

A Simply Regularized Derivation of the Casimir Force

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Received 6 July 2006, Accepted 9 October 2006, Published 20 December 2006

Abstract: We want to calculate the Casimir force between two parallel, uncharged, perfectly conducting plates by a simple automatically regularized approach. Although in the well-known methods one should explicitly subtract the energy term due to the empty space to regularize the calculation, here, the regularization is simply/implicitly achieved by considering only the energy per unit area of each plate.

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*Keywords: Quantum Theory of Electromagnetic Fields, The Casimir Effect
PACS (2006): 13.70.+k*

1. Introduction

There are three well-known technical types of derivation of the Casimir force for different geometries including the simplest geometry of two parallel, uncharged, perfectly conducting plates firstly explored by Casimir [1]. One modern method is the quantum field theoretical approach based on the appropriate Green's function of the geometry of problem [2]. The other technical type is the dimensional regularization method that involves the mathematical complications of the Riemann zeta function and the analytical continuation [2]. The last (the most elementary/the simplest) method is based on modes summation by using the Euler-Maclurian integral formula [3-5].

The problem of finding the Casimir force, not only for the simplest geometry of two plates that we want to study here but also for other more complicated geometries, indispensably/automatically involves some infinities/irregularities; thus, one should regularize the calculation for arriving at the desired finite physical result(s). In the Green' function method, one uses the subtraction of two terms (two Green's functions) to do the required

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regularization. In the dimensional regularization method, although there isn't an explicit subtraction for the regularization of the problem, as is clear from its name, the calculation is regularized dimensionally by going to a complex plane with a mathematically complicated/ambiguous approach. In the simplest method in which the Euler-Maclurian formula is used, the regularization is performed by the subtraction of the zero-point energy of the free space (no plates) from the energy expression under consideration/calculation (e.g. summation of the interior and exterior zero-point energies of the two parallel plates).

Here, as an almost simple method and in an approach near to the methods using the Euler-Maclurian integral formula but of course without explicitly regularizing the calculation (i.e. without subtracting the energy term due to the empty space), we want to derive the Casimir force for the simplest geometry of two parallel, uncharged, perfectly conducting plates. This does not mean that we can omit the regularization by short cutting the problem! In fact, the calculation is automatically/implicitly regularized.

2. The Zero-Point Energy of the Electromagnetic (Em) Fields

The quantization of the operator form of the Hamiltonian

$$H_{EM} = \frac{1}{2} \int (\varepsilon_0 E^2 + \mu_0 B^2) d^3x \quad (1)$$

by considering the electric and magnetic fields as dynamical operators of the system leads to the result [3]

$$H = \sum_m \hbar\omega_m (a_m^+ a_m + \frac{1}{2}I) \quad (2)$$

where ω_m is the angular frequency corresponding to the eigenmodes expansion of the fields operators and the raising and lowering operators, a_m^+ and a_m , satisfy the commutation relation

$$[a_m, a_n^+] = I\delta_{mn} \quad (3)$$

Therefore, the Hamiltonian operator for the EM fields is equivalent to the Hamiltonian operator for a system of infinite number of independent oscillators.

The lowest energy, the zero-point energy (quantum field theoretically: the vacuum energy), for one mode is $\frac{1}{2}\hbar\omega = \frac{1}{2}\hbar ck$; thus, since there are infinitely many modes of arbitrary high frequency in any finite volume, it follows that there should be an infinite zero-point (vacuum) energy in any (finite) volume of space?! Although in many situations it is stated that this zero-point energy is not observable and for this reason the theory should be defined by normal ordering [6], Casimir recognized that such a conclusion is incorrect when he showed that zero-point fluctuations in electromagnetic fields gave rise to an attractive force between parallel, perfectly conducting plates [1].

3. The Casimir Force Between Parallel, Uncharged, Perfectly Conducting Plates

Consider two infinitely large, parallel, uncharged, perfectly conducting plates at a distance (a) from each other (assume each plate is in the xy plane). In order to having the appropriate boundary conditions for the electromagnetic fields, the components of the wave-vectors corresponding to their Fourier transforms should satisfy [3]

$$k_i = \frac{n_i\pi}{x_i} \quad (i = 1, 2, 3) \quad (4)$$

with n_i 's being nonnegative integers and x_i s are the Cartesian dimensions of the problem (here we work with $x_1 = x_2 = 1$ and $x_3 = a$).

Since the dimensions of the plates in the xy plane is infinitely large, the allowed values of k_x, k_y approximate a continuum, the density of modes in the $k_x k_y$ plane is $\frac{1_{area}}{\pi^3}$ and therefore

$$\frac{1}{(area)} \sum_{k_x, k_y} \frac{1}{2} \hbar \omega \rightarrow \frac{1}{2} \hbar c \int \frac{d^2 k}{(2\pi)^2} \sqrt{(k_x^2 + k_y^2)} \quad (\omega = ck) \quad (5)$$

Thus, by means of (2), (4) and (5), we have the following expression for the energy per unit transverse area of each plate ¹

$$E_{area} = 2 \times \hbar c \left[\frac{1}{2} \sum_{n=1}^{\infty} \int \frac{d^2 k}{(2\pi)^2} \sqrt{(k_x^2 + k_y^2) + \left(\frac{n\pi}{a}\right)^2} \right] \quad (6)$$

where the pre-factor 2 has come from summing on two polarization degrees of freedom of the electromagnetic fields.

Changing the (k_x, k_y) system of variables to a system of two dimensional polar variables (θ, k_θ) , makes (6) take the form

$$E_{area} = 2 \times \hbar c \left\{ \frac{1}{2} \sum_{n=1}^{\infty} \left[\int \frac{k_\theta dk_\theta d\theta}{(2\pi)^2} \sqrt{k_\theta^2 + \left(\frac{n\pi}{a}\right)^2} \right] \right\} = \hbar c \left\{ \sum_{n=1}^{\infty} \left[\int \frac{k_\theta dk_\theta}{2\pi} \sqrt{k_\theta^2 + \left(\frac{n\pi}{a}\right)^2} \right] \right\} \quad (7)$$

where the integration on k_θ is from 0 to ∞ .

With another change of variable $u = [k_\theta^2 + (\frac{n\pi}{a})^2]$, we find

$$\begin{aligned} E_{area} &= \frac{1}{2\pi} \hbar c \left\{ \sum_{n=1}^{\infty} \left[\frac{1}{2} \int_{\left(\frac{n\pi}{a}\right)^2}^{\infty} du \sqrt{u} \right] \right\} = \frac{1}{6\pi} \hbar c \left\{ \lim_{(U \rightarrow \infty)} \sum_{n=1}^{\infty} \left[U^{\frac{3}{2}} - \left(\frac{n\pi}{a}\right)^3 \right] \right\} \\ &= \frac{1}{6\pi} \hbar c \left\{ \lim_{(U, N \rightarrow \infty)} \left[U^{\frac{3}{2}} N - \sum_{n=1}^N \left(\frac{n\pi}{a}\right)^3 \right] \right\} \end{aligned} \quad (8)$$

¹ We should mention that just from here (from the equation (6) to the next), the approach we have used here differs from other well-known calculations (e.g. the calculations in [3-5]). Indeed, to the best of our knowledge, the authors who use the Euler-Maclaurian integral formula (particularly L. E. Ballentine [3], C. Itzykson and J. B. Zuber [4], and K. Huang [5]), all work with additional energy terms (the energy terms corresponding to the exterior region of the plates and/or the empty space). Here, simply, it is enough to consider only the energy per unit transverse area of each plate.

By changing $(\frac{a}{\pi})^3 U^{\frac{3}{2}} N = \frac{1}{4} X^4 = \int_0^X y^3 dy$ and considering the limiting process on U , N and thus X , we obtain

$$E_{area} = \left(\frac{\pi^2 \hbar c}{6a^3}\right) \left(\int_0^\infty y^3 dy - \sum_{n=1}^\infty n^3\right) \quad (9)$$

Using Euler-Maclurian integral formula (see **Section A**), we arrive at the result

$$E_{area} = \frac{-\pi^2 \hbar c}{720a^3} \quad (10)$$

The desired force per unit area between the plates is obtained by taking the negative derivative of E_{area} with respect to a

$$F_{area} = -\frac{d}{da} E_{area} = \frac{-\pi^2 \hbar c}{240a^4} \quad (11)$$

This is just the force per unit area between parallel, uncharged, perfectly conducting plates firstly found by Casimir [1].

4. Remark

The method used here for finding the Casimir force may mislead one that there is no need to regularize the calculation. In fact, in the present/standard quantum theory of fields, there are some indispensable infinities/irregularities and we always try to regularize our calculations to find out the physical finite/regular results. But, what has happened here? Is there really no regularization in this work? The answer is clearly negative. Clearly, if we observe the right-hand side of the equation (8) we can see that the two terms in the bracket are individually infinite/irregular and only their subtraction is finite/regular. Thus, we have simply regularized the calculation through the mathematical trick introduced in the equation (8). Of course, one point is important here and should be considered as an advantage. The energy per unit area of each plate (E_{area}) with which we have worked here is a physical/finite/regular quantity (in spite of those absolute energy terms that are infinite/irregular).

5. Section A

The Euler-Maclurian formula is usually used for estimating sums with integrals. The complete version with enough explanation can be found in mathematical physics texts as [7]. A number of examples can be found in [8].

What we need (have used) here is the following formula

$$\sum_{n=1}^{\infty} f(n) - \int_0^{\infty} f(n) dn = \frac{-1}{2} f(0) + \frac{-1}{6 \times 2!} f'(0) + \frac{1}{30 \times 4!} f'''(0) + \dots \quad (A-1)$$

Acknowledgement

The author wishes to acknowledge the efforts of referees, particularly in the mathematical correction of equation 8.

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