

Co-existence of Regular and Chaotic Motions in the Gaussian Map

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Received 12 September 2006, Accepted 29 September 2006, Published 20 December 2006

Abstract: In this communication, the Gaussian map, which has drawn less attention in the past as compare to other one-dimensional maps, has been explored. Particularly, the dynamical behavior of the Gaussian map and the presence of co-existing attractors (which is a rare phenomenon in one-dimensional maps) in the complete parameter space have been investigated. We also suggest a possible geometrical reason for the emergence of co-existing attractors at a particular set of system parameters, which works for all one-dimensional maps. The regions of parameter space, where regular and chaotic motions co-exist, have also been identified.

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Keywords: Gaussian map, Co-existing Attractors, Period-doubling, Chaos

PACS (2006): 05.45.-a, 05.45.Pq

The study of time-evolution of various systems in physics, chemistry, biology or engineering is one of the most frequent and commonly investigated tasks of the natural sciences. It is generally done by using model equations that describe the evolution in time of the system state in a state space. This state space can be characterized by a set of variables which are either continuous or discrete. Also the time may be continuous or discrete integer-valued variable. Generally, all these different possible cases of evolution equations are summarized under the term dynamical system [1]. If the time is considered as a continuous variable then the system model equations may be described by a set of differential equations termed as continuous dynamical system or flows. On the other hand the time can be treated as discrete integer-valued variable; such model equations are described by a set of difference equations called discrete dynamical system or maps. Discreteness of time variable may mean that it is sufficient to measure certain physical variable after a finite interval of time rather than on a continuous basis. In some scien-

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tific cases it is natural to represent time as a discrete variable as in digital electronics, impulsively driven systems etc. One motivation behind the study of such maps is their origin in the description of intersection of state space trajectories with Poincare sections.

A famous prototype example of a one-dimensional map is the logistic map ($x_{n+1} = \lambda x_n(1 - x_n)$, an idealized model for the yearly variation of the population of animals), which has played a key role in the development of chaos theory [2, 3]. It can be considered the simplest example of one-dimensional map, which exhibits wide spectrum of dynamical behaviour including the chaotic motion.

In this communication, we intend to explore another famous one-dimensional map—the Gaussian map, which has drawn less attention due to its similarity with the logistic map. However, Gaussian map exhibits some features (such as co-existing attractors, reverse period doubling etc), which make it distinguishable from logistic map [4]. We particularly analyze the dynamical behaviour of the Gaussian map in the parameter space and investigate the presence of co-existing attractors. We also try to suggest a possible explanation for the presence of co-existing attractors by using geometrical means. The Gaussian map under the consideration is based on the Gaussian function & characterized by two parameters b and c (rather than a single parameter as in the case of logistic map) as follows:

$$x_{n+1} = e^{-bx_n^2} + c$$

It is expected that the behaviour of Gaussian map is similar to the behaviour of logistic map as the Gaussian map function also exhibits a one hump shape similar to the logistic function. There are two parameters (b and c) present in the Gaussian map, so the analysis of the behaviour of its long term iterates becomes little complicated than in the case of logistic map.

In Figure 1, we have plotted the Gaussian map function for different sets of parameters b and c . It is clear that the map function is symmetric about $x = 0$, and has maximum value (always at $x = 0$) equals to $c + 1$. For large values of $|x|$, the function approaches to the minimum value equals to c , however parameter b decides the width of the map function. Now we do the stability analysis of the Gaussian map for some fixed value of parameter b (say $b = 7.5$). In Figure 2, we have plotted the Gaussian map function for $b = 7.5$ and different values of parameter c . We observe that for large negative values of parameter c ($c = -1.0, -0.85, -0.7$), there exist three fixed points (which are evident from the intersection of line $F(x) = x$ and Gaussian map function) for the Gaussian map, however for other values of c ($c = -0.55, -0.40, -0.25$), only one fixed point exists.

These fixed points can be calculated by solving the nonlinear algebraic equation $x = e^{-bx^2} + c$. Since it is a transcendental equation, the analytical solution is not possible. We have solved this equation using the iterative method [5] and the corresponding solution is depicted in Figure 3(a) for $b = 7.5$. Now we analyze the stability of these fixed points. The best way to analyze the stability of fixed points (when no analytical solution for the fixed points exists) is to draw the bifurcation diagram (A plot illustrating the qualitative changes in the dynamical behaviour of the system as a function of system parameter). In Figure 3(b), we have shown such a bifurcation diagram for the Gaussian map by plotting

its long term iterates for several values of initial conditions and same values of parameters as in Figure 3(a). The corresponding Lyapunov exponent (which is a quantitative measure of chaos) curve has been shown in Figure 3(c).

The most important feature we observe from Figure 3 (a), is that in a certain range of parameter c ($-1.03 < c < -0.65$), three fixed points exist (one is positive and two are negative), however for the other values of c only one fixed point exists (which is positive for $c \geq -0.65$ and negative for $c \leq -1.03$ (not shown here)). From the bifurcation diagram shown in Figure 3(b), we may infer that for $-1.03 < c < -0.65$ the fixed point having large negative value is stable however the other negative fixed point is unstable. On the other hand the positive fixed point is stable for $-1.03 < c < -0.897$, then it becomes unstable and bifurcates into two stable fixed points, and further period doubling continues leading to chaos, periodic windows, reverse period doubling etc. for $-0.897 < c < 0.40$ (i.e 2-4-8-16- chaos -periodic windows -chaos-3-6-12. . .chaos. . .12-6-3-chaos-periodic windows-chaos- . . -16-8-4-2). Beyond $c = 0.40$, the positive fixed point again becomes stable and remains so for all values of $c > 0.40$. So for $-1.03 < c < -0.65$, we observe two stable attractors (each corresponds to different set of initial condition) in the bifurcation diagram, one is a period-one attractor however the other is periodic or chaotic depending on the value of parameter c . Two different curves of Lyapunov exponents for $-1.03 < c < -0.65$ in Figure 3(c) confirm the co-existence of two stable attractors.

In Figure 4, we have shown iterates of the Gaussian map function using cobweb diagrams [6] for a fixed value of parameter b (i.e. $b = 7.5$) and different values of c . In Frame (a), we have shown iterates of the Gaussian map for $c = -0.95$ and two different values of initial condition, both converge to different fixed point attractors (infact these are the fixed points of Gaussian map function), hence two stable attractors co-exist. In Frames (b), (c), (d) & (e) similar situations have been depicted for $c = -0.85, -0.79, -0.70, -0.66$ respectively, where period-1 attractor (which is one of the fixed point of Gaussian map) co-exists with period-2, period-4, chaotic & period-3 attractors respectively. In all the cases depicted in Frames (a)-(e), we note a common feature that the Gaussian map function possesses three fixed points or in other words it makes one positive hump and one negative hump with the line $F(x) = x$ (i.e. the 45° line). However in Frame (f), we have shown a cobweb diagram for $c = -0.6$. In this case Gaussian map possesses only one fixed point (unstable) and only one stable attractor (chaotic) exists for all the initial values.

From Figures 1-4, we may conclude that the emergence of co-existing attractors at a particular set of parameters can be understood as a consequence of presence of three fixed points or in other words co-existing attractors are observed only when the map function makes one positive and one negative hump with the line $F(x) = x$ (i.e. the 45° line). We have carried out above analysis for a fixed value of parameter b (i.e. $b = 7.5$) and by varying the parameter c . Now we analyze the dynamical behaviour of the Gaussian map and co-existence of attractors in the complete parameter space (a, b) .

In Figure 5, we have identified different regions of parameter space (a, b) where the dynamics of Gaussian map is chaotic and regular (black and white shades correspond to

chaotic and periodic attractors respectively). The results shown in Figure 5 are based on the extensive numerical calculation of Lyapunov exponent [7] by iterating the Gaussian map 1000 times (after neglecting initial transient behaviour up to 400 iterations) at 2×10^6 different points of the parameter space defined by $0 \leq b \leq 20$ & $-1.0 \leq c \leq 0$. Chaotic situation has been recorded, whenever the Lyapunov exponent becomes greater than 1×10^{-4} . The above calculation has been repeated for several set of initial conditions to include all the co-existing attractors.

In Figure 6(a), we have shown the region of parameter space (dark grey shade), where the Gaussian map possesses three fixed points, however in Figure 6(b) the region of parameter space (light grey region), where co-existing attractors have been observed, is shown. In the light grey shaded region a period-1 attractor co-exists with some other attractor (chaotic, period-1, period-2, period -4., period-3, etc). It is interesting to note that the region of the parameter space, where co-existing attractors exist is a subset of the region shown in Figure 6(a), which conforms our earlier conclusion that the emergence of co-existing attractors at a particular set of parameters can be understood as a consequence of presence of three fixed points. However the converse is not always true as it is clear from Figures 6 (a) & (b) that there is a part of parameter space, where three fixed points exist but co-existing attractors are not present.

Finally, in Figure 7, we have depicted the region of parameter space (black shade), where regular and chaotic motions coexist for different sets of initial condition.

In conclusion, we have analyzed the dynamical behaviour of the Gaussian map and observed that it exhibits a wide spectrum of dynamical behaviours such as regular and chaotic motions, period doubling route to chaos, reverse period doubling, co-existing attractors (which is a rare phenomenon in one-dimensional maps) etc. We also draw an important conclusion that co-existing attractors are observed only when the map function makes one positive and one negative hump with the line $F(x) = x$ (presence of three fixed points). The conclusion made above is also true for a recent study of q-deformed logistic map [8] where co-existing attractors have been observed. Hence it may work for all one-dimensional maps.

Acknowledgement

Author acknowledges to the *Condensed Matter & Statistical Physics Section* of the *Abdus Salam International Centre for Theoretical Physics (ICTP), Trieste, Italy* for sponsoring his Guest Scientist Visit (October 15-November 13, 2005) during which a part of the work, reported in this paper, was completed.

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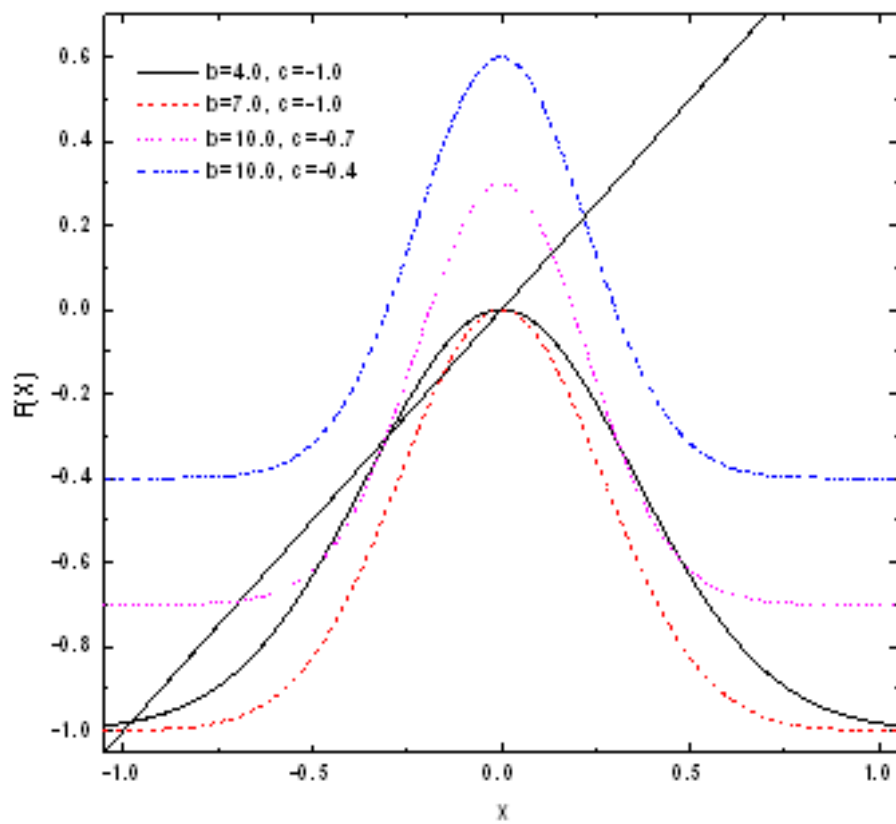


Fig. 1 Gaussian map function ($F(x) = \exp(-bx^2) + c$) for different values of parameters b and c . The diagonal line $F(x) = x$ is also shown.

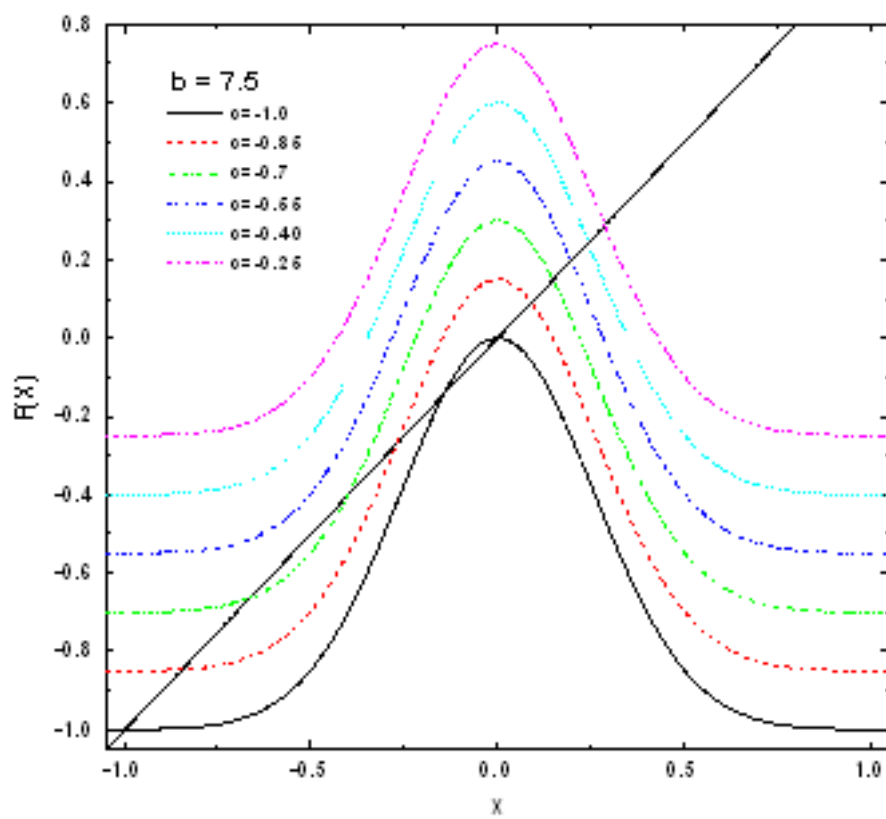


Fig. 2 Gaussian map function ($F(x) = \exp(-bx^2) + c$) for fixed value of parameter b ($b = 7.5$) and different values of parameter c . The diagonal line $F(x) = x$ is also shown.

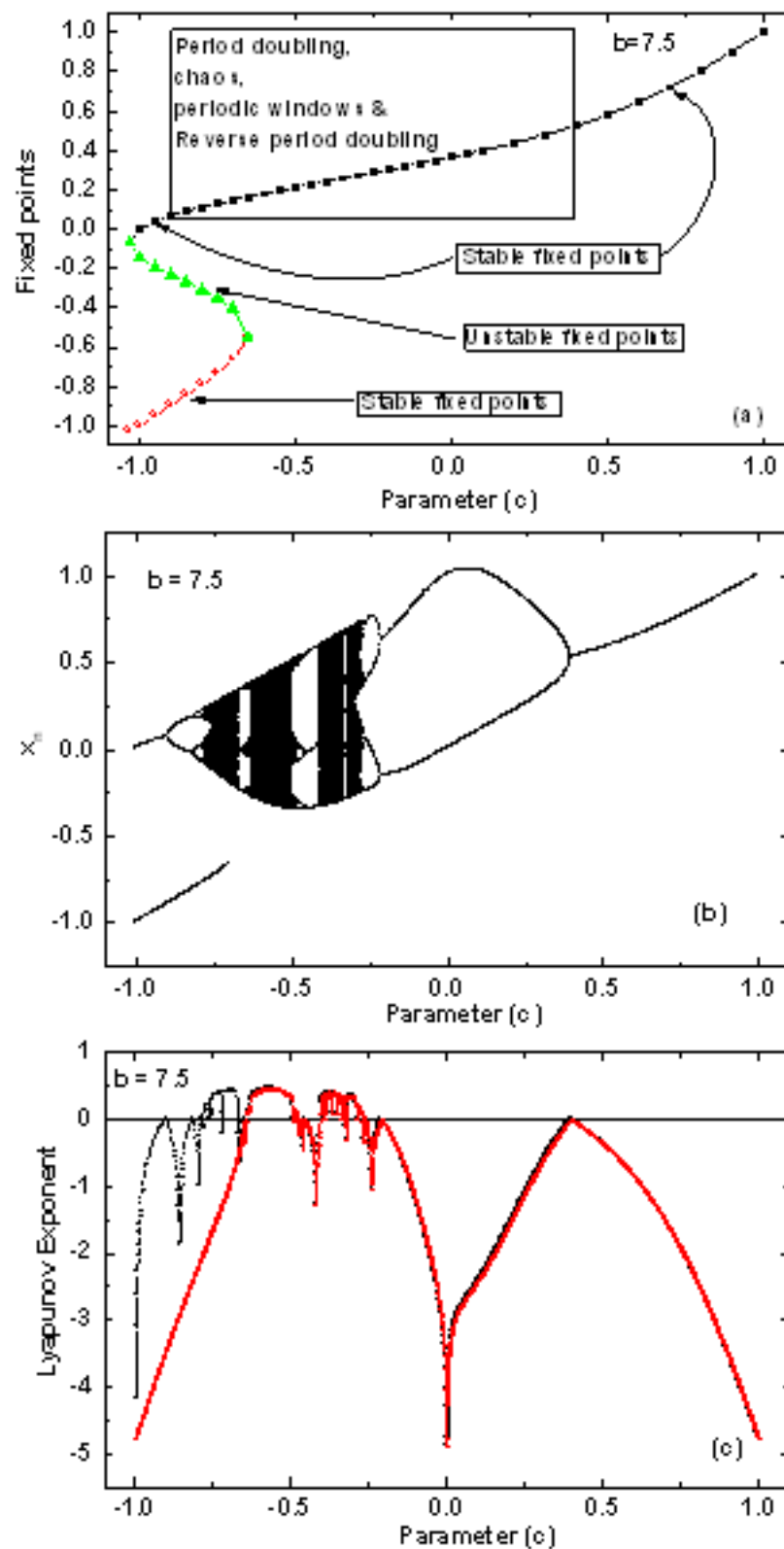


Fig. 3 (a) Fixed points of the Gaussian map as a function of parameter c for fixed value of parameter b ($b = 7.5$), (b) Bifurcation diagram showing the stable attractors, period doubling route to chaos and reverse period doubling as a function of parameter c for fixed value of parameter b ($b = 7.5$) and several values of initial conditions, (c) The Lyapunov exponent corresponding to the situation shown in frame (b).

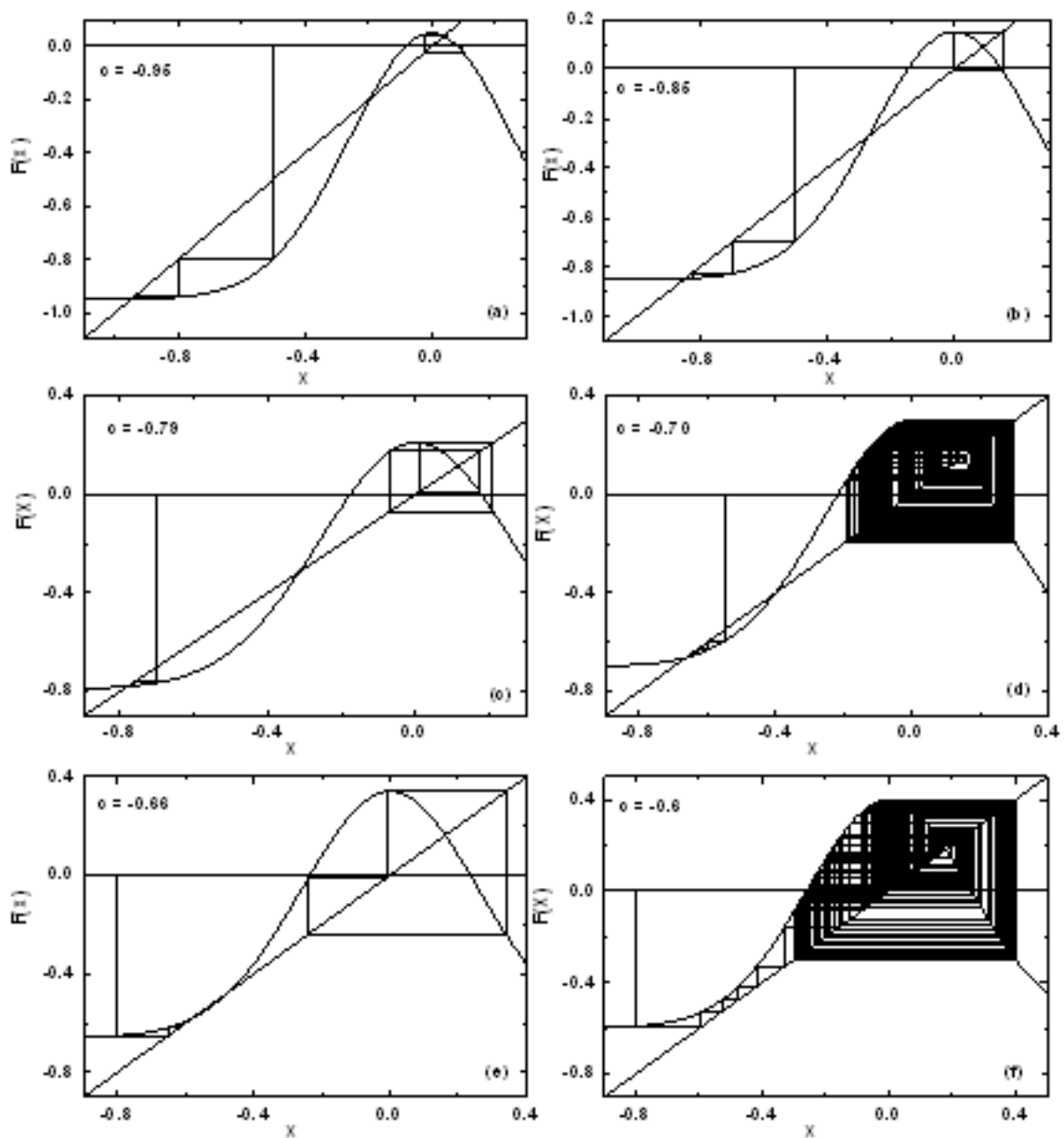


Fig. 4 Cobweb diagrams showing iterates of the Gaussian map for $b = 7.5$ and different values of parameter c .

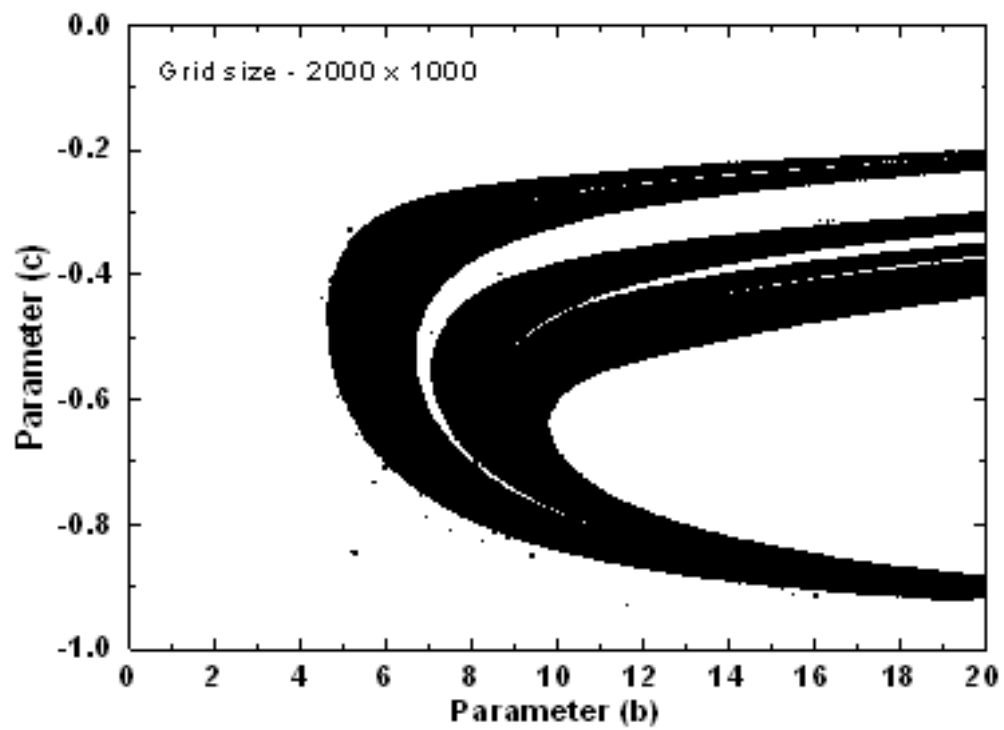


Fig. 5 Parameter space (a, b) of the Gaussian map showing the regions, where chaotic (black shade) and regular (white shade) motions appear.

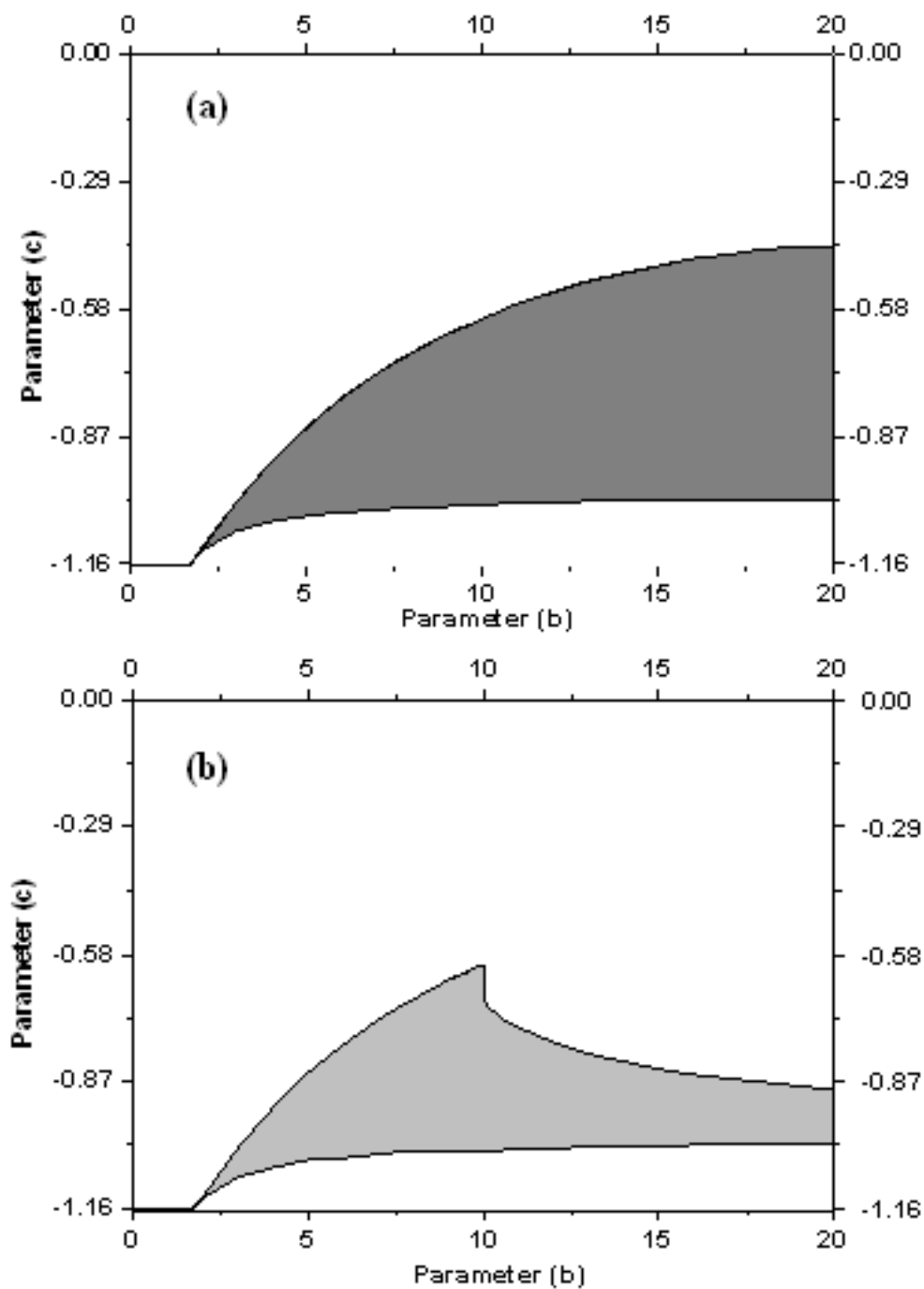


Fig. 6 (a) Parameter space (a, b) showing the region (dark grey shade), where the Gaussian map possesses three fixed points, (b) Parameter space (a, b) showing the region (light grey shade), where the Gaussian map exhibits co-existing attractors.

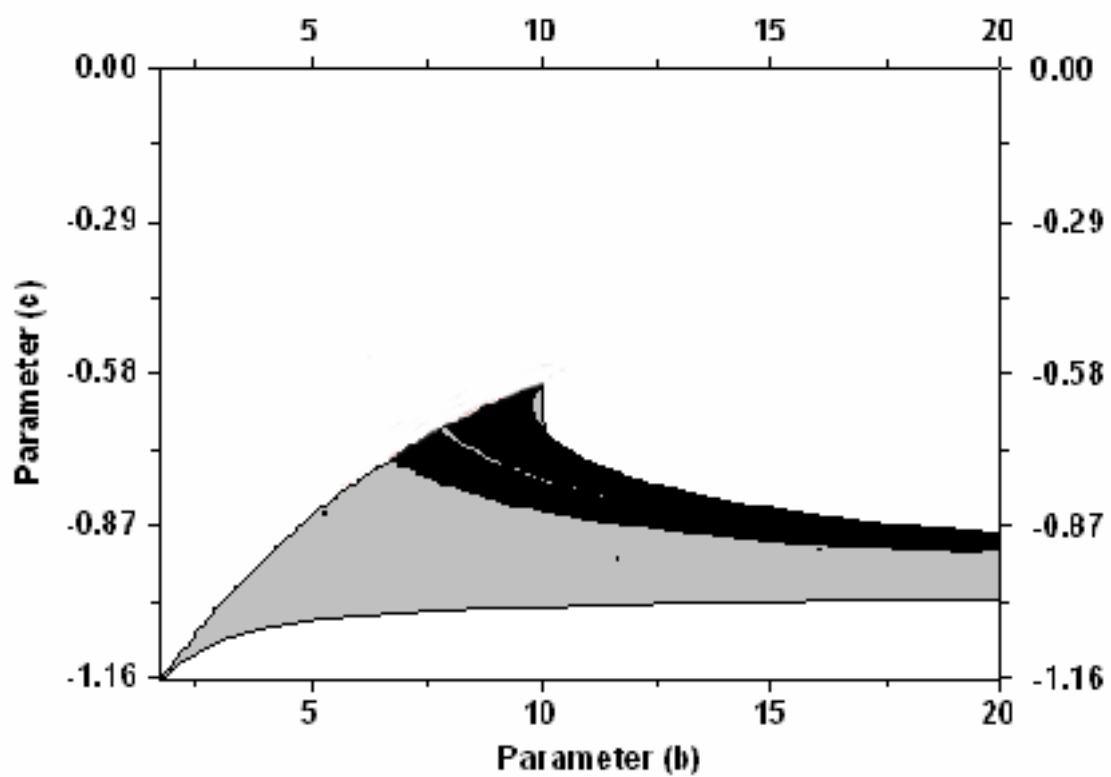


Fig. 7 Parameter space (a, b) showing the region (black shade), where chaotic and regular (period-1) motions co-exist for different sets of initial conditions.