

Exact Solutions for Nonlinear Evolution Equations Via Extended Projective Riccati Equation Expansion Method

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Abstract: By means of a simple transformation, we have shown that the generalized-Zakharov equations, the coupled nonlinear Klein-Gordon-Zakharov equations, the GDS, DS and GZ equations and generalized Hirota-Satsuma coupled KdV system can be reduced to the elliptic-like equations. Then, the extended projective Riccati equation expansion method is used to obtain a series of solutions including new solitary wave solutions, periodic and rational solutions. The method is straightforward and concise, and its applications is promising.

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1. Introduction

The investigation of the exact travelling wave solutions of nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. For example, the wave phenomena observed in fluid dynamics, plasma, elastic media, optical fibers, etc. In the past several decades, both mathematicians and physicists have made significant progression in this direction.

Many effective methods [1 – 13] have been presented such as variational iteration method [6], homotopy perturbation method [3], Exp-function method [8, 12], and others. A complete review on the field is available on [4].

The rest of this paper is organized as follows: In **Section 2**, first we briefly give the steps of the method and apply the method to solve the elliptic-like equation. In **Section**

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3, by using the results obtained in **Section 3**, the corresponding solutions of some class of nonlinear evolution equations in mathematical physics can be obtained. The last section is devoted to the conclusion.

2. Method and its Applications

To illustrate the basic idea of the extended projective Riccati equation expansion method, we consider the nonlinear evolution equation with independent variables, say in two variables x, t ,

$$Q(u, u_x, u_{xx}, \dots) = 0, \quad (1)$$

we consider its travelling wave solutions

$$u(x, t) = u(\xi), \xi = x - \lambda t + \xi_0, \quad (2)$$

then Eq.(1) is reduced to an ordinary differential equation (ODE)

$$Q(u, u', u'', \dots) = 0, \quad (3)$$

where a prime denotes $\frac{d}{d\xi}$.

Step (1). We assume that Eq.(1) has the following formal solution :

$$u(\xi) = a_0 + \sum_{i=1}^M f^{i-1}(\xi)[a_i f(\xi) + b_i g(\xi)], \quad (4)$$

where a_0, a_i and b_i are constants to be determined later. The parameter M can be determined by balancing the highest order derivative term with nonlinear term in Eq.(3),

$$f'(\xi) = pf(\xi)g(\xi), \quad (5)$$

$$g'(\xi) = q + pg^2(\xi) - rf(\xi), \quad (6)$$

$$g^2 = -\frac{1}{p}[q - 2rf + \frac{r^2 + \delta}{q}f^2], \quad (7)$$

where $p \neq 0$ is a real constant, q, r, δ are real constants.

Step (2). Substituting Eq.(4) into (3) and making use of Eqs.(5-7) yields a set of algebraic polynomials for $f^i(\xi)g^j(\xi)$ ($i = 0, 1, \dots; j = 0, 1, \dots$). Eliminating all the coefficients of the power of $f^i(\xi)g^j(\xi)$, yields a series of algebraic equations, from which the parameters a_i, b_i and λ are explicitly determined.

Step (3). It is easy to see that Eqs.(5) and (6) admits the following solutions:

Case(1): $\delta = h^2 - s^2, q \neq 0$, and $pq < 0$,

$$f_1 = \frac{q}{r + s \cosh(\sqrt{-pq\xi}) + h \sinh(\sqrt{-pq\xi})}, \quad (8)$$

$$g_1 = \frac{\sqrt{-pq}}{p} \frac{ssinh(\sqrt{-pq\xi}) + h\cosh(\sqrt{-pq\xi})}{r + scosh(\sqrt{-pq\xi}) + hsinh(\sqrt{-pq\xi})}, \quad (9)$$

$$g_1^2 = -\frac{1}{p} \left[q - 2rf_1 + \frac{r^2 + h^2 - s^2}{q} f_1^2 \right], \quad (10)$$

where h, p, s, q and r are constants.

Case(2): $\delta = -h^2 - s^2, q \neq 0$, and $pq > 0$,

$$f_2 = \frac{q}{r + scos(\sqrt{pq\xi}) + hsin(\sqrt{pq\xi})}, \quad (11)$$

$$g_2 = \frac{\sqrt{pq}}{p} \frac{ssin(\sqrt{pq\xi}) - hcos(\sqrt{pq\xi})}{r + scos(\sqrt{pq\xi}) + hsin(\sqrt{pq\xi})}, \quad (12)$$

$$g_2^2 = -\frac{1}{p} \left[q - 2rf_2 + \frac{r^2 - h^2 - s^2}{q} f_2^2 \right], \quad (13)$$

where h, p, s, q and r are constants.

Case(3): $q = 0$,

$$f_3 = \frac{1}{(pr/2)\xi^2 + m\xi + n}, \quad (14)$$

$$g_3 = \frac{-1}{p} \frac{pr\xi + m}{(pr/2)\xi^2 + m\xi + n}, \quad (15)$$

$$g_3^2 = \frac{2r}{p} f_3 + \left[\frac{m^2}{p^2} - \frac{2rn}{p} \right] f_3^2, \quad (16)$$

where m, n, p, r are arbitrary constants.

Case(4): $p = \pm 1, \delta = -r^2$,

$$f_4(\xi) = \frac{q}{6r} + \frac{2}{pr} \psi(\xi), \quad (17)$$

$$g_4(\xi) = \frac{12\psi'(\xi)}{q + 12\psi'(\xi)}, \quad (18)$$

where $\psi(\xi)$ satisfies

$$\psi'^2(\xi) = 4\psi^3(\xi) - \gamma_2\psi(\xi) - \gamma_3,$$

where $\gamma_2 = \frac{q^2}{12}, \gamma_3 = \frac{pq^3}{216}$,

$$g_4^2 = \frac{2r}{p} f_4 - \frac{p}{q} \quad (19)$$

Case(5): $p = \pm 1, \delta = -\frac{r^2}{25}$,

$$f_5(\xi) = \frac{5q}{6r} + \frac{5pq^2}{72\psi(\xi)}, \quad (20)$$

$$g_5(\xi) = -\frac{q\psi'(\xi)}{\psi(\xi)(pq + 12\psi(\xi))}, \quad (21)$$

$$g_5^2 = -\frac{1}{p}\left[q - 2rf_5 + \frac{24r^2}{25q}f_5^2\right] \quad (22)$$

3. The Exact Solutions of Elliptic-like Equations

Let us consider the elliptic-like equation in [7]

$$A\phi''(\xi) + B\phi(\xi) + D\phi^3(\xi) = 0, \quad (23)$$

where A, B, D are arbitrary constants. In this section, the exact solutions of Eq.(23) are derived using the coupled projective Riccati Eqs.(5) and (6). Considering the homogeneous balance between $\phi''(\xi)$ and $\phi^3(\xi)$ in Eq.(23), the solution of Eq.(23) is given by

$$\phi(\xi) = a_0 + a_1f(\xi) + b_1g(\xi), \quad (24)$$

where a_0, a_1 and b_1 are constants to be determined later, and $f(\xi)$ and $g(\xi)$ satisfy Eqs.(5-7). Substituting Eq.(24) into (23) and making use of Eqs.(5-7), becomes a polynomials for $f_1^i (i = 0, 1, 2, 3)$ and $f_1^j g_1 (j = 0, 1, 2)$, setting the coefficients of the polynomials to zero yields a set of algebraic equations. Solving the system of algebraic equations with the aid of Maple, we have

$$a_0 = 0, a_1^2 = \frac{Ap(r^2 + h^2 - s^2)}{2qD}, b_1^2 = -\frac{Ap^2}{2D} \quad (25)$$

Case(1): $pq < 0, q \neq 0, g_1^2 = -\frac{1}{p}[q - 2rf_1 + \frac{r^2+h^2-s^2}{q}f_1^2]$. Substituting Eq.(25) into Eq.(24) and using Eqs.(5-7), the exact solution of Eq.(23) are derived as

$$\phi_1(\xi) = \frac{a_1q}{r + scosh(\sqrt{-pq\xi}) + hsinh(\sqrt{-pq\xi})} - \frac{b_1\sqrt{-pq}}{p} \frac{ssinh(\sqrt{-pq\xi}) + hcosh(\sqrt{-pq\xi})}{r + scosh(\sqrt{-pq\xi}) + hsinh(\sqrt{-pq\xi})}, \quad (26)$$

$$a_1^2 = \frac{Ap(r^2+h^2-s^2)}{2qD}, AD < 0 \text{ and } b_1^2 = -\frac{Ap^2}{2D}.$$

Case(1.1): $a_0 = a_1 = 0, r = 0$, the exact solution of Eq.(23) are derived as

$$\phi_2(\xi) = -\frac{b_1\sqrt{-pq}}{p} \frac{ssinh(\sqrt{-pq\xi}) + hcosh(\sqrt{-pq\xi})}{r + scosh(\sqrt{-pq\xi}) + hsinh(\sqrt{-pq\xi})}, \quad (27)$$

$$b_1^2 = -\frac{Ap^2}{2D} \text{ and } \frac{A}{D} < 0.$$

Case(1.2): $a_0 = b_1 = 0, r = 0$, the exact solution of Eq.(23) yields

$$\phi_3(\xi) = \frac{a_1q}{scosh(\sqrt{-pq\xi}) + hsinh(\sqrt{-pq\xi})}, \quad (28)$$

$$a_1^2 = \frac{2Ap(h^2-s^2)}{qD}.$$

$$\text{Case(2): } pq > 0, q \neq 0, g_2^2 = -\frac{1}{p}[q - 2rf_2 + \frac{r^2-h^2-s^2}{q}f_2^2].$$

$$\text{Case(2.1): } a_0 = 0, pq > 0,$$

$$\phi_4(\xi) = \frac{a_1q}{r + scos(\sqrt{pq\xi}) + hsin(\sqrt{pq\xi})} + \frac{b_1\sqrt{pq}}{p} \frac{ssin(\sqrt{pq\xi}) - hcos(\sqrt{pq\xi})}{r + scos(\sqrt{pq\xi}) + hsin(\sqrt{pq\xi})}, \quad (29)$$

$$a_1^2 = \frac{Ap(r^2-h^2-s^2)}{2qD} \text{ and } b_1^2 = -\frac{Ap^2}{2D}.$$

$$\text{Case(2.2): } a_0 = a_1 = 0, r = 0, pq > 0,$$

$$\phi_5(\xi) = \frac{b_1\sqrt{pq}}{p} \frac{ssin(\sqrt{pq\xi}) - hcos(\sqrt{pq\xi})}{scos(\sqrt{pq\xi}) + hsin(\sqrt{pq\xi})}, \quad (30)$$

$$b_1^2 = -\frac{2Ap^2}{D}.$$

$$\text{Case(2.3): } a_0 = b_1 = 0, r=0, pq > 0,$$

$$\phi_6(\xi) = \frac{a_1q}{scos(\sqrt{pq\xi}) + hsin(\sqrt{pq\xi})}, \quad (31)$$

$$a_1^2 = \frac{2Ap(-h^2-s^2)}{qD} \text{ and } pq > 0.$$

$$\text{Case(3): } p = \pm 1, \delta = -r^2, g_4^2 = \frac{2r}{p}f_4 - \frac{q}{p}. \text{ The exact solution of Eq.(23) admits}$$

$$\phi_7(\xi) = \frac{12b_1\psi'(\xi)}{q + 12p\psi'(\xi)}, \quad (32)$$

$$b_1^2 = -\frac{Ap^2}{2D} \text{ and } \frac{A}{D} < 0$$

$$\text{Case(4): } p = \pm 1, g_5^2 = -\frac{1}{p}[q - 2rf_5 + \frac{r^2-h^2-s^2}{q}f_5^2]. \text{ The exact solution of Eq.(23) admits}$$

$$\phi_8(\xi) = a_1 \left[\frac{5q}{6r} + \frac{5pq^2}{72r\psi(\xi)} \right] - \frac{b_1q\psi'(\xi)}{\psi(\xi)(pq + 12\psi(\xi))}, \quad (33)$$

$$a_1^2 = \frac{12r^2AP}{25Dq}, b_1^2 = -\frac{AP^2}{2D}, \frac{p}{q} < 0, p = \pm 1 \text{ and } \frac{A}{D} < 0.$$

4. Exact Solutions of Some Class of Nonlinear Evolution Equations

In this section, by using the results obtained in section (3), we will construct the corresponding solutions of the generalized-Zakharov equations, the coupled nonlinear Klein-Gordon-Zakharov equations, the GDS, DS and GZ equations and generalized Hirota-Satsuma coupled KdV system.

4.1 The generalized-Zakharov equations

The generalized Zakharov equations for the complex envelope $\psi(x, t)$ of the high-frequency wave and the real low-frequency field $v(x, t)$ reads [13]

$$i\psi_t + \psi_{xx} - 2\lambda|\psi|^2\psi + 2\psi v = 0, \quad (34)$$

$$v_{tt} - v_{xx} + (|\psi|^2)_{xx} = 0, \quad (35)$$

where the cubic term in Eq.(34) describes the nonlinear-self interaction in the high frequency subsystem, such a term corresponds to a self-focusing effect in plasma physics. The coefficient λ is a real constant that can be a positive or negative number. Let us assume the travelling wave solution of Eqs.(34) and (35) in the form

$$\psi(x, t) = e^{i\eta}\phi(\xi), v = v(\xi),$$

$$\eta = \alpha x + \beta t, \xi = k(x - 2\alpha t), \quad (36)$$

where $\phi(\xi)$ and $v(\xi)$ are real functions, the constants α , β and k are to be determined. Substituting (36) into Eqs.(34) and (35), we have

$$k^2\phi''(\xi) + 2\phi(\xi)v(\xi) - (\alpha^2 + \beta)\phi(\xi) - 2\lambda\phi^3(\xi) = 0, \quad (37)$$

$$k^2(4\alpha^2 - 1)v''(\xi) + k^2(\phi^2)''(\xi) = 0 \quad (38)$$

In order to simplify ODEs (37) and (38), integrating Eq.(38) once and taking integration constant to zero, and integrating yields

$$v(\xi) = \frac{\phi^2(\xi)}{(1 - 4\alpha^2)} + C, \text{ if } \alpha^2 \neq \frac{1}{4}, \quad (39)$$

where C -integration constant. Inserting Eq.(39) into (37), we have

$$A\phi''(\xi) + B\phi(\xi) + D\phi^3(\xi) = 0 \quad (40)$$

Eq.(40) coincides with Eq.(23), where A , B and D are defined by

$$A = k^2,$$

$$B = [2C - \alpha^2 - \beta],$$

$$D = 2\left[\frac{1}{1 - 4\alpha^2} - \lambda\right] \quad (41)$$

Then the solution of Eqs.(34) and (35) are

$$\psi(x, t) = e^{i\eta}\phi(\xi),$$

$$v(x, t) = \frac{\phi^2(\xi)}{(1 - 4\alpha^2)} + C, \quad (42)$$

where $\phi(\xi)$ is given by Eqs.(26-33), $\eta = \alpha x + \beta t$, $\xi = k(x - 2\alpha t)$ and A , B and D are defined by Eq.(41).

4.2 The coupled nonlinear Klein-Gordon-Zakarov equations

The coupled nonlinear Klein-Gordon-Zakarov equations [14] read

$$\begin{aligned} u_{tt} - c_0^2 \nabla^2 u + f_0^2 u + \delta uv &= 0, \\ v_{tt} - c_0^2 \nabla^2 v - \beta \nabla^2 |u|^2 &= 0, \end{aligned} \quad (43)$$

where c_0 , f_0 , β and δ are constants. We seek its following wave packet solution

$$u(x, y, z, t) = \phi(\xi) e^{i(kx + ly + nz - \Omega t)}, v = v(\xi), \xi = px + qy + rz - wt, \quad (44)$$

where $\phi(\xi)$ and $v(\xi)$ are real functions. Substituting Eq.(44) into Eqs.(43) yields

$$[w^2 - c_0^2 P^2] \phi''(\xi) + 2i[w\Omega - c_0^2 K.P] \phi'(\xi) - (w^2 - K^2 c_0^2 - f_0^2) \phi(\xi) + \delta v(\xi) \phi(\xi) = 0,$$

$$[w^2 - c_0^2 P^2] v''(\xi) - \beta P^2 (\phi^2(\xi))'' = 0, \quad (45)$$

$$K = (k, l, n), P = (p, q, r), K.P = kp + lq + nr$$

If we take $w.\Omega = c_0^2 K.P$, then Eqs.(43) leads to

$$[w^2 - c_0^2 P^2] \phi''(\xi) - (w^2 - K^2 c_0^2 - f_0^2) \phi(\xi) + \delta v(\xi) \phi(\xi) = 0, \quad (46)$$

$$[w^2 - c_0^2 P^2] v''(\xi) - \beta P^2 (\phi^2(\xi))'' = 0 \quad (47)$$

Integrating (47) twice with respect to ξ , we get

$$v(\xi) = \frac{c}{w^2 - c_0^2 P^2} + \frac{\beta P^2}{w^2 - c_0^2 P^2} \phi^2(\xi), \quad (48)$$

where c is an integration constant. Substituting (48) into (46) the obtained equation can be expressed as Eq.(23), while the parameters A , B and D are defined by

$$A = [w^2 - c_0^2 P^2]^2,$$

$$B = [(w^2 - c_0^2 P^2)(-w^2 + c_0^2 K^2 c_0^2 + f_0^2) + \delta c],$$

$$D = \delta \beta P^2 \quad (49)$$

Then the solution of Eqs.(43) are defined as follows

$$\begin{aligned}
 u(x, y, z, t) &= \phi(\xi)e^{i(kx+ly+nz-\Omega t)}, \\
 v(x, y, z, t) &= \frac{c}{w^2 - c_0^2 P^2} + \frac{\beta P^2}{w^2 - c_0^2 P^2} \phi^2(\xi), \\
 \Omega &= \frac{c_0^2 K \cdot P}{w},
 \end{aligned} \tag{50}$$

where $\phi(\xi)$ appearing in these solutions is given by Eqs.(26-33) and A , B and D are defined by (49) and $\xi = px + qy + rz - wt$.

4.3 The GDS,DS and GZ equations

We consider a class of NLPDEs with constant coefficients [15]

$$\begin{aligned}
 iu_t + \nu(u_{xx} + D_1 u_{yy}) + E_1 |u|^2 u + C_1 uv &= 0, \\
 D_2 v_{tt} + (v_{xx} - E_2 u_{yy}) + C_2 (|u|^2)_{xx} &= 0,
 \end{aligned} \tag{51}$$

where ν , D_i , E_i , C_i are real constants and $\nu \neq 0$, $D_1 \neq 0$, $C_1 \neq 0$, $C_2 \neq 0$. Eqs.(51) are a class of physically important equations. In fact, if one takes

$$\nu = \frac{1}{2}k^2, D_1 = 2\nu, E_1 = \alpha, C_1 = -1, D_2 = 0, E_2 = D_1, C_2 = -2\alpha, k^2 = \pm 1, \tag{52}$$

then Eqs.(51) represent the DS equations [16]

$$\begin{aligned}
 iu_t + \frac{1}{2}k^2(u_{xx} + k^2 u_{yy}) + \alpha |u|^2 u - uv &= 0, \\
 v_{xx} - k^2 u_{yy} - 2\alpha (|u|^2)_{xx} &= 0
 \end{aligned} \tag{53}$$

If one takes

$$\nu = v(x, t), i.e., v_y = 0, \nu = 1, D_1 = 0, E_1 = -2\sigma, E_2 = -1, C_2 = -1, C_1 = 2, \tag{54}$$

then Eqs.(51) represent the GZ equations [17]

$$\begin{aligned}
 iu_t + u_{xx} - 2\sigma |u|^2 u + 2uv &= 0, \\
 v_{tt} - v_{xx} + (|u|^2)_{xx} &= 0
 \end{aligned} \tag{55}$$

Since u is a complex function, we assume that

$$u(x, y, t) = \phi(\xi)e^{i(kx+ly-\Omega t)}, v(x, y, t) = v(\xi), \xi = px + qy - wt, \quad (56)$$

where both $\phi(\xi)$ and $v(\xi)$ are real functions, and k, l, p, q, Ω and w are constants to be determined later. Substituting Eq.(56) into (51), we have the following ODE for $\phi(\xi)$ and $v(\xi)$

$$\nu(p^2 + D_1q^2)\phi''(\xi) + [\Omega - \nu(k^2 + D_1l^2)]\phi(\xi) + E_1\phi^3(\xi) + i[-w + 2\nu(kp + D_1lq)]\phi'(\xi) + C_1\phi(\xi)v(\xi) = 0, \quad (57)$$

$$[D_2w^2 + p^2 - E_2q^2]v''(\xi) + C_2p^2(\phi^2(\xi))'' = 0 \quad (58)$$

if we set

$$w = 2\nu(kp + D_1lq), \quad (59)$$

then Eq.(57) reduces to

$$\nu(p^2 + D_1q^2)\phi''(\xi) + [\Omega - \nu(k^2 + D_1l^2)]\phi(\xi) + E_1\phi^3(\xi) + C_1\phi(\xi)v(\xi) = 0 \quad (60)$$

Integrating Eq.(58) twice, we get

$$v(\xi) = \frac{c}{D_2w^2 + p^2 - E_2q^2} - \frac{C_2p^2}{D_2w^2 + p^2 - E_2q^2}\phi^2(\xi), \quad (61)$$

where c is an integration constant. Substituting Eq.(61) into (60) yields

$$\nu(p^2 + D_1q^2)(D_2w^2 + p^2 - E_2q^2)\phi''(\xi) + C_1c - (D_2w^2 + p^2 - E_2q^2)[\omega - \nu(k^2 + D_1l^2)]\phi(\xi) + E_1(D_2w^2 + p^2 - E_2q^2) - C_1C_2p^2\phi^3(\xi) = 0, \quad (62)$$

Eq.(62) can be rewritten as Eq.(23), while A, B and D are given by the following equation,

$$A = \nu(p^2 + D_1q^2)(D_2w^2 + p^2 - E_2q^2),$$

$$B = C_1c - (D_2w^2 + p^2 - E_2q^2)[\Omega - \nu(k^2 + D_1l^2)],$$

$$D = E_1(D_2w^2 + p^2 - E_2q^2) - C_1C_2p^2 \quad (63)$$

Then the solution of Eqs.(51) are

$$u(x, y, t) = \phi(\xi)e^{i(kx+ly-\Omega t)}, \quad (64)$$

$$v(x, y, t) = \frac{c}{D_2w^2 + p^2 - E_2q^2} - \frac{C_2p^2}{D_2w^2 + p^2 - E_2q^2}\phi^2(\xi), \quad (65)$$

$$w = 2\nu(kp + D_1lq) \quad (66)$$

The expression $\phi(\xi)$ appearing in these solutions is given by Eqs.(26-33) and $\xi = px + qy - wt$. We may obtain from Eq.(53) that

$$u(x, y, t) = \phi(\xi)e^{i(kx+ly-\Omega t)}, \quad (67)$$

$$v(x, y, t) = \frac{c}{p^2 - k^2q^2} + \frac{2\alpha p^2}{p^2 - k^2q^2}\phi^2(\xi), \quad (68)$$

$$w = k^2(kp + k^2lq), \quad (69)$$

where $\phi(\xi)$ satisfy the elliptic-like Eq.(23) with A , B and D defined as follows

$$A = k^2(p^2 + k^2q^2)(k^2q^2 - p^2),$$

$$B = 2c + (p^2 - k^2q^2)[2\Omega - k^2(k^2 + k^2l^2)],$$

$$D = 2\alpha(p^2 + k^2q^2) \quad (70)$$

The expression $\phi(\xi)$ are defined by Eqs.(26-33) and $\xi = px + qy - wt$. Then From Eq.(55) we have that

$$u(x, y, t) = \phi(\xi)e^{i(kx-\Omega t)}, \quad (71)$$

$$v(x, y, t) = \frac{c}{p^2 - w^2} + \frac{p^2}{p^2 - w^2}\phi^2(\xi), \quad (72)$$

$$w = 2kp, \quad (73)$$

where $\phi(\xi)$ satisfies Eq.(23), while A , B and D are given by

$$A = p^2(p^2 - w^2),$$

$$B = 2c - (p^2 - w^2)[\Omega - k^2],$$

$$D = 2[p^2 - \sigma(p^2 - w^2)] \quad (74)$$

The expression $\phi(\xi)$ appearing in these solutions is given by Eqs.(26-33) and $\xi = px - wt$.

4.4 Generalized Hirota-Satsuma coupled KdV equation

Consider the Hirota-Satsuma coupled KdV system in [18]

$$\begin{aligned}u_t &= \frac{1}{4}u_{xxx} + 3uu_x + 3(w - v^2)_x, \\v_t &= -\frac{1}{2}v_{xxx} - 3uv_x, \\w_t &= -\frac{1}{2}w_{xxx} - 3uw_x\end{aligned}\quad (75)$$

When $w = 0$, Eqs.(75) reduces to be the well-known Hirota-Satsuma coupled KdV system [19]. We seek travelling wave solutions for Eqs.(75) in the form

$$u(x, t) = u(\xi), v(x, t) = v(\xi), w(x, t) = w(\xi), \xi = k(x - ct) \quad (76)$$

Substituting Eq.(76) into (75), we get

$$-cku' = \frac{1}{4}k^3u''' + 3kuu' + 3k(w - v^2)', \quad (77)$$

$$-ckv' = -\frac{1}{2}k^3v''' - 3kuv', \quad (78)$$

$$-ckw' = -\frac{1}{2}k^3w''' - 3kuw' \quad (79)$$

Let

$$u = \alpha v^2 + \beta v + \gamma,$$

$$w = A_0 v + B_0, \quad (80)$$

where $\alpha, \gamma, \beta, A_0$ and B_0 are constants. Inserting Eq.(80) into (78) and (79) integrating once we know that (78) and (79) give rise to the same equation

$$k^2v'' = -2\alpha v^3 - 3\beta v^2 + 2(c - 3\gamma)v + c_1, \quad (81)$$

where c_1 is an integration constant. Integrating (81) we have

$$k^2v'^2 = -\alpha v^4 - 2\beta v^3 + 2(c - 3\gamma)v^2 + 2c_1v + c_2, \quad (82)$$

where c_2 is an integration constant. By means of Eqs.(80-82) we get

$$\begin{aligned}k^2u'' &= 2\alpha k^2v'^2 + k^2(2\alpha v + \beta)v'' = 2\alpha[-\alpha v^4 - 2\beta v^3 \\ &+ 2(c - 3\gamma)v^2 + 2c_1v + c_2] + (2\alpha v + \beta)[-2\alpha v^3 - 3\beta v^2 + 2(c - 3\gamma)v + c_1]\end{aligned}\quad (83)$$

Integrating (77) once we have

$$\frac{1}{4}k^2u'' + \frac{3}{2}u^2 + cu + 3(w - v^2) + c_3 = 0, \quad (84)$$

where c_3 is an integration constant. Inserting (80) and (83) into (84) gives

$$3\alpha c - 3\alpha\gamma + \frac{3}{4}\beta^2 - 3 = 0,$$

$$\frac{1}{2}[\alpha c_1 + \beta c + \gamma\beta] + A_0 = 0,$$

$$\frac{1}{4}(2\alpha c_2 + \beta c_1) + \frac{3}{2}\gamma^2 + c\gamma + 3B_0 + c_3 = 0 \quad (85)$$

Let

$$c_1 = \frac{1}{2\alpha^2}[\beta^3 + 2c\alpha\beta - 6\alpha\beta\gamma],$$

$$v(\xi) = a\phi(\xi) - \frac{\beta}{2\alpha} \quad (86)$$

Therefore from Eq.(81), we have

$$k^2\phi''(\xi) - a\left(\frac{3\beta^2}{2\alpha} + 2c - 6\gamma\right)\phi(\xi) + 2\alpha a^3\phi^3(\xi) = 0, \quad (87)$$

then Eq.(87) can be written as

$$A\phi''(\xi) + B\phi(\xi) + D\phi^3(\xi) = 0 \quad (88)$$

Eq.(88) is the same with Eq.(23) where A , B and D are defined by

$$A = k^2, B = -a\left(\frac{3\beta^2}{2\alpha} + 2c - 6\gamma\right), D = 2\alpha a^3 \quad (89)$$

Then the solutions of Eqs.(75) are given by

$$u(x, t) = \alpha\left[a\phi(\xi) - \frac{\beta}{2\alpha}\right]^2 + \gamma, \quad (90)$$

$$v(x, t) = \left[a\phi(\xi) - \frac{\beta}{2\alpha}\right], \quad (91)$$

$$w(x, t) = A_0\left[a\phi(\xi) - \frac{\beta}{2\alpha}\right] + B_0, \quad (92)$$

the expression $\phi(\xi)$ appearing in these solutions are defined by Eqs.(26-33).

5. Conclusion

In this paper, with the aid of a simple transformation technique, we have shown that the generalized-Zakharov equations, the coupled nonlinear Klein-Gordon-Zakharov equations, the GDS,DS and GZ equations and generalized Hirota-Satsuma coupled KdV system can be reduced to the elliptic-like equation.

The validity of the proposed method has been tested by applying it successfully to the generalized-Zakharov equations, the coupled nonlinear Klein-Gordon-Zakharov equations, the GDS,DS and GZ equations and generalized Hirota-Satsuma coupled KdV system. As a result, many exact wave solutions are obtained which include new solitary wave solutions, periodic and rational solutions.

Finally, it is worthwhile to mention that the proposed method is straightforward and concise, more applications to other nonlinear physical systems should be concerned and deserve further investigation. This is our task in the future work.

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