

Discrete Groups Approach to Non Symmetric Gravitation Theory

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Abstract: A generalized discrete group formalism is obtained and used to describe the Non Symmetric Gravity theory (NGT) coupled to a scalar field. We are able to derive explicitly the various terms of the NGT action including the interaction term without any ad-hoc assumptions. © Electronic Journal of Theoretical Physics. All rights reserved.

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1. Introduction

During the past few years, a renewed interest in the non commutative geometry approach [1], [2], [3], [4] of the standard model and some of the grand unified theories, has appeared among the physicists and mathematicians. The motivation is to find probable answers to the remaining outstanding problems. One of the promising approach is the one using the discrete groups [5], [6], [7] where it is shown that it has an intimate relation to non commutative geometry in which the scalar particles are treated in an equal footing with the usual gauge boson. Recently, this formalism has been applied to the case of General Relativity [8] where it was shown that the gravitational field is completely decoupled from the scalar one.

The purpose of this paper is to generalize this approach based essentially on the work presented in references [9],[10], and derive explicitly the various terms of the Non Symmetric Gravitation theory (NGT) action [11],[12],[13],[14]. In section 2 we present the mathematical formalism, in section 3 we derive the NGT action together with the scalar field interaction terms. Finally, in section 4 we draw our conclusions.

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2. Formalism

An alternative to A. Connes's Non Commutative Geometry [1], [2], [3], [4] is the discrete groups approach [5], [6], [7] based on the algebra of 2×2 matrices having as entries the p -differential forms. In this formulation, a generalized product denoted by \odot is used to define the structure of a \mathbb{Z}_2 graded associative algebra. Thus, the product of two elements of this algebra is given by [8]:

$$\begin{pmatrix} A & C \\ D & B \end{pmatrix} \odot \begin{pmatrix} A' & C' \\ D' & B' \end{pmatrix} = \begin{pmatrix} A \wedge A' + (-)^{\partial C} C \wedge D' & C \wedge B' + (-)^{\partial A} A \wedge C' \\ D \wedge A' + (-)^{\partial B} B \wedge D' & B \wedge B' + (-)^{\partial D} D \wedge C' \end{pmatrix} \quad (1)$$

where $A, B, C, D, A', B', C', D'$ are p -forms, ∂ stands for degree of these p -forms, and \wedge denotes the exterior product.

One can also define a nilpotent differential operator d satisfying a generalized Leibnitz rule as follows [8]:

$$\begin{aligned} \widehat{d}X &= \widehat{d} \begin{pmatrix} A & C \\ D & B \end{pmatrix} = \begin{pmatrix} dA + C + D & -dC - (A - B) \\ -dD + (A - B) & dB + C + D \end{pmatrix} \\ \widehat{d}(X \odot X') &= \widehat{d}X \odot X' + (-)^{\partial X} X \odot \widehat{d}X' \end{aligned} \quad (2)$$

This formulation was applied to describe the Einstein-Hilbert action with a minimal coupling of the gravitation with scalar fields [8].

Concerning NGT, one can define the following generalized spin connection Ω^{ab} :

$$\Omega^{ab} = \begin{pmatrix} \omega^{ab} & \phi^{ab} \\ \bar{\phi}^{ab} & \bar{\omega}^{ab} \end{pmatrix}, \quad a = \begin{cases} i = 1, 2, \dots, n \\ \dot{a} = n + 1, \dots, N \end{cases} \quad (3)$$

where ω^{ab} and $\bar{\omega}^{ab}$ (resp. ϕ^{ab} and $\bar{\phi}^{ab}$) are the generalized hyperbolic complex 1-forms (resp. 0-forms) where their components in the holonomic basis $\{e^i, i = \overline{1, n}\}$ are given by:

$$\omega^{ab} = \omega_{\mu}^{ab} dX^{\mu}, \quad \bar{\omega}^{ab} = \bar{\omega}_{\mu}^{ab} dX^{\mu}, \quad dX^{\mu} = E_i^{\mu} e^i \quad (4)$$

and the generalized vierbein is defined as:

$$E_i^{\mu} = \begin{pmatrix} 0 & \tilde{e}_i^{\mu} \\ e_i^{\mu} & 0 \end{pmatrix}$$

Here e_i^{μ} is the hyperbolic complex and \tilde{e}_i^{μ} its hyperbolic complex conjugate

$$\begin{aligned} e_i^{\mu} &= \alpha_i^{\mu} + \varepsilon \beta_i^{\mu}, \quad \tilde{e}_i^{\mu} = -\varepsilon, \quad \varepsilon^2 = 1 \\ \tilde{e}_i^{\mu} &= \alpha_i^{\mu} - \varepsilon \beta_i^{\mu}, \quad \alpha_i^{\mu}, \beta_i^{\mu} \in C_R^{\infty}(X) \end{aligned} \quad (5)$$

A generalized orthonormal basis can be defined such that:

$$\xi^a = \begin{pmatrix} \rho^a & s^a \\ -\tilde{s}^a & \rho^a \end{pmatrix}, \quad a = \begin{cases} i = 1, 2, \dots, n \\ \dot{a} = n + 1, \dots, N \end{cases} \quad (6)$$

where (resp. s^a and \tilde{s}^a) ρ^a is a 1-form (resp. 0-forms) given by:

$$\begin{aligned} \rho^i &= e_\mu^i dX^\mu, & i &= 1, 2, \dots, n \\ \rho^{\dot{a}} &= 0, & \dot{a} &= n + 1, \dots, N \end{aligned} \quad (7)$$

$$\begin{aligned} s^i &= 0, & \tilde{s}^i &= 0, & i &= 1, 2, \dots, n \\ s^{\dot{a}} &= M\lambda^{\dot{a}}, & \tilde{s}^{\dot{a}} &= \overline{M}\tilde{\lambda}^{\dot{a}}, & \dot{a} &= n + 1, \dots, N \end{aligned} \quad (8)$$

with e_j^μ is the inverse of the vierbein verifying:

$$e_\mu^i e_j^\mu = \delta_j^i, \quad e_\mu^i e_i^\nu = \delta_\mu^\nu \quad (9)$$

and M, \overline{M} are the following 2×2 matrices:

$$M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \overline{M} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (10)$$

here $\lambda^{\dot{a}}$ and its hyperbolic complex conjugate $\tilde{\lambda}^{\dot{a}}$ are arbitrary functions.

The exterior product and differential operator for the generalized spin connection components are defined by:

$$\begin{aligned} \omega^{ab} \wedge \omega^{cd} &= \omega_\mu^{ab} \omega_\nu^{cd} dX^\mu \wedge dX^\nu = E_i^{[\mu} E_j^{\nu]} \omega_\mu^{ab} \omega_\nu^{cd} \cdot e^i \cdot e^j \\ \omega^{ab} \wedge \varphi^{cd} &= E_i^\mu M \omega_\mu^{ab} \varphi^{cd} e^i \\ \varphi^{ab} \wedge \omega^{cd} &= M E_i^\mu \varphi^{ab} \omega_\mu^{cd} \\ \varphi^{ab} \wedge \overline{\varphi}^{cd} &= M \overline{M} \varphi^{ab} \overline{\varphi}^{cd} \end{aligned} \quad (11)$$

and

$$\begin{aligned} d\omega^{ab} &= d(\omega_\mu^{ab} dX^\mu) = (d\omega_\mu^{ab}) dX^\mu = \partial_\mu \omega_\nu^{ab} dX^\mu \wedge dX^\nu \\ d\phi^{ab} &= \partial_\mu \phi^{ab} dX^\mu = M E_i^\mu \partial_\mu \varphi^{ab} e^i \\ d\overline{\phi}^{ab} &= \partial_\mu \overline{\phi}^{ab} dX^\mu = E_i^\mu \overline{M} \partial_\mu \overline{\varphi}^{ab} e^i \end{aligned} \quad (12)$$

with:

$$\phi^{ab} = M \varphi^{ab}, \quad \overline{\phi}^{ab} = \overline{M} \overline{\varphi}^{ab} \quad (13)$$

Now imposing the unitarity condition:

$$(\Omega^{ab})^* = \Omega^{ba} \quad (14)$$

where $*$ is an involution such that:

$$(e^i)^* = -e^i \quad , \quad (dX^\mu)^* = -dX^\mu \quad (15)$$

we obtain the following constraints:

$$\begin{aligned} \tilde{\omega}_\mu^{ab} &= -\omega_\mu^{ba} \quad , \quad \tilde{\bar{\omega}}_\mu^{ab} = -\bar{\omega}_\mu^{ba} \\ \tilde{\varphi}^{ab} &= \bar{\varphi}^{ba} \quad , \quad \tilde{\bar{\varphi}}^{ab} = \varphi^{ba} \end{aligned} \quad (16)$$

As for the 2-form curvature R^{ab} , it is given by [8]:

$$R^{ab} = \widehat{d}\Omega^{ab} + \Omega^{ac} \odot \Omega^{cb}$$

Straightforward calculations lead to:

$$\begin{aligned} R_{11}^{ab} &= d\omega^{ab} + \omega^{ac} \wedge \omega^{cb} + \phi^{ac} \bar{\phi}^{cb} + \phi^{ab} + \bar{\phi}^{ab} = R^{ab} + \tau \varphi^{ac} \bar{\varphi}^{cb} + M \varphi^{ab} + \bar{M} \bar{\varphi}^{ab} \\ R_{12}^{ab} &= -d\phi^{ab} + \phi^{ac} \bar{\omega}^{cb} - \omega^{ac} \phi^{cb} - (\omega^{ab} - \bar{\omega}^{ab}) = -\nabla \phi^{ab} - (\omega^{ab} - \bar{\omega}^{ab}) \\ R_{21}^{ab} &= -d\bar{\phi}^{ab} + \bar{\phi}^{ac} \omega^{cb} - \bar{\omega}^{ac} \bar{\phi}^{cb} + (\omega^{ab} - \bar{\omega}^{ab}) = -\nabla \bar{\phi}^{ab} - (\omega^{ab} - \bar{\omega}^{ab}) \\ R_{22}^{ab} &= d\bar{\omega}^{ab} + \bar{\omega}^{ac} \wedge \bar{\omega}^{cb} + \bar{\phi}^{ac} \phi^{cb} + \phi^{ab} + \bar{\phi}^{ab} = \bar{R}^{ab} + \bar{\tau} \bar{\varphi}^{ac} \varphi^{cb} + M \varphi^{ab} + \bar{M} \bar{\varphi}^{ab} \end{aligned}$$

with

$$\tau = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \bar{\tau} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{aligned} \nabla \phi^{ab} &= e^j e_j^\mu \nabla_\mu \phi^{ab} \tau - e^j e_j^\mu \omega_\mu^{ac} \varphi^{cb} \tau_3 \\ \nabla_\mu \varphi^{ab} &= \partial_\mu \varphi^{ab} - \varphi^{ac} \bar{\omega}_\mu^{cb} + \omega_\mu^{ac} \varphi^{cb} \\ \nabla \bar{\phi}^{ab} &= e^j \tilde{e}_j^\mu \nabla_\mu \bar{\phi}^{ab} \bar{\tau} + e^j \tilde{e}_j^\mu \bar{\omega}_\mu^{ac} \bar{\varphi}^{cb} \tau_3 \\ \nabla_\mu \bar{\varphi}^{ab} &= \partial_\mu \bar{\varphi}^{ab} + \bar{\omega}_\mu^{ac} \bar{\varphi}^{cb} - \bar{\varphi}^{ac} \omega_\mu^{cb} \end{aligned}$$

It is worth mentioning that R^{ab} and \bar{R}^{ab} have the following expressions:

$$\begin{aligned} R^{ab} &= (\partial_\mu \omega_\nu^{ab} + \omega_\mu^{ac} \omega_\nu^{cb}) dX^\mu \wedge dX^\nu = \frac{1}{2} R_{\mu\nu}^{ab} dX^\mu \wedge dX^\nu \\ \bar{R}^{ab} &= (\partial_\mu \bar{\omega}_\nu^{ab} + \bar{\omega}_\mu^{ac} \bar{\omega}_\nu^{cb}) dX^\mu \wedge dX^\nu = \frac{1}{2} \bar{R}_{\mu\nu}^{ab} dX^\mu \wedge dX^\nu \\ R_{\mu\nu}^{ab} &= \partial_\mu \omega_\nu^{ab} + \omega_\mu^{ac} \omega_\nu^{cb} - (\mu \leftrightarrow \nu) = -R_{\nu\mu}^{ab} \\ \bar{R}_{\mu\nu}^{ab} &= \partial_\mu \bar{\omega}_\nu^{ab} + \bar{\omega}_\mu^{ac} \bar{\omega}_\nu^{cb} - (\mu \leftrightarrow \nu) = -\bar{R}_{\nu\mu}^{ab} \end{aligned}$$

The torsion is defined by[8]:

$$T^a = \widehat{d}\xi^a + \Omega^{ab} \odot \xi^b \quad (17)$$

Using the fact that:

$$dX^\mu \wedge dX^\nu = E_i^{[\mu} E_j^{\nu]} e^i \cdot e^j = [\eta_{ij}^{\mu\nu} e^{[i} \cdot e^{j]} + \varepsilon g_{ij}^{\mu\nu} e^{(i} \cdot e^{j)} \tau_3]$$

where $\eta_{ij}^{\mu\nu}$ and $g_{ij}^{\mu\nu}$ are the real and imaginary parts of the product $e_i^\mu \tilde{e}_j^\nu$ that is:

$$G_{ij}^{\mu\nu} = e_i^\mu \tilde{e}_j^\nu = \eta_{ij}^{\mu\nu} - \varepsilon g_{ij}^{\mu\nu} \quad (18)$$

with ε is a pur imaginary hyperbolic complex number ($\varepsilon^2 = 1$) and τ_3 is the usual Pauli matrix:

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the notations $()$ and $[]$ mean symmetric and antisymmetric parts respectively. Direct simplifications lead to:

$$\begin{aligned} (T^a)_{11} &= d\rho^a + \omega^{ab} \wedge \rho^b - \phi^{ab} \tilde{s}^b + s^a - \tilde{s}^a \\ &= (\partial_\mu \rho_\nu^a + \omega_\mu^{ab} \rho_\nu^b) dX^\mu \wedge dX^\nu - \tau \varphi^{ab} \tilde{\lambda}^b + s^a - \tilde{s}^a \\ (T^a)_{22} &= d\rho^a + \bar{\omega}^{ab} \wedge \rho^b + \bar{\phi}^{ab} s^b + s^a - \tilde{s}^a \\ &= (\partial_\mu \rho_\nu^a + \bar{\omega}_\mu^{ab} \rho_\nu^b) dX^\mu \wedge dX^\nu + \bar{\tau} \bar{\varphi}^{ab} \lambda^b + s^a - \tilde{s}^a \\ (T^a)_{12} &= -ds^a + \phi^{ab} \rho^b - \omega^{ab} s^b = -e^j e_j^\mu (\partial_\mu \lambda^a - \varphi^{ab} \rho_\mu^b) \tau - e^j e_j^\mu \omega_\mu^{ab} \lambda^b \bar{\tau} \\ (T^a)_{21} &= d\tilde{s}^a + \bar{\phi}^{ab} \rho^b + \bar{\omega}^{ab} \tilde{s}^b = e^j \tilde{e}_j^\mu (\partial_\mu \tilde{\lambda}^a - \bar{\varphi}^{ab} \rho_\mu^b) \bar{\tau} + e^j \tilde{e}_j^\mu \bar{\omega}_\mu^{ab} \tilde{\lambda}^b \tau \end{aligned}$$

The components of T^i are given by:

$$\begin{aligned} (T^i)_{11} &= (\eta_{kl}^{\mu\nu} e^{[k}.e^{l]} + \varepsilon g_{kl}^{\mu\nu} e^{(k}.e^{l)}) \tau_3 (\partial_\mu e_\nu^i + \omega_\mu^{ij} e_\nu^j) - \tau \varphi^{ia} \tilde{\lambda}^a \\ (T^i)_{22} &= (\eta_{kl}^{\mu\nu} e^{[k}.e^{l]} + \varepsilon g_{kl}^{\mu\nu} e^{(k}.e^{l)}) \tau_3 (\partial_\mu e_\nu^i + \bar{\omega}_\mu^{ij} e_\nu^j) + \bar{\tau} \bar{\varphi}^{ia} \lambda^a \\ (T^i)_{12} &= .e^j \varphi^{ij} \tau - e^j e_j^\mu \omega_\mu^{ia} \lambda^a \bar{\tau} \\ (T^i)_{21} &= e^j \bar{\varphi}^{ij} \bar{\tau} + e^j \tilde{e}_j^\mu \bar{\omega}_\mu^{ia} \tilde{\lambda}^a \tau \end{aligned}$$

while those of $T^{\dot{a}}$ are:

$$\begin{aligned} (T^{\dot{a}})_{11} &= (\eta_{kl}^{\mu\nu} e^{[k}.e^{l]} + \varepsilon g_{kl}^{\mu\nu} e^{(k}.e^{l)}) \tau_3 \omega_\mu^{ak} e_\nu^k - \tau \varphi^{ab} \tilde{\lambda}^b + M \lambda^{\dot{a}} - \bar{M} \tilde{\lambda}^{\dot{a}} \\ (T^{\dot{a}})_{22} &= (\eta_{kl}^{\mu\nu} e^{[k}.e^{l]} + \varepsilon g_{kl}^{\mu\nu} e^{(k}.e^{l)}) \tau_3 \bar{\omega}_\mu^{ak} e_\nu^k + \bar{\tau} \bar{\varphi}^{ab} \lambda^b + M \lambda^{\dot{a}} - \bar{M} \tilde{\lambda}^{\dot{a}} \\ (T^{\dot{a}})_{12} &= -e^j e_j^\mu (\partial_\mu \lambda^{\dot{a}} - \varphi^{ak} e_\mu^k) \tau - e^j e_j^\mu \omega_\mu^{ab} \lambda^b \bar{\tau} \\ (T^{\dot{a}})_{21} &= -e^j \tilde{e}_j^\mu (\partial_\mu \tilde{\lambda}^{\dot{a}} + \bar{\varphi}^{ak} e_\mu^k) \bar{\tau} + e^j \tilde{e}_j^\mu \bar{\omega}_\mu^{ab} \tilde{\lambda}^b \tau \end{aligned}$$

3. The NGT Action

If one defines the scalar product (\cdot, \cdot) as:

$$(X, Y) = \int *tr(X \odot Y) = \int \sqrt{\tilde{e}} d^4x tr X_{i_1 \dots i_p} Y_{j_1 \dots j_q} * (e^{i_1} \dots e^{i_p}) (e^{j_1} \dots e^{j_q}) \quad (19)$$

where $X = X_{i_1 \dots i_p} e^{i_1} \dots e^{i_p}$ and $Y = Y_{j_1 \dots j_q} e^{j_1} \dots e^{j_q}$, and $*$ is the Hodge star operator verifying the following equations:

$$\begin{aligned} * (e^i . e^j) &= -\delta^{ij} = * e^{(i} . e^{j)} \\ * (e^i . e^j . e^k . e^l) &= \delta^{ij} \delta^{kl} - \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} \\ * (e^i . e^j . e^{(k} . e^{l)}) &= \delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} \\ * (e^i) &= 0 = * (e^{j_1} \dots e^{j_{2k+1}}) = 0 \\ * (1) &= 0 \quad , \quad * e^{[i} . e^{j]} = 0 \end{aligned}$$

then the NGT action takes the form:

$$\mathfrak{J} = \frac{1}{2} \int \sqrt{\tilde{e}} d^4x * Tr [E^a \odot E^{b*} - E^{b*} \odot E^a] \odot R^{ba} \quad (20)$$

where E^a are given by:

$$E^a = \begin{pmatrix} \bar{\tau}\rho^a & M\tilde{\lambda}^a \\ -\bar{M}\lambda^a & \bar{\tau}\rho^a \end{pmatrix}, \quad a = \begin{cases} i = 1, 2, \dots, n \\ \dot{a} = n + 1, \dots, N \end{cases} \quad (21)$$

that is:

$$E^i = \begin{pmatrix} \bar{M}e^i & 0 \\ 0 & \bar{M}e^i \end{pmatrix}, \quad E^{\dot{a}} = \begin{pmatrix} 0 & M\tilde{\lambda}^{\dot{a}} \\ -\bar{M}\lambda^{\dot{a}} & 0 \end{pmatrix}$$

After a direct calculation we obtain:

$$\mathfrak{J} = \mathfrak{J}^{(1)} + \mathfrak{J}^{(2)} \quad (22)$$

with

$$\mathfrak{J}^{(1)} = \int \sqrt{e\tilde{e}}d^4x \left(-G^{\mu\nu} (R_{\mu\nu} + \bar{R}_{\mu\nu}) + \frac{1}{2} (\bar{\varphi}^{ia}\varphi^{ai} - \varphi^{ia}\bar{\varphi}^{ai}) \right) \quad (23)$$

$$\begin{aligned} \mathfrak{J}^{(2)} = -\frac{1}{2} \int \sqrt{e\tilde{e}}d^4x \{ & \tilde{\lambda}^{\dot{a}}\tilde{e}_i^\mu \left(\partial_\mu\bar{\varphi}^{\dot{a}i} + \bar{\omega}_\mu^{\dot{a}b}\bar{\varphi}^{bi} - \bar{\varphi}^{\dot{a}b}\omega_\mu^{bi} \right) \\ & + \lambda^{\dot{a}}e_i^\mu \left(\partial_\mu\varphi^{i\dot{a}} + \omega_\mu^{ib}\varphi^{b\dot{a}} - \varphi^{ib}\bar{\omega}_\mu^{b\dot{a}} \right) \} \end{aligned} \quad (24)$$

Now, in order to get dynamical fields, we impose the following weak torsionless conditions:

$$\begin{pmatrix} 0 & \bar{M} \\ -M & 0 \end{pmatrix} \odot T^i = 0 \quad (25)$$

and

$$Tr(\tau_3 \otimes 1) \odot T^{\dot{a}} = 0$$

Here Tr denotes the trace over the 2×2 matrices algebra.

After some straightforward simplifications, the action becomes (see Appendix A):

$$\mathfrak{J} = \int \sqrt{e\tilde{e}}d^4x \mathfrak{L}$$

where

$$\mathfrak{L} = \mathfrak{L}^{(1)} + \mathfrak{L}^{(2)} + \mathfrak{L}^{(3)}$$

with

$$\mathfrak{L}^{(1)} = G^{\mu\nu} (R_{\mu\nu} + \bar{R}_{\mu\nu}) = 2G^{\mu\nu} R_{\mu\nu}$$

$$\mathfrak{L}^{(2)} = -\frac{1}{2} (\bar{\varphi}^{ia}\varphi^{ai} - \varphi^{ia}\bar{\varphi}^{ai}) = 0$$

and

$$\mathfrak{L}^{(3)} = \frac{1}{2}\tilde{\lambda}^{\dot{a}}\tilde{e}_i^\mu \left(\partial_\mu\bar{\varphi}^{\dot{a}i} + \bar{\omega}_\mu^{\dot{a}b}\bar{\varphi}^{bi} - \bar{\varphi}^{\dot{a}b}\omega_\mu^{bi} \right) + \frac{1}{2}\lambda^{\dot{a}}e_i^\mu \left(\partial_\mu\varphi^{i\dot{a}} + \omega_\mu^{ib}\varphi^{b\dot{a}} - \varphi^{ib}\bar{\omega}_\mu^{b\dot{a}} \right)$$

Note that $G^{\mu\nu} = e_i^\mu\tilde{e}_i^\nu$ is the NGT metric.

Setting $\lambda = \exp(\varepsilon\Phi)$, we get:

$$\mathcal{L} = G^{\mu\nu} R_{\mu\nu} + \frac{1}{2} G^{(\mu\nu)} W_\mu W_\nu - \frac{1}{2} G^{[\mu\nu]} \partial_\nu W_\mu + \frac{1}{2} G^{(\mu\nu)} \partial_\mu \Phi \partial_\nu \Phi - G^{(\mu\nu)} \varepsilon W_\mu \partial_\nu \Phi$$

Notice that one can also add the following cosmological term J (see Appendix B):

$$\mathbf{J} = \frac{1}{2} \int *Tr [E^a \odot E^{b*} - E^{b*} \odot E^a] \odot (\xi^b \odot \xi^{a*}) \quad (26)$$

which may be also written as:

$$\mathbf{J} = - \int \sqrt{e\tilde{e}} d^4x \left(-2G^{[\mu\nu]} G_{[\mu\nu]} - 4\lambda\tilde{\lambda} - 8 - G_{ji}^{\nu\mu} G_{\mu\nu}^{ji} \right) \quad (27)$$

4. Conclusions

We have shown that we can consistently generalize the discrete groups formalism to the case of Non Symmetric Gravitation theory, and have obtained in the process a Lagrangian density containing the pure NGT action an interaction term, as well as the kinetic term for the scalar field Φ . Thus, the various terms that Moffat has introduced by hand for mere physical consistency, are here seen to be the result of the generalized discrete group approach. Moreover, a dynamical scalar field was found to be also necessary in this formalism, but contrary to General Relativity, it couples to the gravitational field (term proportional to $G^{(\mu\nu)} \varepsilon W_\mu \partial_\nu \Phi$).

Appendix A

In order to get dynamical fields, we impose a weak torsionless condition:

$$\begin{pmatrix} 0 & \overline{M} \\ -M & 0 \end{pmatrix} \odot T^i = 0 \quad , \quad Tr_\tau \left((\tau_3 \otimes 1) \odot T^a \right) = 0$$

where Tr_τ denotes the trace over $M_2(K)$ ($M, M, \tau, \overline{\tau}, \tau_3$).

We thus get the following constraints:

$$\overline{\omega}_\mu^{ia} = \omega_\mu^{ia} = 0$$

$$(\partial_\mu e_\nu^i + \omega_\mu^{ij} e_\nu^j) = 0$$

$$(\partial_\mu e_\nu^i + \overline{\omega}_\mu^{ij} e_\nu^j) = 0$$

$$\partial_\mu \lambda^a - \varphi^{ak} e_\mu^k - \omega_\mu^{ab} \lambda^b = 0$$

$$\partial_\mu \tilde{\lambda}^a + \overline{\varphi}^{ak} e_\mu^k - \overline{\omega}_\mu^{ab} \tilde{\lambda}^b = 0$$

Consequently we obtain:

$$R_{\mu\nu}^{ij} = \overline{R}_{\mu\nu}^{ij}$$

$$R_{\mu\nu} = \overline{R}_{\mu\nu}$$

Now by imposing also that $Tr(T^i) = 0$, we get:

$$\tilde{\lambda}^a \varphi^{ia} = \lambda^a \overline{\varphi}^{ia}$$

which implies:

$$\overline{\varphi}^{ia} \varphi^{ai} - \varphi^{ia} \overline{\varphi}^{ai} = 0$$

Using the fact that:

$$\bar{\varphi}^{ij}\varphi^{ji} - \varphi^{ij}\bar{\varphi}^{ji} = 0$$

we obtain:

$$\bar{\varphi}^{ia}\varphi^{ai} - \varphi^{ia}\bar{\varphi}^{ai} = 0$$

and thus

$$\mathfrak{L}^{(2)} = -\frac{1}{2} (\bar{\varphi}^{ia}\varphi^{ai} - \varphi^{ia}\bar{\varphi}^{ai}) = 0$$

By taking into account the above constraints, $\mathfrak{L}^{(3)}$ takes the form:

$$\mathfrak{L}^{(3)} = \frac{1}{2}\tilde{\lambda}\tilde{e}_i^\mu (\partial_\mu\bar{\varphi}^{5i} - \bar{\omega}_\mu^{55}\bar{\varphi}^{5i} - \bar{\varphi}^{5j}\omega_\mu^{ji}) + \frac{1}{2}\lambda e_i^\mu (\partial_\mu\varphi^{i5} + \omega_\mu^{ij}\varphi^{j5} - \varphi^{i5}\omega_\mu^{55})$$

Putting $W_\mu = \omega_\mu^{55}$, $\bar{W}_\mu = \bar{\omega}_\mu^{55} = -W_\mu$, and using the compatibility condition:

$$\nabla_\mu e_i^\sigma = \partial_\mu e_i^\sigma - \omega_\mu^{ji} e_j^\sigma + W_{\alpha\mu}^\sigma e_i^\alpha = 0$$

we end up with:

$$2\mathfrak{L}^{(3)} = \tilde{\lambda}G^{\sigma\mu}\partial_\mu(\partial_\sigma(\lambda - W_\sigma)\lambda) - W_\sigma\lambda + G^{\nu\mu}W_{\nu\mu}^\sigma\tilde{\lambda}(\partial_\mu - W_\sigma)\lambda \\ - \tilde{\lambda}G^{\sigma\mu}W_\mu(\partial_\sigma\lambda - W_\sigma\lambda) + h.c.c$$

where here h.c.c. means hyperbolic complex conjugate.

Using the parametrization $\lambda = \exp(\varepsilon\Phi)$, $\mathfrak{L}^{(3)}$ becomes:

$$\mathfrak{L}^{(3)} = G^{(\mu\nu)}W_\mu W_\nu - G^{[\mu\nu]}\partial_\nu W_\mu + G^{(\mu\nu)}\partial_\mu\Phi\partial_\nu\Phi - 2G^{(\mu\nu)}\varepsilon W_\mu\partial_\nu\Phi$$

Finally we get for the action \mathfrak{I} :

$$\mathfrak{I} = \int \sqrt{e}\tilde{e}d^4x\mathfrak{L}$$

with

$$\mathfrak{L} = 2G^{\mu\nu}R_{\mu\nu} + G^{(\mu\nu)}W_\mu W_\nu - G^{[\mu\nu]}\partial_\nu W_\mu + G^{(\mu\nu)}\partial_\mu\Phi\partial_\nu\Phi - 2G^{(\mu\nu)}\varepsilon W_\mu\partial_\nu\Phi$$

Appendix B

The cosmological term can be obtained from the following expression:

$$\mathbf{J} = \frac{1}{2} \int *Tr [E^a \odot E^{b*} - E^{b*} \odot E^a] \odot (\xi^b \odot \xi^{a*}) = \frac{1}{2} (\mathbf{J}^{(1)} - \mathbf{J}^{(2)})$$

where:

$$\mathbf{J}^{(1)} = \int *Tr \{ (E^a \odot E^{b*} - E^{b*} \odot E^a)_{11} \wedge (\xi^b \odot \xi^{a*})_{11} \\ + (E^a \odot E^{b*} - E^{b*} \odot E^a)_{22} \wedge (\xi^b \odot \xi^{a*})_{22} \}$$

and

$$\mathbf{J}^{(2)} = \int *Tr \{ (E^a \odot E^{b*} - E^{b*} \odot E^a)_{12} \wedge (\xi^b \odot \xi^{a*})_{21} \\ + (E^a \odot E^{b*} - E^{b*} \odot E^a)_{21} \wedge (\xi^b \odot \xi^{a*})_{12} \}$$

Straightforward calculations give:

$$\mathbf{J}^{(1)} = 2 \int *Tr \{ (E^i \odot E^{j*} - E^{j*} \odot E^i)_{11} \wedge (\xi^j \odot \xi^{i*})_{11} \\ = -2 \int \sqrt{e}\tilde{e}d^4x (G^{\nu\mu}G_{\mu\nu} - G_{ij}^{\nu\mu}G_{\mu\nu}^{ji} - 12)$$

and:

$$\mathbf{J}^{(2)} = \int *Tr \{ \left((E^{\dot{a}} \odot E^{i*}) - (E^{i*} \odot E^{\dot{a}}) \right)_{12} \wedge (\xi^i \odot \xi^{\dot{a}*})_{21} \\ + \left((E^{\dot{a}} \odot E^{i*}) - (E^{i*} \odot E^{\dot{a}}) \right)_{21} \wedge (\xi^b \odot \xi^{a*})_{12} \\ + ((E^i \odot E^{\dot{b}*}) - (E^{\dot{b}*} \odot E^i))_{21} \wedge (\xi^{\dot{b}} \odot \xi^{i*})_{12} \\ + ((E^i \odot E^{\dot{b}*}) - (E^{\dot{b}*} \odot E^i))_{12} \wedge (\xi^{\dot{b}} \odot \xi^{i*})_{21} \}$$

$$\mathbf{J}^{(2)} = -8 \int \sqrt{e\tilde{e}} d^4x \lambda^{\tilde{a}} \tilde{\lambda}^{\tilde{a}}$$

Finally we obtain:

$$\begin{aligned} \mathbf{J} &= - \int \sqrt{e\tilde{e}} d^4x \left(G^{\nu\mu} G_{\mu\nu} - G_{ij}^{\nu\mu} G_{\mu\nu}^{ji} - 12 - 4\lambda^{\tilde{a}} \tilde{\lambda}^{\tilde{a}} \right) \\ &= - \int \sqrt{e\tilde{e}} d^4x \left(-2G^{[\mu\nu]} G_{[\mu\nu]} - 4\lambda\tilde{\lambda} - 8 - G_{ji}^{\nu\mu} G_{\mu\nu}^{ji} \right) \end{aligned}$$

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