Metric Variation Inside Transitioning Superconducting Shells

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Received 30 August 2006, Accepted 26 September 2006, Published 20 December 2006

Abstract: In this paper, we outline the forward problem of metrical variation due to the Casimir effect in transitioning superconducting shells. We consider a massless scalar quantum field inside a hollow superconducting sphere and a cylinder. Metric equations are developed describing the evolution of the scale factors after the superconducting shells transition to the normal state.

Keywords: General Relativity, Quantum Field, Quantum Vacuum, Casimir Effect, Superconducting Shells (sphere and cylinder)
PACS (2006): 04.20.q, 04.20.Cv, 04.62.+v, 04.90.+e, 02.40.k

1. Introduction

The Casimir effect results in the polarization of the quantum vacuum by conducting boundaries [1, 2]. Vacuum polarization gives rise to a minute force that has been measured experimentally [3, 4] in agreement with the predictions of quantum electrodynamics. In calculating the Casimir force, one properly calculates differences in vacuum pressure

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‡ Supported by RCMI through NIH-NCRR (G12-RR-03045) and Texas Southern University Research Seed Grant 2004/2005.

established by the conducting boundaries.

Boyer first calculated the vacuum modes inside a conducting sphere [5], with a more recent account given by Milton [6]. Applications of the Casimir effect have been studied for massive scalar [7] and Dirac fields [8] confined to the interior of a sphere. The Casimir effect in curved spacetime has been calculated for spherical geometries [9] and for a cylindrical shell in de Sitter space [10] and in the background of a static domain wall [11].

In this paper, we investigate the metrical variations resulting from vacuum energy differences established by hollow superconducting boundaries. We consider the static case when the shells are in the superconducting state and then the dynamical case as the shells pass to the normal state.

2. Transitioning Superconducting Sphere

Our idealized massless, thin sphere of radius R_0 has zero conductivity in the normal state. In the superconducting state, the vacuum inside the hollow is reduced so that there exists a pressure difference Δp inside and outside the sphere. In general, all quantum fields will contribute to the vacuum energy. When the sphere of volume V transitions to the superconducting state, a latent heat of vacuum phase transition ΔpV is exchanged. The distribution of vacuum pressure, energy density and space-time geometry are described by the semi-classical Einstein field equations taking $c = 1$,

$$R_{\mu\nu} - \frac{1}{2}\Re g_{\mu\nu} = 8\pi G \left\langle T_{\mu\nu} \right\rangle \tag{1}$$

where $R_{\mu\nu}$ and $\Re$ are the Ricci tensor and scalar curvature, respectively. $\langle T_{\mu\nu}\rangle$ is the vacuum expectation of the stress energy tensor. Regulation procedures for calculating the renormalized stress energy tensor are given in [12] for various geometries. In spherical coordinates we consider the line element

$$ds^2 = B\left(r,t\right)dt^2 - A\left(r,t\right)dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 \tag{2}$$

where B and A are arbitrary functions of time and the radial coordinate. The corresponding metric tensor is of the form

$$g_{\mu\nu} = Diag\left(B\left(r,t\right), -A\left(r,t\right), -r^2, -r^2 sin^2\right) \tag{3}$$

For a diagonal stress energy tensor, the solutions to equation (1) relating A and B are

$$-\frac{1}{r^2} + \frac{1}{r^2 A} - \frac{A'}{rA^2} = 8\pi G \frac{1}{B}\left\langle T_{00}\right\rangle \tag{4}$$

$$\frac{1}{r^2} - \frac{1}{r^2 A} - \frac{B'}{rAB} = 8\pi G \frac{1}{A}\left\langle T_{11}\right\rangle \tag{5}$$

$$\frac{A'}{2rA^2} - \frac{B'}{2rAB} + \frac{A'B'}{4A^2B} - \frac{B'^2}{4AB^2} - \frac{B''}{2AB} = 8\pi G \frac{1}{r^2}\left\langle T_{22}\right\rangle \tag{6}$$

with a fourth equation identical to (6). The prime denotes ∂_r. Note that all time derivatives cancel from the field equations when the metric is in standard form and the stress energy tensor is diagonal. When the sphere is in the superconducting state, the scalar curvature $\Re = g^{\mu\nu} R_{\mu\nu}$ is given by

$$\Re = \frac{2}{r^2} - \frac{2}{r^2 A} + \frac{2A'}{rA^2} - \frac{2B'}{rAB} + \frac{A'B'}{2A^2B} + \frac{B'^2}{2AB^2} - \frac{B''}{AB} \tag{7}$$

We consider an ideal case where the sphere passes instantaneously from the superconducting to the normal state. The field modes, however, will not relax back instantaneously. The diagonal form of the stress energy tensor results in the cancellation of all time derivatives in the field equations in the static case above. External electromagnetic fields will contribute off-diagonal terms. In this analysis the required time dependence is provided by the zero point field fluctuations and, in particular, their contribution to the off-diagonal stress-energy terms. As the simplest case, we consider the massless scalar quantum field $\phi(r,t)$ with stress energy tensor [12]

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \phi_{,\alpha}\phi_{,\beta} \tag{8}$$

The non-zero components of T are

$$T_{00} = \frac{1}{2}\dot{\phi}^2 + \frac{B}{2A}\phi'^2 \tag{9}$$

$$T_{11} = \frac{1}{2}\phi'^2 + \frac{A}{2B}\dot{\phi}^2 \tag{10}$$

$$T_{22} = r^2 \left(\frac{1}{2B}\dot{\phi}^2 - \frac{1}{2A}\phi'^2 \right) \tag{11}$$

$$T_{33} = r^2 sin^2\theta \left(\frac{1}{2B}\dot{\phi}^2 - \frac{1}{2A}\phi'^2 \right) \tag{12}$$

$$T_{01} = \dot{\phi}\phi' \tag{13}$$

where $T_{01} = T_{10}$. The semi-classical field equations become

$$-\frac{1}{r^2} + \frac{1}{r^2 A} - \frac{A'}{rA^2} = 8\pi G \frac{1}{B} \langle T_{00} \rangle \tag{14}$$

$$\frac{1}{r^2} - \frac{1}{r^2 A} - \frac{B'}{rAB} = 8\pi G \frac{1}{A} \langle T_{11} \rangle \tag{15}$$

$$-\frac{\dot{A}^2}{4A^2B} - \frac{\dot{A}\dot{B}}{4AB^2} + \frac{\ddot{A}}{2AB} + \frac{A'}{2r^2A} - \frac{B'}{2rAB} + \frac{A'B'}{4A^2B} + \frac{B'^2}{4AB^2} - \frac{B''}{2AB} = 8\pi G \frac{1}{r^2} \langle T_{22} \rangle \tag{16}$$

$$-\frac{\dot{A}}{rA} = 8\pi G \langle T_{01} \rangle \tag{17}$$

Equations (14) and (15) are identical to (4) and (5). Two additional equations are identical to (16) and (17). Integral expressions for A and B may be obtained from equation (17) and (14) or (15), respectively. The scalar curvature is given by

$$\Re = \frac{2}{r^2} - \frac{2}{r^2 A} - \frac{\dot{A}^2}{2A^2 B} - \frac{\dot{A}\dot{B}}{AB} + \frac{\ddot{A}}{AB} + \frac{2A'}{rA^2} - \frac{2B'}{rAB} + \frac{A'B'}{2A^2B} + \frac{B'^2}{2AB^2} - \frac{B''}{AB} \tag{18}$$

Comparing equation (18) with (14-16) and (10-12) reveals

$$\Re = 16\pi G \left\langle \frac{\dot{\phi}^2}{2B} - \frac{\phi'^2}{2A} \right\rangle \tag{19}$$

When evaluating changes in scalar curvature, the expression for R in absence of the sphere should be subtracted from that obtained for a given quantum field.

3. Transitioning Superconducting Cylinder

Modeling a superconducting cylinder, we consider the line element in cylindrical coordinates

$$ds^2 = B(r,t)\,dt^2 - A(r,t)\,dr^2 - r^2 d\theta^2 - dz^2 \tag{20}$$

with corresponding metric tensor

$$g_{\mu\nu} = Diag\left(B(r,t), A(r,t), -r^2, -1\right) \tag{21}$$

Solutions to equation (1) relating A and B in the static case, when the cylinder is superconducting state, are

$$-\frac{BA'}{2rA^2} = 8\pi G \langle T_{00} \rangle \tag{22}$$

$$-\frac{B'}{2rB} = 8\pi G \langle T_{11} \rangle \tag{23}$$

$$\frac{A'B'}{2A^2B} + \frac{B'^2}{4AB^2} - \frac{B''}{2AB} = \frac{8\pi G}{r^2} \langle T_{22} \rangle \tag{24}$$

$$\frac{A'}{2rA^2} + \frac{B'}{2rAB} - \frac{A'B'}{4A^2B} + \frac{B'^2}{2AB^2} = 8\pi G \langle T_{33} \rangle \tag{25}$$

Considering a scalar quantum field with stress energy tensor given by equation (8), all components of T are of the same form as the spherical case except equation (12) that becomes

$$T_{33} = \left(\frac{1}{2B}\dot{\phi}^2 - \frac{1}{2A}\phi'^2 \right) \tag{26}$$

The time dependent semi-classical field equations are then

$$-\frac{BA'}{2rA^2} = 8\pi G \langle T_{00} \rangle \tag{27}$$

$$-\frac{\dot{A}}{2rA} = 8\pi G \langle T_{01} \rangle = 8\pi G \langle T_{10} \rangle \tag{28}$$

$$-\frac{B'}{2rB} = 8\pi G \langle T_{11} \rangle \tag{29}$$

$$\frac{A'B'}{4A^2B} + \frac{B'^2}{4AB^2} - \frac{B''}{2AB} - \frac{\dot{A}^2}{4A^2B} + \frac{\dot{A}\dot{B}}{4AB^2} + \frac{\ddot{A}}{2AB} = 8\pi G \langle T_{22} \rangle \tag{30}$$

$$\frac{A'}{2rA^2} - \frac{B'}{2rAB} + \frac{A'B'}{4A^2B} + \frac{B'^2}{4AB^2} - \frac{B''}{2AB} - \frac{\dot{A}^2}{4A^2B} - \frac{\dot{A}\dot{B}}{4AB^2} + \frac{\ddot{A}}{2AB} = 8\pi G \langle T_{33} \rangle \tag{31}$$

Integral expressions for A and B may be obtained from equations (29) and (30) respectively,

$$A(r,t) \sim exp\left(-16\pi G r \int \langle T_{10} \rangle \, dt\right) \tag{32}$$

$$B(r,t) \sim exp\left(-16\pi G \int r \langle T_{11} \rangle \, dr\right) \tag{33}$$

A similar set of equations may be obtained in the spherical case, as mentioned previously. The scalar curvature

$$\Re = \frac{A'}{rA^2} - \frac{B'}{rAB} - \frac{A'B'}{2A^2B} + \frac{B'^2}{2AB^2} - \frac{B''}{AB} - \frac{\dot{A}^2}{2A^2B} - \frac{\dot{A}\dot{B}}{2AB^2} + \frac{\ddot{A}}{AB} \tag{34}$$

becomes comparing equation (31)

$$\Re = 16\pi G \langle T_{33} \rangle \tag{35}$$

which is of the same form as equation (19) obtained for the evolution of scalar curvature inside a superconducting sphere. The quantities $\left\langle \dot{\phi}^2 \right\rangle$ and $\langle \phi'^2 \rangle$ however will not have the same form as in the spherical case. Applying conservation of energy-momentum $\langle T^{\mu\nu} \rangle_{;\nu}$ in the cylindrical case gives the two equations

$$-\frac{rB}{A}\left(\langle T_{00} \rangle \dot{A} + \langle T_{10} \rangle A'\right) = r \langle T_{11} \rangle \dot{A} + 2r \langle T_{00} \rangle \dot{B} + 3r \langle T_{10} \rangle B' + 2B\left(\langle T_{10} \rangle + r \langle T_{00} \rangle_{,t} + r \langle T_{10} \rangle_{,r}\right) \tag{36}$$

$$\frac{rB}{A}\left(2r\langle T_{22}\rangle - 3\langle T_{10}\rangle \dot{A} - 2\langle T_{11}\rangle A' - \langle T_{00}\rangle B'\right) = r\langle T_{10}\rangle \dot{B} + r\langle T_{11}\rangle B'$$
$$+2B\left(\langle T_{11}\rangle + r\langle T_{10}\rangle_{,t} + r\langle T_{11}\rangle_{,r}\right) \qquad (37)$$

A similar set of equations may be obtained in the spherical case. In order to numerically determine the evolution of scale factors and quantum fields in either configuration, one must evaluate renormalized quantities $\left\langle \dot{\phi}^2 \right\rangle$, $\langle \phi'^2 \rangle$, and $\left\langle \dot{\phi}\phi' \right\rangle$. The static solution with Dirichlet boundary conditions applied for the sphere or cylinder in the superconducting state provides the initial condition for the transient case, immediately after the shell becomes normal.

If one assumes the gravitational fields resulting from the Casimir effect are sufficiently weak, it seems plausible to evaluate the renormalized quantities $\left\langle \dot{\phi}^2 \right\rangle$, $\langle \phi'^2 \rangle$, and $\left\langle \dot{\phi}\phi' \right\rangle$ in Minkowski spacetime and then calculate their effect on the scale factors. In 2-d cylindrical coordinate flat space one obtains $\left\langle \dot{\phi}^2 \right\rangle = \langle \phi'^2 \rangle = -\pi/6L^2$ where $L = 2\pi r$ Evidently the quantity $\left\langle \dot{\phi}\phi' \right\rangle = 0$ for scalar fields in Minkowski spacetime. Inspection of the field equations reveals the requirement $\left\langle \dot{\phi}\phi' \right\rangle \neq 0$ to describe the evolution of the scalar quantum field and metrical factors in this analysis.

4. Conclusion

A procedure for describing the evolution of quantum fields and spacetime curvature inside transitioning superconducting shells has been outlined. The static solution with the shell in the superconducting state serves as the initial condition for the transient evolution of field modes. Zero-point field fluctuations are coupled to the time dependent Einstein field equations through the off-diagonal components of the stress energy tensor. Because of the differences in vacuum energy densities, a latent heat of vacuum phase transition is exchanged when a hollow shell transitions between the normal and superconducting state. The analysis presented here may be extended to include massive fields with coupling or spin as well as other superconducting geometries.

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