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# Radiating Shell Supported by a Phantom Energy

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**Abstract:** I describe the evolution of a thin spherically symmetric self-gravitating phantom shell around the radiating shell. The general equations describing the motion of shell with a general form of equation of state are derived. The stability analysis of this phantom shell to linearized spherically symmetric perturbation about static equilibrium solution is carried out.


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## 1. Introduction

Recent astrophysical observations [1,2] related to distant supernovas, cosmic microwave background and galaxy clustering all together essentially changed our view on the evolution of the universe. Now it is generally accepted that the universe at present is expanding with acceleration. The explanation of such cosmological behavior in the framework of general relativity (GR) requires the supposition that a considerable part of the universe consists of a hypothetical dark energy: the exotic matter with a positive energy density $\rho > 0$ and a negative pressure $p = \omega\rho$ with $\omega < -\frac{1}{3}$. In the last few years intensive efforts with a variety of theoretical ideas and models concerning dark gravity and scalar tensor theories, bran world models, dark energy models with negative potentials, tachyon scalar field, scalar field with a negative kinetic energy were discussed (cf [3,4,5]).

The most exotic form of dark energy is a phantom energy with $\omega < -\frac{1}{3}$ [6], for which the weak energy conditions is violated. The exotic nature of phantom energy reveals itself in a number of unusual cosmological consequence (cf. [7]). Therefore, one can consider the phantom energy as a possible candidate for exotic matter. It is possible to extend the

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motion of phantom energy on the case of spherically symmetric space time configurations. Suppose that it is characterized by the equation of state $p = \omega\rho$ with $\omega < -\frac{1}{3}$, where $p$ is the negative radial pressure.

Sushkov [8 ] discussed a model of static spherically symmetric wormholes with phantom energy. The aim of this paper is to describe the thin spherically symmetric phantom shell around a radiating body.

The paper is organized as follows. In Section 2 the general concepts of the dynamics spherically symmetric thin shell are outlined with special attention to Schwarzschild space time. In Section 3 the evolution of thin shell with phantom equation of state is analyzed. Outline of a general linearized stability analysis procedure is given in Section 4. A general conclusion is given in Section 5.

## 2. Dynamics of spherically symmetric thin shell

The line element of any spherically symmetric space time can be written in the form

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = Adt^2 + 2Hdtdq + Bdq^2 + r^2\left(t,q\right)d\Omega^2 \tag{1}$$

Here $t$ and $q$ are the timelike and spacelike coordinates, $A, H$and $B$are functions of $t$ and $q$ only, and $r(t,q)$ is the radius of a two dimensional sphere (in the sense that the area of the sphere is $4\pi r^2$), and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, is the line element of the unit sphere.

For the given space time the coefficients$A, H$and$B$are not uniquely defined. One can transform the line element (1) to the new coordinate system which converses explicitly the spherically symmetric form of the metric

$$\tilde{t} = \tilde{\tau}\left(t,q\right) \quad , \quad \tilde{q} = \tilde{q}\left(t,q\right).$$

The radius $r$is invariant under this transformation. The other very important invariant is

$$\Delta = \gamma^{\alpha\beta}r_{,\alpha}r_{,\beta}$$

where $\gamma^{\alpha\beta}$ is inverse to the two-dimensional metric tensor $\gamma_{\alpha\beta}$ .

In the flat Minkowskian space time $\Delta = -1$, all the surfaces $r = const$ are time like and $r$ can be chosen as spatial coordinate $q = r$. In the curved space time, $\Delta$ can be positive and negative: (1) The region with $\Delta < 0$ is called $R - region$ and the radius can be chosen as a radial coordinate $q$. (2) The region with $\Delta > 0$ is called $T - region$ and the surfaces $r = const$ are spacelike ( the normal vector is timelike) and the radius can be chosen as a time coordinate $t$. In $T - region$ there is no $\dot{r} = 0$ ( where "dot" means a time derivative), hence it must be either $\dot{r} > 0$ ( such region of expansion is called $T_+ - region$) or $\dot{r} < 0$ ( such region of contraction is called $T_- - region$). The same holds for $R - regions$. They are divided into two classes which are called $R_+ - region$ with $r' > 0$ and $R_- - region$ with $r' < 0$ (where prime stands for a spatial derivative ). These $R$ and $T$ regions are separated by the surfaces $\Delta = 0$ which are called the apparent horizons, which can be null, timelike or spacelike apparent horizon.

The metric (1) in a Schwarzschild space time has the form

$$ds^2 = f\left(r\right)dt^2 - f^{-1}\left(r\right)dr^2 - r^2 d\Omega^2 \tag{2}$$

where

$f\left(r\right) = 1 - \frac{2m}{r}$ with $m > 0$.

One of the most important features of GR is that the equations of motion of matter fields are incorporated into the Einstein equations. The Einstein equations of GR are nonlinear partial differential equations. This means that the motion of test particles or fields on the given background will in general be different from that of the matter for the self-consistent solutions. It makes analysis very complicated. To obtain some definite results, choose the simplest possible model, like self-gravitating thin shell. In this section I give a brief history on the equation of motion of a thin shell. When dealing with the time like spherically symmetric thin shell, adjust the covariant formalism derived by Israel [9] to the case of interest.

Let hypersurface $\Sigma$ divided the whole space time into two parts, "in" and "out", and can connect some special coordinate system called Gauss normal coordinates to this hypersurface $\Sigma$. The line element in these coordinates takes the form

$$ds^2 = d\tau^2 - dn^2 - r^2\left(\tau, n\right)d\Omega^2 \tag{3}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, is the line element of the unit sphere, $\tau$ is the proper time of the observer sitting fixed on $\Sigma$. The coordinate $n$ grows from the "in" to the "out" region in the outer normal direction to the hypersurface $\Sigma$ and $r(\tau, n)$ is the radius of the sphere. The hypersurface is situated at $n = 0$ and the intrinsic metric to $\Sigma$ is $ds^2_\Sigma = d\tau^2 - R^2\left(\tau\right)d\Omega^2$, where $R\left(\tau\right) = r\left(\tau, 0\right)$.

Keeping in mind that the metric itself is continuous but some of its derivatives make a jump across the shell. The jump of the extrinsic curvature $K_{ab}$ is $[K_{ab}] = K^{out}_{ab} - K^{in}_{ab}$, where a quantity in square brackets stands for the difference of that quantity evaluated on the outer side, say the "out", minus the quantity evaluated on the inner side, the "in" side.

The hypersurface $\Sigma$ represents the history of a surface layer (a singular hypersurface of order one) if $K^{out}_{ab} \neq K^{in}_{ab}$. This hypersurface $\Sigma$ is called the singular shell if some energy momentum tensor is concentrated on it, namely: $T^k_i = t^k_i\delta(n) + ........$ Otherwise the hypersurface is nonsingular. The Einstein equation determines the relation between the extrinsic curvature $K_{ab}$ and the three dimensional intrinsic energy momentum tensor $t_{ab}$ is given by the Lanczos equation

$$[K_{ab}] = -8\pi\left(t_{ab} - \frac{1}{2}tg_{ab}\right)$$

where $t = t^a_a$. This relation can be written in the form

$$t_{ab} = \frac{-1}{8\pi}\left([K_{ab}] - g_{ab}\left[K\right]\right) \tag{4}$$

where $[K] = g^{ab}[K_{ab}]$ is the trace of the extrinsic curvature. In this case due to spherical symmetry the only nonzero components of $t_a^b$ are $t_0^0$ and $t_2^2 = t_3^3$ [10]. The invariant function $\Delta$ equals

$$\Delta = R_{,r}^2 - R_{,n}^2$$

and

$$R_{,n}|_\Sigma = \varepsilon\sqrt{\dot{R}^2 - \Delta}$$

where $\varepsilon = +1$ if radii increase in the direction of the outer normal, and $\varepsilon = -1$ if radii decrease. Evidently, $\varepsilon = +1$ in $R_+ - region$ and $\varepsilon = -1$ in $R_- - region$. From (4) the components are:

$$\frac{2\varepsilon_{in}}{R}\sqrt{\dot{R}^2 - \Delta_{in}} - \frac{2\varepsilon_{out}}{R}\sqrt{\dot{R}^2 - \Delta_{out}} = 8\pi R t_0^0 \tag{5}$$

$$\begin{aligned}&\tfrac{2\varepsilon_{in}}{R}\sqrt{\dot{R}^2 - \Delta_{in}} - \tfrac{2\varepsilon_{out}}{R}\sqrt{\dot{R}^2 - \Delta_{out}} + \tfrac{\varepsilon_{in}}{\sqrt{\dot{R}^2-\Delta_{in}}}\ddot{R} - \tfrac{\varepsilon_{out}}{\sqrt{\dot{R}^2-\Delta_{out}}}\ddot{R}+ \\ &\tfrac{\varepsilon_{in}}{2R\sqrt{\dot{R}^2-\Delta_{in}}}(1+\Delta_{in}) - \tfrac{\varepsilon_{out}}{2R\sqrt{\dot{R}^2-\Delta_{out}}}(1+\Delta_{out}) + 4\pi R\left({}^{out}T_n^n - {}^{in}\mathrm{T}_n^n\right) = 8\pi t_2^2\end{aligned} \tag{6}$$

The continuity equation for the energy-momentum tensor is transformed to

$$\frac{dt_0^0}{d\tau} + \frac{2\dot{R}}{R}\left(t_0^0 - t_2^2\right) + \left({}^{out}\mathrm{T}_0^n - {}^{in}\mathrm{T}_0^n\right) = 0 \tag{7}$$

This equation is a differential consequence of the first two ones.

In this paper the space-time inside the shell will be the Vaydia metric and outside the shell is the Schwarzschild metric, then the equations (5), (6) and (7) becomes

$$\varepsilon_{in}\sqrt{\dot{R}^2 + F_{in}} - \varepsilon_{out}\sqrt{\dot{R}^2 + F_{out}} = 4\pi R t_0^0 \tag{8}$$

$$\ddot{R} = -4\pi^2 R\left(t_0^0\right)^2 + 8\pi^2 R t_0^0 t_2^2 - \frac{(m_{in} + m_{out})}{2R^2} - \frac{\Delta m^2 t_2^2}{8\pi^2 R^5 \left(t_0^0\right)^3} \tag{9}$$

$$\dot{t}_0^0 + \frac{2\dot{R}}{R}\left(t_0^0 - t_2^2\right) = 0 \tag{10}$$

where

$$\Delta_{out} = -F_{out} = -1 + \frac{2m_{out}}{R}$$

$$\Delta_{in} = -F_{in} = -1 + \frac{2m_{in}(\upsilon)}{R}$$

and $m_{in}(\nu)$ is the mass function in the interior space and depending whether the null coordinate $\upsilon$ is advanced or retarded.

## 3. Dynamics of phantom shell

The equation of state describing phantom energy in cosmology is usually taken as $p = \omega\rho$ where $\omega < -1$ and $p$is a negative spatially homogeneous pressure. By analogy, it is possible to use the same equation of state for a spherically symmetric distribution of phantom energy but with $p$is the negative radial pressure and written in the form $p = -k\rho$ with $k > 1$. Consider a simple linear equation of state by relation

$$t_0^0 = kt_2^2 \tag{11}$$

In the phantom case $k > 1$. The solution of (10) is

$$t_0^0 = CR^{2(k-1)} \tag{12}$$

where $C$is a constant denoted to the shell power.

Using equation (8) and taking into account ($\varepsilon_{in} = -1, \varepsilon_{out} = +1$) one gets the following two equations:

$$\dot{R}^2 = -1 + \frac{1}{R}((m_{in} + m_{out}) + \frac{\delta m^2}{4x} + x) \quad , \tag{13}$$

$$\ddot{R} = -\frac{1}{2R^2}[(m_{in} + m_{out}) + k\frac{\delta m^2}{x} - 2(2k-1)x] \tag{14}$$

where

$$x \equiv 4\pi^2 C^2 R^{4k-1} \tag{15}$$

The sign conditions of the equation of motion of shell will be

$$\varepsilon_{in} = sign\left[\delta m + 8\pi^2 R^3 \left(t_0^0\right)\right] \tag{16}$$

$$\varepsilon_{out} = sign\left[\delta m - 8\pi^2 R^3 \left(t_0^0\right)\right] \tag{17}$$

where $\delta m = m_{out} - m_{in}$. Using (15) to get a more convenient form of the sign conditions

$$\varepsilon_{in} = sign\left(\delta m + 2x\right) \tag{18}$$

$$\varepsilon_{out} = sign\left(\delta m - 2x\right) \tag{19}$$

When the function $\dot{R}^2$ has roots, it is possible to represent both the finite and infinite motion. The change of sign of the acceleration $\ddot{R}$ in (14) occurs when $\ddot{R} = 0$. This corresponds to the quadratic equation whose positive root is

$$x_0 = \frac{m_{in} + m_{out} + \sqrt{(4k-1)^2\delta m^2 + 4m_{in}}m_{out}}{4(2k-1)} \tag{20}$$

It is convenient to define the parameter space of the problem using $m_{in}, m_{out}, k > \frac{1}{4}$ and $R$ as free parameters.

Consider at first the case of $\delta m > 0$. According to (18), $\varepsilon_{in}$ must be positive $\varepsilon_{in} = +1$, the $\varepsilon_{out}$ changes sign at $x = x_1 \equiv \frac{\delta m}{2}$ and $\varepsilon_{out} = -1$ if $x > x_1$. Hence, in the case $x \to \infty$, then $\varepsilon_{out} = -1$ and if $x \to 0$ then $\varepsilon_{out} = +1$. The value of $R$ corresponding to $x_1$ is denoted by $R_1$. Then equation (14) will be $\ddot{R}\,(x_1) = -\frac{m_{out}}{R_1^2}\langle 0$. Therefore $R_1 < R_0$ which corresponds the value of $R$ at $x_0$ in (15). From (13) one obtains $\dot{R}^2\,(x_1) = -1 + \frac{2m_{out}}{R_1}$. For the shell moving from infinity to infinity. Consider now the case of $\delta m < 0$. The density $t_0^0$ is assumed to be always positive. So the negativity of $\delta m$ is caused by the gravitational mass defect. The $\varepsilon_{in}$ changes sign at $x = x_2 \equiv \frac{-\delta m}{2}$ and $\varepsilon_{in} = +1$ if $x > x_2$, $\varepsilon_{out} = -1$ at the same time. In the case of $x \to \infty$ then $\varepsilon_{in} = +1$ and $\varepsilon_{in} = -1$ if $x \to 0$. The value of $R$ corresponding to $x_2$ is

$\ddot{R}\,(x_2) = -\frac{m_{in}}{R_2^2}\langle 0$ and $\dot{R}^2\,(x_2) = -1 + \frac{2m_{in}}{R_2}$.

In the case of $\delta m < 0$ the shell evolves under horizons and cannot reach a distant observer living in $R_+ - region$. But in the case of $\delta m > 0$, the shell can show itself in $R_+ - region$.

## 4. Linearized stability analysis

Rearranging equation (13) into the form

$$\dot{R}^2 = -1 + \frac{1}{R}((m_{in} + m_{out}) + \frac{(m_{out} - m_{in})^2}{4x} + x) \tag{21}$$

where $x \equiv 4\pi^2 C^2 R^{4k-1}$, $C$ is a power constant and $k > 1$. This equation can be written in the dynamical form

$$\dot{R}^2 + V(R) = 0 \tag{22}$$

with the potential given by

$$V(R) = 1 - \frac{1}{R}((m_{in} + m_{out}) + \frac{(m_{out} - m_{in})^2}{4x} + x)$$

The factor $F(R)$and $G(R)$ introduced for computational convenience, are defined by

$$F(R) = 1 - \frac{1}{R}(m_{in} + m_{out})$$

$$G(R) = \frac{(m_{out} - m_{in})}{R}$$

So that the potential $V(R)$ takes the form

$$V(R) = F(R) - 4\pi^2 C^2 R^{4k-2} - \frac{RG^2}{4x} \tag{23}$$

From equation (15) and $M = 4\pi R^2 t_0^0 = 4\pi C R^{2k}$, equation (23) takes the form

$$V(R) = F(R) - (\frac{M}{2R})^2 - (\frac{RG}{M})^2 \tag{24}$$

where $M$ is the surface mass of the shell. Linearized around the stable solution at $R = R_0$, consider a Taylor expansion of $V(R)$ around $R_0$ to second order, provides

$$V(R) = V(R_0) + V'(R_0)(R - R_0) + \frac{1}{2}V''(R_0)(R - R_0)^2 + O\left[(R - R_0)^3\right] \tag{25}$$

Where the prime denotes a derivative with respect to $R$. The first and second derivatives of $V(R)$are given by

$$V'(R) = F'(R) - 2(\frac{M}{2R})(\frac{M}{2R})' - 2(\frac{RG}{M})(\frac{RG}{M})' \tag{26}$$

$$V''(R) = F''(R) - 2[(\frac{M}{2R})']^2 - 2(\frac{M}{2R})(\frac{M}{2R})'' - 2[(\frac{RG}{M})']^2 - 2(\frac{RG}{M})(\frac{RG}{M})'' \quad (27)$$

Evaluated at the static solution ($R = R_0$) and through a long calculation, I find that $V(R_0) = 0$ and $V'(R_0) = 0$. From the condition $V'(R_0) = 0$, one extracts the following useful equilibrium relationship

$$(\frac{M}{2R_0})' \equiv \Gamma = (\frac{R_0}{M})[F'(R) - -2(\frac{R_0 G}{M})(\frac{R_0 G}{M})'] \quad (28)$$

So that,

$$\dot{R}^2 = -\frac{1}{2}V''(R_0)(R - R_0)^2 + O[(R - R_0)^3].$$

If $V''(R_0) < 0$is verified, then the potential $V(R)$has a local maximum at $R_0$, where a small perturbation in the surface radius will produce an irreversible contraction or expansion of the shell. Therefore, the solution is stable if and only if $V(R)$ has a local minimum at $R_0$ and

$V''(R_0) > 0$ is verified. The latter stability condition takes the form

$$(\frac{M}{2R})(\frac{M}{2R})'' < \Psi - \Gamma^2 \quad (29)$$

where $\Psi$is defined as

$\Psi = \frac{F''}{2} - [(\frac{RG}{M})']^2 - (\frac{RG}{M})(\frac{RG}{M})''$ and $\Gamma = (\frac{M}{2R_0})'$.

## 5. Conclusion

The motivation of this work is the fact that in many physically interesting situations in cosmology and astrophysics the essential role was played the full account for gravitational back reaction. In this case of phantom shell such a back reaction may appear crucial for formation of the space time.

The matter is that in GR any type of energy is gravitating, not only energy density but also the tension and pressure are gravitating. The pressure plays a twofold role. The positive pressure causes both repulsion and attraction, the attraction is due to its contribution to the gravitating source. The negative pressure leads to the gravitational repulsion. Hence, the phantom shell is even more repulsive.

In the case of $\delta m > 0$ the distance observer may see the shell but can not register the energy flux of the shell. The stability analysis of this phantom shell to linearized spherically symmetric perturbation about static equilibrium solution is carried out.

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