

Black Scholes Option Pricing with Stochastic Returns on Hedge Portfolio

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Abstract: The Black Scholes model of option pricing constitutes the cornerstone of contemporary valuation theory. However, the model presupposes the existence of several unrealistic and rigid assumptions including, in particular, the constancy of the return on the “hedge portfolio”. There, now, subsists ample justification to the effect that this is not the case. Consequently, several generalisations of the basic model have been attempted. In this paper, we attempt one such generalisation based on the assumption that the return process on the “hedge portfolio” follows a stochastic process similar to the Vasicek model of short-term interest rates. © Electronic Journal of Theoretical Physics. All rights reserved.

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With the rapid advancements in the evolution of financial markets across the globe, the importance of generalisations of the extant mathematical apparatus to enhance its domain of applicability to the pricing of financial products can hardly be overemphasized for further progress and development of the financial microstructure.

Though at an embryonic stage, the unification of physics, mathematics and finance is unmistakably discernible with several fundamental premises of physics and mathematics like quantum mechanics, classical & quantum field theory and related tools of non-commutative probability, functional integration etc. being adopted for pricing of extant financial products and for elucidating on various occurrences of financial markets like stock price patterns, critical crashes etc [1-12].

The Black Scholes formula for the pricing of financial assets [13-17] continues to be the substratum of contemporary valuation theory. However, the model, although of

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immense practical utility is based on several assumptions that lack empirical support. The academic fraternity has attempted several generalisations of the original Black Scholes formula through easing of one or other assumption, in an endeavour to augment its spectrum of applicability.

In this paper, we attempt one such generalisation based on the assumption that the return process on the “hedge portfolio” follows a stochastic process similar to the Vasicek model of short-term interest rates. Section 2 lists out the derivation of the Black-Scholes formula through the partial differential equation based on the construction of the complete “hedge portfolio”. Sec 3, which forms the essence of this paper, attempts a generalisation of the standard Black Scholes pricing formula on the lines aforesaid. Section 4 concludes.

1. The Black Scholes Model

In order to facilitate continuity, we summarize below the original derivation of the Black Scholes model for the pricing of a European call option [13-17 and references therein]. The European call option is defined as a financial contingent claim that enables a right to the holder thereof (but not an obligation) to buy one unit of the underlying asset at a future date (called the exercise date or maturity date) at a price (called the exercise price). Hence, the option contract, has a payoff of $\max(S_T - E, 0) = (S_T - E)^+$ on the maturity date where S_T is the stock price on the maturity date and E is the exercise price.

We consider a non-dividend paying stock, the price process of which follows the geometric Brownian motion with drift $S_t = e^{(\mu t + \sigma W_t)}$. The logarithm of the stock price $Y_t = \ln S_t$ follows the stochastic differential equation

$$dY_t = \mu dt + \sigma dW_t \quad (1)$$

where W_t is a regular Brownian motion representing Gaussian white noise with zero mean and δ correlation in time i.e. $E(dW_t dW_{t'}) = dt dt' \delta(t - t')$ on some filtered probability space $(\Omega, (F_t), P)$ and μ and σ are constants representing the long term drift and the noisiness (diffusion) respectively in the stock price.

Application of Ito’s formula yields the following SDE for the stock price process

$$dS_t = \left(\mu + \frac{1}{2} \sigma^2 \right) S_t dt + \sigma S_t dW_t \quad (2)$$

Let $C(S, t)$ denote the instantaneous price of a call option with exercise price E at any time t before maturity when the price per unit of the underlying is S . It is assumed that $C(S, t)$ does not depend on the past price history of the underlying. Applying the Ito formula to $C(S, t)$ yields

$$dC = \left(\mu S \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t} \right) dt + \frac{\partial C}{\partial S} \sigma S dW, \quad (3)$$

The original option-pricing model propounded by Fischer Black and Myron Scholes envisaged the construction of a “hedge portfolio”, Π , consisting of the call option and

a short sale of the underlying such that the randomness in one cancels out that in the other. For this purpose, we make use of a call option together with $\partial C/\partial S$ units of the underlying stock.

We then have, on applying Ito's formula to the "hedge portfolio", Π ,:-

$$\frac{d\Pi}{dt} = \frac{d}{dt} \left[C(S, t) - S \frac{\partial C(S, t)}{\partial S} \right] = \frac{dC(S, t)}{dt} - \frac{\partial C}{\partial S} \cdot \frac{dS}{dt} \quad (4)$$

where the term involving $\frac{d}{dt} \left(\frac{\partial C}{\partial S} \right)$ has been assumed zero since it envisages a change in the portfolio composition. On substituting from eqs. (2) & (3) in (4), we obtain

$$\frac{d\Pi}{dt} = \frac{dC(S, t)}{dt} - \left(\mu + \frac{1}{2}\sigma^2 \right) S \frac{\partial C(S, t)}{\partial S} - \sigma S \frac{\partial C}{\partial S} \frac{dW}{dt} = \frac{\partial C(S, t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} \quad (5)$$

We note, here, that the randomness in the value of the call price emanating from the stochastic term in the stock price process has been eliminated completely by choosing the portfolio $\Pi = C(S, t) - S \frac{\partial C(S, t)}{\partial S}$. Hence, the portfolio Π is free from any stochastic noise and the consequential risk attributed to the stock price process.

Now $\frac{d\Pi}{dt}$ is nothing but the rate of change of the price of the so-called riskless bond portfolio i.e. the return on the riskless bond portfolio (since the equity related risk is assumed to be eliminated by construction, as explained above) and must, therefore, equal the short-term interest rate r i.e.

$$\frac{d\Pi}{dt} = r \Pi \quad (6)$$

In the original Black Scholes model, this interest rate was assumed as the risk free interest rate r , further, assumed to be constant, leading to the following partial differential equation for the call price:-

$$\frac{d\Pi}{dt} = r \Pi = r \left[C(S, t) - S \frac{\partial C(S, t)}{\partial S} \right] = \frac{\partial C(S, t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2}$$

or equivalently

$$\frac{\partial C(S, t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} + rS \frac{\partial C(S, t)}{\partial S} - rC(S, t) = 0 \quad (7)$$

which is the famous Black Scholes PDE for option pricing with the solution:-

$$C(S, t) = SN(d_1) - Ee^{-r(T-t)}N(d_2) \quad (8)$$

where

$$d_1 = \frac{\log\left(\frac{S}{E}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\log\left(\frac{S}{E}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \text{ and}$$

$$N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{x^2}{2}} dx$$

2. The Black Scholes Model with Stochastic Returns on the Hedge Portfolio

As mentioned earlier, in the above analysis, the interest rate r , which is essentially a proxy for the return on a portfolio that is devoid of any risk emanating from any variables that cause fluctuations and hence risk in stock price process, is taken as constant and equal to the risk free rate. However, this return would, nevertheless, be subject to uncertainties that influence returns on the fixed income securities. It is, now, conventional to model these short term interest rates (that are representative of short term returns on fixed income securities) through a stochastic differential equation of the form [18]

$$dr(t) = -\psi[r(t), t] dt + \eta[r(t), t] dU(t) \quad (9)$$

where $r(t)$ is the short term interest rate at time t , ψ and η are deterministic functions of r, t and $U(t)$ is a Wiener Process.

In our further analysis, we shall assume that this short-term interest rate is represented by the Vasicek model [19] viz.

$$\frac{dr(t)}{dt} + Ar(t) + B - \sum \eta(t) = 0 \quad (10)$$

where $\eta(t)$ is a white noise stochastic process

$$\langle \eta(t) \rangle = 0, \langle \eta(t) \eta(t') \rangle = \sum^2 \delta(t - t') \quad (11)$$

The call price process now becomes a function of two stochastic variables, the stock price process $S(t)$ and the bond return process (interest rate process) $r(t)$. Hence, application of Ito's formula to $C(S, r, t)$ gives

$$dC = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS + \frac{\partial C}{\partial r} dr + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} dt + \frac{1}{2} \sum^2 \frac{\partial^2 C}{\partial r^2} dt \quad (12)$$

where dS is given by eq. (2) and dr by eq. (10) respectively.

As in Section 2, we formulate a "hedge portfolio" Π consisting of a call option $C(S, r, t)$ and a short sale of $\frac{\partial C}{\partial S}$ units of stock $S(t)$ i.e. $\Pi = C - S \frac{\partial C}{\partial S}$. We then have, repeating the same steps as in Section 2 hereof

$$\frac{d\Pi}{dt} = \frac{dC}{dt} - \frac{\partial C}{\partial S} \frac{dS}{dt} = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial r} \frac{dr}{dt} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{1}{2} \sum^2 \frac{\partial^2 C}{\partial r^2} \quad (13)$$

Now, using $\frac{d\Pi}{dt} = r(t) \Pi$, we obtain

$$\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + r(t) S \frac{\partial C}{\partial S} - r(t) C + \frac{1}{2} \sum^2 \frac{\partial^2 C}{\partial r^2} + \frac{\partial C}{\partial r} \frac{dr(t)}{dt} = 0 \quad (14)$$

This equation defies closed form solution with the extant mathematical apparatus. We can, however, obtain explicit expressions for the call price $\bar{C}(S, t)$ averaged over the stochastic part of the interest rate process, as follows:-

$\bar{C}(S, t)$ would, then, be given by substituting $\frac{\int_t^T r(\tau)d\tau}{\int_t^T d\tau}$ for the constant risk free interest rate r in the Black Scholes formula (8).

The averaging process happens to be tedious with extensive computations so we proceed term by term.

We have

$$N(\bar{d}_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\bar{d}_1} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(\bar{d}_1 - x) e^{-\frac{x^2}{2}} dx \tag{15}$$

where $H(x - y)$ is the unit step Heaviside step function defined by [20]

$$H(x, y) = \begin{cases} 0, & x < y \\ 1, & x > y \end{cases}$$

On using the integral representation of $H(x - y)$ as $H(x - y) = \text{Lim}_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega(x-y)}}{\omega - i\epsilon}$ [20]

i.e.

$$H(\bar{d}_1 - x) = \text{Lim}_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega(\bar{d}_1-x)}}{\omega - i\epsilon} \tag{16}$$

we obtain

$$N(\bar{d}_1) = \text{Lim}_{\epsilon \rightarrow 0} \frac{1}{(2\pi)^{\frac{3}{2}} i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{2} + i\omega(\bar{d}_1-x)}}{\omega - i\epsilon} dx d\omega = \text{Lim}_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i\omega(\bar{d}_1 + i\frac{\omega}{2})}}{\omega - i\epsilon} d\omega \tag{17}$$

on performing the Gaussian integration over x in the second step.

Now

$$\bar{d}_1 = \frac{\log\left(\frac{S}{E}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} + \frac{\int_t^T r(\tau) d\tau}{\sigma\sqrt{T-t}} = d_1^0 + \frac{\int_t^T r(\tau) d\tau}{\sigma\sqrt{T-t}} \tag{18}$$

where

$$d_1^0 = \frac{\log\left(\frac{S}{E}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \tag{19}$$

Since the entire stochastic contribution comes from the expression $\int_t^T r(\tau) d\tau$ in $N(\bar{d}_1)$, we have

$$N(\bar{d}_1) = \text{Lim}_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega d_1^0 - \frac{\omega^2}{2}}}{\omega - i\epsilon} I_1 \tag{20}$$

where $I_1 = \left\langle e^{\frac{i\omega}{\sigma\sqrt{T-t}} \int_t^T r(\tau) d\tau} \right\rangle$ and $\langle \rho \rangle$ denotes the average (expectation) of ρ .

Proceeding similarly, we have,

$$N(\bar{d}_2) = \text{Lim}_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega d_2^0 - \frac{\omega^2}{2}}}{\omega - i\epsilon} I_1 \tag{21}$$

$$d_2^0 = \frac{\log\left(\frac{S}{E}\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \quad (22)$$

Similarly the discount factor $e^{-r(T-t)}$ will be replaced by $\left\langle e^{-\int_t^T r(\tau)d\tau} \right\rangle = I_2$ (say).

To evaluate the expectation integrals I_1, I_2 we make use of the functional integral formalism [21]. In this formalism, the expectation I_1 would be given by [22]:-

$$I_1 = \frac{\int_{r(t)}^{r(T)} Dr \exp \left[-\frac{1}{2\Sigma^2} \int_t^T d\tau \left(\frac{dr(\tau)}{d\tau} + Ar(\tau) + B \right)^2 + \frac{i\omega}{\sigma\sqrt{T-t}} \int_t^T d\tau r(\tau) \right]}{\int_{r(t)}^{r(T)} Dr \exp \left[-\frac{1}{2\Sigma^2} \int_t^T d\tau \left(\frac{dr(\tau)}{d\tau} + Ar(\tau) + B \right)^2 \right]} = \frac{P}{Q} \quad (23)$$

where $Dr = \prod_{\tau=t}^T \frac{dr(\tau)}{\sqrt{2\pi}}$ is the functional integration measure.

We first evaluate the functional integral P . Making the substitution $x(\tau) = -\frac{B}{A} - r(\tau)$, we obtain, with a little algebra,

$$\begin{aligned} P &= \int_{x(t)}^{x(T)} Dx \exp \left[-\frac{1}{2\Sigma^2} \int_t^T d\tau \left(\frac{dx(\tau)}{d\tau} + Ax(\tau) \right)^2 + \frac{i\omega}{\sigma\sqrt{T-t}} \int_t^T d\tau \left(-\frac{B}{A} - x(\tau) \right) \right] \\ &= \int_{x(t)}^{x(T)} Dx \exp \left\{ \frac{-1}{2\Sigma^2} \int_t^T d\tau \left[\left(\frac{dx(\tau)}{d\tau} \right)^2 + A^2 x^2(\tau) \right] - \frac{A}{2\Sigma^2} [x^2(T) - x^2(t)] - \frac{i\omega B(T-t)}{A\sigma\sqrt{T-t}} - \frac{i\omega}{\sigma\sqrt{T-t}} \int_t^T x(\tau) d\tau \right\} \\ &= \int_{x(t)}^{x(T)} Dx \exp \left\{ -\frac{A}{2\Sigma^2} [x^2(T) - x^2(t)] - \frac{i\omega B(T-t)}{A\sigma\sqrt{T-t}} - I_3 \right\} \end{aligned} \quad (24)$$

where

$$\begin{aligned} I_3 &= \frac{1}{2\Sigma^2} \int_t^T d\tau \left[\left(\frac{dx(\tau)}{d\tau} \right)^2 + A^2 x^2(\tau) \right] + \frac{i\omega}{\sigma\sqrt{T-t}} \int_t^T x(\tau) d\tau \\ &= \frac{1}{2\Sigma^2} \int_t^T d\tau \left[\left(\frac{dx(\tau)}{d\tau} \right)^2 + A^2 x^2(\tau) + \frac{2i\omega \Sigma^2}{\sigma\sqrt{T-t}} x(\tau) \right] \end{aligned} \quad (25)$$

In order to evaluate I_3 , we perform a shift of the functional variable $x(\tau)$ by some fixed function $y(\tau)$ i.e. $x(\tau) = y(\tau) + z(\tau)$ where $y(\tau)$ is a fixed functional (whose explicit form shall be defined later) but with boundary conditions $y(t) = x(t), y(T) = x(T)$ so that $z(\tau)$, then, has Dirichlet boundary conditions i.e. $z(t) = z(T) = 0$.

Substituting $x(\tau) = y(\tau) + z(\tau)$ in (25), we obtain

$$I_3 = \frac{1}{2\Sigma^2} \int_t^T d\tau \left[\left(\frac{dy(\tau)}{d\tau} \right)^2 + \left(\frac{dz(\tau)}{d\tau} \right)^2 + 2 \left(\frac{dy(\tau)}{d\tau} \right) \left(\frac{dz(\tau)}{d\tau} \right) + A^2 y^2(\tau) + A^2 z^2(\tau) + 2A^2 y(\tau) z(\tau) + \frac{2i\omega \Sigma^2}{\sigma\sqrt{T-t}} y(\tau) + \frac{2i\omega \Sigma^2}{\sigma\sqrt{T-t}} z(\tau) \right] \quad (26)$$

Integrating the second and third term by parts, we get

$$I_3 = \frac{z(\tau)}{2\Sigma^2} \left(\frac{dz(\tau)}{d\tau} + 2 \frac{dy(\tau)}{d\tau} \right) \Big|_t^T + \frac{1}{2\Sigma^2} \int_t^T d\tau \left[-z(\tau) \frac{d^2z(\tau)}{d\tau^2} - 2z(\tau) \frac{d^2y(\tau)}{d\tau^2} + \left(\frac{dy(\tau)}{d\tau} \right)^2 + A^2y^2(\tau) + A^2z^2(\tau) + 2A^2y(\tau)z(\tau) + \frac{2i\omega\Sigma^2}{\sigma\sqrt{T-t}}y(\tau) + \frac{2i\omega\Sigma^2}{\sigma\sqrt{T-t}}z(\tau) \right] \quad (27)$$

Now the boundary terms all vanish since $z(\tau)$ has Dirichlet boundary conditions. Further, if we define the fixed functional $y(\tau)$ in terms of the differential equation

$$-\frac{d^2y(\tau)}{d\tau^2} + A^2y(\tau) + \frac{i\omega\Sigma^2}{\sigma\sqrt{T-t}} = 0 \quad (28)$$

with boundary condition $y(t) = x(t), y(T) = x(T)$ we obtain

$$I_3 = \frac{1}{2\Sigma^2} \int_t^T d\tau \left\{ \left[\left(\frac{dy(\tau)}{d\tau} \right)^2 + A^2y^2(\tau) + \frac{2i\omega\Sigma^2}{\sigma\sqrt{T-t}}y(\tau) \right] + \left[-z(\tau) \frac{d^2z(\tau)}{d\tau^2} + A^2z^2(\tau) \right] \right\} \quad (29)$$

The functional $y(\tau)$ is fixed and is given by the solution of eq (28) as

$$y = \alpha e^{A\tau} + \beta e^{-A\tau} - \gamma \quad (30)$$

where

$$\gamma = \frac{i\omega\Sigma^2}{A^2\sigma\sqrt{T-t}}, \alpha = \frac{x(T)e^{AT} - x(t)e^{At}}{e^{2AT} - e^{2At}} + \gamma \frac{e^{AT} - e^{At}}{e^{2AT} - e^{2At}} \text{ and } \beta = \frac{x(T)e^{-AT} - x(t)e^{-At}}{e^{-2AT} - e^{-2At}} + \gamma \frac{e^{-AT} - e^{-At}}{e^{-2AT} - e^{-2At}}$$

Integrating out the $y(\tau)$ terms in eq. (29) using eq. (30), we obtain

$$I_3 = \frac{1}{2\Sigma^2} \left\{ A[\alpha^2(e^{2AT} - e^{2At}) - \beta^2(e^{-2AT} - e^{-2At}) - A\gamma^2(T-t)] + \int_t^T d\tau \left[-z(\tau) \frac{d^2z(\tau)}{d\tau^2} + A^2z^2(\tau) \right] \right\} \quad (31)$$

Substituting this value of I_3 in eq. (24) we obtain, for P , noting that $Dx = Dz$ since $y(\tau)$ is fixed by eq (28)

$$P = \exp \left\{ -\frac{A^2}{2\Sigma^2} [x^2(T) - x^2(t)] - \frac{i\omega B\sqrt{T-t}}{A\sigma} - \frac{1}{2\Sigma^2} \left[A \left[\left(\frac{x(T)e^{AT} - x(t)e^{At}}{e^{2AT} - e^{2At}} \right)^2 - \left(\frac{x(T)e^{-AT} - x(t)e^{-At}}{e^{-2AT} - e^{-2At}} \right)^2 \right] - \left(\frac{\omega^2\Sigma^4}{A^3\sigma^2(T-t)} \right) \left[\left(\frac{e^{AT} - e^{At}}{e^{2AT} - e^{2At}} \right)^2 - \left(\frac{e^{-AT} - e^{-At}}{e^{-2AT} - e^{-2At}} \right)^2 \right] - A(T-t) \right] + \left(\frac{2i\omega\Sigma^2}{A\sigma\sqrt{T-t}} \right) \left[\frac{(x(T)e^{AT} - x(t)e^{At})(e^{AT} - e^{At})}{(e^{2AT} - e^{2At})^2} - \frac{(x(T)e^{-AT} - x(t)e^{-At})(e^{-AT} - e^{-At})}{(e^{-2AT} - e^{-2At})^2} \right] \right] \right\} \int_{z(t)=0}^{z(T)=0} Dz \exp \left\{ \frac{-1}{2\Sigma^2} \int_t^T d\tau \left[-z(\tau) \frac{d^2z(\tau)}{d\tau^2} + A^2z^2(\tau) \right] \right\} \quad (32)$$

On exactly same lines, we obtain

$$Q = \exp \left\{ -\frac{A^2}{2\Sigma^2} [x^2(T) - x^2(t)] - \frac{1}{2\Sigma^2} \left[A \left[\left(\frac{x(T)e^{AT} - x(t)e^{At}}{e^{2AT} - e^{2At}} \right)^2 - \left(\frac{x(T)e^{-AT} - x(t)e^{-At}}{e^{-2AT} - e^{-2At}} \right)^2 \right] \right] \right\}$$

$$\int_{z(t)=0}^{z(T)=0} Dz \exp \left\{ \frac{-1}{2\Sigma^2} \int_t^T d\tau \left[-z(\tau) \frac{d^2z(\tau)}{d\tau^2} + A^2 z^2(\tau) \right] \right\}$$
(33)

Hence

$$I_1 = \exp \left\{ -\frac{i\omega B\sqrt{T-t}}{A\sigma} - \frac{1}{2\Sigma^2} \left[-\left(\frac{\omega^2 \Sigma^4}{A^3 \sigma^2 (T-t)} \right) \left[\left(\frac{e^{AT} - e^{At}}{e^{2AT} - e^{2At}} \right)^2 - \left(\frac{e^{-AT} - e^{-At}}{e^{-2AT} - e^{-2At}} \right)^2 - A(T-t) \right] + \left(\frac{2i\omega \Sigma^2}{A\sigma\sqrt{T-t}} \right) \left[\frac{(x(T)e^{AT} - x(t)e^{At})(e^{AT} - e^{At})}{(e^{2AT} - e^{2At})^2} - \frac{(x(T)e^{-AT} - x(t)e^{-At})(e^{-AT} - e^{-At})}{(e^{-2AT} - e^{-2At})^2} \right] \right] \right\}$$
(34)

which when substituted in eqs. (20) & (21) shall give the values $N(\bar{d}_1)$ and $N(\bar{d}_2)$ respectively as:-

$$N(\bar{d}_1) = N \left\{ \frac{\frac{\log(\frac{S}{E}) + \frac{1}{2}\sigma^2}{\sigma\sqrt{T-t}} - \frac{B\sqrt{T-t}}{A\sigma} - \frac{Y}{A\sigma\sqrt{T-t}}}{\left[1 - \frac{\Sigma^2 X}{A^3 \sigma^2 (T-t)} \right]^{\frac{1}{2}}} \right\}$$
(35)

and

$$N(\bar{d}_2) = N \left\{ \frac{\frac{\log(\frac{S}{E}) - \frac{1}{2}\sigma^2}{\sigma\sqrt{T-t}} - \frac{B\sqrt{T-t}}{A\sigma} - \frac{Y}{A\sigma\sqrt{T-t}}}{\left[1 - \frac{\Sigma^2 X}{A^3 \sigma^2 (T-t)} \right]^{\frac{1}{2}}} \right\}$$
(36)

where

$$X = \left(\frac{e^{AT} - e^{At}}{e^{2AT} - e^{2At}} \right)^2 - \left(\frac{e^{-AT} - e^{-At}}{e^{-2AT} - e^{-2At}} \right)^2 - A(T-t) \text{ and}$$

$$Y = \frac{(x(T)e^{AT} - x(t)e^{At})(e^{AT} - e^{At})}{(e^{2AT} - e^{2At})^2} - \frac{(x(T)e^{-AT} - x(t)e^{-At})(e^{-AT} - e^{-At})}{(e^{-2AT} - e^{-2At})^2}$$
(37)

To evaluate I_2 , we substitute $\omega = i\sigma\sqrt{T-t}$ in eq. (34) to get

$$I_2 = \exp \left\{ \frac{B(T-t)}{A} - \frac{1}{2\Sigma^2} \left[\left(\frac{\Sigma^4}{A^3} \right) X - \left(\frac{2\Sigma^2}{A} \right) Y \right] \right\}$$
(38)

The closed form solution for the Black Scholes pricing problem with stochastic return on the “hedge portfolio” can now be obtained by substituting the above averages in eq. (8).

3. Conclusion

In this paper, we have obtained closed form expressions for the price of a European call option by modifying the Black Scholes formulation to accommodate a stochastic

return process for the “hedge portfolio” returns. We have modelled this return process on the basis of the Vasicek model for the short-term interest rates. The need for this extension of the Black Scholes model is manifold. Firstly, the construction of the “hedge portfolio” in the Black Scholes theory implies that the fluctuations in the price of the derivative and that of the underlying exactly and immediately cancel each other when combined in a certain proportion viz. one unit of the derivative with a short sale of $\frac{\partial C}{\partial S}$ units of the underlying so that the “hedge portfolio” is devoid of any impact of such fluctuations. This mandates an infinitely fast reaction mechanism of the underlying market dynamics whereby any movement in the price of one asset is instantaneously annulled by reactionary response in the other asset constituting the “hedge portfolio”. This is, obviously strongly unrealistic and there may subsist brief periods or aberrations when the no arbitrage condition may cease to hold and hence, returns on the “hedge portfolio” may be different from the risk free rate. One way of attending to this anomaly is to model the returns on the “hedge portfolio” as a stochastic process as has been done in this study. The parameters defining the process can be obtained through an empirical study of the market dynamics. Another important justification for adopting a stochastic framework for the “hedge portfolio” return process is that the “hedge portfolio” by its very construction, envisages the neutralization of the fluctuations of the two assets inter se i.e. it assumes a perfect correlation between the two assets. In other words, the “hedge portfolio” may be construed as an isolated system that is such that insofar as factors that influence one component of the system, the same factors influence the other component to an equivalent extent and, at the same time, other factors do not impact the system at all. This is another anomaly that distorts the Black Scholes model. The fact is that while the “hedge portfolio” of the Black Scholes model is immunized against price fluctuations of the underlying and its derivative through mutual interaction, other market factors that would impact the portfolio as a whole are not accounted for e.g. factors affecting bond yields and interest rates etc. Consequently, to assume that the “hedge portfolio” is completely risk free is another aberration – it is risk free only to the extent of risk that emanates from factors that impact the underlying and the derivative in like manner and is still subject to risk and uncertainties that originate from factors that either do not effect the underlying and the derivative to equivalent extent or impact the portfolio as a unit entity. Hence, again, it becomes necessary to model the return on the “hedge portfolio” as some short-term interest rate model as has been done here.

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