$

$$-\sum_m D^{(4)}_\lambda (S^z_m)^4 - \sum_{m,n} [\vec{H} - (N_d \vec{S}_n/\mu_0)] \cdot \vec{S}_m - \sum_m K_s \sin 2\theta_m$$

After using spins with unit magnitudes, the equation will be deduced to following form [1].

$$E(\theta) = -\frac{1}{2} \sum_{m,n=1}^{N} \left[ (JZ_{m-n} - \frac{\omega}{4} \Phi_{m-n}) \cos(\theta_m - \theta_n) - \frac{3\omega}{4} \Phi_{m-n} \cos(\theta_m + \theta_n) \right]$$

$$- \sum_{m=1}^{N} \left( D^{(2)}_m \cos^2 \theta_m + D^{(4)}_m \cos^4 \theta_m + H_{in} \sin \theta_m + H_{out} \cos \theta_m \right)$$

$$+ \sum_{m,n=1}^{N} \frac{N_d}{\mu_0} \cos(\theta_m - \theta_n) - K_s \sum_{m=1}^{N} \sin 2\theta_m \quad (1)$$

Here $N$, $m$ (or $n$), $J$, $Z_{m-n}$, $\omega$, $\Phi_{m-n}$, $D^{(2)}_m$, $D^{(4)}_m$, $H_{in}$, $H_{out}$, $N_d$ and $K_s$ are total number of layers, layer index, spin exchange interaction, number of nearest spin neighbors, strength of long range dipole interaction, partial summations of dipole interactions, azimuthal angles of spins, second and fourth order anisotropy constants, in plane and out of plane applied magnetic fields, demagnetization factor and stress induced anisotropy constants, respectively.
By choosing azimuthal angles as $\theta_m = \theta + \varepsilon_m$ and $\theta_n = \theta + \varepsilon_n$, above energy can be expanded up to the third order of $\varepsilon$ as following,

$$E(\theta) = E_0 + E(\varepsilon) + E(\varepsilon^2) + E(\varepsilon^3) \quad (2)$$

Here

$$E_0 = -\frac{1}{2} \sum_{m,n=1}^{N} (JZ_{m-n} - \frac{\omega}{4} \Phi_{m-n}) + \frac{3\omega}{8} \cos 2\theta \sum_{m,n=1}^{N} \Phi_{m-n}$$

$$- \cos^2 \theta \sum_{m=1}^{N} D_{m}^{(2)} - \cos^4 \theta \sum_{m=1}^{N} D_{m}^{(4)} - N(H_{in} \sin \theta + H_{out} \cos \theta - \frac{N_d}{\mu_0} + K_s \sin 2\theta) \quad (3)$$

$$E(\varepsilon) = -\frac{3\omega}{8} \sin 2\theta \sum_{m,n=1}^{N} \Phi_{m-n}(\varepsilon_m + \varepsilon_n) + \sin 2\theta \sum_{m=1}^{N} D_{m}^{(2)} \varepsilon_m + 2 \cos^2 \theta \sin 2\theta \sum_{m=1}^{N} D_{m}^{(4)} \varepsilon_m$$

$$- H_{in} \cos \theta \sum_{m=1}^{N} \varepsilon_m + H_{out} \sin \theta \sum_{m=1}^{N} \varepsilon_m - 2K_s \cos 2\theta \sum_{m=1}^{N} \varepsilon_m$$

$$E(\varepsilon^2) = \frac{1}{4} \sum_{m,n=1}^{N} (JZ_{m-n} - \frac{\omega}{4} \Phi_{m-n})(\varepsilon_m - \varepsilon_n)^2 - \frac{3\omega}{16} \cos 2\theta \sum_{m,n=1}^{N} \Phi_{m-n}(\varepsilon_m + \varepsilon_n)^2$$

$$- (\sin^2 \theta - \cos^2 \theta) \sum_{m=1}^{N} D_{m}^{(2)} \varepsilon_m^2 + 2 \cos^2 \theta(\cos^2 \theta - 3 \sin^2 \theta) \sum_{m=1}^{N} D_{m}^{(4)} \varepsilon_m^2$$

$$+ \frac{H_{in}}{2} \sin \theta \sum_{m=1}^{N} \varepsilon_m^2 + \frac{H_{out}}{2} \cos \theta \sum_{m=1}^{N} \varepsilon_m^2 - \frac{N_d}{2\mu_0} \sum_{m,n=1}^{N} (\varepsilon_m - \varepsilon_n)^2$$

$$+ 2K_s \sin 2\theta \sum_{m=1}^{N} \varepsilon_m^2$$

$$E(\varepsilon^3) = \frac{\omega}{16} \sin 2\theta \sum_{m,n=1}^{N} (\varepsilon_m + \varepsilon_n)^3 \Phi_{m-n} - \frac{4}{3} \cos \theta \sin \theta \sum_{m=1}^{N} D_{m}^{(2)} \varepsilon_m^3$$

$$- 4 \cos \theta \sin \theta(\frac{5}{3} \cos^2 \theta - \sin^2 \theta) \sum_{m=1}^{N} D_{m}^{(4)} \varepsilon_m^3 + \frac{H_{in}}{6} \cos \theta \sum_{m=1}^{N} \varepsilon_m^3$$

$$- \frac{H_{out}}{6} \sin \theta \sum_{m=1}^{N} \varepsilon_m^3 + \frac{4K_s}{3} \cos 2\theta \sum_{m=1}^{N} \varepsilon_m^3$$

After using the constraint $\sum_{m=1}^{N} \varepsilon_m = 0$, $E(\varepsilon) = \vec{\alpha} \cdot \varepsilon$

Here $\vec{\alpha}(\varepsilon) = \vec{B}(\theta) \sin 2\theta$ are the terms of matrices with

$$B_{\lambda}(\theta) = -\frac{3\omega}{4} \sum_{m=1}^{N} \Phi_{|\lambda-m|} + D_{\lambda}^{(2)} + 2D_{\lambda}^{(4)} \cos^2 \theta \quad (4)$$
Also $E(\varepsilon^2) = \frac{1}{2} \vec{e} \cdot C \cdot \vec{e}$  

Here the elements of matrix $C$ can be given as following,

$$C_{mn} = -(JZ_{[m-n]} - \frac{\omega}{4} \Phi_{[m-n]}) - \frac{3\omega}{4} \cos 2\theta \Phi_{[m-n]} + \frac{2N_d}{\mu_0}$$

$$+ \delta_{mn} \left\{ \sum_{\lambda=1}^{N} [JZ_{[m-\lambda]} - \Phi_{[m-\lambda]}(\frac{\omega}{4} + \frac{3\omega}{4} \cos 2\theta)] - 2(sin^2 \theta - \cos^2 \theta)D_m^{(2)} \right.$$  

$$+ 4 \cos^2 \theta(cos^2 \theta - 3 \sin^2 \theta)D_m^{(4)} + H_m \sin \theta + H_{out} \cos \theta - \frac{4N_d}{\mu_0} + 4K_s \sin 2\theta \}$$

If $E(\varepsilon^3) = \varepsilon^2 \beta \cdot \vec{e}$, then matrix elements of matrix $\beta$ can be given as following,

$$\beta_{mn} = \frac{3\omega}{8} \sin 2\theta \Phi_{[m-n]} + \delta_{mn} \left\{ \frac{\omega}{8} \sin 2\theta [A_m - \Phi_0] - \frac{4}{3} \cos \theta \sin \theta D_m^{(2)} \right.$$  

$$- 4 \cos \theta \sin \theta(\frac{5}{3} \cos^2 \theta - \sin^2 \theta)D_m^{(4)} + \frac{H_{in}}{6} \cos \theta - \frac{H_{out}}{6} \sin \theta$$

$$+ \frac{4K_s}{3} \cos 2\theta \}$$

Also $\beta_{mn} = \beta_{nm}$ and matrix $\beta$ is symmetric.

Here $A_m$ values are different for even and odd $N$ values, and can be given as following.

For odd $N$, $A_{\frac{N}{2}+0.5+n} = \sum_{\nu=0}^{\frac{N}{2}-0.5} \Phi_{\nu}$ 

$$+ \sum_{\nu=\frac{N}{2}+0.5-n}^{\frac{N}{2}+0.5} \Phi_{\nu} \text{ for } m > \frac{N}{2}$$

Here $n$ varies from 1 to $\frac{N}{2} - 0.5$.

When $n=0$, $A_{\frac{N}{2}+0.5+n} = 2 \sum_{\nu=0}^{\frac{N}{2}-0.5-n} \Phi_{\nu}$

$A_m$ for $m < \frac{N}{2}$ can be found using $A_{\frac{N}{2}+0.5+n} = A_{\frac{N}{2}+0.5-n}$

For even $N$, $A_{\frac{N}{2}+1+n} = 2 \sum_{\nu=0}^{\frac{N}{2}-1-n} \Phi_{\nu} + \sum_{\nu=\frac{N}{2}-n}^{\frac{N}{2}+n} \Phi_{\nu} \text{ for } m > \frac{N}{2}$

Here $n$ varies from 0 to $\frac{N}{2} - 1$.

$A_m \text{ for } m < \frac{N}{2}$ can be obtained using $A_{\frac{N}{2}+1+n} = A_{\frac{N}{2}-n}$

Therefore, the total magnetic energy given in equation 2 can be deduced to

$$E(\theta) = E_0 + \vec{\alpha} \cdot \vec{e} + \frac{1}{2} \vec{e} \cdot C \cdot \vec{e} + \varepsilon^2 \beta \cdot \vec{e}$$

Because it is difficult to find an equation for $\varepsilon$ in the presence of the third order of $\varepsilon$ in above equation, only the second order of $\varepsilon$ will be considered for following derivation. Then

$$E(\theta) = E_0 + \vec{\alpha} \cdot \vec{e} + \frac{1}{2} \vec{e} \cdot C \cdot \vec{e}$$
Using a proper constraint provides [1] $\bar{\varepsilon} = -C^+\bar{\alpha}$

Here $C^+$ is the pseudo-inverse given by

$$C.C^+ = 1 - \frac{E}{N}.$$  

(8)

$E$ is the matrix with all elements given by $E_{mn}=1$.

After using $\varepsilon$ in equation 7,

$$E(\theta) = E_0 - \frac{1}{2}\bar{\alpha}.C^+\bar{\alpha} - (C^+\alpha)^2\bar{\beta}(C^+\alpha)$$  

(9)

3. Results and Discussion

When $N$ is very large $CC=1$, and $C^+$ is the standard inverse matrix of $C$. For most ferromagnetic materials, $Z_{5>1}=0$ and $\Phi_{5>1} = w$. Therefore, from equation 5, $C_{mn}$ is a constant when the difference between $m$ and $n$ is larger than one. Because this term does not depend on angle, it does not make any effect on physical properties of final equation. Therefore these matrix elements are assumed to be zero for the convenience. But when the difference between $m$ and $n$ is one, $C_{mn} = -JZ_1 + \frac{2}{3}\Phi_1(1 - 3\cos 2\theta) + \frac{2N_d}{\mu_0}$. When $H_{in}, H_{out}$ and $K_s$ are very large, $C_{11} >> C_{12}$. If this $C_{mn}=0$, then the matrix $C$ becomes diagonal, and the elements of inverse matrix $C^+$ is given by $C^+_{nn} = \frac{1}{C_{nn}}$. Therefore all the derivation will be done under above assumption for the convenience.

Also from equation 5,

$$C_{11} = C_{NN} = JZ_1 - \frac{2}{3}\Phi_1(1 - 3\cos 2\theta) - 2(\sin^2 \theta - \cos^2 \theta)D_m^{(2)}$$

$$+4\cos^2 \theta(\cos^2 \theta - 3\sin^2 \theta)D_m^{(4)} + H_{in}\sin \theta + H_{out}\cos \theta - \frac{2N_d}{\mu_0} + 4K_s \sin 2\theta$$

$$C_{22} = C_{33} = \cdots = C_{22} = C_{33} = 2JZ_1 - \frac{\omega}{2}\Phi_1(1 - 3\cos 2\theta) - 2(\sin^2 \theta - \cos^2 \theta)D_m^{(2)}$$

$$+4\cos^2 \theta(\cos^2 \theta - 3\sin^2 \theta)D_m^{(4)} + H_{in}\sin \theta + H_{out}\cos \theta - \frac{2N_d}{\mu_0} + 4K_s \sin 2\theta$$

From equation 4,

$$\alpha_1(\theta) = \alpha_N(\theta) = \left[ -\frac{3\omega}{4}(\Phi_0 + \Phi_1) + D_m^{(2)} + 2D_m^{(4)}\cos^2 \theta \right] \sin(2\theta)$$

$$\alpha_2(\theta) = \alpha_3(\theta) = \cdots = \alpha_{N-1}(\theta) = \left[ -\frac{3\omega}{4}(\Phi_0 + 2\Phi_1) + D_m^{(2)} + 2D_m^{(4)}\cos^2 \theta \right] \sin(2\theta)$$

$$(C^+\alpha)^2\beta(C^+\alpha) = (C_{11}^+\alpha_1)^2(\beta_{11}C_{11}^+\alpha_1 + \beta_{12}C_{22}^+\alpha_2 + \cdots + \beta_{1N}C_{NN}^+\alpha_N)$$

$$+ (C_{22}^+\alpha_2)^2(\beta_{21}C_{22}^+\alpha_1 + \beta_{22}C_{22}^+\alpha_2 + \cdots + \beta_{2N}C_{NN}^+\alpha_N)$$

$$+ (C_{33}^+\alpha_3)^2(\beta_{31}C_{11}^+\alpha_1 + \beta_{32}C_{22}^+\alpha_2 + \cdots + \beta_{3N}C_{NN}^+\alpha_N) + \cdots +$$

$$- \cdots + (C_{NN}^+\alpha_N)^2(\beta_{N1}C_{11}^+\alpha_1 + \beta_{N2}C_{22}^+\alpha_2 + \cdots + \beta_{NN}C_{NN}^+\alpha_N)$$

Also $C_{11}^+ = C_{NN}^+, C_{22}^+ = C_{33}^+ = C_{44}^+ = \cdots = C_{N-1,N-1}^+$ and $\beta_{nn} = \beta_{mn}$. From equation 6a, $\beta_{nn} = 0$ when the difference between $m$ and $n$ is larger than one.
When the difference between \( m \) and \( n \) is one, \( \beta_{12} = \beta_{21} = \beta_{23} = \beta_{32} = - - - - \beta_{N-1,N} \). If \( N \) is very large, then \( N+1 \approx N \). Therefore only even \( N \) will be considered for this derivation. When \( N \) is even, from above equation 6c

\[
\beta_{11} = \beta_{NN}, \beta_{22} = \beta_{N-1,N-1}, \beta_{33} = \beta_{N-2,N-2}, - - - - \beta_{\frac{N}{2},\frac{N}{2}} = \beta_{\frac{N}{2}+1,\frac{N}{2}+1}
\]

\[
(C^{+} \alpha)^2 \beta(C^{+} \alpha) = (\alpha_1)^2[(C_{11}^{+})^3 \beta_{11} + \beta_{12}C_{22}^{+}(C_{11}^{+})^2 \beta_{21} + \beta_{22}(C_{11}^{+})^2 C_{22}^{+} \beta_{22} + \beta_{NN}(C_{11}^{+})^3 \beta_{31}] + (C_{22}^{+} \beta_{22})^2 \{ \alpha_1(C_{11}^{+} \beta_{12} + C_{11}^{+} \beta_{12}) + C_{22}^{+} \alpha_2[\beta_{22} + \beta_{33} + - - - + \beta_{N-1,N-1} + (2N - 8) \beta_{12}] \}
\]

But \( \beta_{22} + \beta_{33} + - - - + \beta_{N-1,N-1} = 2 \beta_{22} + 2 \beta_{33} + - - - + 2 \beta_{\frac{N}{2},\frac{N}{2}} \)

From equation 6a and 6b, when \( Z_{\delta>1} = 0 \) and \( \Phi_{\delta>1} = 0 \)

\[
\beta_{11} = \frac{\omega}{8} \sin 2\theta(4 \Phi_0 + \Phi_1) - \frac{4}{3} \cos \theta \sin \theta D_{m}^{(2)} - 4 \cos \theta \sin \theta \left( \frac{2}{3} \cos^2 \theta - \sin^2 \theta \right) D_{m}^{(4)} + \frac{H_m}{6} \cos \theta - \frac{H_{out}}{6} \sin \theta + \frac{4K_s}{3} \cos 2\theta
\]

\[
\beta_{22} = \frac{\omega}{8} \sin 2\theta(4 \Phi_0 + 2 \Phi_1) - \frac{4}{3} \cos \theta \sin \theta D_{m}^{(2)} - 4 \cos \theta \sin \theta \left( \frac{2}{3} \cos^2 \theta - \sin^2 \theta \right) D_{m}^{(4)} + \frac{H_m}{6} \cos \theta - \frac{H_{out}}{6} \sin \theta + \frac{4K_s}{3} \cos 2\theta = \beta_{33} = - - - = \beta_{\frac{N}{2},\frac{N}{2}}
\]

\[
\beta_{12} = \frac{3\omega}{8} \sin 2\theta \Phi_1
\]

Then \( \beta_{22} + \beta_{33} + - - - + \beta_{N-1,N-1} = 2 \beta_{22} \left( \frac{N}{2} - 1 \right) \)

\[
(C^{+} \alpha)^2 \beta(C^{+} \alpha) = \beta_{12} \left\{ \frac{2 \alpha_1^2 \beta_{22}}{C_{11}^{2}} + \frac{(\alpha_2)^2}{C_{22}^{2}} \left[ \frac{2 \alpha_1}{C_{11}^{2}} + \frac{2 \alpha_2(N - 4)}{C_{22}^{2}} \right] \right\} + \frac{2 \alpha_1^3 \beta_{12}}{C_{11}^{2}} + 2 \left( \frac{\alpha_2}{C_{22}^{2}} \right)^3 \beta_{22} \left( \frac{N}{2} - 1 \right)
\]

But, \( \alpha.C^{+}.\alpha = C_{22}^{+}(N - 2) \alpha_2^2 + C_{11}^{+} \alpha_1^2 \)

From equation 4 and 5,

\[
\alpha.C^{+}.\alpha = \frac{[- \frac{3\omega}{4}(\Phi_0 + 2 \Phi_1) + D_{m}^{(2)} + 2 D_{m}^{(4)} \cos^2 \theta]^2 (N - 2) \sin^2 2\theta}{C_{22}^{2}} + \frac{2}{C_{11}^{2}} [- \frac{3\omega}{4}(\Phi_0 + \Phi_1) + D_{m}^{(2)} + 2 D_{m}^{(4)} \cos^2 \theta]^2 \sin^2 2\theta
\]

From equation 3 and 9,
\[ E(\theta) = -\frac{J}{2} \left[ N Z_0 + 2(N - 1) Z_1 \right] + \left\{ N \Phi_0 + 2(N - 1) \Phi_1 \right\} \left( \frac{\omega}{8} + \frac{3\omega}{8} \cos 2\theta \right) \]
\[ -N \left( \cos^2 \theta D_m^{(2)} + \cos^4 \theta D_m^{(4)} + H_m \sin \theta + H_{out} \cos \theta - \frac{N_d}{\mu_0} + K_s \sin 2\theta \right) \]
\[ + \left[ -\frac{3\omega}{4} (\Phi_0 + 2\Phi_1) + D_m^{(2)} + 2D_m^{(4)} \cos^2 \theta \right] \frac{\left( N - 2 \right) \sin^2 2\theta}{2C_{22}} \]
\[ -\frac{1}{C_{11}} \left[ -\frac{3\omega}{4} (\Phi_0 + \Phi_1) + D_m^{(2)} + 2D_m^{(4)} \cos^2 \theta \right] \frac{\sin^2 2\theta}{2} \]
\[ -\beta_{12} \left\{ \frac{2\alpha_1^2 \alpha_2}{C_{22} C_{11}^2} + \left( \frac{\alpha_2}{C_{22}} \right)^2 \left[ \frac{2\alpha_1}{C_{11}} + \frac{2\alpha_2 \left( N - 4 \right)}{C_{22}} \right] \right\} - \frac{2\alpha_1 \beta_{11}}{C_{11}^2} - 2\left( \frac{\alpha_2}{C_{22}} \right)^3 \beta_{22} \left( \frac{N}{2} - 1 \right) \]

Energy can be found from above equation by substituting \( C_{11}, C_{22}, \alpha_1, \alpha_2, \beta_{11}, \beta_{22} \) and \( \beta_{12} \) in above equation. Because \( \alpha_1 \) and \( \alpha_2 \) depend on \( \sin(2\theta) \), second and third order perturbation terms become zero at \( \theta = 0^0 \) and \( 90^0 \).

This simulation will be carried out for 
\[ \frac{J}{\omega} = \frac{D_m^{(2)}}{\omega} = \frac{H_{in}}{\omega} = \frac{H_{out}}{\omega} = \frac{N_d}{\mu_0 \omega} = \frac{K_s}{\omega} = 10, \text{ and } \frac{D_m^{(4)}}{\omega} = 5. \]

For sc(001) lattice with \( Z_0=4, Z_1=1, \Phi_0 = 9.0336 \) and \( \Phi_1 = -0.3275 \) [1], the 3-D plot of \( \frac{E(\theta)}{\omega} \) versus \( \theta \) and \( N \) is given in figure 1. Although the equation is valid for large values of \( N \) only, the graph has been drawn for even small values of \( N \) too in order to study the variation of energy at small values of \( N \) too. This plot is similar to the 3-D plot of thick sc(001) ferromagnetic film with second order perturbation [6], and the energy is also same. At near \( N=45 \), the energy separation between maximum and minimum is really small, implying that the energy needed to rotate from easy direction to hard direction (anisotropy energy) is really small for the film with this number of layers.

When \( N=10000 \), the graph between \( \frac{E(\theta)}{\omega} \) and \( \theta \) is given in figure 2. Although the some peaks are missing in this graph compared with that of thick sc(001) film (\( N=10000 \)) with second order perturbation [6], the peaks positions of observable two peaks are same. But these two peaks have been inverted and the curve has been flattened compared with the curve of thick sc(001) film with second order perturbation. Easy and hard directions at overshooting part are \( 240.8^0 \) and \( 338.2^0 \), respectively. The angle between easy and hard directions is \( 97.4^0 \).

For fcc(001) lattice with \( Z_0=4, Z_1=4, \Phi_0 = 9.0336 \), and \( \Phi_1 = 1.4294 \) [1], the 3-D plot of \( \frac{E(\theta)}{\omega} \) versus \( \theta \) and \( N \) is given in figure 3. Energy of this curve is slightly less than that of thick sc(001) film (\( N=10000 \)) with second order perturbation [6]. The energy difference between minimum and maximum is very small at near \( N=40 \) and this implies that the energy required to rotate from easy direction to hard direction (anisotropy energy) is very small at this thickness.

When \( N=10000 \), the graph between \( \frac{E(\theta)}{\omega} \) and \( \theta \) is given in figure 4. Peak positions are different from those of thick fcc(001) films with second order perturbation [6]. The curve has been flattened compared with the curve of thick fcc(001) film with second order
perturbation. The angles at minimum and maximum energies of overshooting part are 302.55° and 335°, respectively.

Conclusion

For sc(001) thick ferromagnetic films, it is difficult to find major differences between second order [6] and third order perturbation curves. At near N=45, the anisotropy energy is small in the 3-D plot of sc(001), indicating that the energy required to rotate from easy to hard direction is really small. The energy curve of sc (001) with N=10000 has been flattened by reducing the smooth part of the curve compared with that of sc(001) with second order perturbation [6]. The angle between the easy and hard direction is 97.4° for sc(001) thick film with N=10000. According to 3-D plot of fcc(001), the energy required to rotate the spin from easy to hard direction is very small for films with approximately N=40 thicknesses. For fcc(001) thick film with N=10000, the curve has been flattened compared with that of thick fcc(001) with second order perturbation [6], and the angle between easy and hard direction of overshooting part is 32.45°. The overshooting parts appear after introducing second or third order perturbation, and hence the angle between easy and hard directions is not 90°. But the angle between easy and hard direction was 90° for oriented unperturbed films. The third and second order perturbation vanish at θ=0° and 90° directions. Although this simulation was given here for $\frac{d}{\omega} = \frac{D^{(2)}_m}{\omega} = \frac{6}{\omega}$, $\frac{H_{in}}{\omega} = \frac{H_{out}}{\omega} = \frac{N_d}{\mu_0}, \frac{K_s}{\omega} = 10$, and $\frac{D^{(4)}_m}{\omega} = 5$, this simulation can be carried out for any values of $\frac{d}{\omega}, \frac{D^{(2)}_m}{\omega}, \frac{H_{in}}{\omega}, \frac{H_{out}}{\omega}, \frac{N_d}{\mu_0}, \frac{K_s}{\omega}$ and $\frac{D^{(4)}_m}{\omega}$.

References

Fig. 1 3-D plot of $\frac{E(\theta)}{\omega}$ versus $\theta$ and $N$ for sc(001).

Fig. 2 Graph between $\frac{E(\theta)}{\omega}$ and $\theta$ for $N=10000$ and sc(001).
Fig. 3 3-D plot of $\frac{E(\theta)}{\omega}$ versus $\theta$ and $N$ for fcc(001).

Fig. 4 Graph between $\frac{E(\theta)}{\omega}$ and $\theta$ for $N=10000$ and fcc(001).
Viscous Dusty Fluid Flow with Constant Velocity Magnitude

Siddabasappa\(^1\)*, Y.Venkateshappa\(^2\†, B.Rudraswamy\(^2\), B.J.Gireesha\(^3\‡\) and K.R.Gopinath\(^4\)

\(^1\)Department of P.G studies in mathematics, Government Science college, Bangalore-560001, India
\(^2\)Department of Physics, J.B.campus, Bangalore University, Bangalore-560056, India
\(^3\)Department of Mathematics, Kuvempu University, Shankaraghatta-577451, Shimoga, India
\(^4\)Department of Mathematics, Baldwin Methodist College, Bangalore, India

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Abstract: We consider the viscous dusty fluid, where the velocity of the dust particle is everywhere parallel to that of the fluid with velocity magnitude of the fluid is constant along each individual streamline. Also it is assumed that number density of the dust particle is constant and the dust particles are uniform in size and shape and bulk concentration of the dust is small. Hodograph and Legendre transform of stream function is employed to get the solutions and the geometry of streamlines for these flows by using the resulting partial differential equations when the Jacobian is zero and nonzero cases. In each case the variation of pressure is analyzed graphically.

Keywords: Dusty Fluids; Hodograph and Legendre Transform; Velocity Components; Pressure Function; Streamlines

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1. Introduction

Cosmic dust is widely present in space, where gas and dust clouds are primary precursors for planetary systems. The zodiacal light, seen in the sky on a dark night, is produced by sunlight reflected from particles of dust in orbit around the sun. The tails of comets

* basappagas@yahoo.com
† ananya@rediffmail.com and ananya@rediffmail.com
‡ bjgireesu@rediffmail.com
are produced by emissions of dust and ionized gas from the body of the comet. Dust also covers solid planetary bodies, and vast dust storms can occur on Mars that can cover almost the entire planet. Interstellar dust is found between the stars, and high concentrations can produce diffuse nebulae and reflection nebulae. Dust samples returned from outer space could provide information about conditions in the early solar system. Several spacecraft have been launched in an attempt to gather samples of dust and other materials. Among these was stardust, which flew past comet wild 2 in 2004 and returned a capsule of the remains of the comet to the U.S. in January 2006. The Japanese Hayabusa spacecraft is currently on a mission to collect samples of dust from the surface of an asteroid.

In the present paper some assumptions are made on Saffman model and the basic equations of fluid phase and dust phase are written into a convenient form by using suitable hodograph transformation. Further it is assumed that velocity vector of fluid is everywhere parallel to that of dust velocity and fluid is flowing with constant velocity magnitude. By introducing stream function and Legendre transform of this stream function, flow equations are recasted in the transformed function. By assuming the Jacobian is zero and nonzero, exact solutions to flow variables are obtained. It is shown that the flow is irrotational and streamlines are concentric circles when Jacobian is nonzero and flow is rotational and streamlines are straight lines when it is zero.

P.G.Saffman [1] discussed stability of the laminar flow of a dust gas in which the dust particles are uniformly distributed. Marble [2] discussed the dynamics of dusty gas. R.M.Barron [3, 4] studied two dimensional steady flow of a dusty gas. He obtained solutions to flow variables in orthogonal curvilinear co-ordinate system. Also by considering the dust particle distribution to be variable and the velocity of dust particle is everywhere parallel to velocity of fluid and proved that the possible flows are radial and streamlines are straight lines. He established that the dust particle distribution can not be uniform in radial flow and possible stream lines are only parallel straight lines. He found solutions to flow variables in natural coordinate system where the coordinate axes are the stream lines $\eta$ is a constant and their orthogonal trajectories are curves $\xi = \text{constant}$.

O.P.Chandana et al [5] studied rotational plane flow of viscous fluid in the hodograph plane using Legendre transformation of the stream function. Satter [6] by assuming velocity magnitude is constant along each stream line, given solutions to flow variables of steady plane MHD flow of viscous incompressible fluid of infinite electrical conductivity when magnetic field vector is constantly inclined to velocity vector. M.H.Hadaman et al [7] analyzed the squeezing flow dust fluid and concluded the introduction of dust to fluid squeeze film increase the load carrying capacity of the squeeze. C.S.Bagewadi et al [8] studied the flow of dust gas using Frenet frame field system and found solutions to flow variables using Laplace transformation method. In [9] they studied the geometry of streamlines on spherical surface, inverse surface and parallel surface and that the streamlines are concentric circles on this surface by the method of metric coefficients. Siddabasappa et al [12] obtained the exact solutions to dusty gas flow variables including viscosity and compressibility in different surfaces like spherical, centro, Beltrami surfaces.
using fundamental magnitudes and differential geometry. In [13, 14] the importance and application of flows, helps us to analyze waste water treatment, corrosive particles in engine, oil flows, air pollution, smoke emission from vehicles, emission of fine particles from cement industries, nuclear reactors, filtration, etc. Also, it gives the information about water pollution like rain fall in space, flow of blood, pumping of water in pipes. The possible presence of solid particles such as ash or soot in combustion MHD generators and plasma MHD accelerators and their effect on the performance of such devices led to studies of particulate suspensions in conducting fluids in the presence of magnetic fields. For example in an MHD generator, coal mixed with seed is fed into a combustor. The coal and seed mixture is burned in oxygen and combustion gas expands through a nozzle before it enters the generator section. The gas mixture flowing through the MHD channel consists of a condensable vapour (slag) and a non condensable gas mixed with seed coal combustion products. Both the slag and the non condensable gas are electrically conducting. The presence of slag and seeded particles significantly influences the flow in the MHD channel. This field has important applications in areas as cooling systems, centrifugal separation of matter from fluid, petroleum industry, purification of crude oil, electrostatic precipitation, polymer technology and fluid droplets springs.

2. Governing Equation

Based on Saffman model of the steady motion of a incompressible fluid the governing equations of the flow are

For fluid phase

\[ \nabla \cdot \vec{u} = 0 \quad \text{continuity equation} \] (1)

\[ \rho (\vec{u} \cdot \nabla) \vec{u} + \nabla p = \mu \nabla^2 \vec{u} + KN(\vec{v} - \vec{u}) \quad \text{momentum equation} \] (2)

For dust phase

\[ \nabla \cdot (N \vec{v}) = 0 \quad \text{continuity equation} \] (3)

\[ (\vec{v} \cdot \nabla) \vec{v} = \frac{K}{m} (\vec{u} - \vec{v}) \quad \text{momentum equation} \] (4)

where

\[ \vec{u} \] - velocity of the fluid \(=(u_1, u_2)\)

\[ \vec{v} \] - velocity of the dust particle \(=(v_1, v_2)\)

\[ \rho \] - density of fluid

\[ p \] - fluid pressure

\[ \mu \] - viscosity of fluid

\[ N \] - number density of dust particle per unit volume (constant)

\[ K \] - stock’s coefficient of resistance \((6\pi a \mu)\) for spherical dust particles and ‘a’ is the average radius of dust particles

\[ m \] - average mass of dust particles.

In the present situation the dust particles are assumed to be spherical and uniform in size.
and shape and are uniformly distributed throughout the fluid. The bulk concentration of the dust is small \( \frac{n}{K} = \tau \) may be called the relaxation time, the dust particle \( \frac{nN}{\rho} = f \) mass concentration of dust particle. The last term in equation (2) represents the force due to the relative motion between fluid and dust particles.

Let the velocity of the fluid be every where particle to dust particle velocity so that

\[
\vec{v} = \frac{\alpha}{N} \vec{u} \tag{5}
\]

where \( \alpha \) is to be determined. Using \( \vec{v}(v_1, v_2) \) and \( \vec{u}(u_1, u_2) \) equation (5) reduces to

\[
v_1 = \frac{\alpha}{N} u_1 \quad \text{and} \quad v_2 = \frac{\alpha}{N} u_2. \tag{6}
\]

Equations (1)-(4) using (6) reduces to the following system of six equations.

\[
\begin{align*}
\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} &= 0 \tag{7} \\
\rho(u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_2}{\partial x}) + \frac{\partial p}{\partial x} &= \mu(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2}) + \\
\rho u_2(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}) &= KN(v_1 - u_1) \tag{8} \\
\rho(u_1 \frac{\partial u_1}{\partial y} + u_2 \frac{\partial u_2}{\partial y}) + \frac{\partial p}{\partial y} &= \mu(\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2}) + \\
\rho u_1(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}) &= KN(v_2 - u_2) \tag{9} \\
\frac{\partial (Nv_1)}{\partial x} + \frac{\partial (Nv_2)}{\partial y} &= 0 \tag{10} \\
(v_1 \frac{\partial v_1}{\partial x} + v_2 \frac{\partial v_2}{\partial x}) - v_2(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}) &= \frac{K}{m}(u_1 - v_1) \tag{11} \\
(v_1 \frac{\partial v_1}{\partial y} + v_2 \frac{\partial v_2}{\partial y}) + v_1(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}) &= \frac{K}{m}(u_2 - v_2). \tag{12}
\end{align*}
\]

By assuming number density of the dust particle is constant and introducing vorticity function \( \xi(x, y) \) and energy function \( h(x, y) \) as

\[
\begin{align*}
\xi(x, y) &= \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \tag{13} \\
h(x, y) &= p + \frac{1}{2} \rho u^2 \quad \text{where} \quad u^2 = u_1^2 + u_2^2 \tag{14}
\end{align*}
\]
the system of equations (7)-(12) reduces to the system,

\[ \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0 \]  
(15)

\[ \mu \frac{\partial \xi}{\partial y} - \rho u_2 \xi + KN(u_1 - v_1) = - \frac{\partial h}{\partial x} \]  
(16)

\[ \mu \frac{\partial \xi}{\partial x} - \rho u_1 \xi + KN(v_2 - u_2) = \frac{\partial h}{\partial y} \]  
(17)

\[ u_1 \frac{\partial \alpha}{\partial x} + u_2 \frac{\partial \alpha}{\partial y} = 0 \]  
(18)

\[ m \alpha^2 \left( u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} \right) = KN(N - \alpha)u_1 \]  
(19)

\[ m \alpha^2 \left( u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} \right) = KN(N - \alpha)u_2 \]  
(20)

Equations (6), (13), (14) and (15) - (20) is a system of first order nine partial differential equation in eight unknown functions \(u_1(x, y), u_2(x, y), v_1(x, y), v_2(x, y), \xi(x, y), h(x, y), p(x, y)\) and \(\alpha(x, y)\). where \(\alpha(x, y)\) is calculated from (18) and \(v_1\) and \(v_2\) from (6). Reducing the order of differential equation from two to one is successfully done in (11). Equation (13) gives vorticity and \(h(x, y)\) is form (16) and (17). Lastly the pressure function \(p(x, y)\) is obtained, using equation (14). \(\tau = \frac{\mu}{K}\) may be discussed using equations (19) and (20). Using integrability condition on \(h(x, y)\) from (16) and (17) we have

\[ \mu \left[ \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right] - \rho \left[ u_1 \frac{\partial \xi}{\partial x} + u_2 \frac{\partial \xi}{\partial y} \right] - K(N - \alpha) \left[ \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right] \]

\[ + K \left\{ u_1 \left[ N - \frac{\partial \alpha}{\partial y} \right] - u_2 \left[ N - \frac{\partial \alpha}{\partial x} \right] \right\} = 0 \]  
(21)

Now equations (6), (13), (14), (15), (18), (19), (20) and (21) is system of eight partial differential equations in eight unknowns \(u_1, u_2, v_1, v_2, \xi, h, p\) and \(\alpha\).

**Flows with constant velocity magnitude:**

Now the fluid is flowing with constant velocity magnitude along each individual streamline. We must have

\[ \vec{u} \cdot \text{grad} \ u^2 = 0 \]

so that

\[ u_1^2 \frac{\partial u_1}{\partial x} + u_1 u_2 \left( \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right) + u_2^2 \frac{\partial u_2}{\partial y} = 0 \]  
(22)

Hence in the flow with constant velocity magnitude \(u_1\) and \(u_2\) must satisfy the equation (15) and (22). Once a solution of \(u_1\) and \(u_2\) are determined from (15) and (22), the pressure function is found from the definition of energy function. Now consider the following case

1. \(J \neq 0\) in the entire region of flow
2. \(J = 0\) in the entire region of flow
3. \( J \neq 0 \) in a part of the region and \( J = 0 \) in the remaining part of the region where \( J \) is the Jacobian.

**I when \( J \neq 0 \) in the entire region of flow**

**Hodograph transformations**

Letting the function \( u_1 = u_1(x, y) \) and \( u_2 = u_2(x, y) \) to be such that in the region of flow the Jacobian

\[
J = \frac{\partial u_1}{\partial x} \frac{\partial u_2}{\partial y} - \frac{\partial u_1}{\partial y} \frac{\partial u_2}{\partial x} \neq 0, \quad 0 < |J| < \infty
\]  

(23)

We may consider \( x \) and \( y \) as functions of \( u_1 \) and \( u_2 \) by means of \( x = x(u_1, u_2) \) and \( y = y(u_1, u_2) \), we have the relations

\[
\frac{\partial u_1}{\partial x} = J \frac{\partial y}{\partial u_2}, \quad \frac{\partial u_1}{\partial y} = -J \frac{\partial x}{\partial u_2}, \\
\frac{\partial u_2}{\partial x} = -J \frac{\partial y}{\partial u_1}, \quad \frac{\partial u_2}{\partial y} = J \frac{\partial x}{\partial u_1}.
\]

(24)

With the application of transformation (23) and (24) for the first order partial derivatives appearing in the above system we have the partial differential equations in the \((u, v)\) plane.

\[
\frac{\partial x}{\partial u_1} + \frac{\partial y}{\partial u_2} = 0
\]

(25)

\[
\mu \left[ J \frac{\partial (w_1, y)}{\partial u_1} + J \frac{\partial (x, w_2)}{\partial u_2} \right] - \rho \left[ u_1 J \frac{\partial (\xi, y)}{\partial u_1} + u_2 J \frac{\partial (x, \xi)}{\partial u_2} \right] \\
- \frac{K}{\rho} (N - \alpha) \left( \frac{\partial^2 L}{\partial u_1^2} + \frac{\partial^2 L}{\partial u_2^2} \right) + u_1 \left[ N - J \frac{\partial (x, \alpha)}{\partial (u_1, u_2)} \right] \\
- u_2 \left[ N - J \frac{\partial (\alpha, y)}{\partial (u_1, u_2)} \right] = 0,
\]

(26)

\[
J \left[ u_1 \frac{\partial (\alpha, y)}{\partial (u_1, u_2)} + u_2 \frac{\partial (x, \alpha)}{\partial (u_1, u_2)} \right] = 0
\]

(27)

\[
J \alpha^2 \left[ u_1 \frac{\partial y}{\partial u_2} - u_2 \frac{\partial x}{\partial u_2} \right] = KN(N - \alpha)u_1
\]

(28)

\[
J \alpha^2 \left[ u_2 \frac{\partial x}{\partial u_1} - u_1 \frac{\partial y}{\partial u_1} \right] = KN(N - \alpha)u_2
\]

(29)

\[
\xi = J \left[ \frac{\partial x}{\partial u_2} - \frac{\partial y}{\partial u_1} \right]
\]

(30)

\[
J = \left( \frac{\partial y}{\partial u_2} \frac{\partial x}{\partial u_1} - \frac{\partial x}{\partial u_2} \frac{\partial y}{\partial u_1} \right)^{-1} = j
\]

(31)

\[
u_1^2 \frac{\partial y}{\partial u_2} - u_1 u_2 \left( \frac{\partial x}{\partial u_2} - \frac{\partial y}{\partial u_1} \right) + u_2^2 \frac{\partial x}{\partial u_1} = 0
\]

(32)

**Equations in Legendre transform function**

The equation of continuity (15) implies the existence of a stream function \( \Psi(x, y) \) such that

\[
d\Psi = -u_2 dx + u_1 dy \quad \text{or} \quad \frac{\partial \Psi}{\partial x} = -u_2, \quad \frac{\partial \Psi}{\partial y} = u_1.
\]

(33)
Likewise equation (25) implies the existence of a function \( L(x, y) \) called the Legendre transform function of the stream function \( \Psi(x, y) \) such that

\[
dL = -y du_1 + x du_2 \quad \text{or} \quad \frac{\partial L}{\partial u_1} = -y, \quad \frac{\partial L}{\partial u_2} = x.
\] (34)

Introducing \( L(u_1, u_2) \) into the system (25)-(32), it follows that (25) is identically satisfied and the system may be replaced by

\[
\xi = j \left[ \frac{\partial^2 L}{\partial u_1^2} + \frac{\partial^2 L}{\partial u_2^2} \right]
\] (35)

\[
J^{-1} = \left[ \frac{\partial^2 L}{\partial u_1^2} \frac{\partial^2 L}{\partial u_2^2} - \left( \frac{\partial^2 L}{\partial u_1 \partial u_2} \right)^2 \right] = j
\] (36)

\[
\mu \left( \frac{\partial(u_1, -\frac{\partial L}{\partial u_1})}{\partial(u_1, u_2)} + \frac{\partial L}{\partial(u_1, u_2)} \right) - \left( u_1 \frac{\partial(\xi, -\frac{\partial L}{\partial u_1})}{\partial(u_1, u_2)} + u_2 \frac{\partial L}{\partial u_2} \right) - \frac{K}{\rho} (N - \alpha) \left( \frac{\partial^2 L}{\partial u_1^2} + \frac{\partial^2 L}{\partial u_2^2} \right) + K \left\{ u_1 \left( \frac{N}{J} - \frac{\partial L}{\partial u_2} \right) - \frac{\partial L}{\partial u_1} \right\} = 0
\] (37)

\[
J \left( u_1 \frac{\partial(\alpha, -\frac{\partial L}{\partial u_1})}{\partial(u_1, u_2)} + u_2 \frac{\partial L}{\partial u_2} \right) = 0
\] (38)

\[
-Jm\alpha^2 \left( u_1 \frac{\partial^2 L}{\partial u_2 \partial u_1} + u_2 \frac{\partial^2 L}{\partial u_2 \partial u_1} \right) = KN(N - \alpha) u_1
\] (39)

\[
-Jm\alpha^2 \left( u_1 \frac{\partial^2 L}{\partial u_2 \partial u_1} + u_2 \frac{\partial^2 L}{\partial u_2 \partial u_1} \right) = KN(N - \alpha) u_2
\] (40)

\[
u_1 u_2 \frac{\partial^2 L}{\partial u_1^2} + (u_2^2 - u_1^2) \frac{\partial^2 L}{\partial u_1 \partial u_2} - u_1 u_2 \frac{\partial^2 L}{\partial u_2^2} = 0
\] (41)

**Polar coordinates:**

By employing Polar coordinates \((q, \theta)\) in the hodograph plane, defined by the relations

\[
u_1 = q \cos \theta, \quad u_2 = q \sin \theta, \quad q = \sqrt{u_1^2 + u_2^2}, \quad \theta = \tan^{-1} \left( \frac{u_2}{u_1} \right)
\] (42)

and defining \( L^*(q, \theta) \), \( \xi^*(q, \theta) \), \( J^*(q, \theta) \) to be the Legendre transform, vorticity and jacobian functions in \((q, \theta)\) coordinates. Using

\[
\frac{\partial(F, G)}{\partial(u_1, u_2)} = \frac{\partial(F^*, G^*)}{\partial(q, \theta)} \quad \text{and} \quad \frac{\partial(q, \theta)}{\partial(u_1, u_2)} = \frac{1}{q} \frac{\partial(F^*, G^*)}{\partial(q, \theta)}
\] (43)
where \( F(u_1, u_2) = F^*(q, \theta) \), \( G(u_1, u_2) = G^*(q, \theta) \) are continuously differentiable functions, the equations (37) becomes

\[
\mu \left( \frac{\partial (w_1, -\cos \theta \frac{\partial L}{\partial q} + \sin \theta \frac{\partial L}{\partial \theta})}{\partial (u_1, u_2)} + \frac{\partial (\sin \theta \frac{\partial L}{\partial q} + \cos \theta \frac{\partial L}{\partial \theta}, w_2)}{\partial (u_1, u_2)} \right) - \left( \frac{\partial (\xi, -\cos \theta \frac{\partial L}{\partial q} + \sin \theta \frac{\partial L}{\partial \theta})}{\partial (u_1, u_2)} + u_2 \frac{\partial (\sin \theta \frac{\partial L}{\partial q} + \cos \theta \frac{\partial L}{\partial \theta}, \xi)}{\partial (u_1, u_2)} \right)
- qK \left[ \frac{\partial^2 L}{\partial q^2} + \frac{1}{q} \frac{\partial L}{\partial q} + \frac{1}{q^2} \frac{\partial^2 L}{\partial \theta^2} \right] + K \left[ u_1 \left( \frac{\partial (\sin \theta \frac{\partial L}{\partial q} + \cos \theta \frac{\partial L}{\partial \theta}, \alpha)}{\partial (u_1, u_2)} \right) \right. \\
\left. - u_2 \left( \frac{\partial (\alpha, -\cos \theta \frac{\partial L}{\partial q} + \sin \theta \frac{\partial L}{\partial \theta})}{\partial (u_1, u_2)} \right) \right] = 0
\]

Since the equation (44) holds identically for all values of q, equating the coefficients of different powers of q, we have

\[
\mu \left( \frac{\partial (w_1, -\cos \theta \frac{\partial L}{\partial q} + \sin \theta \frac{\partial L}{\partial \theta})}{\partial (u_1, u_2)} + \frac{\partial (\sin \theta \frac{\partial L}{\partial q} + \cos \theta \frac{\partial L}{\partial \theta}, w_2)}{\partial (u_1, u_2)} \right) - \left( \frac{\partial (\xi, -\cos \theta \frac{\partial L}{\partial q} + \sin \theta \frac{\partial L}{\partial \theta})}{\partial (u_1, u_2)} + u_2 \frac{\partial (\sin \theta \frac{\partial L}{\partial q} + \cos \theta \frac{\partial L}{\partial \theta}, \xi)}{\partial (u_1, u_2)} \right)
+ k \left[ u_1 \left( \frac{\partial (\sin \theta \frac{\partial L}{\partial q} + \cos \theta \frac{\partial L}{\partial \theta}, \alpha)}{\partial (u_1, u_2)} \right) \right. \\
\left. - u_2 \left( \frac{\partial (\alpha, -\cos \theta \frac{\partial L}{\partial q} + \sin \theta \frac{\partial L}{\partial \theta})}{\partial (u_1, u_2)} \right) \right] = 0
\]

and the equation (41) in \((q, \theta)\) form is

\[
\frac{\partial^2 L^*}{\partial q \partial \theta} - \frac{\partial L^*}{\partial \theta} = 0
\]

Now we use (44) and (47) to get the expression for L. Once \( L(q, \theta) \) is defined we can express it in \((u, v)\) from with the help of (42), J is evaluated from (36) satisfies \( 0 < |J| < \infty \). The solutions for the velocity components \( u_1 \) and \( u_2 \) are obtained by solving equations \( x = \frac{\partial L}{\partial u_2}, \quad y = -\frac{\partial L}{\partial u_1} \). After obtaining velocity components \( \alpha(x, y) \) can be evaluated by solving (18), then the velocity component of dust are from equation (6). Having obtained the velocity components in the physical plane vorticity and energy functions are determined from the vorticity and linear momentum equations in the system of equations (13), (16) and (17). Finally the pressure function is evaluated from (14).

**Solutions to flow variables**

Assuming the most general solution of (47) in the form

\[
L^*(q, \theta) = q\phi(\theta) + \chi(q)
\]
where $\phi$ and $\chi$ are arbitrary functions of their arguments. Now equation (48) in (46) gives us

$$q\chi'' + \chi' + (\phi'' + \phi) = 0.$$  \hfill (49)

Set

$$q\chi'' + \chi' = \lambda \quad \text{and} \quad \phi'' + \phi = -\lambda$$ \hfill (50)

where $\lambda$ is constant and primes denote differentiation with respect to the arguments. Now from (50) we have

$$\chi = \lambda q - \lambda_1 \ln q + \lambda_2 \quad \text{and} \quad \phi = A \cos \theta + B \sin \theta - \lambda$$ \hfill (51)

where $A, B, \lambda_1, \lambda_2$ are arbitrary constants. Equations (51) in (48) gives us

$$L^*(q, \theta) = Aq \cos \theta + Bq \sin \theta - \lambda_1 \ln q + \lambda_2.$$  \hfill (52)

The above equation in $u_1$ and $u_2$ plane using (42) is

$$L(u_1, u_2) = Au_1 + Bu_2 - \frac{\lambda_1}{2} \ln(u_1^2 + u_2^2) + \lambda_2.$$ \hfill (52)

From (34) and (52) we have

$$x = B - \frac{u_2 \lambda_1}{u_1^2 + u_2^2} \quad \text{and} \quad y = -A + \frac{u_1 \lambda_1}{u_1^2 + u_2^2}$$

and hence

$$u_1 = \frac{\lambda_3 (y + A)}{\lambda_1} \quad \text{and} \quad u_2 = \frac{\lambda_3 (B - x)}{\lambda_1}$$ \hfill (53)

provided

$$u_1^2 + u_2^2 = \lambda_1^2 \left[ (y + A)^2 + (B - x)^2 \right] = \lambda_3 (\text{constant})$$

Using (53) in (13) and (23) the vorticity and jacobian are

$$\xi = \frac{-2\lambda_3}{\lambda_1} \quad \text{and} \quad J = \frac{2\lambda_3}{\lambda_1}$$ \hfill (54)

This shows that the flow is irrotational. From (18) and (53) we have

$$(y + A) \frac{\partial \alpha}{\partial x} + (B - x) \frac{\partial \alpha}{\partial y} = 0$$ \hfill (55)

the general solution of this equation is

$$\alpha(x, y) = a_1 x + \frac{(y + A)}{B - x} a_1 xy + a_2$$ \hfill (56)
where $a_1$ and $a_2$ are constants. From (53), (56) and (6), velocity components of dust particles are given by

$$v_1 = \frac{\lambda_3(y + A)}{N\lambda_1} \left[(a_1x + a_2) + a_1xy(y + a)(B - x)^{-1}\right]$$

$$v_2 = \frac{\lambda_3}{N\lambda_1} (a_1x + a_2)(B - x) + \frac{\lambda_2a_1xy(y + A)}{N\lambda_1}$$

(57)

The integrability condition on $h$ from (16), (17) and using (6), the pressure function $p(x, y)$ is given by

$$p(x, y) = \rho \frac{\lambda_3^2}{\lambda_1^2} \left(y^2 + 2Ay - Bx + \frac{x^2}{2}\right) - K \frac{\lambda_3}{\lambda_1} [(y + A)(N - C)x$$

$$-(y + A)\frac{x^3}{3} - \alpha x^2 - (C - N) - (N - C)x + \alpha \frac{x^3}{3} + (3x$$

$$+ 2B \log(x - B)) \left(a \frac{y^3}{3} + aA \frac{y^2}{2}\right) + (x + B \log(x - B))\right]$$

$$+ K (y + A)^2 \frac{\lambda_3}{\lambda_1} \alpha y \left[x + B \log(x - B)\right]$$

$$- \frac{1}{2} \rho \frac{\lambda_3^2}{\lambda_1^2} [(y + A)^2 + (B - x)^2]$$

(58)

The streamlines are given by $(x - B)^2 + (y + A)^2 = \text{constant}$. Hence streamlines are concentric circles. The variation of pressure for different densities is shown in figure (1) and (2).

II When the Jacobian is zero

Consider $J = \frac{\partial u_2}{\partial x} \frac{\partial u_1}{\partial y} - \frac{\partial u_2}{\partial y} \frac{\partial u_1}{\partial x} = 0$ in the entire region of flow. In this case $u_1$ is a function of $u_2$ or $u_2$ is a function of $u_1$. Consider the case when $u_2$ is a function of $u_1$

Let

$$u_2 = f(u_1)$$

(59)

where 'f' is an arbitrary of $u_1$. Using (59) in (15) and (22) we get

$$\frac{\partial u_1}{\partial x} + f'(u_1) \frac{\partial u_1}{\partial y} = 0$$

(60)

$$(u_1^2 + u_1ff') \frac{\partial u_1}{\partial x} + (u_1f + f^2f') \frac{\partial u_1}{\partial y} = 0$$

(61)

Eliminating $\frac{\partial u_1}{\partial x}$ from (60) and (61), we have

$$(u_1f' - f)(u_1 + f)f' \frac{\partial u_1}{\partial y} = 0$$

(62)

implies that either

$$i) \frac{\partial u_1}{\partial y} = 0 \text{ or } ii) (u_1f' - f) \text{ or } iii) (u_1 + ff') = 0$$
(i) If $\frac{\partial u_1}{\partial y} = 0$ gives us $\frac{\partial u_1}{\partial x} = 0$. Therefore $u_1 = \text{constant}$, $u_2 = \text{constant}$.

Let $u_1 = c_1$ and $u_2 = c_2$  \hspace{1cm} (63)

From (18)

$$\alpha = b_1 \left[ x - \frac{c_1}{c_2} y \right] + b_2$$  \hspace{1cm} (64)

and therefore

$$\xi = 0 \quad \text{and} \quad j = 0$$

$$v_1 = \frac{b_1 c_1 (c_2 x - c_1 y) + c_2 b_2}{N c_2}, \quad v_2 = \frac{b_1 c_2 x - c_1 y + b_2}{N}$$  \hspace{1cm} (65)

$$h(x, y) = akc_2 xy + K(c_3 - N)(c_1 x + c_2 y) + \frac{Kc_1}{2} a(x^2 - y^2) - c_5$$

and hence pressure function is

$$p(x, y) = aKc_2 xy + K(c_3 - N)(c_1 x + c_2 y) + \frac{Kc_1}{2} a(x^2 - y^2) - c_5 - \frac{\rho}{2}(c_1^2 + c_2^2).$$  \hspace{1cm} (66)

The variation of $p$ is graphed in figure (3) and (4). In this case streamlines are straight lines given by

$$c_2 x - c_1 y = \text{constant}$$

(ii) Next $u_1 f' - f = 0$ this gives $f = d_1 u_1$. Put $f' = d_1$ in (63), we have

$$\frac{\partial u_1}{\partial x} + d_1 \frac{\partial u_1}{\partial y} = 0$$  \hspace{1cm} (67)

The general solution of this equation is

$$u_1 = g(d_1 x - y)$$  \hspace{1cm} (68)
where \(g\) is an arbitrary. By taking one particular value \(u_1 = d_1x - y\) we have the following exact solutions to flow variables,

\[
\begin{align*}
    u_1 &= d_1x - y, \\
    v_1 &= \frac{d_2(d_1x - y)^2}{d_1N} + \frac{d_3}{N}(d_1x - y), \\
    v_2 &= \frac{d_2(d_1x - y)^2}{d_1N} + \frac{d_3d_1}{N}(d_1x - y) \\
    h(x, y) &= -\rho u_2(d_1^2 + 1)x - K\left(d_1\frac{x^2}{2} - xy\right)N + K\left(d_1\frac{x^2}{2} - yx\right) \\
    &= \left\{d_2\left(x - \frac{y}{d_1}\right) + d_3\right\} + K(d_1x - y)\left(\frac{x^2}{2} - \frac{xy}{d_1}\right) - (d_3x - \rho d_1^2 + \rho - \frac{Kx}{d_1} - ax - d_3 + N)\left(d_1xy - \frac{y^2}{2}\right) - \left(d_1\frac{xy^2}{2} - \frac{y^3}{3}\right) d_2 \\
    &= -Kx\left(N - 3d_2x\frac{2}{2} - 2d_3 - \frac{x}{2}\right) y - \frac{Kx}{d_1}(d_2 + 1)\frac{y^2}{2} \\
\end{align*}
\]

The pressure expression is

\[
p(x, y) = -\rho u_2(d_1^2 + 1)x - K\left(d_1\frac{x^2}{2} - xy\right)N + K\left(d_1\frac{x^2}{2} - yx\right) \\
    &= \left\{d_2\left(x - \frac{y}{d_1}\right) + d_3\right\} + K(d_1x - y)\left(\frac{x^2}{2} - \frac{xy}{d_1}\right) - (d_3x - \rho d_1^2 + \rho - \frac{Kx}{d_1} - ax - d_3 + N)\left(d_1xy - \frac{y^2}{2}\right) - \left(d_1\frac{xy^2}{2} - \frac{y^3}{3}\right) d_2 \\
    &= -Kx\left(N - 3d_2x\frac{2}{2} - 2d_3 - \frac{x}{2}\right) y - \frac{Kx}{d_1}(d_2 + 1)\frac{y^2}{2} \\
    &= \frac{1}{2}\left((d_1x - y)^2 + d_1^2(d_1x - y)^2\right)
\]

In this case also, the streamlines are straight lines are given by \(y - d_1x = \text{constant}\). The variation of pressure is graphically shown in figure (5) and (6). (iii) In the last case \(u_1 + f f' = 0\) we have \(u_1 = d_1x - y, u_2 = d_1(d_1x - y)\), \(\xi = 0, j = 0\) are the solutions and the geometry of streamlines is similar to (ii).

**III** \(J \neq 0\) a part of the region and \(J = 0\) in the remaining part of the region

From I and II, we see that the streamlines are concentric circles, when \(J \neq 0\) and they are parallel straight lines when \(J = 0\). Therefore there exists no common streamline pattern. Also if \(J \neq 0\) the flow is irrotational and if \(J = 0\) the flow is rotational as shown in case (i) and in (ii) it is rotational if \(d_1^2 = -1\), otherwise it is irrotational. The velocity vector of fluid when \(J \neq 0\) is \(u_1 = \frac{\lambda_1(y + A)}{x_1}\) and \(u_2 = \frac{\lambda_1(B - x)}{x_1}\) and when \(J = 0\) is \(u_1 = C_1, u_2 = C_2\) which are constants. We see that there is a discontinuity in the velocity field as we cross from one region to another. Therefore such flows cannot exists.
3. Conclusion

B.J.Gireesha et al., [10] discussed the flow of an unsteady dusty fluid under varying pressure gradient using differential geometry technique. But in our work it is used Hodograph method and it is taken that the magnitude of velocity of fluid is constant along each individual streamline. In [10] analytical solutions to velocity components of both fluid and dust phase are obtained using laplace transform technique. But in this article it is used the technique of Legendre transformation. In their paper solutions are obtained in terms of binormal vector but ours is the physical plane. In this article we also found solutions to pressure and vorticity function. We also discussed variation of velocity components, further it is analyzed the pressure variation graphically. When Jacobian is non zero, the pressure variation is shown in figure (1) and (2). It is observed that the density increases with increase of pressure. Such situations may be seen in oceans etc., When the Jacobian is zero, we have three possibilities. The variation of pressure in the first possibilities is parabolic when y=constant and it is inverted when x=constant, as in figure (3) and (4). Here as density increases pressure decreases as we observe in air and some gases. The second possibility is graphed in figure (5) and (6). Here also pressure increases with variation of density. In the third possibility the solutions and pressure variation is similar to second possibility. When the Jacobian is non zero the streamlines are concentric circles and when it is zero the streamlines are straight lines. From equations (57), (65) and (69) it is observed that as the number density of the dust particle increases the velocity of the dust phase decreases. When Jacobian is non zero the flow is irrotational. When it is zero the flow is rotational. We have such situations arised in free air, oceans etc.,

References


Fig. 1

Fig. 2

Fig. 3
The Influence of Long-Range Interaction on Critical Behavior of Some Alloys

S. V. Belim*

Omsk State University, pr. Mira, 55-a, Omsk, Russia

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Abstract: The critical behavior of some alloys are analyzed within the framework of Heisenbergs model with long-range interaction. On based experimental values of the critical exponent \( \gamma \) we calculate the value of parameter of long-range interaction.

Keywords: Long-Range Interaction; Critical Exponents; Heisenbergs Model; Alloys

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1. Introduction

In series of paper [1, 2, 3, 4, 5] experimental critical exponents are differ from results of theoretic-fields approach for 3D models of Heisenberg (\( \gamma = 1.386, \beta = 0.364 \)), 3D XY-model (\( \gamma = 1.316, \beta = 0.345 \)) and 3D Ising model (\( \gamma = 1.241, \beta = 0.325 \)) [6]. Authors of these papers are indicate that next-nearest-neighbor interaction must be take into account for explanation the difference of experimental results from theoretic results. This interaction may be take into account by means of term of hamiltonian of the form \( J(r) \sim r^{-D-\sigma} \), where \( D \) is dimension of system and \( \sigma \) is parameter of long-range interaration [7].

In [1] the critical magnetic behavior of EuO is investigated. Critical exponents of this system are \( \gamma = 1.29 \pm 0.01, \beta = 0.368 \pm 0.005 \). In this article swoun that next-nearest-neighbor interaction \( J_2 \) must be take into account. The next-nearest-neighbor interaction is equal \( (0.5 \pm 0.2)J_1 \) (\( J_1 \) – nearest interaction).

In paper[2] critical exponents of \( La_{0.5}Sr_{0.5}CoO_3 \) are measured (\( \gamma = 1.351 \pm 0.009, \beta = 0.321 \pm 0.002 \)). Earlier in [3] the critical behavior was investigated in alloys \( La_{1-x}Sr_xCoO_3 \) (\( 0.2 \leq x \leq 0.3 \)). Critical exponents of these alloys have values \( 0.43 \leq \beta \leq 0.46 \), \( 1.39 \leq \gamma \leq 1.43 \).

* belim@univer.omsk.su
The differ of critical exponents from theoretic results for short-range systems was finding for ferromagnetic phase transition in La_{0.1}Ba_{0.9}VS_3 [4]. Critical exponents of these alloys have values $\gamma = 1.366$, $\beta = 0.501$. Similar results were found in [5] for alloys Fe_{90-x}Mn_xZr_{10} (0 \leq x \leq 16).

Critical exponents of three-dimensional system with long-range intaraction for varios value $\sigma$ were calculate in [8]. In this paper was shown that value of exponent $\gamma$ increase and exponent $\beta$ decrease with increasing of parameter $\sigma$. If $\sigma$ is greater then 2, then Heisenberg exponents are valid. If $\sigma$ is greater then 1.5, then mean feild exponents are valid. In interval $1.5 < \sigma < 2$ there are new classes of universality.

The aim of this paper is calculation value of parameter of long-range intaraction from experimental values of critical exponenets for varias systems.

2. The Theoretic-field Description

The Hamiltonian of a system with long-range effects can be written as

$$H = \int d^D q \left\{ \frac{1}{2} (\tau_0 + q^\sigma) \varphi^2 + u_0 \varphi^4 \right\}, \quad (1)$$

where $\varphi$ is the $n$-dimension order-parameter fluctuations, $D$ is the space dimensionality, $\tau_0 \sim |T - T_c|$, $T_c$ is the critical temperature, and $u_0$ is a positive constant. The critical behavior depends essentially on the parameter $\sigma$ that determines the rate of interaction decay with increasing distance. As was shown in [7], the influence of long-range effects is appreciable for $0 < \sigma < 2$, while the critical behavior at $\sigma \geq 2$ is equivalent to the behavior of short-range systems. For this reason, we restrict ourselves in what follows to the case $0 < \sigma < 2$.

The standard renormalization-group procedure based on the Feynman diagrams [6] with the $G(\vec{k}) = 1/(\tau + |\vec{k}|^\sigma)$ propagator yields the following expressions for the functions $\beta, \gamma_\varphi$ and $\gamma_t$, specifying the differential renormalization-group equation:

$$\beta = -(2\sigma - D) \left[ 1 - 4(n + 8)v + \left( 64(5n + 22)(2\tilde{J}_1 - 1) - 128(n + 2)\tilde{G} \right) v^2 \right],$$

$$\gamma_t = (2\sigma - D) \left[ -2(n + 2)v + 48(n + 2)\left( 2\tilde{J}_1 - 1 - \frac{1}{3} \tilde{G} \right) v^2 \right], \quad (2)$$

$$\gamma_\varphi = 64(n + 2)\tilde{G} v^2,$$

$$v = u \cdot J_0, \quad \tilde{J}_1 = \frac{J_1}{J_0^2}, \quad \tilde{G} = \frac{G}{J_0^3}.$$

$$J_1 = \int \frac{d^D q d^D p}{(1 + |\vec{q}|^\alpha)(1 + |\vec{p}|^\alpha)(1 + |\vec{q}|^2 + p^2 + 2\vec{p}\vec{q}^\alpha/2)};$$

$$J_0 = \int \frac{d^D q}{(1 + |\vec{q}|^\alpha)^2} ;$$

$$G = -\frac{\partial}{\partial |\vec{k}|^\alpha} \int \frac{d^D q d^D p}{(1 + |\vec{q}|^2 + k^2 + 2\vec{k}\vec{q}^\alpha)(1 + |\vec{p}|^\alpha)(1 + |\vec{q}|^2 + p^2 + 2\vec{p}\vec{q}^\alpha/2)}.$$

The expressions obtained for the $\beta$-functions are asymptotic series and summation methods must be used to extract necessary physical information from these series. In
this work, the following Borel-Leroy transformation, which provides adequate results for series appearing in the theory of critical phenomena [9], is used

\[ f(v) = \sum_i c_i v^i = \int_0^\infty e^{-t} t^b F(vt) dt, \]

\[ F(v) = \sum_i \frac{c_i}{(i+b)!} v^i. \] (3)

The \([2/1]\) approximants with variation of the parameter \(b\) are used to calculate the \(\beta\)-functions in the two-loop approximation. As shown in [9], such variation of \(b\) makes it possible to determine the range of variation of the vertex functions and to estimate the accuracy of the critical exponents obtained.

The critical behavior regime is fully determined by the stable fixed points of the renormalization-group transformation; these points can be found from the condition that the \(\beta\) functions vanish:

\[ \beta(v^*) = 0. \] (4)

The condition for stability reduces to the requirement that the \(\beta\)-function derivative at the fixed point be positive:

\[ \lambda = \frac{\partial \beta(v^*)}{\partial v} > 0. \] (5)

The index \(\nu\) characterizing the growth of correlation radius in the vicinity of critical point \((R_c \sim |T - T_c|^{-\nu})\) is found from the expression:

\[ \nu = \nu(\sigma + \gamma_t)^{-1}. \]

The Fisher index \(\eta\) describing the behavior of correlation function in the vicinity of critical point in the wave-vector space \((G \sim k^{\sigma+\eta})\) is determined by the scaling function \(\gamma_\varphi: \eta = 2 - \sigma + \gamma_\varphi\). Other critical indices can be determined from the scaling relations:

\[ \gamma = \nu(\sigma - \eta), \quad \beta = \frac{\nu}{2}(D - \sigma + \eta). \] (6)

It is worth noting that the Pade-Leroy summation procedure is possible not for any \(b\) values and this significantly limits the possibility of applying the method. This limitation is associated with the appearance of the poles of the approximants near the solutions of the system of Eqs. (2); for this reason, it is impossible to determine the position of the fixed points. In this work, the parameter \(b\) varies from 0 to a value beginning with which the determination of the stable fixed point becomes impossible. In this range, 20 values of the parameter \(b\) are taken for which the fixed points are searched. Average values with a certain accuracy determined by the spread in the values for various \(b\) values are taken as the effective charges at the fixed point.

3. The Value of Parameter \(\sigma\)

Let us consider the applicability models with long-range interaction for explanation of experimental data. The value of parameter \(\sigma\) shell calculated on based of experimental value of critical exponent \(\gamma\).
For \textit{EuO} \cite{1} experimental critical exponents have values $\gamma = 1.29 \pm 0.01$, $\beta = 0.368 \pm 0.005$. Within the framework of Heisenbergs model ($n = 3$) the value of critical exponent $\gamma = 1.290 \pm 0.002$ correspond the value $\sigma = 1.941$ and critical exponent $\beta = 0.376 \pm 0.008$. Within the framework of XY-model ($n = 2$) the value of critical exponent $\gamma = 1.29 \pm 0.03$ correspond the value $\sigma = 1.991$ and critical exponent $\beta = 0.354 \pm 0.007$. As can be seen from comparison theoretical and experimental results the Heisenbergs model with long-range interaction demonstrate satisfactory conformance to experiment.

For \textit{La}$_{0.5}$\textit{Sr}$_{0.5}$\textit{CoO}$_3$ \cite{2} experimental critical exponents have values $\gamma = 1.351 \pm 0.009$, $\beta = 0.321 \pm 0.002$. Within the framework of Heisenbergs model ($n = 3$) the value of critical exponent $\gamma = 1.351 \pm 0.002$ correspond the value $\sigma = 1.980$ and critical exponent $\beta = 0.368 \pm 0.004$. Within the framework of XY-model ($n = 2$) the value of critical exponent $\gamma = 1.351$ don’t exist, the maximum value of $\gamma$ is equal 1.316 for $\sigma = 2$. As can be seen the differ theoretical results from experimental results is significant. However in \cite{2} the critical exponent $\gamma$ was measured for the critical temperature $T_c = 223.18$ K, and the critical exponent $\beta$ for the critical temperature $T_c = 222.82$ K. As is well known that values of critical exponents are very strong depends from selection of the critical temperature.

For \textit{La}$_{0.1}$\textit{Ba}$_{0.9}$\textit{VS}$_3$ \cite{4} experimental critical exponents have values $\gamma = 1.366$, $\beta = 0.501$. Within the framework of Heisenbergs model ($n = 3$) the value of critical exponent $\gamma = 1.366 \pm 0.002$ correspond the value $\sigma = 1.990$ and critical exponent $\beta = 0.369 \pm 0.009$. In this case the XY-model are not valid. It is shown that theoretical value of critical exponent $\beta$ don’t agree the result of experiment. But the experimental value $\beta = 0.501$ is very strange, because limit $\beta \leq 0.5$ is always valid. The equality $\beta = 0.5$ is valid only for mean field theory, in which $\gamma = 1$.

Critical exponents of alloys \textit{Fe}$_{90-x}$\textit{Mn}$_x$\textit{Zr}$_{10}$ ($0 \leq x \leq 16$ depend on parameter $x$ \cite{5}. In the table there are experimental values of critical exponents from paper \cite{5}. Also in this table there are parameters $\sigma$ and critical exponents $\beta_H$, which calculated Within the framework of Heisenbergs model with long-range interaction on the grounds of values of critical exponents $\gamma$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\gamma$</th>
<th>$\beta$</th>
<th>$\sigma$</th>
<th>$\beta_H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.376</td>
<td>0.369</td>
<td>1.995</td>
<td>0.36 ± 0.01</td>
</tr>
<tr>
<td>4</td>
<td>1.383</td>
<td>0.373</td>
<td>1.998</td>
<td>0.37 ± 0.01</td>
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<td>0.358</td>
<td>1.984</td>
<td>0.37 ± 0.02</td>
</tr>
<tr>
<td>8</td>
<td>1.364</td>
<td>0.355</td>
<td>1.987</td>
<td>0.36 ± 0.02</td>
</tr>
<tr>
<td>10</td>
<td>1.406</td>
<td>0.356</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>12</td>
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<tr>
<td>16</td>
<td>1.412</td>
<td>0.362</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

For values $x > 8$ the parameter $\sigma$ don’t calculated, because the maximum value of
exponent $\gamma$ is 1.386 for value parameter $\sigma = 2$. It is shown that in interval $0 \leq x \leq 8$ the theory demonstrate agreement with experiment.

As can be seen from calculation the use of Heisenbergs model with long-range interaction is valid for explanation of experimental data.

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References
