

On the Dynamics of a n-D Piecewise Linear Map

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Abstract: This paper, derives sufficient conditions for the existence of chaotic attractors in a general n-D piecewise linear discrete map, along the exact determination of its dynamics using the standard definition of the largest Lyapunov exponent.

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1. Introduction

There are many works that focus on the topic of the rigorous mathematical proof of chaos in a discrete mapping (continuous or not). For example it has been studied rigorously from a control and anti-control schemes or from the use of Lyapunov exponents, see for example [1-2-3-4-5-6], to prove the existence of chaos in n -dimensional dynamical discrete system, since a large number of physical and engineering systems have been found to exhibit a class of continuous or discontinuous piecewise linear maps [12-13] where the discrete-time state space is divided into two or more compartments with different functional forms of the map separated by borderlines [14-15-16-17-18]. The theory for discontinuous maps is in the preliminary stage of development, with some progress reported for 1-D and n-D discontinuous maps in [19-20-21-22-23], these results are restrictive, and cannot be obtained in the general n-dimensional context [23].

This paper, derives sufficient conditions for the existence of chaotic attractors in a general n-D piecewise linear discrete map, along the exact determination of its dynamics using the standard definition of the Lyapunov exponents as the usual test for chaos.

In the following, we present the standard definition of the Lyapunov exponents for a discrete n-D mapping.

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Theorem 1. (Lyapunov exponent): Considered the following n-D discrete dynamical system:

$$x_{k+1} = f(x_k), x_k \in \mathbb{R}^n, k = 0, 1, 2, \dots \quad (1)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, is the vector field associated with system (1), let $J(x)$ be its Jacobian evaluated at x , let also the matrix:

$$T_r(x_0) = J(x_{r-1}) J(x_{r-2}) \dots J(x_1) J(x_0). \quad (2)$$

Moreover, let $J_i(x_0, l)$ be the module of the i^{th} eigenvalue of the l^{th} matrix $T_r(x_0)$, where $i = 1, 2, \dots, n$ and $r = 0, 1, 2, \dots$

Now, the Lyapunov exponents of a n-D discrete time systems are defined by:

$$\omega_i(x_0) = \ln \left(\lim_{r \rightarrow +\infty} J_i(x_0, r)^{\frac{1}{r}} \right), i = 1, 2, \dots, n. \quad (3)$$

2. The main result

Let us consider the following n-D map of the form: $f : D \rightarrow D$, $D \subset \mathbb{R}^n$, defined by:

$$x_{k+1} = f(x_k) = A_i x_k + b_i, \text{ if } x_k \in D_i, i = 1, 2, \dots, m. \quad (4)$$

where $A_i = (a_{jl}^i)_{1 \leq j, l \leq n}$ and $b_i = (b_i^j)_{1 \leq j \leq n}$, are respectively $n \times n$ and $n \times 1$ real matrices, for all $i = 1, 2, \dots, m$, and $x_k = (x_k^j)_{1 \leq j \leq n} \in \mathbb{R}^n$ is the state variable, and m is the number of disjoint domains on which D is partitioned. Due to the shape of the vector field f of the map (4) the plane can be divided into m regions denoted by $(D_i)_{1 \leq i \leq m}$, and in each of these regions the map (4) is linear.

The Jacobian matrix of the map (4) is:

$$J(x_k) = \begin{cases} A_1, & \text{if } x_k \in D_1, \\ A_2, & \text{if } x_k \in D_2, \\ \dots & \\ A_m, & \text{if } x_k \in D_m, \end{cases} \quad (5)$$

In the following we will compute analytically all the Lyapunov exponents of the map (1) and we will show that these exponents are the same in each linear regions $(D_i)_{1 \leq i \leq m}$ defined above. The essential idea of our proof is the assumption that the matrices $(A_i)_{1 \leq i \leq m}$ has the same eigenvalues, i.e. they are equivalent, then, if one compute analytically a Lyapunov exponent of the map (4) in a region D_i (which is the logarithm of the absolute value of an eigenvalue of a matrix A_i) then, one can find that these exponents are identical in each linear region D_i , for all $i \in \{1, 2, \dots, m\}$. Thus, one can consider the Jacobian matrix $J(x_k)$ of the map (4) as any matrix A_i , denoted by $A = (a_{jl})_{1 \leq j, l \leq n}$.

Assume that the eigenvalues of A are listed in order as follow:

$$\left| \lambda_1 \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| \geq \left| \lambda_2 \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| \geq \dots \geq \left| \lambda_n \left((a_{jl})_{1 \leq j, l \leq n} \right) \right|, \quad (6)$$

where the notation $\lambda_i \left((a_{jl})_{1 \leq j, l \leq n} \right)$ indicate that the eigenvalue λ_i depend only the coefficients $(a_{jl})_{1 \leq j, l \leq n}$, then $T_r(x_0) = A^r$, and its eigenvalues are $\lambda_1^r \left((a_{jl})_{1 \leq j, l \leq n} \right), \dots, \lambda_n^r \left((a_{jl})_{1 \leq j, l \leq n} \right)$, then the Lyapunov exponents of the map (4) are:

$$\omega_i(x_0) = \ln \left(\lim_{r \rightarrow +\infty} \left(\left| \lambda_i \left((a_{jl})_{1 \leq j, l \leq n} \right) \right|^r \right)^{\frac{1}{r}} \right) = \ln \left| \lambda_i \left((a_{jl})_{1 \leq j, l \leq n} \right) \right|, i = 1, 2, \dots, n. \quad (7)$$

Hence, according to (6) all the Lyapunov exponents are listed as follow:

$$\omega_1 \left((a_{jl})_{1 \leq j, l \leq n} \right) \geq \omega_2 \left((a_{jl})_{1 \leq j, l \leq n} \right) \geq \dots \geq \omega_n \left((a_{jl})_{1 \leq j, l \leq n} \right), \quad (8)$$

Define the following subsets of \mathbb{R}^{n^2} in term of the vector $(a_{jl})_{1 \leq j, l \leq n}$ as follow:

$$\Omega_1 = \left\{ (a_{jl})_{1 \leq j, l \leq n} \in \mathbb{R}^{n^2}, \left| \lambda_n \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| > 1 \right\}, \quad (9)$$

$$\Omega_2 = \left\{ (a_{jl})_{1 \leq j, l \leq n} \in \mathbb{R}^{n^2}, \left| \lambda_1 \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| < 1 \right\}, \quad (10)$$

$$\Omega_3 = \left\{ (a_{jl})_{1 \leq j, l \leq n} \in \mathbb{R}^{n^2}, \left| \lambda_1 \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| = 1 \right\}, \quad (11)$$

$$\Omega_4 = \left\{ (a_{jl})_{1 \leq j, l \leq n} \in \mathbb{R}^{n^2}, \left| \lambda_i \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| < 1, i = 2, \dots, n \right\}, \quad (12)$$

$$\Omega_5 = \left\{ (a_{jl})_{1 \leq j, l \leq n} \in \mathbb{R}^{n^2}, \left| \lambda_2 \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| = 1 \right\}, \quad (13)$$

$$\Omega_6 = \left\{ (a_{jl})_{1 \leq j, l \leq n} \in \mathbb{R}^{n^2}, \left| \lambda_i \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| < 1, i = 3, \dots, n \right\}, \quad (14)$$

$$\Omega_7 = \left\{ (a_{jl})_{1 \leq j, l \leq n} \in \mathbb{R}^{n^2}, \left| \lambda_i \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| = 1, i = 1, 2, \dots, K, \text{ where } 1 \leq K \leq n \right\}, \quad (15)$$

$$\Omega_8 = \left\{ (a_{jl})_{1 \leq j, l \leq n} \in \mathbb{R}^{n^2}, \left| \lambda_i \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| < 1, i = K + 1, \dots, n \right\}, \quad (16)$$

$$\Omega_9 = \left\{ (a_{jl})_{1 \leq j, l \leq n} \in \mathbb{R}^{n^2}, \left| \lambda_1 \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| > 1, \text{ and } \prod_{i=2}^{i=n} \left| \lambda_i \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| < 1 \right\}, \quad (17)$$

Finally, one obtain the following results:

(1) The map (4) is super chaotic when all its Lyapunov exponents are positive, i.e. $\omega_n \left((a_{jl})_{1 \leq j, l \leq n} \right) > 0$, according to inequalities (6) and (8). Thus, one may obtain $(a_{jl})_{1 \leq j, l \leq n} \in \Omega_1$.

(2) The map (4) converges to a stable fixed point when all the Lyapunov exponents are negative, i.e. $\left| \lambda_1 \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| < 1$, according to inequalities (6) and (8). Thus, one may obtain $(a_{jl})_{1 \leq j, l \leq n} \in \Omega_2$.

(3) The map (4) converges to a circle attractor when $\omega_1 = 0$, and $0 > \omega_2 \geq \dots \geq \omega_n$, i.e. $\left| \lambda_1 \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| = 1$, and $\left| \lambda_i \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| < 1$, for $i = 2, \dots, n$. Thus, one may obtain $(a_{jl})_{1 \leq j, l \leq n} \in \Omega_3 \cap \Omega_4$.

(4) The map (4) converges to a torus attractor when $\omega_1 = \omega_2 = 0$, and $0 > \omega_3 \geq \dots \geq \omega_n$, i.e. $\left| \lambda_1 \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| = \left| \lambda_2 \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| = 1$, and $\left| \lambda_i \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| < 1$, for $i = 3, \dots, n$. Thus, one may obtain $(a_{jl})_{1 \leq j, l \leq n} \in \Omega_3 \cap \Omega_5 \cap \Omega_6$.

(5) The map (4) converges to a K-torus attractor when $\omega_1 = \omega_2 = \dots = \omega_K = 0$, and $0 > \omega_{K+1} \geq \dots \geq \omega_n$, i.e. $\left| \lambda_1 \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| = \left| \lambda_2 \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| = \dots = \left| \lambda_K \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| = 1$, and $\left| \lambda_i \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| < 1$, $i = K + 1, \dots, n$. Thus, one may obtain $(a_{jl})_{1 \leq j, l \leq n} \in \Omega_7 \cap \Omega_8$.

(6) The map (4) converges to a chaotic attractor when $\omega_1 > 0$, and $\sum_{i=2}^n \omega_i < 0$, i.e. $\left| \lambda_1 \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| > 1$, and $\prod_{i=2}^{i=n} \left| \lambda_i \left((a_{jl})_{1 \leq j, l \leq n} \right) \right| < 1$. Thus, one may obtain $(a_{jl})_{1 \leq j, l \leq n} \in \Omega_9$.

Generally, for a continuous map positive Lyapunov exponent indicate chaos, negative exponent indicate fixed points, and if the Lyapunov exponent is equal to 0, then the dynamics is periodic, while for a discontinuous map a zero Lyapunov exponent is not indicate periodic behavior, in this case the map generates a symbolic sequence $\hat{s} = \{\hat{s}_0; \hat{s}_1; \dots; \hat{s}_j; \dots\}$ composed of symbols $\hat{s}_j = i$ if $x_j = f^j(x_0) \in D_i$, $i = 1, \dots, m$. Each of those symbolic sequences is called “admissible” and its symbols describe the order in which trajectories, starting from any initial condition x_0 , visit the various sub regions D_i , $i = 1, \dots, m$. In [7] a general approach for finding periodic trajectories in piecewise-linear maps, this procedure is based on the decomposition of the initial state via the eigenvectors of their jacobian and it is applied to digital filters with two’s complement overflow and $\Sigma\Delta$ modulators [8-9-10-11]. Finally, one conclude that there is some cases (depend mainly on position of the initial conditions) where the behavior of a map is not periodic in spite of its Lyapunov exponent is zero.

Hence, the following theorem is proved.

Theorem 2. Considered a general n-D piecewise linear map of the form:

$$f(x_k) = x_{k+1} = A_i x_k + b_i, \text{ if } x_k \in D_i \subset \mathbb{R}^n, \text{ } i = 1, 2, \dots, m, \quad (18)$$

and assume the following:

(a) The map (18) is piecewise linear. i.e the integer m verify $m \geq 2$, and there exist

$i, j \in \{1, 2, \dots, m\}$ such that $b_i \neq 0$ and $b_i \neq b_j$.

(b) The map (18) has a set of fixed point. i.e. There is a set of integers i in $\{1, 2, \dots, m\}$ such that the equations $A_i x + b_i = x$, has at least a zero x in the subregion D_i .

(c) All the matrices A_i and A_j are equivalent. i.e. there exist invertible matrices P_{ij} such that: $A_i = P_{ij} A_j P_{ij}^{-1}$, for all $i, j \in \{1, 2, \dots, m\}$.

Then, the dynamics of the map (18) is known in term of the vector $(a_{jl})_{1 \leq j, l \leq n} \in \mathbb{R}^{n^2}$ in the following cases:

- (1) if $(a_{jl})_{1 \leq j, l \leq n} \in \Omega_1$, then the map (18) is super chaotic.
- (2) if $(a_{jl})_{1 \leq j, l \leq n} \in \Omega_2$, then the map (18) converges to a stable fixed point.
- (3) if $(a_{jl})_{1 \leq j, l \leq n} \in \Omega_3 \cap \Omega_4$, then the map (18) converges to a circle attractor.
- (4) if $(a_{jl})_{1 \leq j, l \leq n} \in \Omega_3 \cap \Omega_5 \cap \Omega_6$, then the map (18) converges to a torus attractor.
- (5) if $(a_{jl})_{1 \leq j, l \leq n} \in \Omega_7 \cap \Omega_8$, then the map (18) converges to a K-torus attractor.
- (6) if $(a_{jl})_{1 \leq j, l \leq n} \in \Omega_9$, then the map (18) is chaotic.

3. Conclusion

We have reported a rigorous proof of chaos in a general n-D piecewise linear map, along the exact determination of its dynamics using the standard definition of the largest Lyapunov exponent.

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