

# Building of Heat Kernel on Non-Compact Homogeneous Spaces

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**Abstract:** Method of the solution of the main problem of homogeneous spaces thermodynamics on non-compact spaces in the case of non-compact homogeneous spaces is presented in the article. The method is based on the formalism of coadjoint orbits. In that article we present algorithm that allows efficiently evaluate heat kernel on non-compact homogeneous spaces. The method is illustrated with non-trivial example.

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## 1. Introduction

The goal to be achieved in present work is to demonstrate how method of orbits can be applied to the problems of quantum statistic mechanics (thermodynamics of homogeneous spaces) on non-compact homogeneous space [2]. Method of coadjoint orbits appeared to be quite powerful tool in theory of representations, Fourier analysis on homogeneous spaces, geometric quantization and integration of PDE's. As it was shown in [1] the use of the orbits method is the most fruitful if not the ultimate way to solve the main problem of homogeneous spaces thermodynamics for non-compact unimodular Lie groups with left-invariant riemannian metric. Non-compact Lie groups were chosen there as the object for the investigation because they give

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us simple but transparent enough example of spaces for which heat kernel and and statistic sum hardly can be found in the framework of methods existed before, for instance widely used separation of variables. Below we will consider much wider class of spaces then Lie groups with riemannian metric and we will describe the algorithm for solution of the main problem of quantum statistic mechanics on arbitrary non-compact homogeneous space.

The methods traditionally used for solution of PDE's are hardly applicable for integration of heat-kernel equation on homogeneous spaces because of the same reasons as for Lie groups. We do not tend to describe them here since we discussed all of necessary details of the problem in [1].

Problem to be discussed in that article is interesting and important because for non-compact space is not yet possible way to represent statistic sum as series with factorized volume of the manifold for  $n$ -dimensional compact space

$$Z_\beta = \frac{Vol_M}{(4\pi\beta)^{n/2}} \sum a_i \beta^i, \quad (1)$$

were coefficients  $a_i$  represent spectral invariants, which may be expressed through functions on the manifold, symbol of operator  $H$  and it's derivatives. All existing results in that field were related to the compact manifolds and non-compact manifolds of finite volume ([3],[4],[5]). Below we shall show the algorithm to build heat kernel on arbitrary non-compact homogeneous space. Solution of wave and heat kernel equation for different classes of manifolds is the problem of great interest in modern mathematical physics. In works by Chalykh and Veselov ([6],[7]) that problem was solved for some compact spaces and non-compact spaces as well and authors obtained explicit formulas for heat kernel on these spaces. That became possible because of special structure of groups of motions and since that spaces themselves (most part of considered spaces were symmetric with simple or semisimple group of motion).

In general our task here is evaluation of the statistic sum (distribution function)  $Z_\beta$  as a sum over the spectrum of energy operator  $H = -\Delta$  (Laplace-Beltrami operator) on homogeneous space

$$Z_\beta = \sum_n d_n \exp(-\beta E_n), \quad (2)$$

where  $d_n$  is a degeneration of  $E_n$ ,  $\beta$  - inverse temperature. In case of elliptic operator  $H$  on compact space series (2) is always convergent [2].

Statistic sum (2) on homogeneous space can be expressed as a trace of density matrix (heat kernel)  $\rho_\beta(x, x')$

$$Z_\beta = \int \rho_\beta(x, x) d\mu(x), \quad d\mu(x) = \sqrt{|g|} dx, \quad (3)$$

being a solution of Bloch equation (heat kernel equation) on corresponding manifold

$$\frac{\partial \rho_\beta(x, x')}{\partial \beta} + H(x) \rho_\beta(x, x') = 0, \quad \rho_\beta(x, x')|_{\beta=0} = \delta(x - x') / \sqrt{|g|}. \quad (4)$$

In case of arbitrary homogeneous space integration of heat kernel equation (3) is not that obvious. Below we shall develop the method that would allow us to get global solution of that equation.

## 2. Harmonic Analysis on Homogeneous Spaces

Size required for the article can not allow us to give detailed description of harmonic analysis on homogeneous spaces based on method of orbits. Since that we will mention here only the most necessary of it's principal constructions. Details of the method interested reader may find in [8].

Let real group  $G$  acts on smooth manifold  $M$ , i.e.  $M = G/H$  — homogeneous space, where  $H$  — isotropy subgroup of marked point  $x_0$ ,  $\{e_A\}$  — basic vectors of Lie algebra  $\mathfrak{g}$  of group  $G$ . With use of local coordinates  $(x)$  action of transformation group on manifold  $M$  can be expressed through the action of vector fields  $X_A$ , which form basis of Lie algebra  $\mathfrak{g}$ :

$$X_A = X_A^a(x) \frac{\partial}{\partial x^a}, \quad \text{rank } X_A^a(x_0) = \dim M; \quad [X_A, X_B] = C_{AB}^C X_C. \quad (5)$$

Any algebra  $\mathfrak{g}$  of generators  $X_A$  of transformation group can be put into correspondence associate algebra  $\mathbf{D}(M)$  of invariant and pseudo-invariant operators  $Y$  on  $M$  with condition

$$[Y, X_A] = 0, \quad A = 1, \dots, \dim \mathfrak{g}.$$

Algebra  $\mathbf{D}(M)$  is finitely generated (has finite number of generators). Let's note it's basis through  $\{Y_\mu\}$  — set of functionally independent operators which means that any of invariant operators can be expressed as a function of operators  $Y_\mu$ . Obviously commutator of any pair of invariant operators will be invariant operator. Since that there are such symmetrized operator functions  $\Omega_{\mu\nu}(Y_1, Y_2, \dots)$ , that

$$[Y_\mu, Y_\nu] = \Omega_{\mu\nu}(Y). \quad (6)$$

Associate algebra with basis commutation rules (6) is called *functional* algebra or shortly  $\mathcal{F}$ -algebra. In particular, if  $\Omega_{\mu\nu}$  are quadratic polynomes such algebra is called quadratic algebra. *Index* ( $\text{ind } \mathcal{F}$ ) of  $\mathcal{F}$ -algebra is called number of elements that generate it's center.

Dimension and index of  $\mathcal{F}$ -algebra of invariant operators are easy to count knowing structural constants of algebra  $\mathfrak{g}$  and isotropy subalgebra  $\mathfrak{h}$ :

$$\dim \mathcal{F} = \dim \mathfrak{g} + \dim \mathfrak{h}^\lambda - 2 \dim \mathfrak{h}, \quad \lambda \in \mathfrak{h}^\perp = \{f \in \mathfrak{g}^* \mid \langle f, \mathfrak{h} \rangle = 0\}; \quad (7)$$

$$\text{ind } \mathcal{F} = \dim \mathfrak{g}^\lambda / \mathfrak{h}^\lambda, \quad \mathfrak{g}^\lambda = \{X \in \mathfrak{g} \mid \langle \lambda, [X, \mathfrak{g}] \rangle = 0\}, \quad \mathfrak{h}^\lambda = \mathfrak{g}^\lambda \cap \mathfrak{h}. \quad (8)$$

Here and below  $\lambda$  is generally situated linear functional that belongs to the subspace  $\mathfrak{h}^\perp$ . The way formulas (7), (8) were derived and algorithm to find algebra of invariant operators is described in work [9].

Lets introduce symbols of operators as functions on cotangent bundle  $T^*M$ :

$$X_A(x, \partial_x) \rightarrow X_A(x, p) = X_A^a(x) p_a, \quad Y_\mu(x, \partial_x) \rightarrow Y_\mu^{cl}(x, p).$$

One can show that any invariant operator  $Y(x, \partial_x)$  can be corresponded with invariant function  $Y^{cl}(x, p)$  cotangent bundle  $T^*M$  and vice versa .

Symbols of operators satisfy following relations in respect to Poisson bracket defined using canonic symplectic 2-form  $\omega = dp \wedge dx$ :

$$\begin{aligned} \{X_A(x, p), X_B(x, p)\} &= C_{AB}^C X_C(x, p), \quad \{Y_\mu^{cl}(x, p), Y_\nu^{cl}(x, p)\} = \Omega_{\mu\nu}(Y^{cl}(x, p)), \\ \{X_A(x, p), Y_\mu^{cl}(x, p)\} &= 0. \end{aligned}$$

Moments mappings

$$\mu : T^*M \rightarrow \mathfrak{g}^*, \quad X(x, p) = f \in \mathfrak{g}^*; \quad \tilde{\mu} : T^*M \rightarrow \mathcal{F}^*, \quad Y^{cl}(x, p) = g \in \mathcal{F}^*$$

are Poisson mappings of Poisson algebra of functions on  $T^*M$  at Poisson algebras on  $\mathfrak{g}^*$  and  $\mathcal{F}^*$  with brackets:

$$\begin{aligned} \{\varphi, \psi\}^{\mathcal{F}}(g) &= \Omega_{\mu\nu}(g) \frac{\partial\varphi(g)}{\partial g_\mu} \frac{\partial\psi(g)}{\partial g_\nu}; \quad g = g_\mu E^\mu \in \mathcal{F}^*; \quad \varphi, \psi \in C^\infty(\mathcal{F}^*); \\ \{\varphi, \psi\}(f) &= C_{AB}^C f_C \frac{\partial\varphi(f)}{\partial f_A} \frac{\partial\psi(f)}{\partial f_B}; \quad f = f_A e^A \in \mathfrak{g}^*; \quad \varphi, \psi \in C^\infty(\mathfrak{g}^*). \end{aligned}$$

Even more, symplectic sheets  $\Omega \subset \mathfrak{g}^*$  and  $\tilde{\Omega} \subset \mathcal{F}^*$  are in mutual correspondence [10]:

$$\Omega = \mu(\tilde{\mu}^{-1}(\tilde{\Omega})), \quad \tilde{\Omega} = \tilde{\mu}(\mu^{-1}(\Omega)); \quad \text{codim } \Omega = \text{codim } \tilde{\Omega}. \tag{9}$$

Formula (9) means that centers of Poisson algebras coincide in the sense that bases of Casimir functions in  $C^\infty(\mathfrak{g}^*)$  and  $C^\infty(\mathcal{F}^*)$  may be chosen following way:

$$K_\alpha(X(x, p)) = \tilde{K}_\alpha(Y^{cl}(x, p)), \quad \alpha = 1, \dots, \text{ind } \mathcal{F}.$$

Last equation is satisfied even if we change Casimir functions for symmetrized functions of operators:

$$K_\alpha(iX(x, \partial_x)) = \tilde{K}_\alpha(Y(x, \partial_x)), \quad \alpha = 1, \dots, \text{ind } \mathcal{F}. \tag{10}$$

Let's note as  $U$  Lagrange submanifold to the symplectic sheet  $\tilde{\Omega}$ . *Defect*  $d(M)$  of homogeneous space  $M$  is defined as dimension of Lagrange submanifold to the symplectic sheet on coalgebra of invariant operators [9]:

$$d(M) = \dim U = \frac{1}{2} \dim \tilde{\Omega}.$$

Following equality definitely takes place

$$\dim \mathcal{F} - \text{ind } \mathcal{F} = \dim \tilde{\Omega} = 2d(M).$$

If we substitute in it (7), (8) for dimension and index of  $\mathcal{F}$ -algebra we shall acquire following expression

$$d(M) = \frac{1}{2} \dim \mathfrak{g}/\mathfrak{g}^\lambda - \dim \mathfrak{h}/\mathfrak{h}^\lambda, \quad \lambda \in \mathfrak{h}^\perp. \tag{11}$$

To make calculation procedure more convenient we rewrite (11) as follows

$$d(M) = \frac{1}{2} \text{rank}\langle \lambda, [\mathfrak{g}, \mathfrak{g}] \rangle - \text{rank}\langle \lambda, [\mathfrak{g}, \mathfrak{h}] \rangle, \quad \lambda \in \mathfrak{h}^\perp. \quad (12)$$

Homogeneous spaces with  $d(M) = 0$  are called *commutative*. For commutative spaces  $\dim \mathcal{F} = \text{ind } \mathcal{F}$ , i.e. algebra of invariant operators is commutative and since that (10) is generated by Cazimir operators of algebra  $\mathfrak{g}$ . For instance all symmetric and weakly symmetric spaces are commutative [12].

Let  $Q$  be Lagrange submanifold to the symplectic sheet  $\Omega$  (coadjoint orbit) in  $\mathfrak{g}^*$ . Manifold  $Q$  is of dimension

$$\dim Q = \frac{1}{2} \dim \mathfrak{g}/\mathfrak{g}^\lambda, \quad \lambda \in \mathfrak{h}^\perp. \quad (13)$$

Let's represent algebra  $\mathfrak{g}$  as algebra of differential operators  $l(q, \partial_q, J)$ , that give exact irreducible representation of algebra  $\mathfrak{g}$  in space of functions on  $Q$  ( $\lambda$ -representation [11]):

$$[l_A(q, \partial_q; J), l_B(q, \partial_q; J)] = C_{AB}^C l_C(q, \partial_q; J), \quad K_\alpha(-il(q, \partial_q, J)) = \kappa_\alpha(J), \quad \det \frac{\partial \kappa_\alpha(J)}{\partial J_\beta} \neq 0. \quad (14)$$

Here  $q$  are local coordinates on  $Q$ ,  $J$  are parameters enumerating orbits in  $\mathfrak{g}$  which are integer.

Let's build irreducible representation of algebra  $\mathcal{F}$  by differential operators  $\zeta(u, \partial_u; J)$  on Lagrange submanifold  $U$  of the symplectic sheet  $\tilde{\Omega}$  in  $\mathcal{F}^*$  being in agreement with  $\lambda$ -representation.

$$[\zeta_\mu(u, \partial_u; J), \zeta_\nu(u, \partial_u; J)] = -\Omega_{\mu\nu}(\zeta(u, \partial_u; J)), \quad \tilde{K}_\alpha(\zeta(u, \partial_u, J)) = \kappa_\alpha(J). \quad (15)$$

Set of generalized functions  $D_{qu}^J(x)$  is to be defined from the equations

$$(X_A(x, \partial_x) + l_A(q, \partial_q; J))D_{qu}^J(x) = 0, \quad (Y_\mu(x, \partial_x) - \zeta_\mu(u, \partial_u; J))D_{qu}^J(x) = 0 \quad (16)$$

and is full and orthogonal  $C^\infty(M)$  [8]:

$$\int_M D_{\tilde{q}\tilde{u}}^{\tilde{J}}(x) \overline{D_{qu}^J(x)} d\mu(x) = \delta(J, \tilde{J})\delta(q, \tilde{q})\delta(u, \tilde{u}); \quad (17)$$

$$\int D_{qu}^J(x) \overline{D_{qu}^J(\tilde{x})} d\mu(J)d\mu(q)d\mu(u) = \delta(x, \tilde{x}). \quad (18)$$

Here  $d\mu(J)$  is spectral measure of Casimir operators,  $\delta(J, \tilde{J})$  is  $\delta$ -function in respect to that measure,  $d\mu(x)$   $d\mu(q)$ ,  $d\mu(u)$  are quasi-invariant measures on homogeneous space  $M$  and on Lagrange manifolds  $Q$  and  $U$ ,  $\delta(x, \tilde{x})$ ,  $\delta(q, \tilde{q})$ ,  $\delta(u, \tilde{u})$  are  $\delta$ -function in respect to corresponding measures.

Set of functions  $D_{qu}^J(x)$  is full and orthogonal and allows direct and inverse Fourier transform to be performed:

$$\varphi(x) = \int \overline{\psi(q, u, J)} D_{qu}^J(x) d\mu(J)d\mu(q)d\mu(u), \quad \psi(q, u, J) = \int \overline{\varphi(x)} D_{qu}^J(x) d\mu(x). \quad (19)$$

If functions  $\varphi(x)$  and  $\psi(q, u, J)$  are connected by the relations (19) ( $\varphi \sim \psi$ ) one easily can prove that the same relations connect functions:

$$X_A(x, \partial_x)\varphi(x) \sim l_A(q, \partial_q, J)\psi(q, u, J), \quad Y_\mu(x, \partial_x)\varphi(x) \sim \zeta_\mu(u, \partial_u, J)\psi(q, u, J). \quad (20)$$

We assume here that the measures are chosen the way that operators  $X_A$ ,  $l_A$  are anti-hermitian and operators  $Y_\mu$ ,  $\zeta_\mu$  are hermitian. That assumption was done to make the text of that article easier and even more, that case is quite widespread. No obstacles stop us from consideration of the general case when such measures do not exist.

Let operator  $H(x, \partial_x)$  is invariant under action of group  $G$ . Then it can be presented as a function of invariant operators  $H = H(Y(x, \partial_x))$  and after fourier transform is done (19) it goes into  $\tilde{H}(\zeta(u, \partial_u, J))$  which depends of  $d(M)$  (see formula (11)) independent variables  $u$ . If operator  $H(x, \partial_x)$  is element of enveloping algebra  $U(\mathfrak{g})$ , i.e.  $H(x, \partial_x) = H(X(x, \partial_x))$  then after fourier transform is done it will take form  $\tilde{H}(l(q, \partial_q, J))$  and it will depend on  $\dim Q$  (see formula (13)) independent variables  $q$ .

So the variant of harmonic analysis on homogeneous spaces based on method of orbits, presented above, allows us to effectively integrate linear differential equations with non-commuting symmetries [13]. We shall apply that formalism to find heat kernel on homogeneous riemannian spaces.

### 3. Integration of Bloch Equation on Homogeneous Spaces

Below we shall consider Bloch equation (4) on homogeneous space and apply the formalism of harmonic analysis described above to integrate it.

Let group  $G$  acts on homogeneous space  $M = G/H$ . We supply that space with the structure of riemannian manifold introducing metric tensor on the manifold according to the rule

$$g^{ij} = G^{ab} X_a^i X_b^j, \quad (21)$$

where  $G^{ab}$  is symmetric matrix. Here  $X_a^i$  are generators of group action given by (5). That metric is so-called central metric and integration of geodesic flows on homogeneous spaces with such metrics was discussed in [14].

Firstly we must evaluate defect  $d(M)$  of homogeneous space according to (11) or (12) and dimension and index of  $\mathcal{F}$ —algebra. Then one must find explicit formulas for operators  $X_i(x)$  — generators of action of transformation group on homogeneous space, which are obtained as left-invariant vector fields restricted on the space  $M$ . The method to build operators  $X_i(x)$  on  $M = G/H$  quite easily knowing structural constants of Lie algebras of Lie groups  $G$  and  $H$  was described in [15] and may be realized as a computer program.

Operator  $H(x)$  in Bloch equation (4) on homogeneous space  $M$  we obtain as a quadratic function of operators  $X_i(x)$ :

$$H(-i\hbar X) = -\hbar^2 G^{ab} X_a X_b = -\hbar^2 \Delta, \quad (22)$$

where  $\Delta$  is Laplace-Beltrami operator on manifold with riemannian metric (21).

The main idea of presented method is to make number of independent variables in Bloch equation smaller using non-commuting symmetry operators. That can be done using formalism of harmonic analysis when solution of Bloch equation is presented as Fourier decomposition according to (19). That gives us possibility to represent heat kernel  $\rho_\beta(x, x')$  on entire homogeneous space by expression

$$\rho_\beta(x, x') = \int \mathcal{R}_\beta(q, u, q', u', J, J') \overline{D_{qu}^J}(x') D_{q'u'}^{J'}(x) d\mu(q) d\mu(q') d\mu(u) d\mu(u') d\mu(J) d\mu(J'), \quad (23)$$

where (23) function  $\mathcal{R}_\beta(q, u, q'u', J, J')$  can be treated as a heat kernel on a Lagrange submanifold to the symplectic sheet  $Q$  (coadjoint orbit).

Bloch equation on homogeneous space after Fourier transform is done goes into Bloch equation on corresponding orbit

$$\frac{\partial \mathcal{R}_\beta(q, u, q'u', J, J')}{\partial \beta} + H(-i\hbar l) \mathcal{R}_\beta(q, u, q', u', J, J') = 0 \quad (24)$$

with initial condition

$$\mathcal{R}_\beta(q, u, q', u', J, J')|_{\beta=0} = \delta(q, q') \delta(u, u') \delta(J, J'), \quad (25)$$

where number of independent variables  $q$  is sufficiently reduced and equals  $n' = \dim Q$ . We must point that in one interesting case when  $n' = 1$  reduced Bloch equation appears to be ordinary differential equation and since that integrable in quadratures. Although mentioned situation is not common the method allows to sufficiently simplify Bloch equation and as a result to integrate it.

Since operators of  $\lambda$  – representation depend only of variables  $q$ , we can represent heat kernel on Lagrange submanifold  $\mathcal{R}_\beta(q, u, q'u', J, J')$  following way

$$\mathcal{R}_\beta(q, u, q', u', J, J') = \mathcal{R}_\beta(q, q') \delta(u, u') \delta(J, J'), \quad \mathcal{R}_\beta(q, q')|_{\beta=0} = \delta(q, q'), \quad (26)$$

where  $\mathcal{R}_\beta(q, q')$  satisfies the same Bloch equation (24) with operator  $H(-i\hbar l)$ , which follows from (20). The way we considered function  $\mathcal{R}_\beta(q, u, q'u', J, J')$  in (26) makes possible to do integrations in (23) over  $d\mu(u')$  and  $d\mu(J')$ . After integration we will obtain following simplified formula for  $\rho_\beta(x, x')$ :

$$\rho_\beta(x, x') = \int \mathcal{R}_\beta(q, q') \overline{D_{qu}^J}(x') D_{q'u}^J(x) d\mu(q) d\mu(q') d\mu(u) d\mu(J), \quad (27)$$

So we finally have to solve the equation for function  $\mathcal{R}_\beta(q, q')$

$$\frac{\partial \mathcal{R}_\beta(q, q')}{\partial \beta} + H(-i\hbar l) \mathcal{R}_\beta(q, q') = 0, \quad \mathcal{R}_\beta(q, q')|_{\beta=0} = \delta(q, q') \quad (28)$$

After solution of equation (28) and substituting it in (26) we obtain a solution of reduced Bloch equation (24) on Lagrange submanifold of a symplectic sheet to the orbit that corresponds to the homogeneous space. The transition to the solution on the entire space  $\rho_\beta(x, x')$  can be obtained using transformation (27).

### 4. Example

As an example of presented method we chose the solution of the main problem of thermodynamics tree-dimensional homogeneous space with four-dimensional Lie group acting on it. The algebra of Lie group  $G$  is determined by following commutation rules:

$$[e_2, e_3] = e_1, [e_2, e_4] = e_3, [e_3, e_4] = -e_2. \tag{29}$$

Casimir functions of that algebra are  $K_1 = f_1, K_2 = f_2^2 + f_3^2 - 2f_1f_4$ .

Let's consider 3-dimensional homogeneous space  $M = G/H$ , where Lie algebra of one-dimensional subgroup  $H$  is  $\{e_3\}$ . We choose linear functional  $\lambda(J) = \{-2j^2, 0, 0, n\}, n \in N$ .

It's easy to calculate that the defect of that space  $d(M) = 0$  and since that considered space is symmetric and we don't have here generators of  $\mathcal{F}$  – algebra except for Casimir operators.

Generators  $X_a$  of group action on  $M$  look as follows

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x_1}, & X_2 &= -x_2 \sin(x_4) \frac{\partial}{\partial x_1} + \cos(x_4) \frac{\partial}{\partial x_2}, \\ X_3 &= x_2 \cos(x_4) \frac{\partial}{\partial x_1} + \sin(x_4) \frac{\partial}{\partial x_2}, & X_4 &= \frac{\partial}{\partial x_4}. \end{aligned} \tag{30}$$

Operators of  $\lambda$ -representation are

$$l_1 = -2ij^2, \quad l_2 = j(\partial_q - q), \quad l_3 = -ij(\partial_q + q), \quad l_4 = -i(\partial_q + n), \tag{31}$$

with polarization chosen as  $\{e_1, e_2 - ie_3, e_4\}$ .

Functions  $D_{qu}^\lambda(x)$  can easily be found from equations (16)

$$D_{qu}^J(x) = D_q^J(x) = \exp(2jqx_2e^{ix_3} - x_2^2j^2 + 2ijx_1 - \frac{q^2}{2}e^{2ix_2} - inx_3), \tag{32}$$

and since defect  $d(M)$  of homogeneous space  $M$  is zero we have here only dependence on variable  $q$ . Measure on  $q$ -space is  $d\mu(q) = \exp(-q\bar{q})d^2q/\pi$  and  $\delta$ -function in respect to that measure takes form  $\delta(\bar{q}', q) = \exp(q\bar{q}')$ .

Matrix  $G$  was chosen as

$$G^{ab} = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & B & -C \\ 0 & 0 & C & 0 \end{pmatrix}$$

with condition  $A, B, C > 0, B > 2C$ .

Operator of Bloch equation on considered homogeneous space (22) looks as follows

$$H(x) = (A + Bx_2^2) \frac{\partial^2}{\partial x_1^2} + B \frac{\partial^2}{\partial x_2^2} + C(x_2 \sin(x_4) \frac{\partial}{\partial x_1} - \cos(x_4) \frac{\partial}{\partial x_2}). \tag{33}$$

That operator coincides with Laplace-Beltrami operator built according to the well known rule for riemannian manifold using metric (21).

Since that Bloch equation on that space is

$$\frac{\partial \rho_\beta(x, x')}{\partial \beta} - \hbar^2 \left( (A + Bx_2^2) \frac{\partial^2}{\partial x_1^2} + B \frac{\partial^2}{\partial x_2^2} + C(x_2 \sin(x_4)) \frac{\partial}{\partial x_1} - \cos(x_4) \frac{\partial}{\partial x_2} \right) \rho_\beta(x, x') = 0. \quad (34)$$

Bloch equation on corresponding Lagrange submanifold of a symplectic sheet to the orbit contains only one independent variable and takes following form

$$\frac{\partial \mathcal{R}_\beta(q, q')}{\partial \beta} - (-4j^4 A - 2j^2 B + Cjq) \mathcal{R}_\beta(q, q') + (-4Bj^2 q - Cj) \frac{\partial}{\partial q} \mathcal{R}_\beta(q, q') = 0. \quad (35)$$

Solution of (35) which represents density matrix on orbit and satisfies special initial condition is

$$\begin{aligned} \mathcal{R}_\beta(q, q') = & \exp(-4j^2 B \beta \left( \frac{C^2}{B^2 j^2} + \frac{j^2 A}{B} + \frac{1}{2} \right)) \exp\left( \frac{Cq}{4Bj} - \frac{C((4Bqj + C) \exp(-4Bj^2 \beta) - C)}{16B^2 j^2} \right) \times \\ & \times \exp\left( \frac{q'((4Bqj + C) \exp(-4Bj^2 \beta) - C)}{4Bj} \right). \end{aligned} \quad (36)$$

Finally solution of Bloch equation on entire homogeneous space we obtain as integral over variable  $j$

$$\begin{aligned} \mathcal{R}_\beta(x, x') = & \int \frac{j}{2\pi^2} \delta(x_3 - x'_3) \exp(-4j^4 \beta A - \frac{\beta C^2}{4B} - 2i(x_1 - x'_1)j - (2B\beta + x_2^2 + x_2'^2) - \\ & - \frac{Cx_2 e^{-ix_3}}{2B} (e^{-4Bj^2 \beta} - 1) - \frac{C^2 e^{-2ix_3} (e^{4Bj^2 \beta} - 1)}{32B^2 j^2} (e^{4Bj^2 \beta} (2e^{2ix_3} - 1) + 1)) dj. \end{aligned} \quad (37)$$

## 5. Conclusion

Above we have shown clear algorithm that partly allows to solve the main problem of homogeneous spaces thermodynamics for arbitrary homogeneous space. Solution of Bloch equation (4) and building of density matrix are just the first but important step to the solution of much more complex task — to find statistic sum for arbitrary non-compact homogeneous space, the problem that was stated many years ago but yet to be solved. We must point out that in mentioned works were obtained exact explicit formulas for heat kernel. Method presented above does not give answer immediately at the moment we define the structure of the manifold. Our method is an algorithm that must be applied to a given homogeneous space to find the heat kernel on it.

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