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Particle Interference without Waves

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Abstract: After eighty years of Quantum Mechanics (QM) we have learned to live with wave functions without worrying about their physical nature. This attitude is certainly justified by the extraordinary success of the theory in predicting and explaining not only all the phenomena encountered in the domain of microphysics, but also some spectacular nonclassical macroscopic behaviors of matter. Nevertheless one cannot ignore that the *wave–particle duality* of quantum objects not only still raises conceptual problems among the members of the small community of physicists who are still interested in the foundations of our basic theory of matter, but also induces thousands and thousands of physics students all around the world to ask each year, at their first impact with Quantum Mechanics, embarrassing questions to their teachers without receiving really convincing answers. Remember that Feynman once said "It is fair to say that nobody understands Quantum Mechanics". My purpose is to show that these difficulties can only be faced by pursuing a line of research which takes for granted the irreducible nature of randomness in the quantum world. This can be done by eliminating *from the beginning* the unphysical concept of wave function. I believe that this elimination is conceptually similar to the elimination of the aether, together with its paradoxical properties, from classical electrodynamics, accomplished by relativity theory. In our case the lesson sounds: No wave functions, no problems about their physical nature. Furthermore, the adoption of a statistical approach from the beginning for the description of the physical properties of quantum systems sounds methodologically better founded than the conventional *ad hoc* hybrid procedure of starting with the determination of a system's wave function of unspecified nature followed by a "hand made" construction of the probability distributions of its .physical variables.

Keywords: Wigner-Feynman approach of Quantum Mechanics, Quantum Interpretations and Representations, Classical Ensemble Theory, Foundations of Quantum Mechanics
PACS (2006): 03.65.w, 05.20.Gg, 03.65.Ta

1. Introduction

After eighty years of Quantum Mechanics (QM) we have learned to live with wave functions without worrying about their physical nature. This attitude is certainly justified by the extraordinary success of the theory in predicting and explaining not only all the phenomena encountered in the domain of microphysics, but also some spectac-

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ular nonclassical macroscopic behaviors of matter. Nevertheless one cannot ignore that the *wave–particle duality* of quantum objects not only still raises conceptual problems among the members of the small community of physicists who are still interested in the foundations of our basic theory of matter, but also induces thousands and thousands of physics students all around the world to ask each year, at their first impact with Quantum Mechanics, embarrassing questions to their teachers without receiving really convincing answers. Remember that Feynman once said "It is fair to say that nobody understands Quantum Mechanics"[1].

Typical examples of this unsatisfaction are the nonseparable (or nonlocal) character of long distance correlated two-particle systems [2] and the dubious meaning of the superposition of state vectors of measuring instruments [3], and in general of all macroscopic objects (Schrödinger's cat).

In the former case experiments have definitely established that Einstein was wrong in claiming that QM has to be completed by introducing extra "hidden" variables in order to specify the "objective" physical state of each particle, but have shed no light on the nature of the entangled two-particle state vector responsible for the peculiar quantum correlation between them, a correlation which exceeds the classical one expected from the constraints of conservation laws.

In the latter case, generations of theoretical physicists in neoplatonist mood have insisted in claiming that the realistic aspect of macroscopic objects is only an illusion valid For All Practical Purposes (in jargon FAPP). The common core of their views is the belief that the only entity existing behind any object, be it small or large, is its wave function, which rules the random occurrence of the object's potential physical properties.

The most extravagant and bold version of this approach is undoubtedly the one known as the Many Worlds Interpretation of QM [4], which goes a step further by eliminating the very founding stone on which QM has been built, namely the essential randomness of quantum events. Chance disappears: the evolution of the whole Universe is written – a curious revival of Laplace - in the deterministic evolution of its wave function.

"The Many-Worlds Interpretation (MWI) – in the words of Lev Vaidman, one of its most eminent supporters [5] - is an approach to quantum mechanics according to which, in addition to the world we are aware of directly, there are many other similar worlds which exist in parallel at the same time and in the same space. The existence of the other worlds makes it possible to *remove randomness* and action at a distance from quantum theory and thus from all physics."

I believe that it is grossly misleading to attribute the epistemological status of "consistent physical theory" to this sort of science fiction, which postulates the existence of myriads and myriads of *physical objects* (indeed entire worlds!) which are *in principle undetectable*. My purpose is to show that these difficulties can only be faced by pursuing a line of research which goes in the opposite direction, namely which takes for granted the irreducible nature of randomness in the quantum world.

This can be done by eliminating *from the beginning* the unphysical concept of wave function. I believe that this elimination is conceptually similar to the elimination of the aether, together with its paradoxical properties, from classical electrodynamics, accomplished by relativity theory. In our case the lesson sounds: No wave functions, no problems about their physical nature.

Furthermore, the adoption of a statistical approach from the beginning for the description of the physical properties of quantum systems sounds methodologically better founded than the conventional *ad hoc* hybrid procedure of starting with the determination of a system's wave function of unspecified nature followed by a "hand made" construction of the probability distributions of its physical variables.

2. The Phase Space Approach to the Formulation of Quantum Mechanics

This statistical approach goes back to Wigner's formulation [6] of QM in phase space. Wigner functions replace the classical probability distributions as a perfectly valid tool for calculating all the statistical properties of any ensemble of quantum particles, in spite of the fact that they lack the fundamental property of being non-negative. This is indeed the price one has to pay in order to introduce in the classical expressions the constraint of the uncertainty principle.

Almost fifty years later, Feynman [7] reconsidered the possibility of giving up the assumption that the "probability" for an event must always be a positive number. This extension of the probability concept does not lead to absurd consequences. On the contrary, Feynman argued that, once the "strong mental block against negative probabilities" has been overcomed, "they are entirely rational, and their use simplifies calculations and thought in a number of applications in physics." One should add also that the validity of Feynman's approach has been strikingly supported by the recent direct experimental determination of the Wigner functions for the states of quantum harmonic oscillators [8]

This line of thought has been developed by myself in the last few years [9]. I have in fact shown that the Wigner representation of quantum mechanics in phase space can be derived from two basic physical principles (the Heisenberg uncertainty principle for canonically conjugated variables and the Planck quantization of variables such as energy or angular momentum) *without need of introducing Schrödinger waves*. An important consequence of these postulates is that the random variables obeying these constraints *must* be represented by non commuting mathematical entities whose explicit expression, however, never enters in the theory, which deals only with their ensemble averages.[1]

The aim of the present paper, however, is not to argue further on general grounds in favor of this new approach. In principle, in fact, the new formalism allows the solution of any quantum mechanical problem in terms of Wigner functions without ever having

[1] A generalization of this approach to field quantization ([10]) has further shown that this approach leads also to the clarification of the notion of *wave/particle duality*, which still holds but does not involve Schrdinger waves. It refers instead to the dual nature of the quantum field as a unique physical entity *objectively existing in ordinary three dimensional space*. I will not be concerned with this question in this paper.

to introduce Schrödinger waves. However, since the most elementary evidence of the existence of Schrödinger waves is usually considered to be the “interference pattern” of a beam of particles passing through a Mach Zender interferometer, it may be interesting to show that the notion of irreducible randomness allows a straightforward explanation of this phenomenon without ever having to introduce this puzzling and elusive entity.

3. The Statistical Properties of a Classical Ensemble of Particles with Random Dichotomic Variables

To achieve this goal it is necessary to start by considering a beam of classical particles (fig.1) impinging on a beam splitter A which randomly may be either reflected to proceed along a path L_1 or transmitted to proceed along a path L_2. In fact one should at this point stress that, classically, *randomness is a purely epistemic phenomenon*, namely that it is a consequence of the impossibility of reconstructing by means of a suitable detection apparatus the ‘real’ deterministic path of each particle which, depending on the initial conditions, will lead it to be either reflected or transmitted in the interaction with the beam splitter’s atoms. The actual physical state of each particle of a classical ensemble *mus*t be different from that of the other ones: if they all had exactly the same state they could not behave differently.

In order to describe the statistical aspects of the phenomenon we define a random variable **A** with the value a=1 in the case of reflection and the value a=-1 in the case of transmission. At the end of L_1 the particles impinge on the upper side of a second beam splitter B and may be either reflected (value b=1 of the random variable **B**) and detected by a detector D_1 or transmitted (b=-1) and detected by a detector D_2. The particles arriving from L_2 impinge on the opposite side of B to be either transmitted (b=-1) to reach the counter D_1, or reflected (b=1) to reach the counter D_2. To summarize: the counter D_1 receives both the particles with a=-1, b=-1 and those ones with a=+1, b=+1, and the counter D_2 receives the particles with a=+1, b=-1 and those with a=-1 and b=+1.

We can define therefore this classical statistical ensemble of particles by means of three numbers: x, y, z representing respectively the ensemble average $<\mathbf{A}>$ of **A**, the ensemble average $<\mathbf{B}>$ of **B**, and the ensemble average $<\mathbf{AB}>$ of their product (correlation coefficient). It is useful for later reference to introduce an independent notation **C** for the product **AB** because, in spite of the fact that for each particle the value of **C** is determined by the product of the values of **A** and **B**, the ensemble average of the values of **C** is statistically independent from the ensemble averages of the other two variables, because it depends on physical factors other than those of **A** and **B** taken separately: x and y represent the average values of the random events of reflection or transmission of each particle in the interaction with the splitters, while z depends on the physical features of the two paths. No direct connection exists however in this description between these features (length difference, physical nature of the phase shifter S or relative time delay) and the value of z. In a classical particle ensemble there are no waves, of course, and in a purely probabilistic description, the connection can only show up in the relation between the ensemble average values of the random events. We will come back to this point when we will discuss the difference between the classical and the quantum ensemble.

In a given physical ensemble, given the values of x,y,z, the probabilities for the corresponding four alternatives ($a = \pm 1$, $b = \pm 1$) are given by:

$$p_{ab} = (\frac{1}{4})[1 + ax + by + abz] \tag{1}$$

where

$$x =< \mathbf{A} >= \sum ap_{ab}; \quad y =< \mathbf{B} >= \sum bp_{ab}; \quad z =< \mathbf{C} >= \sum abp_{ab} \tag{2}$$

If we impose that $0 \leq \mathrm{p}_{ab} \leq 1$, as it must be for classical probabilities, then one finds that the values of x, y, z should satisfy the condition of being the coordinates of a point inside the equilateral octahedron with vertices

$$x = \pm 1, z = 0, y = 0; \quad y = \pm 1, z = 0, x = 0; \quad z = \pm 1, x = 0, y = 0 \tag{3}$$

These limiting values correspond respectively to: (i) pure reflection (or pure transmission) by A followed by equally probable reflection *and* transmission by B, and zero correlation; (ii) equally probable reflection *and* transmission by A followed by pure reflection (or pure transmission) by B, again with zero correlation; (iii) complete correlation or autocorrelation between the two beam splitters with equiprobable transmission and reflection by both A and B. *Apart from these limiting cases, all other possible ensembles have average values of the random variables with values less than one.* This means that always particles with both values ± 1 of **A, B, C** are present.

The above constraint leads to the relation between x, y, z

$$-1 \leq x + y + z \leq +1 \tag{4}$$

The counting rates of the counters D_1 and D_2 are therefore given by

$$P_1 = p_{++} + p_{--} = (\frac{1}{2})(1 + z); \quad P_2 = p_{+-} + p_{-+} = (\frac{1}{2})(1 - z) \tag{5}$$

This procedure of defining an ensemble by giving the three values of x, y, z with the constraint (4), is hower too general for our purpose of preparing the ground to the formulation of the constraints wich define the corresponding quantum ensemble.

To pursue this aim it is convenient to select a class of ensembles, characterized by the values of three new random variables, **U**, **V**, **W**, which satisfy a sort of classical *uncertainty principle* - a property that will become the actual uncertainty principle of the corresponding quantum ensemble – defined by the properties

$$< \mathbf{U} >= u; \quad < \mathbf{V} >= 0; \quad < \mathbf{W} >= 0 \tag{6}$$

U is the variable whose average value u labels the ensemble while the other two variables satisfy the constraint of being completely undetermined, namely that all their possible values are equally probable. This program is fulfilled by choosing an arbitrary unit vector of components α, β, γ in the x, y, z space and defining **U** in terms of the variables **A**, **B**, **C** as a linear combination

$$\mathbf{U} \ = \alpha\mathbf{A} \ +\beta\mathbf{B} + \gamma\mathbf{C} \ ; \quad \alpha^2 + \beta^2 + \gamma^2 = 1 \tag{7}$$

In order to satisfy the two constraints (6) for **V** and **W** we have to define, in addition to (α, β, γ) two other mutually orthogonal unit vectors (λ, μ, ν) and $((\beta\nu - \gamma\mu), (\gamma\lambda - \alpha\nu), (\alpha\mu - \beta\lambda))$ such that

$$\mathbf{V} \ = \lambda\mathbf{A} \ +\mu\mathbf{B} + \nu\mathbf{C} \ ; \quad \lambda^2 + \mu^2 + \nu^2 = 1; \quad \alpha\lambda + \beta\mu + \gamma\nu = 0 \tag{8}$$

$$\mathbf{W} \ = (\beta\nu - \gamma\mu) \ \mathbf{A} + (\gamma\lambda - \alpha\nu)\mathbf{B} + (\alpha\mu - \beta\lambda)\mathbf{C} \tag{9a}$$

The definitions (7) (8) (9) can be transformed into equations for x, y, z by taking the average values of the variables **A**, **B**, **C**, involved and making use of the relations between α, β, γ and λ, μ, ν. The result is

$$x = \alpha u; \quad y = \beta u; \quad z = \gamma u \tag{9b}$$

The constraint (4) leads therefore to the constraint for u

$$-1 \leq u(\alpha + \beta + \gamma) \leq +1 \tag{10}$$

which implies that the **absolute value of u is always smaller than one,** because $\alpha + \beta + \gamma$ is always greater than one (except in the limiting cases of the vertices of the octahedron). This confirms the above statement that *particles with both values* ± 1 *of* **A, B, C** *are always present in any ensemble.*

In terms of the new parameters defining the ensemble the probabilities p_{ab} become

$$p_{ab} = (\frac{1}{4})[1 + u(a\alpha + b\beta + ab\gamma)] \tag{11}$$

and the counting rates of the counters

$$P_1 = p_{++} + p_{--} = (\frac{1}{2})(1 + u\gamma); \quad P_2 = p_{+-} + p_{-+} = (\frac{1}{2})(1 - u\gamma) \tag{12}$$

4. The Quantum Ensemble

We now try to construct the quantum mechanical theory of our experimental setup by introducing the necessary changes to the definitions of the classical ensemble. The first step is to assume as a **postulate** that *randomness is an irreducible property of quantum events.* After all, this is exactly what all physics students learn in their introductory courses when they are taught that the actual decay of a single unstable radioactive nucleus is a purely random event which may occur at any time within the limit of a typical lifetime.

This postulate means that the different behavior of each particle going through the interferometer is not due to *hidden variables* having a priori different values, but the effect of individual intrinsically random events triggered by chance occurring in a dispersion-free quantum ensemble in which ***all the particles*** *have the* ***same*** *physical state. This common state should be either one or the other of the two states corresponding to the values* ± 1 *of a suitable dichotomic random variable* $\mathbf{U}$.

If we assume the ***uncertainty principle*** *as the founding principle of quantum theory we must further assume that* ***for each particle*** *the value of the two other statistically independent variables* $\mathbf{V}$, $\mathbf{W}$ *is completely undetermined, namely that their values* ± 1 *are equally probable.*

This principle justifies our choice of the variables $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ with their definitions (7) (8) (9) and their properties (6), as the proper classical framework to start with for the construction of the corresponding quantum ensemble

The quantum constraints imply therefore that the absolute value of the ensemble average value u of $\mathbf{U}$ (which, accordingly to eq. (10) was in the classical case ***always*** smaller than one) must be now be ***always*** equal to one (for all values of α, β, γ), because ***all the particles have the value*** $+1$ **(*or*** -1) of $\mathbf{U}$. Eqs. (9) are therefore now replaced by

$$x = \alpha; \quad y = \beta; \quad z = \gamma \tag{13}$$

and the constraints (4) and (10) by

$$x^2 + y^2 + z^2 = 1 \tag{14}$$

The immediate consequence of the replacement of (4) with (14) (*which implies that x, y, z are always outside of the octahedron*) is clearly that the *probabilities* p_{ab} cannot be any more constrained within the interval $0 \leq \mathrm{p}_{ab} \leq 1$. This is not unexpected, in the light of the considerations exposed in the introductory sections.

There is however an even more striking consequence of our quantum postulate. It follows from the observation that, if all the particles have the same value +1 (or alternatively -1) of $\mathbf{U}$ then *the ensemble average* of $\mathbf{U}^2$ must necessarily be

$$< \mathbf{U}^2 >= 1 \tag{15}$$

Until now we have implicitly assumed that $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are the same random variables of the classical ensemble, satisfying the commutative property of multiplication of ordinary numbers. However, under this assumption, if one introduces for $\mathbf{U}$ the expression (7) and takes into account eqs. (13), the ensemble average of $\mathbf{U}^2$ turns out to be

$$< \mathbf{U}^2 >= 1 + 6\alpha\beta\gamma \tag{16}$$

This means that the quantum random variables $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ cannot satisfy the commutative law of multiplication of ordinary numbers. Also this result, of course is not unexpected.

Therefore, the only way to fulfill the physical requirement of the uncertainty principle is in fact to assume that the variables **A**, **B** and their correlation **C** are represented by mathematical objects which do not satisfy the commutative law of multiplication. Typical objects of this kind are Hermitian matrices with eigenvalues ± 1. It should be kept in mind that In order to ensure that the correlation variable **C** be an Hermitian matrix (with real eigenvalues), it should be related to the product of the two Hermitian matrices **A** and **B** by (i is the imaginary unit);

$$\mathbf{AB} = i\mathbf{C} \tag{17}$$

With this assumption the ensemble averages of the variables coincide with the expectation values (traces) of the corresponding matrices. It is immediate to see that if one assumes for them the commutation and anticommutation relations of the three Pauli spin matrices

$$\mathbf{AB} - \mathbf{BA} = 2i\mathbf{C}; \quad \mathbf{AB} + \mathbf{BA} = 0 \quad \text{and cyclic permutations} \tag{18}$$

one obtains

$$\mathbf{UV} - \mathbf{VU} = 2i\mathbf{W}; \quad \mathbf{UV} + \mathbf{VU} = 0 \tag{19}$$

and therefore

$$< \mathbf{U} >=< \mathbf{U}^2 >= 1; \quad < \mathbf{V} >= 0; \quad < \mathbf{W} >= 0 \tag{20}$$

as it should be.

It is however important to stress that in this approach one never needs to use the explicit form of the matrices, or to make reference to their matrix elements and eigenvectors. *Only expectation values enter in this formulation of the theory.* The classical relations (11) and (12) are now replaced by the quantum ones

$$p_{ab} = (\frac{1}{4})[1 + (a\alpha + b\beta + ab\gamma)] \tag{21}$$

and

$$P_1 = p_{++} + p_{--} = (\frac{1}{2})(1 + \gamma); \quad P_2 = p_{+-} + p_{-+} = (\frac{1}{2})(1 - \gamma) \tag{22}$$

5. Conclusions

Our quantization procedure is therefore accomplished with the result that the mathematical nature of the quantum random variables (Hermitian matrices) turns out to be, rather than an arbitrary *a priori* assumption, a *consequence* of the physical postulate that *randomness is an essential irreducible feature of quantum events* (uncertainty principle). It is not only a matter of taste. In the conventional approach, in fact, one always risks to look invain for nonexisting physical aspects of purely mathematical concepts.

We have seen that a direct consequence of the quantum postulate is that the condition (14) implies that the quantum statistical probability distributions p_{ab} defined in eq. (11)

are no longer positive definite because x, y, z are always chosen outside the octahedron (3). This is often considered to be an obstacle to any direct probabilistic formulation of QM.

However, once that, following Feynman's suggestion, the strong prejudice against treating the "pseudoprobabilities" as legitimate expressions of the statistical properties of the quantum ensemble is overcome, it is easy to show that they lead to the experimentally well known "interference" patterns of a beam of quantum particles going through a Mach Zender interferometer. In fact the essential result of this reformulation of the phenomenon is that the quantum counting rates (22) of the counters D_1 and D_2 differ from the classical ones (12) for the absence of the factor u which, as repeatedly underlined, is always smaller than one.

The effects of the variation of the physical parameters of the setup (represented in fig.1 by the "shifter" S) shows up in fact both in the classical and in the quantum model through the functional dependence $\gamma = cos\theta$ on the polar angle θ of the unit vector representing the ensemble on the sphere (14)*his means therefore that in quantum ensembles the counting rate of each counter always varies between 0 and 1 depending on the value of* γ *while in the corresponding classical ensemble it varies only in the limited range between* $(1/2)(1-u)$ *and* $(1/2)(1+u)$.

We expect therefore that a beam made of identical atoms with different values of internal degrees of freedom (excited states) would show the reduced interference pattern of the classical model.

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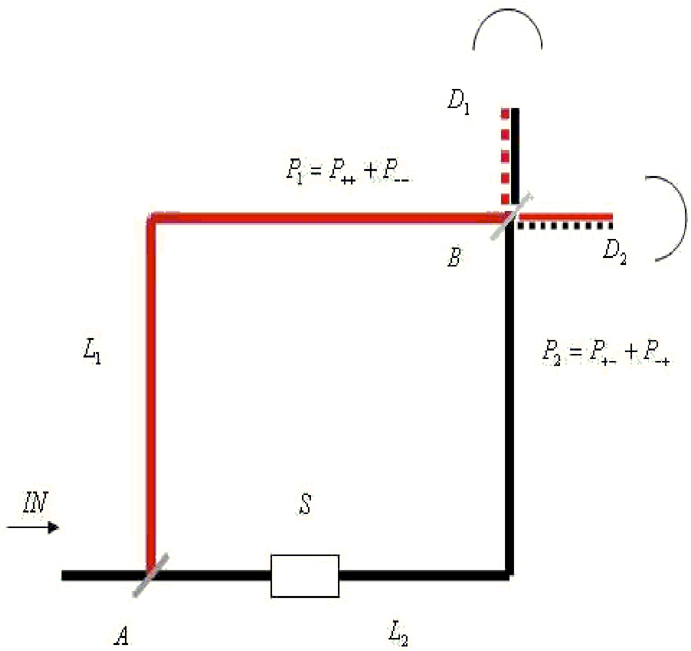

Fig. 1

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Metric Variation Inside Transitioning Superconducting Shells

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Abstract: In this paper, we outline the forward problem of metrical variation due to the Casimir effect in transitioning superconducting shells. We consider a massless scalar quantum field inside a hollow superconducting sphere and a cylinder. Metric equations are developed describing the evolution of the scale factors after the superconducting shells transition to the normal state.

Keywords: General Relativity, Quantum Field, Quantum Vacuum, Casimir Effect, Superconducting Shells (sphere and cylinder)
PACS (2006): 04.20.q, 04.20.Cv, 04.62.+v, 04.90.+e, 02.40.k

1. Introduction

The Casimir effect results in the polarization of the quantum vacuum by conducting boundaries [1, 2]. Vacuum polarization gives rise to a minute force that has been measured experimentally [3, 4] in agreement with the predictions of quantum electrodynamics. In calculating the Casimir force, one properly calculates differences in vacuum pressure

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established by the conducting boundaries.

Boyer first calculated the vacuum modes inside a conducting sphere [5], with a more recent account given by Milton [6]. Applications of the Casimir effect have been studied for massive scalar [7] and Dirac fields [8] confined to the interior of a sphere. The Casimir effect in curved spacetime has been calculated for spherical geometries [9] and for a cylindrical shell in de Sitter space [10] and in the background of a static domain wall [11].

In this paper, we investigate the metrical variations resulting from vacuum energy differences established by hollow superconducting boundaries. We consider the static case when the shells are in the superconducting state and then the dynamical case as the shells pass to the normal state.

2. Transitioning Superconducting Sphere

Our idealized massless, thin sphere of radius R_0 has zero conductivity in the normal state. In the superconducting state, the vacuum inside the hollow is reduced so that there exists a pressure difference Δp inside and outside the sphere. In general, all quantum fields will contribute to the vacuum energy. When the sphere of volume V transitions to the superconducting state, a latent heat of vacuum phase transition ΔpV is exchanged. The distribution of vacuum pressure, energy density and space-time geometry are described by the semi-classical Einstein field equations taking $c = 1$,

$$R_{\mu\nu} - \frac{1}{2}\Re g_{\mu\nu} = 8\pi G \left\langle T_{\mu\nu} \right\rangle \tag{1}$$

where $R_{\mu\nu}$ and $\Re$ are the Ricci tensor and scalar curvature, respectively. $\left\langle T_{\mu\nu} \right\rangle$ is the vacuum expectation of the stress energy tensor. Regulation procedures for calculating the renormalized stress energy tensor are given in [12] for various geometries. In spherical coordinates we consider the line element

$$ds^2 = B\left(r,t\right)dt^2 - A\left(r,t\right)dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 \tag{2}$$

where B and A are arbitrary functions of time and the radial coordinate. The corresponding metric tensor is of the form

$$g_{\mu\nu} = Diag\left(B\left(r,t\right), -A\left(r,t\right), -r^2, -r^2 sin^2\right) \tag{3}$$

For a diagonal stress energy tensor, the solutions to equation (1) relating A and B are

$$-\frac{1}{r^2} + \frac{1}{r^2 A} - \frac{A'}{rA^2} = 8\pi G \frac{1}{B}\left\langle T_{00} \right\rangle \tag{4}$$

$$\frac{1}{r^2} - \frac{1}{r^2 A} - \frac{B'}{rAB} = 8\pi G \frac{1}{A}\left\langle T_{11} \right\rangle \tag{5}$$

$$\frac{A'}{2rA^2} - \frac{B'}{2rAB} + \frac{A'B'}{4A^2 B} - \frac{B'^2}{4AB^2} - \frac{B''}{2AB} = 8\pi G \frac{1}{r^2}\left\langle T_{22} \right\rangle \tag{6}$$

with a fourth equation identical to (6). The prime denotes ∂_r. Note that all time derivatives cancel from the field equations when the metric is in standard form and the stress energy tensor is diagonal. When the sphere is in the superconducting state, the scalar curvature $\Re = g^{\mu\nu}R_{\mu\nu}$ is given by

$$\Re = \frac{2}{r^2} - \frac{2}{r^2A} + \frac{2A'}{rA^2} - \frac{2B'}{rAB} + \frac{A'B'}{2A^2B} + \frac{B'^2}{2AB^2} - \frac{B''}{AB} \tag{7}$$

We consider an ideal case where the sphere passes instantaneously from the superconducting to the normal state. The field modes, however, will not relax back instantaneously. The diagonal form of the stress energy tensor results in the cancellation of all time derivatives in the field equations in the static case above. External electromagnetic fields will contribute off-diagonal terms. In this analysis the required time dependence is provided by the zero point field fluctuations and, in particular, their contribution to the off-diagonal stress-energy terms. As the simplest case, we consider the massless scalar quantum field $\phi(r,t)$ with stress energy tensor [12]

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} \tag{8}$$

The non-zero components of T are

$$T_{00} = \frac{1}{2}\dot{\phi}^2 + \frac{B}{2A}\phi'^2 \tag{9}$$

$$T_{11} = \frac{1}{2}\phi'^2 + \frac{A}{2B}\dot{\phi}^2 \tag{10}$$

$$T_{22} = r^2\left(\frac{1}{2B}\dot{\phi}^2 - \frac{1}{2A}\phi'^2\right) \tag{11}$$

$$T_{33} = r^2 sin^2\theta\left(\frac{1}{2B}\dot{\phi}^2 - \frac{1}{2A}\phi'^2\right) \tag{12}$$

$$T_{01} = \dot{\phi}\phi' \tag{13}$$

where $T_{01} = T_{10}$. The semi-classical field equations become

$$-\frac{1}{r^2} + \frac{1}{r^2A} - \frac{A'}{rA^2} = 8\pi G\frac{1}{B}\langle T_{00}\rangle \tag{14}$$

$$\frac{1}{r^2} - \frac{1}{r^2A} - \frac{B'}{rAB} = 8\pi G\frac{1}{A}\langle T_{11}\rangle \tag{15}$$

$$-\frac{\dot{A}^2}{4A^2B} - \frac{\dot{A}\dot{B}}{4AB^2} + \frac{\ddot{A}}{2AB} + \frac{A'}{2r^2A} - \frac{B'}{2rAB} + \frac{A'B'}{4A^2B} + \frac{B'^2}{4AB^2} - \frac{B''}{2AB} = 8\pi G\frac{1}{r^2}\langle T_{22}\rangle \tag{16}$$

$$-\frac{\dot{A}}{rA} = 8\pi G\langle T_{01}\rangle \tag{17}$$

Equations (14) and (15) are identical to (4) and (5). Two additional equations are identical to (16) and (17). Integral expressions for A and B may be obtained from equation (17) and (14) or (15), respectively. The scalar curvature is given by

$$\Re = \frac{2}{r^2} - \frac{2}{r^2 A} - \frac{\dot{A}^2}{2A^2 B} - \frac{\dot{A}\dot{B}}{AB} + \frac{\ddot{A}}{AB} + \frac{2A'}{rA^2} - \frac{2B'}{rAB} + \frac{A'B'}{2A^2 B} + \frac{B'^2}{2AB^2} - \frac{B''}{AB} \tag{18}$$

Comparing equation (18) with (14-16) and (10-12) reveals

$$\Re = 16\pi G \left\langle \frac{\dot{\phi}^2}{2B} - \frac{\phi'^2}{2A} \right\rangle \tag{19}$$

When evaluating changes in scalar curvature, the expression for R in absence of the sphere should be subtracted from that obtained for a given quantum field.

3. Transitioning Superconducting Cylinder

Modeling a superconducting cylinder, we consider the line element in cylindrical coordinates

$$ds^2 = B(r,t)\,dt^2 - A(r,t)\,dr^2 - r^2 d\theta^2 - dz^2 \tag{20}$$

with corresponding metric tensor

$$g_{\mu\nu} = Diag\left(B(r,t), A(r,t), -r^2, -1\right) \tag{21}$$

Solutions to equation (1) relating A and B in the static case, when the cylinder is superconducting state, are

$$-\frac{BA'}{2rA^2} = 8\pi G \langle T_{00} \rangle \tag{22}$$

$$-\frac{B'}{2rB} = 8\pi G \langle T_{11} \rangle \tag{23}$$

$$\frac{A'B'}{2A^2 B} + \frac{B'^2}{4AB^2} - \frac{B''}{2AB} = \frac{8\pi G}{r^2} \langle T_{22} \rangle \tag{24}$$

$$\frac{A'}{2rA^2} + \frac{B'}{2rAB} - \frac{A'B'}{4A^2 B} + \frac{B'^2}{2AB^2} = 8\pi G \langle T_{33} \rangle \tag{25}$$

Considering a scalar quantum field with stress energy tensor given by equation (8), all components of T are of the same form as the spherical case except equation (12) that becomes

$$T_{33} = \left(\frac{1}{2B}\dot{\phi}^2 - \frac{1}{2A}\phi'^2 \right) \tag{26}$$

The time dependent semi-classical field equations are then

$$-\frac{BA'}{2rA^2} = 8\pi G \langle T_{00}\rangle \tag{27}$$

$$-\frac{\dot{A}}{2rA} = 8\pi G \langle T_{01}\rangle = 8\pi G \langle T_{10}\rangle \tag{28}$$

$$-\frac{B'}{2rB} = 8\pi G \langle T_{11}\rangle \tag{29}$$

$$\frac{A'B'}{4A^2B} + \frac{B'^2}{4AB^2} - \frac{B''}{2AB} - \frac{\dot{A}^2}{4A^2B} + \frac{\dot{A}\dot{B}}{4AB^2} + \frac{\ddot{A}}{2AB} = 8\pi G \langle T_{22}\rangle \tag{30}$$

$$\begin{aligned}\frac{A'}{2rA^2} - \frac{B'}{2rAB} + \frac{A'B'}{4A^2B} + \frac{B'^2}{4AB^2} - \frac{B''}{2AB} - \frac{\dot{A}^2}{4A^2B} - \frac{\dot{A}\dot{B}}{4AB^2} + \frac{\ddot{A}}{2AB}& \\ = 8\pi G \langle T_{33}\rangle&\end{aligned} \tag{31}$$

Integral expressions for A and B may be obtained from equations (29) and (30) respectively,

$$A(r,t) \sim exp\left(-16\pi G r \int \langle T_{10}\rangle \, dt\right) \tag{32}$$

$$B(r,t) \sim exp\left(-16\pi G \int r \langle T_{11}\rangle \, dr\right) \tag{33}$$

A similar set of equations may be obtained in the spherical case, as mentioned previously. The scalar curvature

$$\Re = \frac{A'}{rA^2} - \frac{B'}{rAB} - \frac{A'B'}{2A^2B} + \frac{B'^2}{2AB^2} - \frac{B''}{AB} - \frac{\dot{A}^2}{2A^2B} - \frac{\dot{A}\dot{B}}{2AB^2} + \frac{\ddot{A}}{AB} \tag{34}$$

becomes comparing equation (31)

$$\Re = 16\pi G \langle T_{33}\rangle \tag{35}$$

which is of the same form as equation (19) obtained for the evolution of scalar curvature inside a superconducting sphere. The quantities $\left\langle \dot{\phi}^2 \right\rangle$ and $\langle \phi'^2 \rangle$ however will not have the same form as in the spherical case. Applying conservation of energy-momentum $\langle T^{\mu\nu}\rangle_{;\nu}$ in the cylindrical case gives the two equations

$$\begin{aligned}-\frac{rB}{A}\left(\langle T_{00}\rangle \dot{A} + \langle T_{10}\rangle A'\right) = r\langle T_{11}\rangle \dot{A} + 2r\langle T_{00}\rangle \dot{B} + 3r\langle T_{10}\rangle B'& \\ +2B\left(\langle T_{10}\rangle + r\langle T_{00}\rangle_{,t} + r\langle T_{10}\rangle_{,r}\right)&\end{aligned} \tag{36}$$

$$\frac{rB}{A}\left(2r\left\langle T_{22}\right\rangle - 3\left\langle T_{10}\right\rangle \dot{A} - 2\left\langle T_{11}\right\rangle A' - \left\langle T_{00}\right\rangle B'\right) = r\left\langle T_{10}\right\rangle \dot{B} + r\left\langle T_{11}\right\rangle B' + 2B\left(\left\langle T_{11}\right\rangle + r\left\langle T_{10}\right\rangle_{,t} + r\left\langle T_{11}\right\rangle_{,r}\right) \quad (37)$$

A similar set of equations may be obtained in the spherical case. In order to numerically determine the evolution of scale factors and quantum fields in either configuration, one must evaluate renormalized quantities $\left\langle \dot{\phi}^2 \right\rangle$, $\langle \phi'^2 \rangle$, and $\left\langle \dot{\phi}\phi' \right\rangle$. The static solution with Dirichlet boundary conditions applied for the sphere or cylinder in the superconducting state provides the initial condition for the transient case, immediately after the shell becomes normal.

If one assumes the gravitational fields resulting from the Casimir effect are sufficiently weak, it seems plausible to evaluate the renormalized quantities $\left\langle \dot{\phi}^2 \right\rangle$, $\langle \phi'^2 \rangle$, and $\left\langle \dot{\phi}\phi' \right\rangle$ in Minkowski spacetime and then calculate their effect on the scale factors. In 2-d cylindrical coordinate flat space one obtains $\left\langle \dot{\phi}^2 \right\rangle = \langle \phi'^2 \rangle = -\pi/6L^2$ where $L = 2\pi r$ Evidently the quantity $\left\langle \dot{\phi}\phi' \right\rangle = 0$ for scalar fields in Minkowski spacetime. Inspection of the field equations reveals the requirement $\left\langle \dot{\phi}\phi' \right\rangle \neq 0$ to describe the evolution of the scalar quantum field and metrical factors in this analysis.

4. Conclusion

A procedure for describing the evolution of quantum fields and spacetime curvature inside transitioning superconducting shells has been outlined. The static solution with the shell in the superconducting state serves as the initial condition for the transient evolution of field modes. Zero-point field fluctuations are coupled to the time dependent Einstein field equations through the off-diagonal components of the stress energy tensor. Because of the differences in vacuum energy densities, a latent heat of vacuum phase transition is exchanged when a hollow shell transitions between the normal and superconducting state. The analysis presented here may be extended to include massive fields with coupling or spin as well as other superconducting geometries.

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Black Scholes Option Pricing with Stochastic Returns on Hedge Portfolio

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Abstract: The Black Scholes model of option pricing constitutes the cornerstone of contemporary valuation theory. However, the model presupposes the existence of several unrealistic and rigid assumptions including, in particular, the constancy of the return on the "hedge portfolio". There, now, subsists ample justification to the effect that this is not the case. Consequently, several generalisations of the basic model have been attempted. In this paper, we attempt one such generalisation based on the assumption that the return process on the "hedge portfolio" follows a stochastic process similar to the Vasicek model of short-term interest rates.

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With the rapid advancements in the evolution of financial markets across the globe, the importance of generalisations of the extant mathematical apparatus to enhance its domain of applicability to the pricing of financial products can hardly be overemphasized for further progress and development of the financial microstructure.

Though at an embryonic stage, the unification of physics, mathematics and finance is unmistakably discernible with several fundamental premises of physics and mathematics like quantum mechanics, classical & quantum field theory and related tools of non-commutative probability, functional integration etc. being adopted for pricing of extant financial products and for elucidating on various occurrences of financial markets like stock price patterns, critical crashes etc [1-12].

The Black Scholes formula for the pricing of financial assets [13-17] continues to be the substratum of contemporary valuation theory. However, the model, although of

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immense practical utility is based on several assumptions that lack empirical support. The academic fraternity has attempted several generalisations of the original Black Scholes formula through easing of one or other assumption, in an endeavour to augment its spectrum of applicability.

In this paper, we attempt one such generalisation based on the assumption that the return process on the "hedge portfolio" follows a stochastic process similar to the Vasicek model of short-term interest rates. Section 2 lists out the derivation of the Black-Scholes formula through the partial differential equation based on the construction of the complete "hedge portfolio". Sec 3, which forms the essence of this paper, attempts a generalisation of the standard Black Scholes pricing formula on the lines aforesaid. Section 4 concludes.

1. The Black Scholes Model

In order to facilitate continuity, we summarize below the original derivation of the Black Scholes model for the pricing of a European call option [13-17 and references therein]. The European call option is defined as a financial contingent claim that enables a right to the holder thereof (but not an obligation) to buy one unit of the underlying asset at a future date (called the exercise date or maturity date) at a price (called the exercise price). Hence, the option contract, has a payoff of $\max(S_T - E, 0) = (S_T - E)^+$ on the maturity date where S_T is the stock price on the maturity date and E is the exercise price.

We consider a non-dividend paying stock, the price process of which follows the geometric Brownian motion with drift $S_t = e^{(\mu t + \sigma W_t)}$. The logarithm of the stock price $Y_t = In\ S_t$ follows the stochastic differential equation

$$dY_t = \mu dt + \sigma dW_t \tag{1}$$

where W_t is a regular Brownian motion representing Gaussian white noise with zero mean and δ correlation in time i.e. $E(dW_t dW_{t'}) = dtdt'\delta(t - t')$ on some filtered probability space $(\Omega, (F_t), P)$ and μ and σ are constants representing the long term drift and the noisiness (diffusion) respectively in the stock price.

Application of Ito's formula yields the following SDE for the stock price process

$$dS_t = \left(\mu + \frac{1}{2}\sigma^2\right) S_t dt + \sigma S_t dW_t \tag{2}$$

Let $C(S,t)$ denote the instantaneous price of a call option with exercise price E at any time t before maturity when the price per unit of the underlying is S . It is assumed that $C(S,t)$ does not depend on the past price history of the underlying. Applying the Ito formula to $C(S,t)$ yields

$$dC = (\mu S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2})dt + \frac{\partial C}{\partial S}\sigma S dW, \tag{3}$$

The original option-pricing model propounded by Fischer Black and Myron Scholes envisaged the construction of a "hedge portfolio", $\prod$, consisting of the call option and a short sale of the underlying such that the randomness in one cancels out that in the other. For this purpose, we make use of a call option together with $\partial C/\partial S$ units of the underlying stock.

We then have, on applying Ito's formula to the "hedge portfolio", $\prod$,:-

$$\frac{d\prod}{dt} = \frac{d}{dt}\left[C\left(S,t\right) - S\frac{\partial C\left(S,t\right)}{\partial S}\right] = \frac{dC\left(S,t\right)}{dt} - \frac{\partial C}{\partial S}.\frac{dS}{dt} \tag{4}$$

where the term involving $\frac{d}{dt}\left(\frac{\partial C}{\partial S}\right)$ has been assumed zero since it envisages a change in the portfolio composition. On substituting from eqs. (2) & (3) in (4), we obtain

$$\frac{d\prod}{dt} = \frac{dC\left(S,t\right)}{dt} - \left(\mu + \frac{1}{2}\sigma^2\right)S\frac{\partial C\left(S,t\right)}{\partial S} - \sigma S\frac{\partial C}{\partial S}\frac{dW}{dt} = \frac{\partial C\left(S,t\right)}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C\left(S,t\right)}{\partial S^2} \tag{5}$$

We note, here, that the randomness in the value of the call price emanating from the stochastic term in the stock price process has been eliminated completely by choosing the portfolio $\prod = C\left(S,t\right) - S\frac{\partial C(S,t)}{\partial S}$. Hence, the portfolio $\prod$ is free from any stochastic noise and the consequential risk attributed to the stock price process.

Now $\frac{d\prod}{dt}$ is nothing but the rate of change of the price of the so-called riskless bond portfolio i.e. the return on the riskless bond portfolio (since the equity related risk is assumed to be eliminated by construction, as explained above) and must, therefore, equal the short-term interest rate r i.e.

$$\frac{d\prod}{dt} = r\prod \tag{6}$$

In the original Black Scholes model, this interest rate was assumed as the risk free interest rate r , further, assumed to be constant, leading to the following partial differential equation for the call price:-

$$\frac{d\prod}{dt} = r\prod = r\left[C\left(S,t\right) - S\frac{\partial C\left(S,t\right)}{\partial S}\right] = \frac{\partial C\left(S,t\right)}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C\left(S,t\right)}{\partial S^2}$$

or equivalently

$$\frac{\partial C\left(S,t\right)}{\partial t} + \frac{1}{2}\sigma^2 S\frac{\partial^2 C\left(S,t\right)}{\partial S^2} + rS\frac{\partial C\left(S,t\right)}{\partial S} - rC\left(S,t\right) = 0 \tag{7}$$

which is the famous Black Scholes PDE for option pricing with the solution:-

$$C\left(S,t\right) = SN\left(d_1\right) - Ee^{-r(T-t)}N\left(d_2\right) \tag{8}$$

where
$d_1 = \frac{\log\left(\frac{S}{E}\right)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\log\left(\frac{S}{E}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$ and

$$N\left(y\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{x^2}{2}}dx$$

2. The Black Scholes Model with Stochastic Returns on the Hedge Portfolio

As mentioned earlier, in the above analysis, the interest rate r , which is essentially a proxy for the return on a portfolio that is devoid of any risk emanating from any variables that cause fluctuations and hence risk in stock price process, is taken as constant and equal to the risk free rate. However, this return would, nevertheless, be subject to uncertainties that influence returns on the fixed income securities. It is, now, conventional to model these short term interest rates (that are representative of short term returns on fixed income securities) through a stochastic differential equation of the form [18]

$$dr\left(t\right) = -\psi\left[r\left(t\right), t\right]dt + \eta\left[r\left(t\right), t\right]dU\left(t\right) \tag{9}$$

where $r\left(t\right)$ is the short term interest rate at time t , ψ and η are deterministic functions of r, t and $U\left(t\right)$ is a Wiener Process.

In our further analysis, we shall assume that this short-term interest rate is represented by the Vasicek model [19] viz.

$$\frac{dr\left(t\right)}{dt} + Ar\left(t\right) + B - \sum \eta\left(t\right) = 0 \tag{10}$$

where $\eta\left(t\right)$ is a white noise stochastic process

$$\left\langle \eta\left(t\right)\right\rangle = 0, \left\langle \eta\left(t\right)\eta\left(t^{'}\right)\right\rangle = \sum^{2}\delta\left(t-t^{'}\right) \tag{11}$$

The call price process now becomes a function of two stochastic variables, the stock price process $S\left(t\right)$ and the bond return process (interest rate process) $r\left(t\right)$. Hence, application of Ito's formula to $C\left(S, r, t\right)$ gives

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial r}dr + \frac{1}{2}\sigma^{2}S^{2}\frac{\partial^{2}C}{\partial S^{2}}dt + \frac{1}{2}\sum^{2}\frac{\partial^{2}C}{\partial r^{2}}dt \tag{12}$$

where dS is given by eq. (2) and dr by eq. (10) respectively.

As in Section 2, we formulate a "hedge portfolio" Π consisting of a call option $C\left(S, r, t\right)$ and a short sale of $\frac{\partial C}{\partial S}$ units of stock $S\left(t\right)$ i.e. $\prod = C - S\frac{\partial C}{\partial S}$. We then have, repeating the same steps as in Section 2 hereof

$$\frac{d\prod}{dt} = \frac{dC}{dt} - \frac{\partial C}{\partial S}\frac{dS}{dt} = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial r}.\frac{dr}{dt} + \frac{1}{2}\sigma^{2}S^{2}\frac{\partial^{2}C}{\partial S^{2}} + \frac{1}{2}\sum^{2}\frac{\partial^{2}C}{\partial r^{2}} \tag{13}$$

Now, using $\frac{d\prod}{dt} = r\left(t\right)\prod$, we obtain

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^{2}S^{2}\frac{\partial^{2}C}{\partial S^{2}} + r\left(t\right)S\frac{\partial C}{\partial S} - r\left(t\right)C + \frac{1}{2}\sum^{2}\frac{\partial^{2}C}{\partial r^{2}} + \frac{\partial C}{\partial r}\frac{dr\left(t\right)}{dt} = 0 \tag{14}$$

This equation defies closed form solution with the extant mathematical apparatus. We can, however, obtain explicit expressions for the call price $\overline{C}\left(S, t\right)$ averaged over the stochastic part of the interest rate process, as follows:-

$\overline{C}(S,t)$ would, then, be given by substituting $\frac{\int_t^T r(\tau)d\tau}{\int_t^T d\tau}$ for the constant risk free interest rate r in the Black Scholes formula (8).

The averaging process happens to be tedious with extensive computations so we proceed term by term.

We have

$$N\left(\bar{d}_1\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\bar{d}_1} e^{-\frac{x^2}{2}}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H\left(\bar{d}_1 - x\right)e^{-\frac{x^2}{2}}dx \tag{15}$$

where $H(x-y)$ is the unit step Heaviside step function defined by [20]

$$H(x,y) = \begin{cases} 0, & x < y \\ 1, & x > y \end{cases}$$

On using the integral representation of $H(x-y)$ as $H(x-y) = Lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega \frac{e^{i\omega(x-y)}}{\omega - i\varepsilon}$ [20]

i.e.

$$H\left(\bar{d}_1 - x\right) = Lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega \frac{e^{i\omega\left(\bar{d}_1 - x\right)}}{\omega - i\varepsilon} \tag{16}$$

we obtain

$$N\left(\overline{d_1}\right) = Lim_{\varepsilon\to 0}\frac{1}{(2\pi)^{\frac{3}{2}}i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-\frac{x^2}{2}+i\omega\left(\bar{d}_1 - x\right)}}{\omega - i\varepsilon}dxd\omega = Lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\omega\left(\bar{d}_1 + i\frac{\omega}{2}\right)}}{\omega - i\varepsilon}d\omega \tag{17}$$

on performing the Gaussian integration over x in the second step.

Now

$$\bar{d}_1 = \frac{\log\left(\frac{S}{E}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} + \frac{\int_t^T r(\tau)\,d\tau}{\sigma\sqrt{T-t}} = d_1^0 + \frac{\int_t^T r(\tau)\,d\tau}{\sigma\sqrt{T-t}} \tag{18}$$

where

$$d_1^0 = \frac{\log\left(\frac{S}{E}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \tag{19}$$

Since the entire stochastic contribution comes from the expression $\int_t^T r(\tau)\,d\tau$ in $N\left(\overline{d_1}\right)$, we have

$$N\left(\overline{d_1}\right) = Lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega \frac{e^{i\omega d_1^0 - \frac{\omega^2}{2}}}{\omega - i\varepsilon}I_1 \tag{20}$$

where $I_1 = \left\langle e^{\frac{i\omega}{\sigma\sqrt{T-t}}\int_t^T r(\tau)d\tau}\right\rangle$ and $\langle\rho\rangle$ denotes the average (expectation) of ρ.

Proceeding similarly, we have,

$$N\left(\overline{d_2}\right) = Lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega \frac{e^{i\omega d_2^0 - \frac{\omega^2}{2}}}{\omega - i\varepsilon}I_1 \tag{21}$$

$$d_2^0 = \frac{\log\left(\frac{S}{E}\right) - \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{T-t}} \tag{22}$$

Similarly the discount factor $e^{-r(T-t)}$ will be replaced by $\left\langle e^{-\int_t^T r(\tau)d\tau}\right\rangle = I_2$ (say).

To evaluate the expectation integrals I_1, I_2 we make use of the functional integral formalism [21]. In this formalism, the expectation I_1 would be given by [22]:-

$$I_1 = \frac{\int_{r(t)}^{r(T)} Dr \exp\left[-\frac{1}{2\sum^2}\int_t^T d\tau \left(\frac{dr(\tau)}{d\tau} + Ar(\tau) + B\right)^2 + \frac{i\omega}{\sigma\sqrt{T-t}}\int_t^T d\tau r(\tau)\right]}{\int_{r(t)}^{r(T)} Dr \exp\left[-\frac{1}{2\sum^2}\int_t^T d\tau \left(\frac{dr(\tau)}{d\tau} + Ar(\tau) + B\right)^2\right]} = \frac{P}{Q} \tag{23}$$

where $Dr = \prod_{\tau=t}^T \frac{dr(\tau)}{\sqrt{2\pi}}$ is the functional integration measure.

We first evaluate the functional integral P . Making the substitution $x(\tau) = -\frac{B}{A} - r(\tau)$, we obtain, with a little algebra,

$$P=\int_{x(t)}^{x(T)} Dx \exp\left[-\frac{1}{2\sum^2}\int_t^T d\tau\left(\frac{dx(\tau)}{d\tau}+Ax(\tau)\right)^2+\frac{i\omega}{\sigma\sqrt{T-t}}\int_t^T d\tau\left(-\frac{B}{A}-x(\tau)\right)\right]$$

$$=\int_{x(t)}^{x(T)} Dx \exp\left\{\frac{-1}{2\sum^2}\int_t^T d\tau\left[\left(\frac{dx(\tau)}{d\tau}\right)^2+A^2x^2(\tau)\right]-\frac{A}{2\sum^2}\left[x^2(T)-x^2(t)\right]-\frac{i\omega B(T-t)}{A\sigma\sqrt{T-t}}-\frac{i\omega}{\sigma\sqrt{T-t}}\int_t^T x(\tau)d\tau\right\}$$

$$=\int_{x(t)}^{x(T)} Dx \exp\left\{-\frac{A}{2\sum^2}\left[x^2(T)-x^2(t)\right]-\frac{i\omega B(T-t)}{A\sigma\sqrt{T-t}}-I_3\right\} \tag{24}$$

where

$$I_3 = \frac{1}{2\sum^2}\int_t^T d\tau\left[\left(\frac{dx(\tau)}{d\tau}\right)^2 + A^2x^2(\tau)\right] + \frac{i\omega}{\sigma\sqrt{T-t}}\int_t^T x(\tau)\,d\tau$$

$$= \frac{1}{2\sum^2}\int_t^T d\tau\left[\left(\frac{dx(\tau)}{d\tau}\right)^2 + A^2x^2(\tau) + \frac{2i\omega\sum^2}{\sigma\sqrt{T-t}}x(\tau)\right] \tag{25}$$

In order to evaluate I_3 , we perform a shift of the functional variable $x(\tau)$ by some fixed function $y(\tau)$ i.e. $x(\tau) = y(\tau) + z(\tau)$ where $y(\tau)$ is a fixed functional (whose explicit form shall be defined later) but with boundary conditions $y(t) = x(t), y(T) = x(T)$ so that $z(\tau)$, then, has Drichlet boundary conditions i.e. $z(t) = z(T) = 0$.

Substituting $x(\tau) = y(\tau) + z(\tau)$ in (25), we obtain

$$I_3 = \frac{1}{2\sum^2}\int_t^T d\tau \begin{bmatrix} \left(\frac{dy(\tau)}{d\tau}\right)^2 + \left(\frac{dz(\tau)}{d\tau}\right)^2 + 2\left(\frac{dy(\tau)}{d\tau}\right)\left(\frac{dz(\tau)}{d\tau}\right) + A^2y^2(\tau) + \\ A^2z^2(\tau) + 2A^2y(\tau)z(\tau) + \frac{2i\omega\sum^2}{\sigma\sqrt{T-t}}y(\tau) + \frac{2i\omega\sum^2}{\sigma\sqrt{T-t}}z(\tau) \end{bmatrix} \tag{26}$$

Integrating the second and third term by parts, we get

$$I_3 = \frac{z(\tau)}{2\sum^2}\left(\frac{dz(\tau)}{d\tau} + 2\frac{dy(\tau)}{d\tau}\right)\Big|_t^T + \frac{1}{2\sum^2}\int_t^T d\tau \begin{bmatrix} -z(\tau)\frac{d^2z(\tau)}{d\tau^2} - 2z(\tau)\frac{d^2y(\tau)}{d\tau^2} + \left(\frac{dy(\tau)}{d\tau}\right)^2 + A^2y^2(\tau) + \\ A^2z^2(\tau) + 2A^2y(\tau)z(\tau) + \frac{2i\omega\sum^2}{\sigma\sqrt{T-t}}y(\tau) + \frac{2iw\sum^2}{\sigma\sqrt{T-t}}z(\tau) \end{bmatrix} \tag{27}$$

Now the boundary terms all vanish since $z(\tau)$ has Drichlet boundary conditions. Further, if we define the fixed functional $y(\tau)$ in terms of the differential equation

$$-\frac{d^2y(\tau)}{d\tau^2} + A^2y(\tau) + \frac{i\omega\sum^2}{\sigma\sqrt{T-t}} = 0 \tag{28}$$

with boundary condition $y(t) = x(t), y(T) = x(T)$ we obtain

$$I_3 = \frac{1}{2\sum^2}\int_t^T d\tau\left\{\left[\left(\frac{dy(\tau)}{d\tau}\right)^2 + A^2y^2(\tau) + \frac{2i\omega\sum^2}{\sigma\sqrt{T-t}}y(\tau)\right] + \left[-z(\tau)\frac{d^2z(\tau)}{d\tau^2} + A^2z^2(\tau)\right]\right\} \tag{29}$$

The functional $y(\tau)$ is fixed and is given by the solution of eq (28) as

$$y = \alpha e^{A\tau} + \beta e^{-A\tau} - \gamma \tag{30}$$

where
$\gamma = \frac{i\omega\Sigma^2}{A^2\sigma\sqrt{T-t}}$, $\alpha = \frac{x(T)e^{AT} - x(t)e^{At}}{e^{2AT} - e^{2At}} + \gamma\frac{e^{AT} - e^{At}}{e^{2AT} - e^{2At}}$ and $\beta = \frac{x(T)e^{-AT} - x(t)e^{-At}}{e^{-2AT} - e^{-2At}} + \gamma\frac{e^{-AT} - e^{-At}}{e^{-2AT} - e^{-2At}}$

.

Integrating out the $y(\tau)$ terms in eq. (29) using eq. (30), we obtain

$$I_3 = \frac{1}{2\sum^2}\left\{A\left[\alpha^2\left(e^{2AT} - e^{2At}\right) - \beta^2\left(e^{-2AT} - e^{-2At}\right) - A\gamma^2(T-t)\right] + \int_t^T d\tau\left[-z(\tau)\frac{d^2z(\tau)}{d\tau^2} + A^2z^2(\tau)\right]\right\} \tag{31}$$

Substituting this value of I_3 in eq. (24) we obtain, for P , noting that $Dx = Dz$ since $y(\tau)$ is fixed by eq (28)

$$P = \exp\left\{-\frac{A^2}{2\sum^2}\left[x^2(T) - x^2(t)\right] - \frac{i\omega B\sqrt{T-t}}{A\sigma} - \frac{1}{2\sum^2}\begin{bmatrix} A\left[\left(\frac{x(T)e^{AT} - x(t)e^{At}}{e^{2AT} - e^{2AT}}\right)^2 - \left(\frac{x(T)e^{-AT} - x(t)e^{-At}}{e^{-2AT} - e^{-2At}}\right)^2\right] \\ -\left(\frac{\omega^2\Sigma^4}{A^3\sigma^2(T-t)}\right)\left[\left(\frac{e^{AT} - e^{At}}{e^{2AT} - e^{2At}}\right)^2 - \left(\frac{e^{-AT} - e^{-At}}{e^{-2AT} - e^{-2At}}\right)^2 - A(T-t)\right] \\ +\left(\frac{2i\omega\Sigma^2}{A\sigma\sqrt{T-t}}\right)\begin{bmatrix} \frac{\left(x(T)e^{AT} - x(t)e^{At}\right)\left(e^{AT} - e^{At}\right)}{\left(e^{2AT} - e^{2At}\right)^2} - \\ \frac{\left(x(T)e^{-AT} - x(t)e^{-At}\right)\left(e^{-AT} - e^{-At}\right)}{\left(e^{-2AT} - e^{-2At}\right)^2} \end{bmatrix} \end{bmatrix}\right\}$$

$$\int_{z(t)=0}^{z(T)=0} Dz\exp\left\{\frac{-1}{2\sum^2}\int_t^T d\tau\left[-z(\tau)\frac{d^2z(\tau)}{d\tau^2} + A^2z^2(\tau)\right]\right\} \tag{32}$$

On exactly same lines, we obtain

$$Q = \exp\left\{-\frac{A^2}{2\sum^2}\left[x^2(T) - x^2(t)\right] - \frac{1}{2\sum^2}\left[A\left[\left(\frac{x(T)e^{AT}-x(t)e^{At}}{e^{2AT}-e^{2AT}}\right)^2 - \left(\frac{x(T)e^{-AT}-x(t)e^{-At}}{e^{-2AT}-e^{-2At}}\right)^2\right]\right]\right\}$$
$$\int_{z(t)=0}^{z(T)=0} Dz \exp\left\{\frac{-1}{2\sum^2}\int_t^T d\tau\left[-z(\tau)\frac{d^2z(\tau)}{d\tau^2} + A^2z^2(\tau)\right]\right\} \tag{33}$$

Hence

$$I_1 = \exp\left\{-\frac{i\omega B\sqrt{T-t}}{A\sigma} - \frac{1}{2\sum^2}\left[\begin{array}{l} -\left(\frac{\omega^2\Sigma^4}{A^3\sigma^2(T-t)}\right)\left[\left(\frac{e^{AT}-e^{At}}{e^{2AT}-e^{2At}}\right)^2 - \left(\frac{e^{-AT}-e^{-At}}{e^{-2AT}-e^{-2At}}\right)^2 - A(T-t)\right] \\ +\left(\frac{2i\omega\Sigma^2}{A\sigma\sqrt{T-t}}\right)\left[\begin{array}{l}\frac{\left(x(T)e^{AT}-x(t)e^{At}\right)\left(e^{AT}-e^{At}\right)}{(e^{2AT}-e^{2At})^2} - \\ \frac{\left(x(T)e^{-AT}-x(t)e^{-At}\right)\left(e^{-AT}-e^{-At}\right)}{(e^{-2AT}-e^{-2At})^2}\end{array}\right]\end{array}\right]\right\} \tag{34}$$

which when substituted in eqs. (20) & (21) shall give the values $N\left(\bar{d}_1\right)$ and $N\left(\bar{d}_2\right)$ respectively as:-

$$N\left(\bar{d}_1\right) = N\left\{\frac{\frac{\log\left(\frac{S}{E}\right)+\frac{1}{2}\sigma^2}{\sigma\sqrt{T-t}} - \frac{B\sqrt{T-t}}{A\sigma} - \frac{Y}{A\sigma\sqrt{T-t}}}{\left[1 - \frac{\Sigma^2 X}{A^3\sigma^2(T-t)}\right]^{\frac{1}{2}}}\right\} \tag{35}$$

and

$$N\left(\bar{d}_2\right) = N\left\{\frac{\frac{\log\left(\frac{S}{E}\right)-\frac{1}{2}\sigma^2}{\sigma\sqrt{T-t}} - \frac{B\sqrt{T-t}}{A\sigma} - \frac{Y}{A\sigma\sqrt{T-t}}}{\left[1 - \frac{\Sigma^2 X}{A^3\sigma^2(T-t)}\right]^{\frac{1}{2}}}\right\} \tag{36}$$

where
$X = \left(\frac{e^{AT}-e^{At}}{e^{2AT}-e^{2At}}\right)^2 - \left(\frac{e^{-AT}-e^{-At}}{e^{-2AT}-e^{-2At}}\right)^2 - A(T-t)$ and

$$Y = \frac{\left(x(T)e^{AT} - x(t)e^{At}\right)\left(e^{AT} - e^{At}\right)}{\left(e^{2AT} - e^{2At}\right)^2} - \frac{\left(x(T)e^{-AT} - x(t)e^{-At}\right)\left(e^{-AT} - e^{-At}\right)}{\left(e^{-2AT} - e^{-2At}\right)^2} \tag{37}$$

To evaluate I_2 , we substitute $\omega = i\sigma\sqrt{T-t}$ in eq. (34) to get

$$I_2 = \exp\left\{\frac{B(T-t)}{A} - \frac{1}{2\sum^2}\left[\left(\frac{\Sigma^4}{A^3}\right)X - \left(\frac{2\Sigma^2}{A}\right)Y\right]\right\} \tag{38}$$

The closed form solution for the Black Scholes pricing problem with stochastic return on the “hedge portfolio” can now be obtained by substituting the above averages in eq. (8).

3. Conclusion

In this paper, we have obtained closed form expressions for the price of a European call option by modifying the Black Scholes formulation to accommodate a stochastic return process for the "hedge portfolio" returns. We have modelled this return process on the basis of the Vasicek model for the short-term interest rates. The need for this extension of the Black Scholes model is manifold. Firstly, the construction of the "hedge portfolio" in the Black Scholes theory implies that the fluctuations in the price of the derivative and that of the underlying exactly and immediately cancel each other when combined in a certain proportion viz. one unit of the derivative with a short sale of $\frac{\partial C}{\partial S}$ units of the underlying so that the "hedge portfolio" is devoid of any impact of such fluctuations. This mandates an infinitely fast reaction mechanism of the underlying market dynamics whereby any movement in the price of one asset is instantaneously annulled by reactionary response in the other asset constituting the "hedge portfolio". This is, obviously strongly unrealistic and there may subsist brief periods or aberrations when the no arbitrage condition may cease to hold and hence, returns on the "hedge portfolio" may be different from the risk free rate. One way of attending to this anomaly is to model the returns on the "hedge portfolio" as a stochastic process as has been done in this study. The parameters defining the process can be obtained through an empirical study of the market dynamics. Another important justification for adopting a stochastic framework for the "hedge portfolio" return process is that the "hedge portfolio" by its very construction, envisages the neutralization of the fluctuations of the two assets inter se i.e. it assumes a perfect correlation between the two assets. In other words, the "hedge portfolio" may be construed as an isolated system that is such that insofar as factors that influence one component of the system, the same factors influence the other component to an equivalent extent and, at the same time, other factors do not impact the system at all. This is another anomaly that distorts the Black Scholes model. The fact is that while the "hedge portfolio" of the Black Scholes model is immunized against price fluctuations of the underlying and its derivative through mutual interaction, other market factors that would impact the portfolio as a whole are not accounted for e.g. factors affecting bond yields and interest rates etc. Consequently, to assume that the "hedge portfolio" is completely risk free is another aberration – it is risk free only to the extent of risk that emanates from factors that impact the underlying and the derivative in like manner and is still subject to risk and uncertainties that originate from factors that either do not effect the underlying and the derivative to equivalent extent or impact the portfolio as a unit entity. Hence, again, it becomes necessary to model the return on the "hedge portfolio" as some short-term interest rate model as has been done here.

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Co-existence of Regular and Chaotic Motions in the Gaussian Map

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Abstract: In this communication, the Gaussisn map, which has drawn less attention in the past as compare to other one-dimensional maps, has been explored. Particularly, the dynamical behavior of the Gaussian map and the presence of co-existing attractors (which is a rare phenomenon in one-dimensional maps) in the complete parameter space have been investigated. We also suggest a possible geometrical reason for the emergence of co-existing attractors at a particular set of system parameters, which works for all one-dimensional maps. The regions of parameter space, where regular and chaotic motions co-exist, have also been identified.

Keywords: Gaussian map, Co-existing Attractors, Period-doubling, Chaos
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The study of time-evolution of various systems in physics, chemistry, biology or engineering is one of the most frequent and commonly investigated tasks of the natural sciences. It is generally done by using model equations that describe the evolution in time of the system state in a state space. This state space can be characterized by a set of variables which are either continuous or discrete. Also the time may be continuous or discrete integer-valued variable. Generally, all these different possible cases of evolution equations are summarized under the term dynamical system [1]. If the time is considered as a continuous variable then the system model equations may be described by a set of differential equations termed as continuous dynamical system or flows. On the other hand the time can be treated as discrete integer-valued variable; such model equations are described by a set of difference equations called discrete dynamical system or maps. Discreteness of time variable may mean that it is sufficient to measure certain physical variable after a finite interval of time rather than on a continuous basis. In some scien-

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tific cases it is natural to represent time as a discrete variable as in digital electronics, impulsively driven systems etc. One motivation behind the study of such maps is their origin in the description of intersection of state space trajectories with Poincare sections.

A famous prototype example of a one-dimensional map is the logistic map ($x_{n+1} = \lambda x_n(1 - x_n)$, an idealized model for the yearly variation of the population of animals), which has played a key role in the development of chaos theory [2, 3]. It can be considered the simplest example of one-dimensional map, which exhibits wide spectrum of dynamical behaviour including the chaotic motion.

In this communication, we intend to explore another famous one-dimensional map- the Gaussian map, which has drawn less attention due to its similarity with the logistic map. However, Gaussian map exhibits some features (such as co-existing attractors, reverse period doubling etc), which make it distinguishable from logistic map [4]. We particularly analyze the dynamical behaviour of the Gaussian map in the parameter space and investigate the presence of co-existing attractors. We also try to suggest a possible explanation for the presence of co-existing attractors by using geometrical means. The Gaussian map under the consideration is based on the Gaussian function & characterized by two parameters b and c (rather than a single parameter as in the case of logistic map) as follows:

$$x_{n+1} = e^{-bx_n^2} + c$$

It is expected that the behaviour of Gaussian map is similar to the behaviour of logistic map as the Gaussian map function also exhibits a one hump shape similar to the logistic function. There are two parameters (b and c) present in the Gaussian map, so the analysis of the behaviour of its long term iterates becomes little complicated than in the case of logistic map.

In Figure 1, we have plotted the Gaussian map function for different sets of parameters b and c. It is clear that the map function is symmetric about $x = 0$, and has maximum value (always at $x = 0$) equals to $c + 1$. For large values of $|x|$, the function approaches to the minimum value equals to c, however parameter b decides the width of the map function. Now we do the stability analysis of the Gaussian map for some fixed value of parameter b (say $b = 7.5$). In Figure 2, we have plotted the Gaussian map function for $b = 7.5$ and different values of parameter c. We observe that for large negative values of parameter c (c = *-1.0, -0.85, -0.7*), there exist three fixed points (which are evident from the intersection of line $F(x) = x$ and Gaussian map function) for the Gaussian map, however for other values of c (c = *-0.55,-0.40,-0.25),* only one fixed point exists.

These fixed points can be calculated by solving the nonlinear algebraic equation $x = e^{-bx^2} + c$. Since it is a transcendental equation, the analytical solution is not possible. We have solved this equation using the iterative method [5] and the corresponding solution is depicted in Figure 3(a) for $b = 7.5$. Now we analyze the stability of these fixed points. The best way to analyze the stability of fixed points (when no analytical solution for the fixed points exists) is to draw the bifurcation diagram (A plot illustrating the qualitative changes in the dynamical behaviour of the system as a function of system parameter). In Figure 3(b), we have shown such a bifurcation diagram for the Gaussian map by plotting its long term iterates for several values of initial conditions and same values of parameters as in Figure 3(a). The corresponding Lyapunov exponent (which is a quantitative measure of chaos) curve has been shown in Figure 3(c).

The most important feature we observe from Figure 3 (a), is that in a certain range of parameter c ($-1.03 < c < -0.65$), three fixed points exist (one is positive and two are negative), however for the other values of c only one fixed point exists (which is positive for $c \geq -0.65$ and negative for $c \leq -1.03$(not shown here)). From the bifurcation diagram shown in Figure 3(b), we may infer that for $-1.03 < c < -0.65$ the fixed point having large negative value is stable however the other negative fixed point is unstable. On the other hand the positive fixed point is stable for $-1.03 < c < -0.897$, then it becomes unstable and bifurcates into two stable fixed points, and further period doubling continues leading to chaos, periodic windows, reverse period doubling etc. for $-0.897 < c < 0.40$ (i.e 2-4-8-16- . . . . chaos -periodic windows -chaos-3-6-12. . .chaos. . . .12-6-3-chaos-periodic windows-chaos-. . . -16-8-4-2). Beyond $c = 0.40$, the positive fixed point again becomes stable and remains so for all values of $c > 0.40$. So for $-1.03 < c < -0.65$, we observe two stable attractors (each corresponds to different set of initial condition) in the bifurcation diagram, one is a period-one attractor however the other is periodic or chaotic depending on the value of parameter c. Two different curves of Lyapunov exponents for $-1.03 < c < -0.65$ in Figure 3(c) confirm the co-existence of two stable attractors.

In Figure 4, we have shown iterates of the Gaussian map function using cobweb diagrams [6] for a fixed value of parameter b (i.e. $b = 7.5$) and different values of c. In Frame (a), we have shown iterates of the Gaussian map for $c = -0.95$ and two different values of initial condition, both converge to different fixed point attractors (infect these are the fixed points of Gaussian map function), hence two stable attractors co-exist. In Frames (b), (c), (d) & (e) similar situations have been depicted for c= -0.85, -0.79, -0.70, -0.66 respectively, where period-1 attractor (which is one of the fixed point of Gaussian map) co-exists with period-2, period-4, chaotic & period-3 attractors respectively. In all the cases depicted in Frames (a)-(e), we note a common feature that the Gaussian map function possesses three fixed points or in other words it makes one positive hump and one negative hump with the line $F(x) = x$ (i.e. the 45^o line). However in Frame (f), we have shown a cobweb diagram for c=-0.6. In this case Gaussian map possesses only one fixed point (unstable) and only one stable attractor (chaotic) exists for all the initial values.

From Figures 1-4, we may conclude that the emergence of co-existing attractors at a particular set of parameters can be understood as a consequence of presence of three fixed points or in other words co-existing attractors are observed only when the map function makes one positive and one negative hump with the line $F(x) = x$ (i.e. the 45^o line). We have carried out above analysis for a fixed value of parameter b (i.e. $b = 7.5$) and by varying the parameter c. Now we analyze the dynamical behaviour of the Gaussian map and co-existence of attractors in the complete parameter space (a, b).

In Figure 5, we have identified different regions of parameter space (a, b) where the dynamics of Gaussian map is chaotic and regular (black and white shades correspond to chaotic and periodic attractors respectively). The results shown in Figure 5 are based on the extensive numerical calculation of Lyapunov exponent [7] by iterating the Gaussian map 1000 times (after neglecting initial transient behaviour up to 400 iterations) at 2×10^6different points of the parameter space defined by $0 \leq b \leq 20$ & $-1.0 \leq c \leq 0$. Chaotic situation has been recorded, whenever the Lyapunov exponent becomes greater than 1×10^{-4}. The above calculation has been repeated for several set of initial conditions to include all the co-existing attractors.

In Figure 6(a), we have shown the region of parameter space (dark grey shade), where the Gaussian map possesses three fixed points, however in Figure 6(b) the region of parameter space (light grey region), where co-existing attractors have been observed, is shown. In the light grey shaded region a period-1 attractor co-exists with some other attractor (chaotic, period-1, period-2, period -4. . . . . . , period-3, etc). It is interesting to note that the region of the parameter space, where co-existing attractors exist is a subset of the region shown in Figure 6(a), which conforms our earlier conclusion that the emergence of co-existing attractors at a particular set of parameters can be understood as a consequence of presence of three fixed points. However the converse is not always true as it is clear from Figures 6 (a) & (b) that there is a part of parameter space, where three fixed points exist but co-existing attractors are not present.

Finally, in Figure 7, we have depicted the region of parameter space (black shade), where regular and chaotic motions coexist for different sets of initial condition.

In conclusion, we have analyzed the dynamical behaviour of the Gaussian map and observed that it exhibits a wide spectrum of dynamical behaviours such as regular and chaotic motions, period doubling route to chaos, reverse period doubling, co-existing attractors (which is a rare phenomenon in one-dimensional maps) etc. We also draw an important conclusion that co-existing attractors are observed only when the map function makes one positive and one negative hump with the line $F(x) = x$ (presence of three fixed points). The conclusion made above is also true for a recent study of q-deformed logistic map [8] where co-existing attractors have been observed. Hence it may work for all one-dimensional maps.

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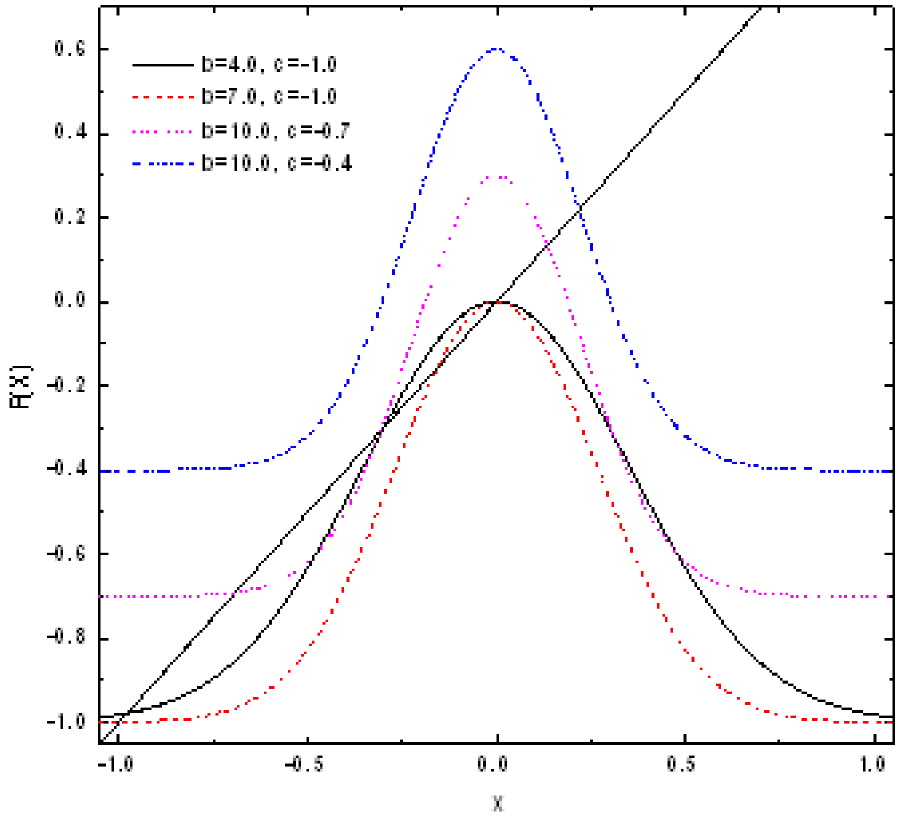

Fig. 1 Gaussian map function ($F(x) = \exp(-bx^2) + c$) for different values of parameters b and c. The diagonal line $F(x) = x$ is also shown.

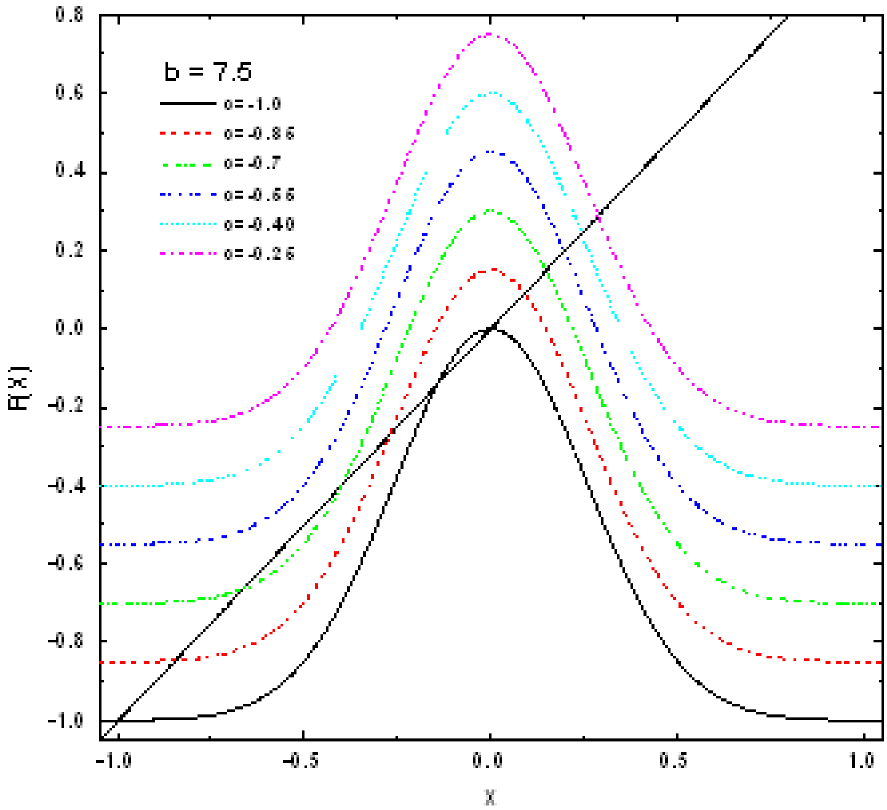

Fig. 2 Gaussian map function ($F(x) = \exp(-bx^2) + c$) for fixed value of parameter b ($b = 7.5$) and different values of parameter c. The diagonal line $F(x) = x$ is also shown.

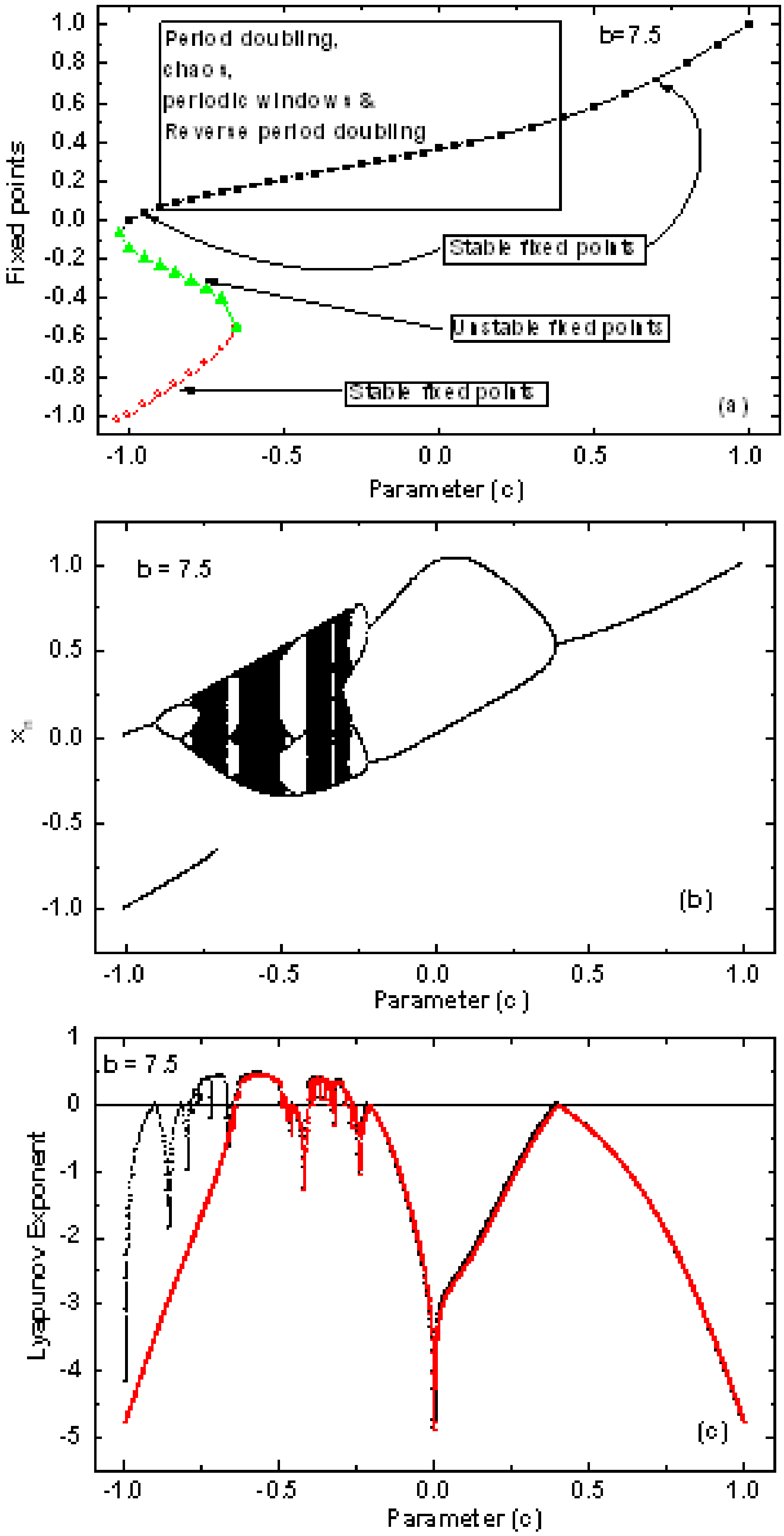

Fig. 3 (a) Fixed points of the Gaussian map as a function of parameter c for fixed value of parameter b ($b = 7.5$), (b) Bifurcation diagram showing the stable attractors, period doubling route to chaos and reverse period doubling as a function of parameter c for fixed value of parameter b ($b = 7.5$) and several values of initial conditions, (c) The Lyapunov exponent corresponding to the situation shown in frame (b).

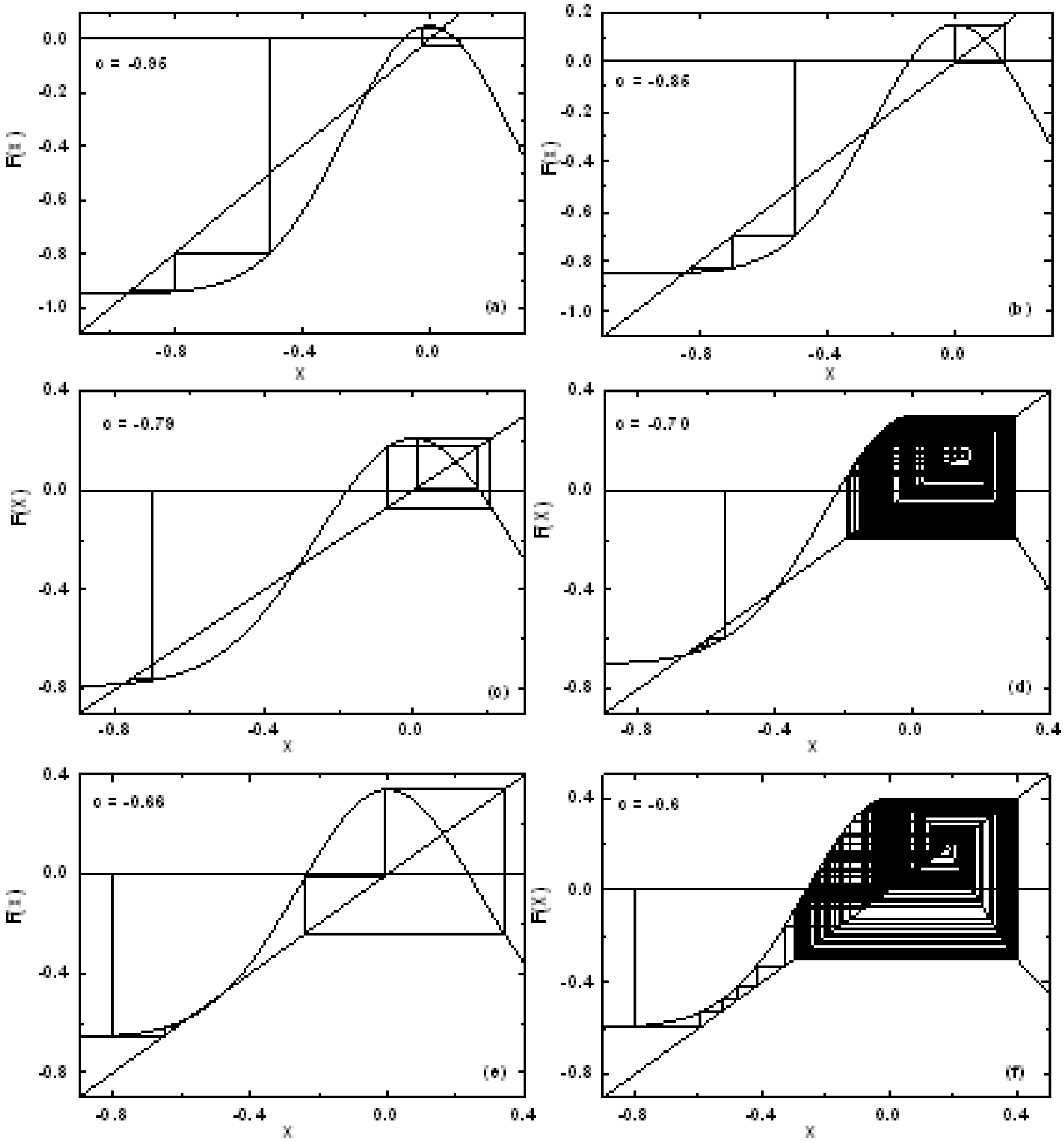

Fig. 4 Cobweb diagrams showing iterates of the Gaussian map for $b = 7.5$ and different values of parameter c.

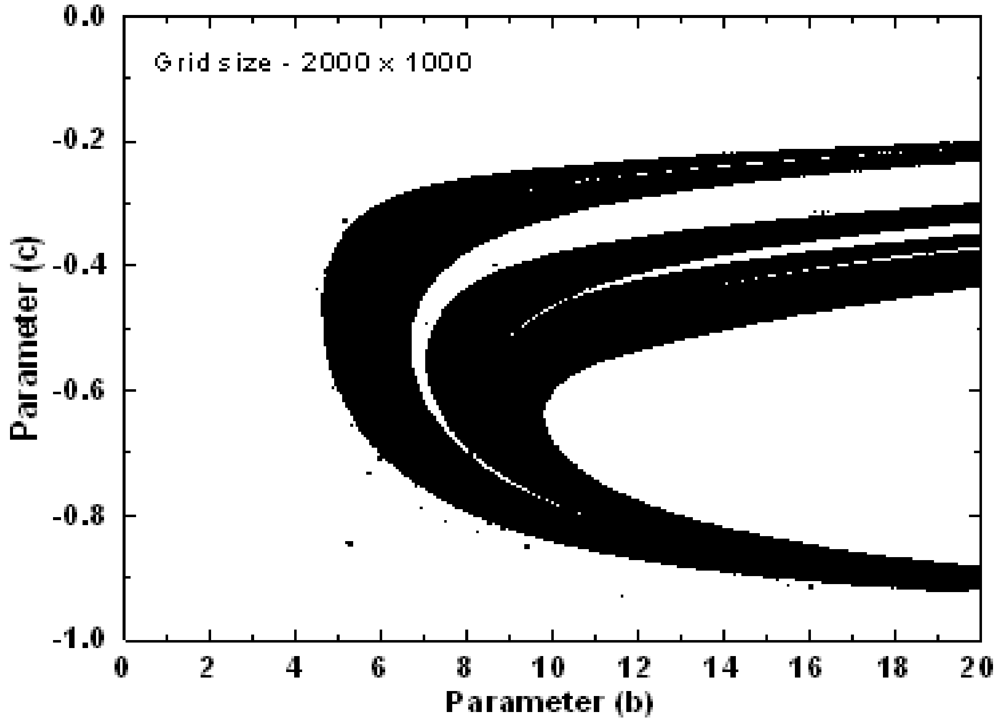

Fig. 5 Parameter space (a, b) of the Gaussian map showing the regions, where chaotic (black shade) and regular (white shade) motions appear.

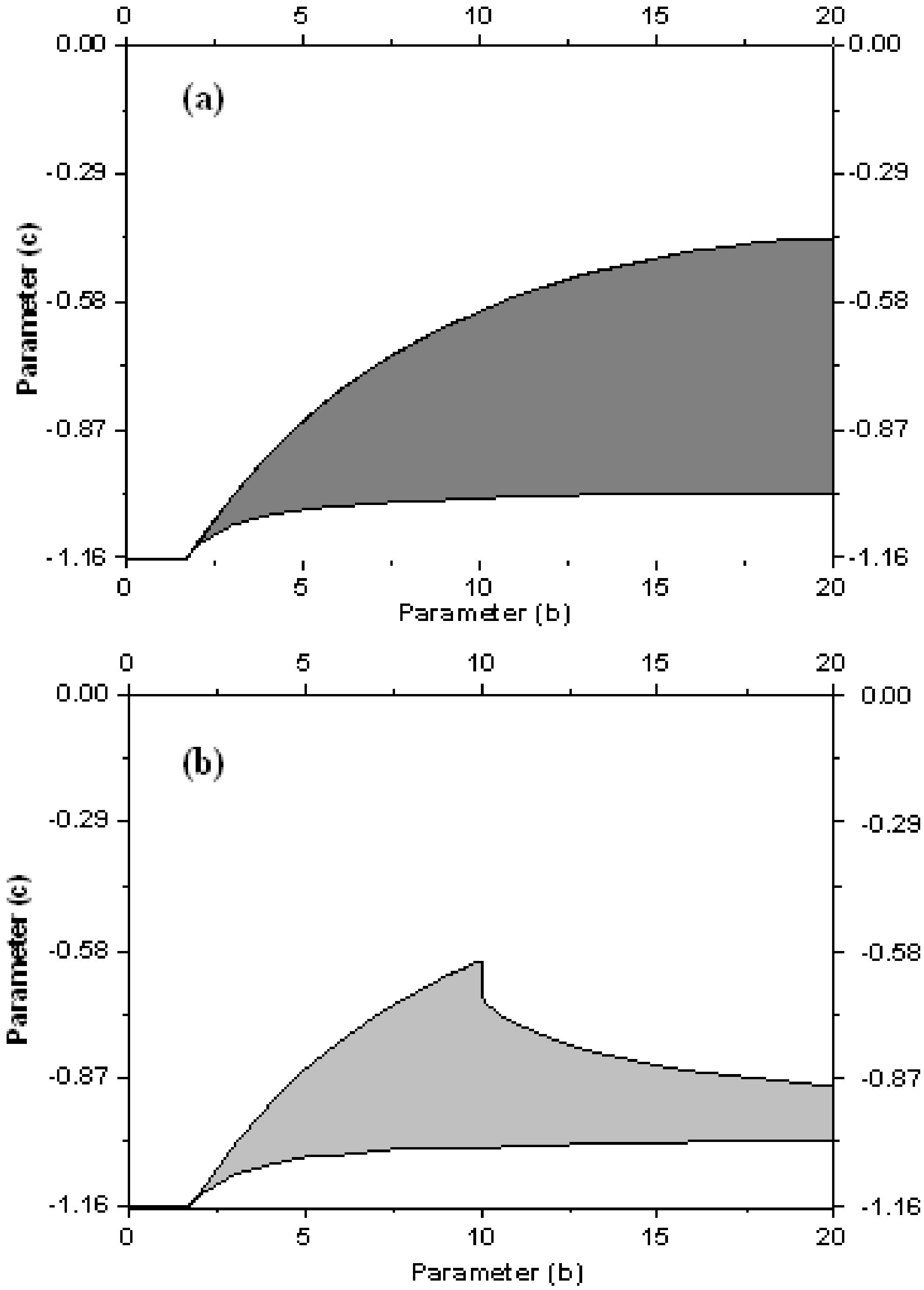

Fig. 6 (a) Parameter space (a, b) showing the region (dark grey shade), where the Gaussian map possesses three fixed points, (b) Parameter space (a, b) showing the region (light grey shade), where the Gaussian map exhibits co-existing attractors.

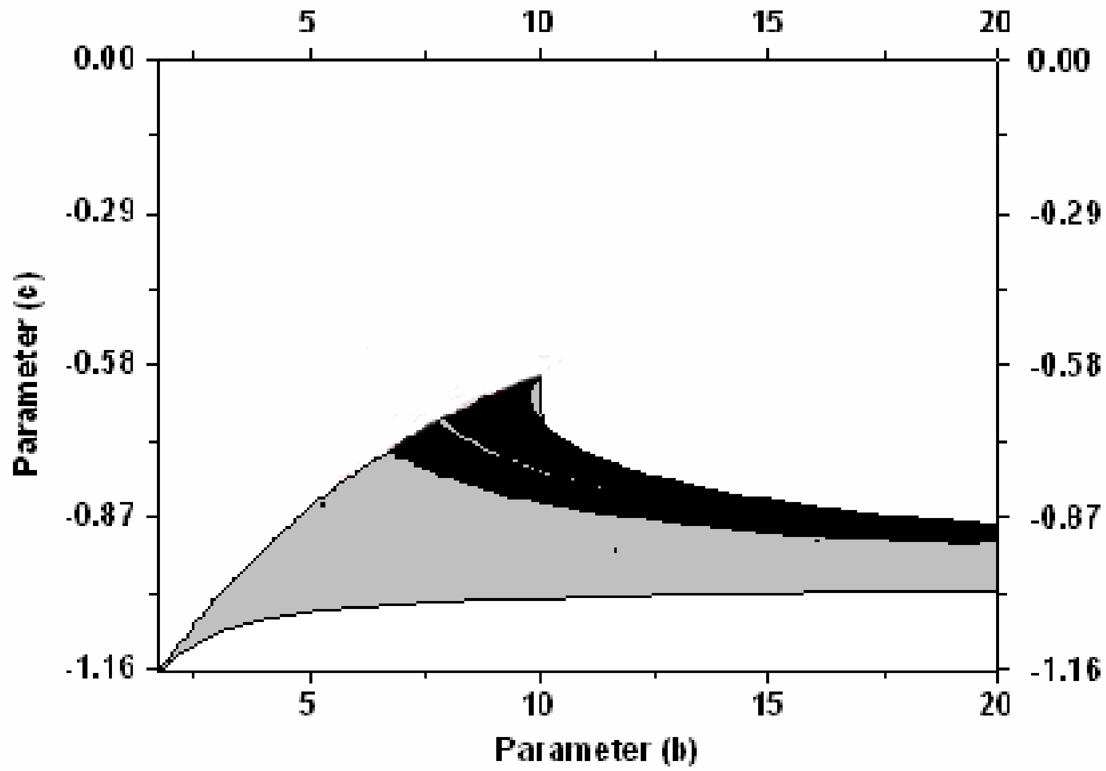

Fig. 7 Parameter space (a, b) showing the region (black shade), where chaotic and regular (period-1) motions co-exist for different sets of initial conditions.

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Does the Formation of Temperature Dependence of Axion Walls Help Delineate a Regime Where the Wheeler De Witt Equation Holds?

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Abstract: We examine from first principles the implications of the 5^{th} Randall Sundrum Brane world dimension in terms of setting initial conditions for chaotic inflationary physics. Our model pre supposes that the inflationary potential pioneered by Guth is equivalent in magnitude in its initial inflationary state to the effective potential presented in the Randall -Sundrum model We also consider an axion contribution to chaotic inflation (which may have a temperature dependence) which partly fades out up to the point of chaotic inflation being matched to a Randall – Sundrum effective potential. If we reject an explicit axion mass drop off to infinitesimal values at high temperatures, we may use the Bogomolnyi inequality to re scale and re set initial conditions for the chaotic inflationary potential. Then the Randall-Sundrum brane world effective potential delineates the end of the dominant role of di quarks, and the beginning of inflation. It also leads to a new region where the Wheeler De Witt equation holds.

Keywords: Field Theory, Early Universe, Randall-Sundrum Theory, Tunneling
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1. Introduction

This investigation is attempting to show that the fifth dimension postulated by Randall-Sundrum theory helps give us an action integral which leads to a minimum physical potential we can use to good effect in determining initial conditions for the onset of inflation. The 5^{th} dimension of the Randall-Sundrum brane world is of the genre, for

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$-\pi \leq \theta \leq \pi$

$$x_5 \equiv R \cdot \theta \tag{1}$$

This lead to an additional embedding structure for typical GR fields, assuming as one may write up a scalar potential 'field 'with $\phi_0(x)$ real valued, and the rest of it complex valued as [1]:

$$\phi(x^\mu, \theta) = \frac{1}{\sqrt{2 \cdot \pi \cdot R}} \cdot \left\{ \phi_0(x) + \sum_{n=1}^{\infty} [\phi_n(x) \cdot \exp(i \cdot n \cdot \theta) + C.C.] \right\} \tag{2}$$

This scalar field makes its way to an action integral structure which will be discussed later on, which Sundrum used to forming an effective potential. Our claim in this analysis can also be used as a way of either embedding a Bogomolyni inequality, perhaps up to five dimensions [2], or a straight forward reduction in axion mass due to a rise in temperature [3] helped reduced effective potential in this structure , with the magnitude of the Sundrum potential forming an initial condition for the second potential of the following phase transition .Note that we are referring to a different form of the scalar potential , which we will call $\tilde{\phi}$, which has the following dynamic [4].

$$\begin{aligned} &\tilde{V}_1 \to \tilde{V}_2 \\ &\tilde{\phi}\,(increase) \leq 2 \cdot \pi \to \tilde{\phi}\,(decrease) \leq 2 \cdot \pi \\ &t \leq t_P \to t \geq t_P + \delta \cdot t \end{aligned} \tag{3a}$$

The potentials $\tilde{V}_1$, and $\tilde{V}_2$ were described in terms of **S-S'** di quark pairs nucleating and then contributing to a chaotic inflationary scalar potential system. Here, $m^4 \approx (1/100) \cdot M_P^4$

$$\tilde{V}_1(\phi) = \frac{M_P^4}{2} \cdot \left(1 - \cos\left(\tilde{\phi}\right)\right) + \frac{m^4}{2} \cdot \left(\tilde{\phi} - \phi^*\right)^2 \tag{3b}$$

$$\tilde{V}_2(\phi) \propto \frac{1}{2} \cdot \left(\tilde{\phi} - \phi_C\right)^2 \tag{3c}$$

We should keep in mind that ϕ_C in Eqn 3a is an equilibrium value of a true vacuum minimum of Eqn. 3 a after tunneling. In the potential system given as Eqn, (3b) we see a steadily rising scalar field value which is consistent with the physics of Figure 1 . In the potential system given by Eqn. (3c) we see a reduction of the 'height of a scalar field which is consistent with the chaotic inflationary potential overshoot phenomena We should note that ϕ^* in Eq (3a)is a measure of the onset of quantum fluctuations. **Appendix I** is a discussion of Axion potentials which we claim is part of the contribution of the potential given in Eqn. (3b) Note that the tilt to the potential given in Eqn. (3b) is due to a quantum fluctuation. As explained by Guth for quadratic potentials [5],

$$\phi^* \equiv \left(\frac{3}{16 \cdot \pi}\right)^{\frac{1}{4}} \cdot \frac{M_P^{3/2}}{m^{\frac{1}{2}}} \cdot M_P \to \left(\frac{3}{16 \cdot \pi}\right)^{\frac{1}{4}} \cdot \frac{1}{m^{\frac{1}{2}}} \tag{3d}$$

This in the context of the fluctuations having an upper bound of

$$\tilde{\tilde{\phi}} > \sqrt{\frac{60}{2 \cdot \pi}} M_P \approx 3.1 M_P \equiv 3.1 \tag{3e}$$

Here,$\tilde{\tilde{\phi}} > \phi_C$. Also, the fluctuations Guth had in mind were modeled via [6]

$$\tilde{\phi} \equiv \tilde{\tilde{\phi}}^{-} \frac{m}{\sqrt{12 \cdot \pi \cdot G}} \cdot t \tag{3f}$$

In the potential system given by Eqn. (3c) we see a reduction of the 'height' or magnitude of a scalar field which is consistent with the chaotic inflationary potential overshoot phenomena mentioned just above. This leads us to use the Randal-Sundrum effective potential [1], in tandem with tying in baryogenesis [7] to the formation of chaotic inflation initial conditions for Eqn. (3c) , with the Randall-Sundrum brane world effective potential delineating the end of the dominant role of di quarks, due to baryogenesis, and the beginning of inflation. The role of the Bogomolnyi inequality is to introduce, from a topoplogical domain wall stand point a mechanism for the introduction of baryogenesis in early universe models, and the combination of that analysis, plus matching conditions with the Randal-Sundrum effective potential sets us up for chaotic inflation.

2. How to Form the Randall-Sundrum Effective Potential

The consequences of the fifth dimension mentioned in Eqn. (1) above show up in a simple warped compactification involving two branes, i.e. a Planck world brane, and an IR brane. This construction with the physics of this 5 dimensional system allow for solving the hierarchy problem of particle physics, and in addition permits us to investigate the following five dimensional action integral [1].

$$S_5 = \int d^4x \cdot \int_{-\pi}^{\pi} d\theta \cdot R \cdot \left\{ \frac{1}{2} \cdot (\partial_M \phi)^2 - \frac{m_5^2}{2} \cdot \phi^2 - K \cdot \phi \cdot [\delta(x_5) + \delta(x_5 - \pi \cdot R)] \right\} \tag{4}$$

This integral, will lead to the following equation to solve

$$-\partial_\mu \partial^\mu \phi + \frac{\partial_\theta^2}{R^2} \phi - m_5^2 \phi = K \cdot \frac{\delta(\theta)}{R} + K \cdot \frac{\delta(\theta - \pi)}{R} \tag{5}$$

Here, what is called m_5^2 can be linked to Kalusa Klein "excitations" [1] via (for $n \quad > 0$)

$$m_n^2 \equiv \frac{n^2}{R^2} + m_5^2 \tag{6a}$$

This uses [8] (assuming l is the curvature radius of AdS$_5$)

$$m_5^3 \equiv \frac{M_P^2}{l} \tag{6b}$$

This is for a compactification scale, for $m_5 << \frac{1}{R}$, and after an ansatz of the following is used:

$$\phi \equiv A \cdot [\exp(m_5 \cdot R \cdot |\theta|) + \exp(m_5 \cdot R \cdot (\pi - |\theta|))] \tag{7}$$

We then obtain after a non trivial vacuum averaging

$$\langle \phi(x, \theta) \rangle = \Phi(\theta) \tag{8}$$

$$S_5 = -\int d^4x \cdot V_{eff}(R_{phys}(x)) \tag{9}$$

This is leading to an initial formulation of

$$V_{eff}(R_{phys}(x)) = \frac{K^2}{2 \cdot m_5} \cdot \frac{1 + \exp(m_5 \cdot \pi \cdot R_{phys}(x))}{1 - \exp(m_5 \cdot \pi \cdot R_{phys}(x))} \tag{10}$$

Now , if one is looking at an addition of a 2^{nd} scalar term of opposite sign, but of equal magnitude [1]

$$S_5 = -\int d^4x \cdot V_{eff}(R_{phys}(x)) \rightarrow -\int d^4x \cdot \tilde{V}_{eff}(R_{phys}(x)) \tag{11}$$

This is for when we set up an effective Randall – Sundrum potential looking like [1]

$$\tilde{V}_{eff}(R_{phys}(x)) = \frac{K^2}{2 \cdot m_5} \cdot \frac{1 + \exp(m_5 \cdot \pi \cdot R_{phys}(x))}{1 - \exp(m_5 \cdot \pi \cdot R_{phys}(x))} + \frac{\tilde{K}^2}{2 \cdot \tilde{m}_5} \cdot \frac{1 - \exp(\tilde{m}_5 \cdot \pi \cdot R_{phys}(x))}{1 + \exp(\tilde{m}_5 \cdot \pi \cdot R_{phys}(x))} \tag{12}$$

This above system has a meta stable vacuum for a given special value of $R_{phys}(x)$ We will from now on use this as a 'minimum' to compare a similar action integral for the potential system given by Eqn. (3a) above. Note that this is done, while assuming that

3. How to Compare the Randall-Sundrum Effective Potential Minimum With an Effective Potential Minimum Involving the Potential of EQN. (3a) Above

We are forced to consider two possible routes to the collapse of a complex potential system to the chaotic inflationary model promoted by Guth [5].

The first such model involves a simple reduction of the axion wall potential [9] as given by, especially when N = 1

$$V(a) = m_a^2 \cdot (f_{PQ}/N)^2 \cdot (1 - \cos[a/(f_{PQ}/N])) \tag{13}$$

The simplest way to deal with Eqn.(13) is to set $m_a^2(T) \underset{T \to \infty}{\longrightarrow} \varepsilon^+$, when Kolb [9] writes

$$m_{axion}(T) \cong .1 \cdot m_{axion}(T = 0) \cdot (\Lambda_{QCD}/T)^{3.7} \tag{14}$$

i.e. to declare that the axion 'mass' vanishes, and to let this drop off in value give a simple truncated version of chaotic inflationary potentials along the lines given by a transition from Eqn (3b) to Eqn . (3c) We should note that Λ_{QCD} is the enormous value of the cosmological constant which is 10^{120} larger than what it is observed to be today [10, 11, 12], and for now we are side steeping the question of if or not the negative valued Randall-Sundrum cosmological constant [8].

$$\Lambda_5 = -\frac{6}{l^2} \tag{15}$$

has a bearing on this situation. Not to mention the problems inherent in several proposed fixes to the cosmological constant problem [13].

Now if we want an equivalent explanation, which may involve baryogenesis, we need to look at the component behavior of each of the terms in Eqn. (13) without assuming $m_a^2(T) \underset{T\to\infty}{\longrightarrow} \varepsilon^+$. Then, we need to re define several of the variables presented above. Now, in the typical theory presented by

$$\frac{M_P^4}{2} \cdot \left(1 - \cos\left(\tilde{\phi}\right)\right) \propto m_a^2 \cdot (f_{PQ}/N)^2 \cdot (1 - \cos\left[a/(f_{PQ}/N\right]) \tag{16}$$

We then have to present a varying in magnitude value for the 'scalar'$\tilde{\phi}$ involving ultimately the Bogolmolnyi inequality. I have done several of these for condensed matter current problems, but for our cosmology situation, we first have to work with

$$[a/(f_{PQ}/N)] \approx \tilde{\phi} \tag{17}$$

There has been credible work with instantons in higher dimensions, starting with Hawkings 1999 article [14] This, however, addresses a way of linking an instanton structure with baryogenesis, dark energy, and issues of how Randall-Sundrum brane structrure can be used to formation of initial conditions of inflationary cosmology.

Clarifying what can be done with an instanton style quantum nucleation in multiple dimensions [15] may help us with more acceptable models [16, 17] as to estimating, roughly, a quantum value for the cosmological constant, as an improvement in recent calculations. I refer interested readers to **Appendix II** on this matter, but for now will restrict this discussion to a qualitative derivation done for condensed matter currents for motivational purposes only. Start with a wave functional

$$\Psi \propto \exp(-\int d^3x_{space} d\tau_{Euclidian} L_E) \equiv \exp\left(-\int d^4x \cdot L_E\right) \tag{18a}$$

$$L_E \geq |Q| + \frac{1}{2} \cdot \left(\tilde{\phi} - \phi_0\right)^2 \{\} \underset{Q\longrightarrow 0}{\longrightarrow} \frac{1}{2} \cdot \left(\tilde{\phi} - \phi_0\right)^2 \cdot \{\} \tag{18b}$$

Where

$$\{\} = 2 \cdot \Delta \cdot E_{gap} \tag{18c}$$

This leads, if done correctly to the quadratic sort of potential contribution as given by [18] $\psi_\mu(\phi) \equiv \psi_\mu \cdot \exp(\alpha_\mu \cdot \phi^2)$, At the same time it raises the question of if or not when there is a change from the 1^{st} to the 2^{nd} potential system,

This is for his chaotic inflation model using his potential; I call the 2^{nd} potential

Let us now view a toy problem involving use of a S-S' pair which we may write as [19]

$$\tilde{\phi} \approx \pi \cdot [\tanh b\,(x - x_a) + \tanh b\,(x_b - x)] \tag{19}$$

This is for a di quark pair along the lines given when looking at the first potential system, which is a take off upon Zhitinisky's color super conductor model [20]

4. The Comparison this Sort of Model Building Leaves for Investigators

Now for the question the paper is raising, Can we realistically state the following for initial conditions of a nucleating universe ? If so, then what are the consequences ?

$$S_5 = -\int d^4x \cdot \tilde{V}_{eff}\,(R_{phys}\,(x)) \propto (-\int d^3x_{space} d\tau_{Euclidian} L_E) \equiv \left(-\int d^4x \cdot L_E\right) \tag{20}$$

The right hand side of Eqn (20) can be stated as having

$$L_E \geq \frac{1}{2} \cdot \left(\tilde{\phi} - \phi_0\right)^2 \cdot \{\}\,. \tag{21}$$

We can insist that this ΔE_{gap}between a false and a true vacuum minimum [21], that

$$\{\} \equiv 2 \cdot \Delta E_{gap} \tag{22}$$

So, this leads to the following question. Does a reduction of axion wall mass for the first potential system given in Eqn.(3b) being transformed to Eqn.(3c) above give us consistent physics, due to temperature dependence in axion 'mass' , or should we instead look at what can be done with S-S' instanton physics and the Bogolmyi inequality [22], in order to perhaps take into account Baryogenesis ? Also, can this shed light upon the Wheeler De Witts equations [23] modification by Ashtekar [24] in early universe quantum bounce conditions ?

Finally, does this process of baryogenesis,if it occurs lend then to the regime where there is a bridge between classical applications of the Wheeler De Witt equation to the quantum bounce condition raised by Ashtekar [24] ?

5. Tie in with Di Quark Potential Systems, and the Classical Wheeler De-Witt Equation

Abbay Ashtekar's quantum bounce [24] gives a discretized version of the Wheeler De Witt equation. Let us first review classical De Witt theory which incidently ties in with inflationary n= 2 scalar potential field cosmology.This will be useful in analyzing consequences of the wave functional so formed in Eqn. (18a) and suggest quantum bounce analogies we will comment upon later.

In the common versions of Wheeler De Witt theory a potential system using a scale radius $R(t)$, with R_0 as a classical turning point value [23]

$$U(R) = \left(\frac{3 \cdot \pi \cdot c^3 \cdot R_0}{2 \cdot G}\right)^2 \cdot \left[\left(\frac{R}{R_0}\right)^2 - \left(\frac{R}{R_0}\right)^4\right] \tag{23a}$$

Here we have that

$$R_0^{\sim} c \cdot t_0 \equiv l_P \equiv c \cdot \sqrt{\frac{3}{\Lambda}} \sim 7.44 \times 10^{-36} meters \tag{23b}$$

As well as

$$\sqrt{\frac{3}{\Lambda}} = t_p \sim 2.48 \times 10^{-44} sec \tag{24}$$

Now, Alfredo B. Henriques [16] presents a way in which one can obtain a Wheeler De Witt equation based upon

$$\tilde{\hat{H}} \cdot \Psi\left(\phi\right) = \left[\frac{1}{2} \cdot \left(A_\mu \cdot p_\phi^2 + B_\mu \cdot m^2 \cdot \phi^2\right) \cdot \Psi\left(\phi\right)\right] \tag{25}$$

Using a momentum operator as give by

$$\hat{p}_\iota = -i \cdot \hbar \cdot \frac{\partial}{\partial \cdot \phi} \tag{26}$$

This is assuming a real scalar field ϕ as well as a 'scalar mass 'm 'based upon a derivation originally given by Thieumann [25]. The above equation as given by Theumann, and secondarily by Henriques [16] lead directly to considering the real scalar field ϕ as leading to a prototype wave functional for the ϕ^2 potential term as given by

$$\psi_\mu\left(\phi\right) \equiv \psi_\mu \cdot \exp(\alpha_\mu \cdot \phi^2) \tag{27a}$$

As well as an energy term

$$E_\mu = \sqrt{A_\mu \cdot B_\mu} \cdot m \cdot \hbar \tag{27b}$$

$$\alpha_\mu = \sqrt{B_\mu / A_\mu} \cdot m \cdot \hbar \tag{27c}$$

This is for a 'cosmic' Schrodinger equation as given by

$$\tilde{\hat{H}} \cdot \psi_\mu\left(\phi\right) = E_\mu\left(\phi\right) \tag{27d}$$

This has

$$A_\mu = \frac{4 \cdot m_{pl}}{9 \cdot l_{pl}^9} \cdot \left(V_{\mu+\mu_0}^{1/2} - V_{\mu-\mu_0}^{1/2}\right)^6 \tag{27e}$$

And

$$B_\mu = \frac{m_{pl}}{l_{pl}^3} \cdot \left(V_\mu\right) \tag{27f}$$

Here V_μ is the eignvalue of a so called volume operator [6], and the interested readers are urged to consult with the cited paper to go into the details of this, while at the time noting m_{pl} is for Planck mass, and l_{pl} is for Planck length, and keep in mid that the main point made above, is that a potential operator based upon a quadratic term leads to a Gaussian wavefunctional with an exponential similarly dependent upon a quadratic ϕ^2exponent. We do approximate solitons via the evolution of Eqn. (27a) and Eqn. (27d) above, and so how we reconcile higher order potential terms in this approximation of wave functionals is extremely important.

Now Ashtekar in his longer arXIV article [26] make reference to a revision of this momentum operation along the lines of basis vectors $|\mu\rangle$by

$$\hat{p}_\iota \, |\mu\rangle = \frac{8 \cdot \pi \cdot \gamma \cdot l_{PL}^2}{6} \cdot \mu \, |\mu\rangle \tag{28a}$$

With the advent of this re definition of momentum we are seeing what Ashtekar works with as a sympletic structure with a revision of the differential equation assumed in Wheeler – De Witt theory to a form characterized by [26]

$$\frac{\partial^2}{\partial \phi^2} \cdot \Psi \equiv -\Theta \cdot \Psi \tag{28b}$$

Θ in this situation is such that

$$\Theta \neq \Theta\left(\phi\right) \tag{28c}$$

Also, and more importantly this Θ is a difference operator, allowing for a treatment of the scalar field as an 'emergent time' , or 'internal time' so that one can set up a wave functional built about a Gaussian wavefunctional defined via

$$\max \tilde{\Psi}\left(k\right) = \left.\tilde{\Psi}\left(k\right)\right|_{k \equiv k^*} \tag{28d}$$

This is for a crucial 'momentum' value

$$p_\phi^* = -\left(\sqrt{16 \cdot \pi \cdot G \cdot \hbar^2/3}\right) \cdot k^* \tag{28e}$$

And

$$\phi^* = -\sqrt{3/16 \cdot \pi G} \cdot \ln|\mu^*| + \phi_0 \tag{29}$$

Which leads to, for an initial point in 'trajectory space' given by the following relation $(\mu^*, \phi_0) =$ (initial degrees of freedom [dimensionless number] $\sim$'eignvalue of 'momentum', initial 'emergent time ')

So that if we consider eignfunctions of the De Witt (difference) operator, as contributing toward

$$e_k^s\left(\mu\right) = \left(1/\sqrt{2}\right) \cdot \left[e_k(\mu) + e_k(-\mu)\right] \tag{30a}$$

With each $e_k(\mu)$ an eignfunction of Eqn. (12a) above, with eignvalues of Eqn. (12a) above given by $\omega(k)$, we have a potentially numerically treatable early universe wave functional data set which can be written as

$$\Psi\left(\mu, \phi\right) = \int_{-\infty}^{\infty} dk \cdot \tilde{\Psi}\left(k\right) \cdot e_k^s\left(\mu\right) \cdot \exp\left[i\omega\left(k\right) \cdot \phi\right] \tag{30b}$$

This equation above has a 'symmetry' as seen in Figure 1 of Ashtekar's PRL article [6] about ϕ, reflecting upon a quantum bounce for a pre ceding universe prior to the 'big bang' contracting to the singularity and a 'rebirth ' as seen by a different 'branch of Eqn. (30b) emerging for a 'growing' set of values of ϕ..

6. Conclusion

We are presenting a question which may be of relevance to JDEM research. Namely if Ashtekar is correct in his quantum geometry [26], and the break down of early universe conditions not permitting the typical application of the Wheeler De Witt equation, then what do we have to verify it experimentally? The axion wall dependence so indicated above may provide an answer to that, and may be experimentally measurable via Kadotas pixel reconstructive scheme [27].

Furthermore, we also argue that the semi classical analysis of the initial potential system as given by Eqn (**3**) above and its subsequent collapse is de facto evidence for a phase transition to conditions allowing for dark energy to be created at the beginning of inflationary cosmology.. [28, 29].This builds upon an earlier paper done by Kolb in minimum conditions for reconstructing scalar potentials [30, 31, 32, 33].It also will necessitate reviewing other recent derivation bound to the cosmological constant in cosmology model in a more sophisticated manner than has been presently done [34] In doing so, it may be appropriate to try to reconcile A. Ashtekar's approach involving a discretization of the Wheeler De Witt equation with the bounce calculations in general cosmology pioneered by Hackworth and Weinberg [35]... Needless to say, the work so presented above leaves open the question if or not baryogenesis, is involved in involving a collapse of the first term of Eqn. (3b) along the lines of the Bogomolnyi inequality, or else we have to skp this and to adhere to the topological defect models pioneered by Trodden,et al [36, 37].

Appendix I: Forming an Axion Potential Term as Part of The Contribution to Equation 2A

Kolb's book [7] has a discussion of an Axion potential given in his Eqn. (10.27)

$$V\left(a\right) = m_a^2 \cdot \left(f_{PQ}/N\right)^2 \cdot \left(1 - \cos\left[a/(f_{PQ}/N\right]\right) \tag{1}$$

Here, he has the mass of the Axion potential as given by m_a as well as a discussion of symmetry breaking which occurs with a temperature$T \approx f_{PQ}$. Furthermore, he states that the Axion goes to a massless regime for high temperatures, and becomes massive as the temperature drops. Due to the fact that Axions were cited by Zhitinisky in his QCD ball formation [20], this is worth considering, and I claim that this potential is part of Eqn. (6b) with the added term giving a tilt to this potential system, due to the role quantum fluctuations play in inflation. Here, N>1 leads to tipping of the wine bottle potential, and N degenerate CP-conserving minimal values. The interested reader is urged to consult section 10.3 of Kolb's Early universe book for additional details [9].

Appendix II: Estimation of Tunneling Time for New Potential System Given in EQN. (5)

We calculate tunneling time in the case of a false vacuum is to use a WKB type bounce calculation for forming an energy based tunneling [38]

$$\tau_{tunneling} \approx \left| \frac{\partial \cdot S_{WKB}(E)}{\partial \left| E \right|} \right| \tag{1a}$$

We need now to do this for a potential system given in part by Eqn. (3a) to Eqn. (3c) in the main text above above, and to do it consistently. Assuming that $S_{WKB}(E) \approx S_I\left(E\right)$ via a Coleman thin wall approximation for a bubble of space time, this leads to [8]

$$S_I\left(E\right) \approx \frac{27 \cdot \pi^2 \cdot \tilde{\sigma}^4}{2 \cdot \left| E \right|^3} \tag{1b}$$

Here, $\tilde{\sigma}$ is the surface tension of a bubble, and

$$E \equiv V_{\min} \tag{1c}$$

If one defines the minimum of the potential as being due to the 1^{st} tilted washboard potential $E \equiv V_{\min}$ is not going to be a zero quantity, and we will have a non zero but not huge value for tunneling time. This explicitly uses [8]

$$V(\phi^*) \approx \frac{-3 \cdot \tilde{\sigma}}{R_{crit}} \tag{1d}$$

If $R_{crit} \propto l_P$, i.e is on the order of Planck length, and $V(\phi^*) \propto V_{\min}$ of the 1^{st} tilted washboard potential given in Eqn. (5), this leads to a non zero, finite tunneling time for instantons in the bubble of space time used om early universe configuration, leading to

$$\Lambda_{total}\big|_{R_{critical}} = \lambda_{other} + V_{\min} \approx \Lambda_{observed} \tag{1e}$$

Initial configuration of the domain wall nucleation potential used in Eqn. (6b) which we claim eventually becomes in sync with Eqn. (6b) due to the phase transition alluded to by Dr. Edward Kolb's model of how the initial degrees of freedom declined from over 100 to something approaching what we see today in given flat Euclidian space models of space time (i.e. the FRW metric used in standard cosmology)

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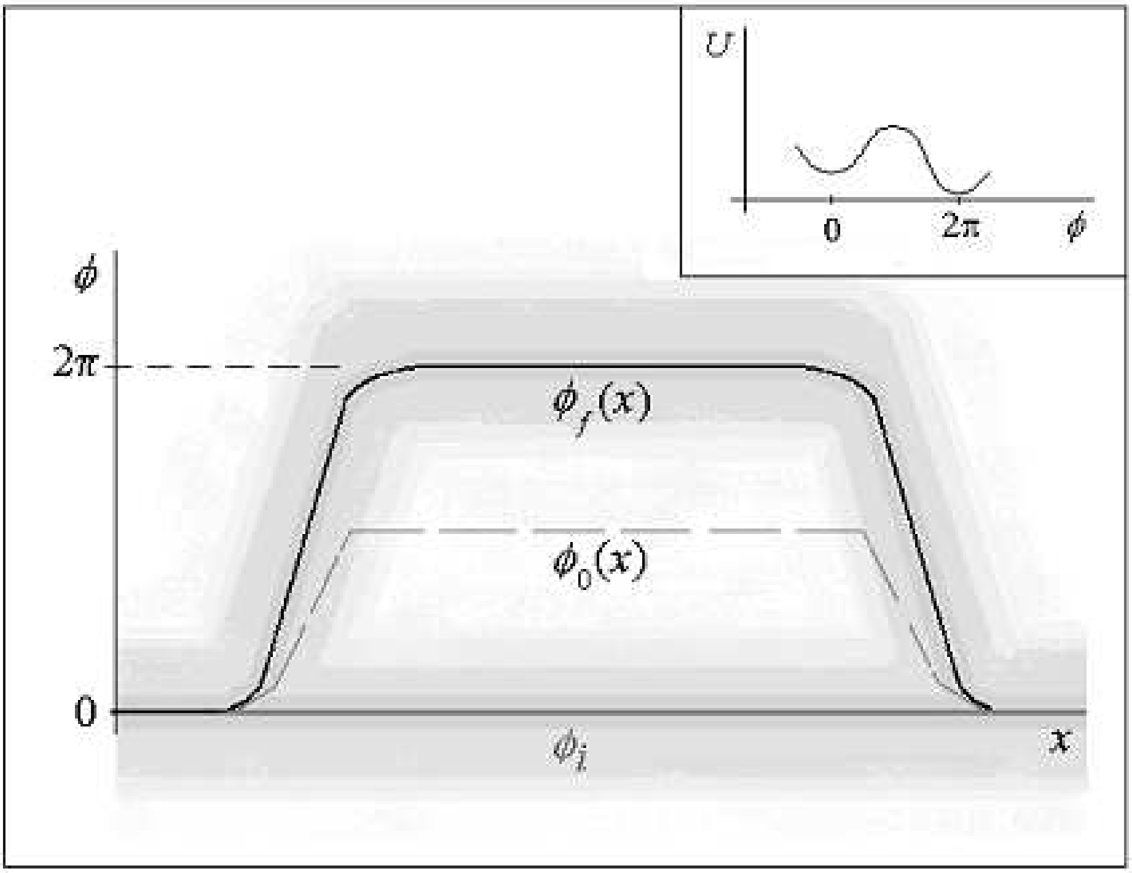

Fig. 1

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Extended Non Symmetric Gravitation Theory with a Scalar Field in Non Commutative Geometry

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Abstract: An extended method to reformulate the non symmetric gravitation theory in the non commutative geometry formalism is presented where all the lagrangian terms, including the various interaction ones with scalar fields, emerge naturally.

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1. Introduction

Recently, a new geometrical picture describing the various fundamental interactions has been proposed by A.Connes [1], [2], [3], [4]. It consists in the generalization of the classical differential geometry using a more profound mathematical formalism based on the discrete spaces non commutative geometry (NCG).

The success of the latter comes from the fact that it gives a geometric interpretation of the Higgs fields origin used in the in the standard model.

In this context, Chamseddine and collaborators [5], [6], [7] have reformulated General Relativity by considering a composed space- time consisting of a tensor product of a 4-dimensional manifold and a two points discrete space.

On the other hand, there exist many others theories inspired by General Relativity, which are based on a general non-symmetric metric $g_{\mu\nu}$,and in particular the Non Symmetric Gravitation theory (NGT) [8], [9], [10].

Yet, NGT as it was initially formulated lacked self-consistency; in particular the non-physical modes in the skewon sector are coupled with the physical ones.

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Recently, a new more consistent version of NGT was proposed by Moffat and Legaré [11], where these problems have been circumvented by adding new terms by hand to the action, but without any geometrical motivation.

In [12], we have derived an NGT action where the new terms added by Moffat in his new version of NGT are now a consequence of the discrete structure of space-time, with an additional interaction term $\varepsilon g_{ab}^{\mu\nu} R_{\mu\nu}^{ba}$.

In this work, we propose a new action for NGT without this interaction term. This was possible by generalizing the trace and introducing an operator P which permutes the indices of the Dirac γ matrices.

Moreover, the construction of this action with the scalar field coupled to NGT was possible by taking a general form of the generators of the 1-forms space $\Omega_D^1(\mathcal{B})$.

2. Formalism

In [6], the Hilbert-Einstein action was reformulated from the following Dirac operator:

$$D = \begin{pmatrix} \gamma^a \otimes e_a^\mu \partial_\mu \otimes 1 & \gamma^5 \otimes M_{12} \otimes K_{12} \\ \gamma^5 \otimes M_{21} \otimes K_{21} & \gamma^a \otimes e_a^\mu \partial_\mu \otimes 1 \end{pmatrix} = \begin{pmatrix} \gamma^a e_a^\mu \partial_\mu & \gamma^5 M_{12} K_{12} \\ \gamma^5 M_{21} K_{21} & \gamma^a e_a^\mu \partial_\mu \end{pmatrix}$$

where e_a^μ are the General Relativity (GR) vierbeins.

NGT as an extension of GR is based on the non-symmetric metric:

$$g^{\mu\nu} = e_a^\mu \widetilde{e}_b^\nu \eta^{ab}$$

where here e_a^μ is the NGT vierbein and $\widetilde{e}_a^\mu$ it's hyperbolic complex conjugate.

To generate non-symmetric terms and get an NGT action, we generalize the Dirac operator in the following form:

$$D = \begin{pmatrix} \gamma^a \otimes E_a^\mu \partial_\mu \otimes 1 & \gamma^5 \otimes M_{12} \otimes K_{12} \\ \gamma^5 \otimes M_{21} \otimes K_{21} & \gamma^a \otimes E_a^\mu \partial_\mu \otimes 1 \end{pmatrix} = \begin{pmatrix} \gamma^a E_a^\mu \partial_\mu & \gamma^5 M_{12} K_{12} \\ \gamma^5 M_{21} K_{21} & \gamma^a E_a^\mu \partial_\mu \end{pmatrix}$$

where E_a^μ is a 2×2 matrix (the generalized vierbein) defined by:

$$E_a^\mu = (E_a^\mu)^\star = \begin{pmatrix} 0 & e_a^\mu \\ \widetilde{e}_a^\mu & 0 \end{pmatrix}$$

and M_{12}, M_{21} (resp. KK$_{12}$, K_{21}) are 2×2 (resp. NN$\times$N) matrices. Here the γ^a 's are the ordinary Dirac matrices in the flat four-dimensional space-time and redefined such that [6]:

$$\gamma^{a*} = -\gamma^a, \left\{\gamma^a, \gamma^b\right\} = \gamma^a\gamma^b + \gamma^b\gamma^a = -2\delta^{ab} \tag{1}$$

$$\gamma^{ab} = \frac{1}{2}\left[\gamma^a, \gamma^b\right] \quad , \quad \gamma^{(ab)} = \frac{1}{2}\left\{\gamma^a, \gamma^b\right\} = -\delta^{ab}$$

In order to have a self-adjoint Dirac operator as it is required, one has to set the conditions:

$$K_{21}^{*} = K_{12} = K \tag{2}$$

and

$$M_{12} = M_{21}^{*} = M$$

The basic algebra $\mathcal{A}$ is defined as:

$$\mathcal{A} = C_{\mathbb{R}}^{\infty}(X) \otimes (M_2(\mathbb{K}) \oplus M_2(\mathbb{K})) = C_{\mathbb{R}}^{\infty}(X, M_2(\mathbb{K})) \oplus C_{\mathbb{R}}^{\infty}(X, M_2(\mathbb{K})) \tag{3}$$

$$\mathcal{A} = \left\{ \alpha^{(1)} + \alpha^{(2)}; \alpha^{(i)} = \begin{pmatrix} a^{(i)} & 0 \\ 0 & b^{(i)} \end{pmatrix}; a^{(i)}, b^{(i)} \in C_{\mathbb{R}}^{\infty}(X, \mathbb{K}), i = 1, 2 \right\} \tag{4}$$

with

$$C_{\mathbb{R}}^{\infty}(X, M_2(\mathbb{K})) = C_{\mathbb{R}}^{\infty}(X) \otimes M_2(\mathbb{K})$$

where $C_{\mathbb{R}}^{\infty}(X)$ denotes the space of an infinite differentiable real function on a manifold X, and $M_2(\mathbb{K})$ is the set of the 2×2 hyperbolic complex matrices.

In what follows, we restrict ourselves to a sub-algebra $\mathcal{B}$ of $\mathcal{A}$ such that:

A representation of this sub-algebra on a Hilbert space $\mathcal{H}$ is given by:

$$\pi(\alpha) = \pi\left(\alpha^{(1)} + \alpha^{(2)}\right) = \begin{pmatrix} 1 \otimes \alpha^{(1)} \otimes 1 & 0 \\ 0 & 1 \otimes \alpha^{(2)} \otimes 1 \end{pmatrix} = \begin{pmatrix} \alpha^{(1)} & 0 \\ 0 & \alpha^{(2)} \end{pmatrix} \tag{5}$$

where $\mathcal{H}$ is defined as[6]:

$$\mathcal{H} = L^2(S_1, dv_1) \oplus L^2(S_2, dv_2) \tag{6}$$

with $L^2(S_i, dv_i)$ is the square integrable functions over S_i such that

$$S_i = S_0 \otimes \mathbb{K} \quad , \quad i = 1, 2 \tag{7}$$

where S_0 is the spinors space, and dv_i the volume element on X.

For the space of 1-forms denoted by $\Omega_D^1(\mathcal{B})$, one has as a representation:

$$\Omega_D^1(\mathcal{B}) = \pi\left(\Omega^1(\mathcal{B})\right) = \left\{ \pi(\omega) = \pi\left(\sum_i \alpha_i \delta\beta_i\right) = \sum_i \pi(\alpha_i)\left[D, \pi(\beta_i)\right] \right\} \tag{8}$$

Straightforward calculations give:

$$\pi(\omega) = \begin{pmatrix} \gamma^a E_a^{\mu} \omega_{\mu}^{(1)} & \gamma^5 K_{12} \Phi_{12} \\ \gamma^5 K_{21} \Phi_{21} & \gamma^a E_a^{\mu} \omega_{\mu}^{(2)} \end{pmatrix} \tag{9}$$

where

$$\omega_\mu^{(m)} = \sum_i \alpha_i^{(m)} \partial_\mu \beta_i^{(m)} \qquad m = 1, 2 \tag{10}$$

and

$$\Phi_{mn} = \phi_{mn} M_{mn} \tag{11}$$

with

$$\phi_{mn} = \left(\sum_i \left(\alpha_i^{(m)} \beta_i^{(n)} - 1 \right) \right) \qquad m \neq n = 1, 2 \tag{12}$$

where we have used the normalization condition:

$$\sum_i \alpha_i^{(1)} \beta_i^{(1)} = \sum_i \alpha_i^{(2)} \beta_i^{(2)} = 1 \tag{13}$$

Now, in order to get a representation of the 2-forms space $\Omega^2(\mathcal{B})$ without the junk forms (auxiliary fields), one has to take:

$$\Omega_D^2(\mathcal{B}) = \pi\left(\Omega_D^2(\mathcal{B})\right) / Aux^2 \tag{14}$$

where Aux^2 is the space of the auxiliary fields defined as:

$$Aux^2 = \{\pi(\delta\omega) \quad / \quad \pi(\omega) = 0\} \quad , \quad \omega \in \Omega^1(\mathcal{B}) \tag{15}$$

with

$$\pi(\delta\omega) = \sum_i \pi(\delta\alpha_i \delta\beta_i) = \sum_i [D, \pi(\alpha_i)] [D, \pi(\beta_i)] \tag{16}$$

Direct but lengthy calculations lead to:

$$\pi(\delta\omega)_{11} = \gamma^a \gamma^b E_a^\mu E_b^\nu \left(\partial_\mu \omega_\nu^{(1)} - X_{\mu\nu}^{(1)} \right) + K_{12} K_{21} M_{12} M_{21} (\phi_{12} + \phi_{21})$$
$$\pi(\delta\omega)_{22} = \gamma^a \gamma^b E_a^\mu E_b^\nu \left(\partial_\mu \omega_\nu^{(2)} - X_{\mu\nu}^{(2)} \right) + K_{21} K_{12} M_{21} M_{12} (\phi_{21} + \phi_{12})$$
$$\pi(\delta\omega)_{12} = K_{12} \gamma^a \gamma^5 \left(E_a^\mu M_{12} \left(\partial_\mu \phi_{12} + \omega_\mu^{(1)} \right) - M_{12} E_a^\mu \omega_\mu^{(2)} - [E_a^\mu, M_{12}] Y_\mu^{(12)} \right)$$
$$\pi(\delta\omega)_{21} = K_{21} \gamma^a \gamma^5 \left(E_a^\mu M_{21} \left(\partial_\mu \phi_{21} + \omega_\mu^{(2)} \right) - M_{21} E_a^\mu \omega_\mu^{(1)} - [E_a^\mu, M_{21}] Y_\mu^{(21)} \right)$$

where the hyperbolic complex functions $X_{\mu\nu}^{(m)}$ and $Y_\mu^{(mn)}$ are given by:

$$X_{\mu\nu}^{(m)} = \sum_i \alpha_i^{(m)} \partial_\mu \partial_\nu \beta_i^{(m)} \qquad m = 1, 2$$

and

$$Y_\mu^{(mn)} = \sum_i \alpha_i^{(m)} \partial_\mu \beta_\mu^{(n)} \qquad m \neq n = 1, 2 \tag{17}$$

After some simplifications, we obtain for Aux^2 the following expression:

$$Aux^2 = \begin{pmatrix} \gamma^a \gamma^b E_a^\mu E_b^\nu X_{\mu\nu}^{(1)} & K \gamma^a \gamma^5 [E_a^\mu, M_{12}] Y_\mu \\ K^* \gamma^a \gamma^5 [E_a^\mu, M_{21}] Z_\mu & \gamma^a \gamma^b E_a^\mu E_b^\nu X_{\mu\nu}^{(2)} \end{pmatrix} \tag{18}$$

where $X_{\mu\nu}^{(1)}=X_{\nu\mu}^{(1)}, X_{\mu\nu}^{(2)}=X_{\nu\mu}^{(2)}, Y_\mu, Z_\mu$ are arbitrary hyperbolic complex functions.

The curvature tensor R^{AB} $\left(A,B=\overline{1,5}\right)$ is given by the Cartan structure equations [6],[7]:

$$R^{AB}=d\Omega^{AB}+\sum_C \Omega^{AC}\Omega^{CB}$$

where $\Omega^{AB}\in\Omega_D^1(\mathcal{B})$are the components of the connection such that:

$\left(\Omega^{AB}\right)_{mm}=\gamma^a E_a^\mu \omega_\mu^{(m)AB} \qquad m=1,2$

$\left(\Omega^{AB}\right)_{mn}=\gamma^5 K_{mn}M_{mn}\phi_{mn}^{AB} \qquad m\neq n=1,2.$

and are subject to the unitarity condition:

$\left(\Omega^{AB}\right)^*=\Omega^{BA}$

This leads to the following constraints on the fields:

$$\widetilde{\omega}_\mu^{(1)AB}=-\omega_\mu^{(1)BA}\ ,\ \ \widetilde{\omega}_\mu^{(2)AB}=-\omega_\mu^{(2)BA}$$
$$\widetilde{\phi}_{12}^{AB}=\phi_{21}^{BA}\ ,\quad \widetilde{\phi}_{21}^{AB}=\phi_{12}^{BA}$$

In order to get an explicit expression for R^{AB} and Aux^2, we make the following choice for the matrix M:

$$M=\mu\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \tag{19}$$

We note that this choice is not arbitrary; indeed when we go from the space $\pi\left(\Omega_D^2(\mathcal{B})\right)$ to the quotient space $\Omega_D^2(\mathcal{B})=\pi\left(\Omega_D^2(\mathcal{B})\right)/Aux^2$, the fields ϕ_{mn}^{AB} vanish unless the matrix M obeys equation $(A1)$. For simplicity, μ is taken to be equal to one.

With this choice, we get the following expressions for the elements of the curvature tensor R^{AB}:

$R_{11}^{AB}=\gamma^a\gamma^b\left(\eta_{ab}^{\mu\nu}\mathbf{1}-\varepsilon g_{ab}^{\mu\nu}\tau_3\right)\left(R_{\mu\nu}^{(1)AB}-X_{\mu\nu}^{(1)AB}\right)+K.K^*\mathrm{H}_{12}^{AB}\tau$

$R_{22}^{AB}=\gamma^a\gamma^b\left(\eta_{ab}^{\mu\nu}\mathbf{1}-\varepsilon g_{ab}^{\mu\nu}\tau_3\right)\left(R_{\mu\nu}^{(2)AB}-X_{\mu\nu}^{(2)AB}\right)+K^*.K\mathrm{H}_{21}^{AB}\overline{\tau}$

$R_{12}^{AB}=K.\gamma^a\gamma^5\widetilde{e}_a^\mu\left(\nabla_\mu\varphi_{12}^{AB}\overline{\tau}-\left(\varphi_{12}^{AC}\omega_\mu^{(2)CB}+\omega^{(2)AB}\right)\tau+Y_\mu^{AB}\tau_3\right)$

$R_{21}^{AB}=K.^*\gamma^a\gamma^5 e_a^\mu\left(\nabla_\mu\varphi_{21}^{AB}\cdot\tau-\left(\varphi_{21}^{AC}\omega_\mu^{(1)CB}+\omega_\mu^{(1)AB}\right)\overline{\tau}-Z_\mu^{AB}\tau_3\right)$

where:

$R_{\mu\nu}^{(m)AB}=\partial_\mu\omega_\nu^{(m)AB}+\sum\limits_C\omega_\mu^{(m)AC}\omega_\nu^{(m)CB} \qquad m=1,2$

and:

$\mathrm{H}_{12}^{AB}=\varphi_{12}^{AC}\varphi_{21}^{CB}+\varphi_{12}^{AB}+\varphi_{21}^{AB}$

$\mathrm{H}_{21}^{AB}=\varphi_{21}^{AC}\varphi_{12}^{CB}+\varphi_{21}^{AB}+\varphi_{12}^{AB}$

$\nabla_\mu\varphi_{12}^{AB}=\partial_\mu\varphi_{12}^{AB}+\omega_\mu^{(1)AC}\varphi_{12}^{CB}+\omega_\mu^{(1)AB}$

$\nabla_\mu\varphi_{21}^{AB}=\partial_\mu\varphi_{21}^{AB}+\omega_\mu^{(2)AC}\varphi_{21}^{CB}+\omega_\mu^{(2)AB}$

We obtain for Aux^2 the following expression:

$$Aux^2=\begin{pmatrix}\gamma^a\gamma^b\left(\eta_{ab}^{\mu\nu}-\varepsilon g_{ab}^{\mu\nu}\tau_3\right)X_{\mu\nu}^{(1)} & -K.\gamma^a\gamma^5\widetilde{e}_a^\mu\tau_3 Y_\mu\\ K^*.\gamma^a\gamma^5 e_a^\mu\tau_3 Z\mu & \gamma^a\gamma^b\left(\eta_{ab}^{\mu\nu}-\varepsilon g_{ab}^{\mu\nu}\tau_3\right)X_{\mu\nu}^{(2)}\end{pmatrix} \tag{20}$$

where we have used the fact that:

$$M_{12} = M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_{21} = M_{12}^{*} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$\tau = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \overline{\tau} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$E_a^{\mu} E_b^{\nu} = \eta_{ab}^{\mu\nu} \mathbf{1} - \varepsilon g_{ab}^{\mu\nu} \tau_3$$

with:

$\eta_{ab}^{\mu\nu} = (\alpha_a^{\mu}\alpha_b^{\nu} - \beta_a^{\mu}\beta_b^{\nu}) = +\eta_{ba}^{\nu\mu}$
$g_{ab}^{\mu\nu} = \alpha_a^{\mu}\beta_b^{\nu} - \beta_a^{\mu}\alpha_b^{\nu} = -g_{ba}^{\nu\mu}$
$\tau - \overline{\tau} = \tau_3$
and
$\alpha_a^{\mu} = \frac{1}{2}\left(e_a^{\mu} + \widetilde{e}_a^{\mu}\right)$
$\beta_a^{\mu} = \frac{1}{2}\varepsilon\left(e_a^{\mu} - \widetilde{e}_a^{\mu}\right)$
ε is the hyperbolic complex number satisfying:
$\varepsilon^2 = 1$,and, $\widetilde{\varepsilon} = -\varepsilon$

The expression of the curvature tensor R^{AB} orthogonal to the space of the auxiliary fields Aux^2 is given by:

$R_{11}^{AB} = \frac{1}{2}\left(\gamma^{ab}\eta_{ab}^{\mu\nu}\mathbf{1} + \varepsilon g_{ab}^{\mu\nu}\tau_3\right)\left(R_{\mu\nu}^{(1)AB}\right) - \frac{1}{2}\left(Tr\left(K.K^{*}\right) - 2\left(K.K^{*}\right)\tau\right)\mathrm{H}_{12}^{AB}$
$R_{22}^{AB} = \frac{1}{2}\left(\gamma^{ab}\eta_{ab}^{\mu\nu}\mathbf{1} + \varepsilon g_{ab}^{\mu\nu}\tau_3\right)R_{\mu\nu}^{(2)AB} - \frac{1}{2}\left(Tr\left(K.K^{*}\right) - 2\left(K^{*}.K\right)\overline{\tau}\right)\mathrm{H}_{21}^{AB}$
$R_{12}^{AB} = \frac{1}{2}K.\gamma^{a}\gamma^{5}\widetilde{e}_a^{\mu}\left(\nabla_{\mu}\varphi_{12}^{AB} - \left(\varphi_{12}^{AC}\omega_{\mu}^{(2)CB} + \omega^{(2)AB}\right)\right)$
$R_{21}^{AB} = \frac{1}{2}K.^{*}\gamma^{a}\gamma^{5}e_a^{\mu}\left(\nabla_{\mu}\varphi_{21}^{AB} - \left(\varphi_{21}^{AC}\omega_{\mu}^{(1)CB} + \omega^{(1)AB}\right)\right)$
with $R_{\mu\nu}^{(m)AB} = \partial_{\mu}\omega_{\nu}^{(m)AB} + \sum\limits_{C}\omega_{\mu}^{(m)AC}\omega_{\nu}^{(m)CB} - (\mu \leftrightarrow \nu)$
$\qquad = -R_{\nu\mu}^{(m)AB}$ with $m = 1, 2$

Concerning the torsion $T^A\,(A = \overline{1,5})$, it is given by the Cartan structure equations [6],[7]:

$$T^A = d\xi^A + \sum_B \Omega^{AB}.\xi^B$$

where ξ^A are the generators of the space of 1-forms $\Omega_D^1(\mathcal{B})$ and which have the following expressions:

$$\xi^a = \pi\left(\widetilde{e}_{\mu}^{a} \oplus \widetilde{e}_{\mu}^{a}\delta \cdot x^{\mu} \oplus x^{\mu}\right) = \begin{pmatrix} \gamma^b E_b^{\mu}\widetilde{e}_{\mu}^{a} & 0 \\ 0 & \gamma^b E_b^{\mu}\widetilde{e}_{\mu}^{a} \end{pmatrix} \quad , a = 1, 2, 3, 4$$

$$\xi^{a*} = -\gamma^b E_b^{\mu} e_{\mu}^{a} \otimes \mathbf{1} = -\widetilde{\xi}$$

$$\xi^5 = \pi\left(\lambda \oplus \widetilde{\lambda}\delta \cdot 0 \oplus 1\right) = \begin{pmatrix} 0 & \gamma^5 K M_{21}\lambda \\ -\gamma^5 K^{*} M_{21}\widetilde{\lambda} & 0 \end{pmatrix} = -\xi^{5*} = -\widetilde{\xi}^5$$

with:

$$d\xi^a = \pi\left(\delta.\widetilde{e}^a_\mu \oplus \widetilde{e}^a_\mu.\delta.x^\mu \oplus x^\mu\right) = \gamma^b\gamma^c E^\mu_b E^\nu_c \partial_\mu \widetilde{e}^a_\mu \otimes \mathbf{1}$$

$$d\xi^5 = \pi\left(\delta\lambda \oplus \widetilde{\lambda}\delta\cdot 0 \oplus 1\right) = \begin{pmatrix} KK^* M_{12}M_{21}\left(\lambda-\widetilde{\lambda}\right) & K\gamma^a\gamma^5 E^\mu_a M_{12}\partial_\mu\lambda \\ -K^*\gamma^a\gamma^5 E^\mu_a M_{21}\partial_\mu\widetilde{\lambda} & K^*KM_{21}M_{12}\left(\lambda-\widetilde{\lambda}\right)\end{pmatrix}.$$

where λ is an hyperbolic complex function.

Then, the components of T^A orthogonal to the auxiliary fields space take the form:

$(T^a)_{11} = \left(\gamma^{cd}\eta^{\mu\nu}_{cd}\mathbf{1} + \varepsilon g^{\mu\nu}\tau_3\right)\left(\partial_\mu\widetilde{e}^a_\nu + \omega^{(1)ab}_\mu\widetilde{e}^b_\nu\right) + \frac{1}{2}\left(Tr\left(K.K^*\right) - 2\left(K.K^*\right)\tau\right)\widetilde{\lambda}\varphi^{a5}_{12}$

$(T^a)_{22} = \left(\gamma^{cd}\eta^{\mu\nu}_{cd}\mathbf{1} + \varepsilon g^{\mu\nu}\tau_3\right)\left(\partial_\mu\widetilde{e}^a_\nu + \omega^{(2)ab}_\mu\widetilde{e}^b_\nu\right) - \frac{1}{2}\left(Tr\left(K.K^*\right) - 2\left(K^*.K\right)\overline{\tau}\right)\lambda\varphi^{a5}_{21}$

$(T^a_{12}) = K.\gamma^d\gamma^5\widetilde{e}^\mu_d\lambda\omega^{(1)a5}_\mu$

$(T^a)_{21} = -K.^*\gamma^d\gamma^5 e^\mu_d\widetilde{\lambda}\omega^{(2)a5}_\mu$

and:

$(T^5)_{11} = \left(\gamma^{cd}\eta^{\mu\nu}_{cd}\mathbf{1} + \varepsilon g^{\mu\nu}\tau_3\right)\omega^{(1)5b}_\mu\widetilde{e}^b_\nu + \frac{1}{2}\left(Tr\left(K.K^*\right) - 2\left(K.K^*\right)\tau\right)\left(\varphi^{55}_{12}\widetilde{\lambda} - \lambda + \widetilde{\lambda}\right)$

$(T^5)_{22} = \left(\gamma^{cd}\eta^{\mu\nu}_{cd}\mathbf{1} + \varepsilon g^{\mu\nu}\tau_3\right)\omega^{(2)5b}_\mu\widetilde{e}^b_\nu - \frac{1}{2}\left(Tr\left(K.K^*\right) - 2\left(K^*.K\right)\overline{\tau}\right)\left(\varphi^{55}_{21}\lambda + \lambda - \widetilde{\lambda}\right)$

$(T^5)_{12} = K.\gamma^d\gamma^5\widetilde{e}^\mu_d\left(\partial_\mu\lambda + \omega^{(1)55}_\mu\lambda\right)$

$(T^5)_{21} = -K.^*\gamma^d\gamma^5 e^\mu_d\left(\partial_\mu\widetilde{\lambda} + \omega^{(2)55}_\mu\widetilde{\lambda}\right)$

The orthogonality condition of the T^A to the space Aux^2 leads to the following constraints on the fields:

$$\omega^{(2)a5}_\mu\widetilde{\lambda} - \widetilde{e}^b_\mu\varphi^{ab}_{21} = 0 \tag{21}$$

$$\omega^{(1)a5}_\mu\lambda + \widetilde{e}^b_\mu\varphi^{ab}_{12} = 0 \tag{22}$$

$$\partial_\mu\widetilde{\lambda} + \omega^{(2)55}_\mu\widetilde{\lambda} - \widetilde{e}^b_\mu\varphi^{5b}_{21} = 0 \tag{23}$$

$$\partial_\mu\lambda + \omega^{(1)55}_\mu\lambda + \widetilde{e}^b_\mu\varphi^{5b}_{12} = 0 \tag{24}$$

The action $\mathfrak{I}$ has the same form as that given in [5] namely:

$$\mathfrak{I} = \frac{1}{2}\left(E^A E^{B*} - E^{B*}E^A, R^{BA}\right) \tag{25}$$

where here $(,)$ denotes the generalized scalar product defined as:

$$\left(E^A E^{B*} - E^{B*}E^A, R^{BA}\right) = \int d^4x\sqrt{e\widetilde{e}}Tr_K\cdot\mathfrak{Tr}\left(\left(E^B E^{A*} - E^{A*}E^B\right)R^{BA}\right) \tag{26}$$

with Tr_K is the trace over the K matrices, while $\mathfrak{Tr}$ is the generalized trace defined in the appendix. The $E^{A'}$ are the generators of a subspace of the1-form space $\Omega^1_D(A)$ defined by:

$$\left\{\begin{matrix} \pi(\omega) = \pi\left(\sum\limits_i \alpha_i\delta\beta_i\right) = \sum\limits_i \pi(\alpha_i)\left[D, \pi(\beta_i)\right] \\ \alpha_i \in A\,, \quad \beta_i \in \boldsymbol{\mathcal{B}}, \alpha_i = \alpha^{(1)}_i \oplus \alpha^{(2)}_i, \quad \beta_i = \beta^{(1)}_i \oplus \beta^{(2)}_i \end{matrix}\right\}$$

A representation of this space is given by:

$$\pi(\omega) = \begin{pmatrix} \gamma^a \omega_\mu^{(1)} E_a^\mu & \gamma^5 K \Phi_{12} M_{12} \\ \gamma^5 K^* \Phi_{21} M_{12} & \gamma^a \omega_\mu^{(2)} E_a^\mu \end{pmatrix}$$

and the generators are:

$E^a = \pi\left(e_\mu^a \tau \oplus e_\mu^a \tau \cdot \delta x^\mu \oplus x^\mu\right) = \gamma^b e_\mu^a \tau E_b^\mu \otimes \mathbf{1} = \gamma^a M_{12} \otimes \mathbf{1}$

$\widetilde{E}^a = \pi\left(\widetilde{e}_\mu^a \overline{\tau} \oplus \widetilde{e}_\mu^a \overline{\tau} \cdot \delta x^\mu \oplus x^\mu\right) = \gamma^b \widetilde{e}_\mu^a \overline{\tau} E_b^\mu \otimes \mathbf{1} = \gamma^a M_{21} \otimes \mathbf{1}$

$E^5 = \pi\left(\lambda \oplus \widetilde{\lambda} \cdot \delta 0 \oplus 1\right) = \begin{pmatrix} 0 & \gamma^5 K M_{12} \lambda \\ -\gamma^5 K^* M_{21} \widetilde{\lambda} & 0 \end{pmatrix} = \widetilde{E}^5$

In fact, if we use the generators ξ^A of the space $\Omega_D^1(A)$

$$\xi^a = \gamma^b E_b^\mu \widetilde{e}_\mu^a \otimes \mathbf{1} = \begin{matrix} 0 & \gamma^b e_b^\mu \widetilde{e}_\mu^a \\ \gamma^a & 0 \end{matrix} \otimes \mathbf{1}$$

to define the action, we get two contributions to the action: one coming from the term $\gamma^b \widetilde{e}_b^\mu \widetilde{e}_\mu^a = \gamma^a$ and which gives the NGT action, and the other one from the term $\gamma^b e_b^\mu \widetilde{e}_\mu^a$ giving a meaningless contribution. This is however not the case for the generators E^A where the matrix $\tau = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in the expression of E^a insures the absence of such terms in the action.

A straightforward calculation gives the following expression:

$\mathfrak{I} = \mathfrak{I}^{(1)} + \mathfrak{I}^{(2)} + \mathfrak{I}^{(3)} + \mathfrak{I}^{(4)}$

with

$\mathfrak{I}^{(1)} = \frac{1}{2}\left(E^a E^{b*} - E^{b*} E^a, R^{ba}\right) = \frac{1}{2}\int d^4x \sqrt{e\widetilde{e}}\, Tr \cdot \mathfrak{Tr}\left(\left(E^b E^{a*} - E^{a*} E^b\right) R^{ba}\right)$

$\mathfrak{I}^{(2)} = \frac{1}{2}\left(E^5 E^{b*} - E^{b*} E^5, R^{b5}\right) = \frac{1}{2}\int d^4x \sqrt{e\widetilde{e}}\, Tr \cdot \mathfrak{Tr}\left(\left(E^5 E^b - E^b E^5\right) R^{b5}\right)$

$\mathfrak{I}^{(3)} = \frac{1}{2}\left(E^a E^{5*} - E^{5*} E^a, R^{5a}\right) = \frac{1}{2}\int d^4x \sqrt{e\widetilde{e}}\, Tr \cdot \mathfrak{Tr}\left(\left(\widetilde{E}^a E^5 - E^5 \widetilde{E}^a\right) R^{5a}\right)$

$\mathfrak{I}^{(4)} = \frac{1}{2}\left(E^5 E^{5*} - E^{5*} E^5, R^{55}\right) = \frac{1}{2}\int d^4x \sqrt{e\widetilde{e}}\, Tr \mathfrak{Tr}\left(\left(E^5 E^5 - E^5 E^5\right) R^{55}\right)$

Now by using the properties of the generalized trace $\mathfrak{Tr} = \mathcal{P}tr$ (defined in the appendix), we obtain:

$\mathfrak{I}^{(1)} = \int d^4x \sqrt{e\widetilde{e}} \left\{-\left(e_a^\mu \widetilde{e}_b^\nu + \frac{1}{2}\delta_{ab} G^{\mu\nu}\right)\left(R_{\mu\nu}^{(1)ba} + R_{\mu\nu}^{(2)ba}\right) + \frac{1}{2} Tr\left(KK^*\right)\left(H_{12}^{aa} - H_{21}^{aa}\right)\right\}$

and by using the compatibility conditions for $\widetilde{e}_\mu^a$:

$\nabla_\mu \widetilde{e}_\mu^a = \partial_\mu \widetilde{e}_\nu^a + \omega_\mu^{ab} \widetilde{e}_\mu^b - W^\lambda{}_{\mu\nu} \widetilde{e}_\lambda^a = 0$

we get:

$\left(e_a^\mu \widetilde{e}_b^\nu + \frac{1}{2}\delta_{ab} G^{\mu\nu}\right) R^{ba}{}_{\mu\nu} = -G^{\mu\nu}\left(R_{\sigma\lambda\nu}^\lambda - \frac{1}{2}\widetilde{R}_{\lambda\mu\nu}^\lambda\right) = -G^{\mu\nu} R_{\mu\nu}.$

Therefore $\mathfrak{I}^{(1)}$ becomes:

$$\mathfrak{I}^{(1)} = \int d^4x \sqrt{e\widetilde{e}} \left\{G^{\mu\nu}\left(R_{\mu\nu}^{(1)} + R_{\mu\nu}^{(2)}\right) + \frac{1}{2} Tr\left(KK^*\right)\left(\mathrm{H}_{12}^{aa} - \mathrm{H}_{21}^{aa}\right)\right\} \tag{27}$$

Similarly, we get the following expressions for $\mathfrak{I}^{(2)}$,$\mathfrak{I}^{(3)}$and $\mathfrak{I}^{(4)}$:
$\mathfrak{I}^{(2)}= \int d^4x\sqrt{e\widetilde{e}}\left\{\frac{1}{4}Tr\left(K.K^*\right)\widetilde{\lambda}\widetilde{e}_a^{\mu}\left(\nabla_\mu\varphi_{12}^{a5}-\varphi_{12}^{aB}\omega_\mu^{(2)B5}-\omega_\mu^{(2)a5}\right)\right\}$
$\mathfrak{I}^{(3)}= \int d^4x\sqrt{e\widetilde{e}}\frac{1}{4}Tr\left(K.K^*\right)\lambda e_a^{\mu}\left(\nabla_\mu\varphi_{21}^{5a}-\varphi_{21}^{5B}\omega_\mu^{(1)Ba}-\omega_\mu^{(1)5a}\right)$
$\mathfrak{I}^{(4)}=0$
The action $\mathfrak{I}$ now takes the form:

$$\begin{aligned}\mathfrak{I}=\int d^4x\sqrt{e\widetilde{e}}\{G^{\mu\nu}&\left(R_{\mu\nu}^{(1)}+R_{\mu\nu}^{(2)}\right)+\tfrac{1}{2}Tr\left(KK^*\right)\left(\varphi_{12}^{aB}\varphi_{21}^{Ba}-\varphi_{21}^{aB}\varphi_{12}^{Ba}\right)\\ &+\tfrac{1}{4}Tr\left(K.K^*\right)\widetilde{\lambda}\widetilde{e}_a^{\mu}\left(\nabla_\mu\varphi_{12}^{a5}-\varphi_{12}^{aB}\omega_\mu^{(2)B5}-\omega_\mu^{(2)a5}\right)\\ &+\tfrac{1}{4}Tr\left(K.K^*\right)\lambda e_a^{\mu}\left(\nabla_\mu\varphi_{21}^{5a}-\varphi_{21}^{5B}\omega_\mu^{(1)Ba}-\omega_\mu^{(1)5a}\right)\}\end{aligned}$$

If we impose the strong condition $T^A=0$, the fields ϕ_{mn}^{AB} vanish and the contribution of the discrete space is trivial. So, in order to get a dynamical contribution of the discrete space, we must impose the following weak conditions:

$$Tr_K\left(T^a\right)=0 \quad \text{and} \quad T^5\neq 0 \tag{28}$$

where Tr_K denotes the trace over the matrices K_{12},K_{21}.

By using the constraints on the fields given by the condition $(A2)$,and the equations (21), (22), (23), we obtain the following expression for the action:

$$\begin{aligned}&\mathfrak{I}=\int d^4x\sqrt{e\widetilde{e}}\mathfrak{L} \qquad \mathfrak{L}=\mathfrak{L}^{(1)}+\mathfrak{L}^{(2)}+\mathfrak{L}^{(3)}\\ &\mathfrak{L}^{(1)}=G^{\mu\nu}\left(R_{\mu\nu}^{(1)}+R_{\mu\nu}^{(2)}\right)=2G^{\mu\nu}R_{\mu\nu}\\ &\mathfrak{L}^{(2)}=\tfrac{1}{2}Tr\left(KK^*\right)\left(\varphi_{12}^{aB}\varphi_{21}^{Ba}-\varphi_{21}^{aB}\varphi_{12}^{Ba}\right)=0\\ &\mathfrak{L}^{(3)}=\tfrac{1}{4}Tr\left(K.K^*\right)\widetilde{\lambda}\widetilde{e}_a^{\mu}\left(\nabla_\mu\varphi_{12}^{a5}-\varphi_{12}^{aB}\omega_\mu^{(2)B5}-\omega_\mu^{(2)a5}\right)\\ &\qquad+\tfrac{1}{4}Tr\left(K.K^*\right)\lambda e_a^{\mu}\left(\nabla_\mu\varphi_{21}^{5a}-\varphi_{21}^{5B}\omega_\mu^{(1)Ba}-\omega_\mu^{(1)5a}\right)\end{aligned}$$

If we put $\omega_\mu^{(2)55}=W_\mu$, $\widetilde{W}_\mu=-W_\mu$, the expression for $\mathfrak{L}^{(3)}$ becomes:

$$\begin{aligned}\mathfrak{L}^{(3)}=&\frac{1}{4}Tr\left(K.K^*\right)\left(\widetilde{\lambda}G^{\sigma\mu}\partial_\mu\left(\left(\partial_\sigma-W_\sigma\right)\lambda\right)-\widetilde{\lambda}G^{\nu\mu}\left(\left(\partial_\sigma-W_\sigma\right)\lambda\right)\widetilde{W}_{\mu\nu}^{\sigma}\right.\\ &\left.-\widetilde{\lambda}G^{\sigma\mu}\left(\left(\partial_\sigma-W_\sigma\right)\lambda\right)W_\mu\right)+h.c.c.\end{aligned}$$

where we have used the compatibility conditions on e_a^σ .
In what follows, we consider several cases of special significance:
(i) The case $\lambda=\overline{\lambda}=1$
Here, the expression for $\mathfrak{L}^{(3)}$ becomes:

$$\begin{aligned}\mathfrak{L}^{(3)}&=\tfrac{1}{4}Tr\left(K.K^*\right)\left\{-G^{\sigma\mu}\partial_\mu W_\sigma+G^{\nu\mu}W_\sigma\widetilde{W}^{\sigma}{}_{\mu\nu}+G^{\sigma\mu}W_\sigma W_\mu+h.c.c\right\}\\ &=\tfrac{1}{4}Tr\left(K.K^*\right)\left\{2G^{(\mu\nu)}W_\mu W_\nu-2G^{[\mu\nu]}\partial_\nu W_\mu\right\}\end{aligned}$$

and therefore $\mathfrak{L}$ takes the form:

$$\mathfrak{L}=G^{\mu\nu}R_{\mu\nu}-\frac{1}{4}xG^{(\mu\nu)}W_\mu W_\nu+\frac{1}{4}xG^{[\mu\nu]}\partial_\nu W_\mu \tag{29}$$

where $x=-Tr\left(K.K^*\right)$.We remind the reader that K being an hyperbolic complex matrix, x can be a positive number.

By using the following decomposition [10]:

$$\begin{aligned} W^{\lambda}_{\mu\nu} &= \Gamma^{\lambda}_{\mu\nu} - \tfrac{2}{3}\delta^{\lambda}_{\mu} W_{\nu} \\ R_{\mu\nu}(W) &= R_{\mu\nu}(\Gamma) + \tfrac{2}{3} W_{[\mu,\nu]} \end{aligned} \tag{30}$$

where

$$\begin{aligned} W_{\mu} &= \frac{1}{2}\left(W^{\alpha}_{\mu\alpha} - W^{\alpha}_{\alpha\mu}\right) \\ W_{[\mu,\nu]} &= \frac{1}{2}\left(W_{\mu,\nu} - W_{\nu,\mu}\right) \end{aligned}$$

and redefining W_{μ} such that:

$$\left(\frac{2}{3} + \frac{1}{4}x\right) W_{\mu} = \frac{2}{3}\overline{W}_{\mu}$$

together with:

$$\overline{W}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \frac{2}{3}\delta^{\lambda}_{\mu}\overline{W}_{\nu}$$

we get:

$$G^{\mu\nu}R_{\mu\nu}(W) + \frac{1}{4}xG^{[\mu\nu]}\partial_{\nu}W_{\mu} = G^{\mu\nu}R_{\mu\nu}\left(\overline{W}\right)$$

This leads to the following expression for the action:

$$\mathfrak{L}{=}G^{\mu\nu}R_{\mu\nu}\left(\overline{W}\right) - f(x)\, G^{(\mu\nu)}_{\mu}\overline{W}_{\mu}\overline{W}_{\nu}$$

where now the interaction constant $f(x)$ is given by:

$$f(x) = \frac{x}{\left(2 + \frac{3}{4}x\right)^{2}}$$

It is worth noting that this function has an extremum (maximum) at the point:

$$x = \frac{8}{3},\ \max f(x) \ = \ f\left(\frac{8}{3}\right) = \frac{1}{6}$$

so, the choice of this optimal value leads exactly to the modified Moffat's action of NGT [11]:

$$\mathfrak{L}{=}G^{\mu\nu}R_{\mu\nu}\left(\overline{W}\right) + \frac{1}{2}\sigma G^{(\mu\nu)}_{\mu}\overline{W}_{\mu}\overline{W}_{\nu}$$

where $\sigma = -\frac{1}{3}$.

Concerning the skew term added by hand by Moffat in his new action [11]:

$$\mathfrak{L}_{skew} = -\frac{1}{4}\mu^{2}\left(-g\right)^{\frac{1}{2}} G^{[\mu\nu]}G_{[\mu\nu]}$$

it can also be generated from the cosmological term:

$$\mathbf{J} = \frac{1}{2}\left(E^{A}E^{B*} - E^{B*}E^{A}, \xi^{A}\xi^{B*}\right) = \int \sqrt{e\widetilde{e}}\, d^{4}x\,\mathfrak{L}_{\cos}$$

which gives after some straightforward calculations (see the appendix):

$$\mathfrak{L}_{\cos} = -2G^{[\mu\nu]}G_{[\mu\nu]} - 4Tr\left(K.K^{*}\right)\lambda\widetilde{\lambda} - 8 - G_{ba}^{\mu\nu}G_{\nu\mu}^{ab}$$

(ii) The case $\lambda = \exp\left(\varepsilon\Phi\right)$, $\widetilde{\lambda} = \exp\left(-\varepsilon\Phi\right)$ (Φ is a real field).

By following the same steps as in case (i) , we obtain:

$$\mathfrak{L}{=}G^{\mu\nu}R_{\mu\nu}\left(\overline{W}\right) + \frac{1}{2}\sigma G_{\mu}^{(\mu\nu)}\overline{W}_{\mu}\overline{W}_{\nu} - \frac{2}{3}G^{(\mu\nu)}\partial_{\mu}\Phi\partial_{\nu}\Phi + \frac{2}{3}G^{(\mu\nu)}\varepsilon\overline{W}_{\mu}\partial_{\nu}\Phi$$

and

$\mathfrak{L}_{\cos} = -2G^{[\mu\nu]}G_{[\mu\nu]} + \frac{8}{3} - G_{ba}^{\mu\nu}G_{\nu\mu}^{ab}$

We thus have obtained an action in which NGT is coupled to a massless scalar field.

(iii) The general case $\lambda = \Phi, \widetilde{\lambda} = \widetilde{\Phi}$:

We have:

$\mathfrak{L} = G^{\mu\nu}R_{\mu\nu}\left(W\right) - \frac{1}{8}x\left\{G^{\mu\sigma}\left(\widetilde{\Phi}D_{\sigma}D_{\mu}\Phi + \Phi D_{\mu}D_{\sigma}\widetilde{\Phi}\right) - \frac{1}{2}G^{\mu\nu}W^{\sigma}{}_{\mu\nu}\left(\widetilde{\Phi}D_{\sigma}\Phi - \Phi D_{\sigma}\widetilde{\Phi}\right)\right\}$

and:

$\mathfrak{L}_{\cos} = -2G^{[\mu\nu]}G_{[\mu\nu]} + \frac{32}{3}\Phi\widetilde{\Phi} - 8 - G_{ba}^{\mu\nu}G_{\nu\mu}^{ab}$

where

$D_{\sigma}\Phi = \left(\partial_{\sigma} - W_{\sigma}\right)\Phi$

$D_{\sigma}\widetilde{\Phi} = \left(\partial_{\sigma} + W_{\sigma}\right)\widetilde{\Phi}$

$D_{\mu}D_{\sigma}\Phi = \left(\partial_{\mu} - W_{\mu}\right)\left(\partial_{\sigma} - W_{\sigma}\right)\Phi$

$D_{\mu}D_{\sigma}\widetilde{\Phi} = \left(\partial_{\mu} + W_{\mu}\right)\left(\partial_{\sigma} + W_{\sigma}\right)\widetilde{\Phi}$

We have thus obtained an action in which NGT is now coupled to a massive scalar field.

3. Conclusions

In this work, we have generalized the ordinary formalism of non commutative geometry to derive the various terms of the new Moffat's version of the NGT Lagrangian..

Moreover, and as a consequence of the discrete structure of space time, an additional unwanted interaction term has arised. In order to get rid of it, we were lead to generalize the notion of trace and introduce a γ matrices ordering operator P.

Furthermore, the construction of the action with the scalar field coupled to NGT was possible by taking a general form of the generators of the 1-forms space.

It is worth noting that in the term $\frac{1}{2}\sigma G_{\mu}^{(\mu\nu)}\overline{W}_{\mu}\overline{W}_{\nu}$, the value of the coupling constant σ comes naturally as the maximum of the function $f\left(x\right)$. This suggests that even the couplings may have a geometrical interpretation.

4. Appendix

4.1 The Cosmological Term

One can add to the action the following cosmological term:

$$\mathbf{J} = \frac{1}{2}\left(E^A E^{B*} - E^{B*} E^A, \xi^A \xi^{B*}\right)$$

A straightforward calculation gives us the following expressions:
$\frac{1}{2} Tr\mathfrak{Tr}\left(E^b E^{a*} - E^{a*} E^b\right)\left(\xi^b \xi^{a*}\right) = Tr\mathfrak{Tr}\left(\gamma^b \gamma^a \tau - \gamma^a \gamma^b \overline{\tau}\right)\left(\gamma^c \gamma^d e^a_\nu \widetilde{e}^b_\mu\right)\left(\eta^{\mu\nu}_{cd} - \varepsilon g^{\mu\nu}_{cd} \tau_3\right)$
$= G^{\mu\nu} G_{\nu\mu} - G^{\mu\nu}_{ab} G^{ba}_{\nu\mu} - 12$
and:
$\frac{1}{2}\left(E^5 E^{b*} - E^{b*} E^5, \xi^b \xi^{5*}\right) = \frac{1}{2} Tr\mathfrak{Tr}\left(E^5 E^b - E^b E^5\right) \xi^b \xi^{5*}$
$= \frac{1}{2} Tr\mathfrak{Tr}\left(\gamma^a \gamma^5 K^* \widetilde{\lambda}\right)\left(-K \gamma^a \gamma^5 \overline{\tau} \lambda\right) = -2Tr\left(K.K^*\right) \lambda \widetilde{\lambda}$
$\frac{1}{2}\left(E^a E^{5*} - E^{5*} E^a, R^{5a}\right) = \frac{1}{2} Tr\mathfrak{Tr}\left(\left(\widetilde{E}^a E^5 - E^5 \widetilde{E}^a\right) \xi^5 \xi^{a*}\right)$
$= \frac{1}{2} Tr\mathfrak{Tr}\left(\gamma^a \gamma^5 K \lambda\right)\left(-K^* \gamma^a \gamma^5 \overline{\tau} \widetilde{\lambda}\right) = -2Tr\left(K.K^*\right) \lambda \widetilde{\lambda}$
The term $G^{\mu\nu} G_{\nu\mu}$ is written as:

$$G^{\mu\nu} G_{\nu\mu} = \eta^{\mu\nu} \eta_{\nu\mu} + g^{\mu\nu} g_{\nu\mu} = \eta^{\mu\nu} \eta_{\mu\nu} - g^{\mu\nu} g_{\mu\nu}$$

and

$$G^{\mu\nu} G_{\mu\nu} = e^\mu_a \widetilde{e}^\nu_a e^b_\mu \widetilde{e}^b_\nu = \eta^{\mu\nu} \eta_{\mu\nu} + g^{\mu\nu} g_{\mu\nu} = 4$$

so we get

$$G^{\mu\nu} G_{\nu\mu} = 4 - 2g^{\mu\nu} g_{\mu\nu} = 4 - 2G^{[\mu\nu]} G_{[\mu\nu]}$$

$$\mathbf{J} = \int \sqrt{e\widetilde{e}} d^4x \left(-2G^{[\mu\nu]} G_{[\mu\nu]} - 4Tr\left(K.K^*\right) \lambda \widetilde{\lambda} - 8 - G^{\mu\nu}_{ba} G^{ab}_{\nu\mu}\right)$$

4.2 The Condition $Tr_k T^a = 0$

If we impose the condition $Tr_k T^a = 0$ and $T^5 \neq 0$ we get:

$$Tr_k\left(T^a\right)_{11} = Tr_k\left(T^a\right)_{22} = 0 \Rightarrow Tr_\tau Tr_k\left(T^a\right)_{11} = Tr_\tau Tr_k\left(T^a\right)_{22} = 0$$

so:

$$\partial_\mu \widetilde{e}^a_\nu + \omega^{(2)ab}_\mu \widetilde{e}^b_\nu = 0 \quad (31)$$

$$\partial_\mu \widetilde{e}^a_\nu + \omega^{(1)ab}_\mu \widetilde{e}^b_\nu = 0 \quad (32)$$

One solution of (31) and (32) is given by :

$$\omega^{(1)ab}_\mu = \omega^{(2)ab}_\mu$$

Now the condition:

$$Tr_k\left(T^a_{12}\right) = Tr_k\left(T^a\right)_{21} = 0$$

implies:
$\omega_\mu^{(2)a5} = \omega_\mu^{(2)5a} = \omega_\mu^{(1)a5} = \omega_\mu^{(1)5a} = 0$

From the above results, we obtain:

$$R_{\mu\nu}^{(1)ab} = R_{\mu\nu}^{(2)ab} = R_{\mu\nu}^{ab}$$
$$R_{\mu\nu} = R_{\mu\nu} = R_{\mu\nu}$$

while from (21) and (22) we get:

$$\varphi_{12}^{ab} = \varphi_{21}^{ab} = 0 \tag{33}$$

Similarly, the constraint:

$$Tr_k\left((T^a)_{11} - (T^a)_{22}\right) = 0$$

leads to:

$$-\tau_3\widetilde{\lambda}\varphi_{12}^{a5} + \tau_3\lambda\varphi_{21}^{a5} = 0$$

and therefore:

$$\widetilde{\lambda}\varphi_{12}^{a5} = \lambda\varphi_{21}^{a5} \Leftrightarrow \widetilde{\lambda}\varphi_{12}^{5a} = \lambda\varphi_{21}^{5a} \tag{34}$$

Consequently:

$$\varphi_{12}^{a5}\varphi_{21}^{5a} = \varphi_{21}^{a5}\varphi_{12}^{5a}$$

Taking now into account the equation (33) one has:

$$\varphi_{12}^{aB}\varphi_{21}^{Ba} = \varphi_{21}^{aB}\varphi_{12}^{Ba}$$

Finally from (23) and (24), we end up with:

$$\varphi_{21}^{5b} = \widetilde{e}_b^\mu\partial_\mu\widetilde{\lambda} + \widetilde{e}_b^\mu\omega_\mu^{(2)55}\widetilde{\lambda}$$

$$\varphi_{12}^{b5} = e_b^\mu\partial_\mu\lambda - e_b^\mu\omega_\mu^{(2)55}\lambda$$

4.3 The Unitarity Condition

The unitarity condition $\left(\Omega^{AB}\right)^* = \left(\Omega^{BA}\right)$ which takes the form:

$$\begin{pmatrix} -\gamma^a E_a^\mu\widetilde{\omega}_\mu^{(1)AB} & \gamma^5 K M_{12}\widetilde{\phi}_{21}^{AB} \\ \gamma^5 K^* M_{21}\widetilde{\phi}_{12}^{AB} & -\gamma^a E_a^\mu\widetilde{\omega}_\mu^{(2)AB} \end{pmatrix} = \begin{pmatrix} \gamma^a E_a^\mu\omega_\mu^{(1)BA} & \gamma^5 K M_{12}\phi_{12}^{BA} \\ \gamma^5 K^* M_{21}\phi_{21}^{BA} & \gamma^a E_a^\mu\omega_\mu^{(2)BA} \end{pmatrix}$$

leads to the following constraints:

$$\widetilde{\omega}_\mu^{(1)AB} = -\omega_\mu^{(1)BA}\ ,\ \ \widetilde{\omega}_\mu^{(2)AB} = -\omega_\mu^{(2)BA}$$
$$\widetilde{\phi}_{12}^{AB} = \phi_{21}^{BA}\ ,\ \ \widetilde{\phi}_{21}^{AB} = \phi_{12}^{BA}$$

$$\widetilde{R}^{AB}_{\ \ \mu\nu} = -R^{BA}_{\ \ \mu\nu}$$

4.4 The Generalized Trace

In the Euclidean case, the γ^a Dirac matrices satisfy:

$$\gamma^{a*} = -\gamma^a \quad , \quad \left\{\gamma^a, \gamma^b\right\} = \gamma^a\gamma^b + \gamma^b\gamma^a = -2\delta^{ab}$$
$$\gamma^{ab} = \frac{1}{2}\left[\gamma^a, \gamma^b\right] \ , \ \gamma^{(ab)} = \frac{1}{2}\left\{\gamma^a, \gamma^b\right\} = -\delta^{ab}$$

Now, the generalized trace $\mathfrak{Tr} = Ptr$ is defined as:

$\mathfrak{Tr}\gamma^a\gamma^b = Ptr\gamma^a\gamma^b = tr\gamma^a\gamma^b = tr\gamma^{(ab)} = -\delta^{ab}$

$\mathfrak{Tr}\gamma^a\gamma^b\gamma^c\gamma^d = Ptr\gamma^a\gamma^b\gamma^c\gamma^d = \delta^{ab}\delta^{cd} - \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}$

$\mathfrak{Tr}\gamma^a\gamma^b\gamma^{(cd)} = Ptr\gamma^a\gamma^b\gamma^{(cd)} = \delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}$

$\mathfrak{Tr}\gamma^a\gamma^b\gamma^{[cd]} = Ptr\gamma^a\gamma^b\gamma^{cd} = -2\delta^{ac}\delta^{bd}$

where P is an operator which permutes the indices of the γ^a matrices, and "tr" holds for the trace over the Clifford algebra.

Moreover, one can consider $\mathfrak{Tr}$ as an operator acting on the tensors $\varkappa_{cd}$ such that:

$\mathfrak{Tr}\gamma^a\gamma^b\varkappa_{cd} = -\delta^{ab}\varkappa_{cd}$

$\mathfrak{Tr}\gamma^a\gamma^b\gamma^c\gamma^d\varkappa_{cd} = \delta^{ab}\varkappa_{cc} - \varkappa_{ab} + \varkappa_{ba}$

$\mathfrak{Tr}\gamma^a\gamma^b\gamma^{(cd)}\varkappa_{cd} = \delta^{ab}\varkappa_{cc} + \varkappa_{ab} + \varkappa_{ba}$

$\mathfrak{Tr}\gamma^a\gamma^b\gamma^{[cd]}\varkappa_{cd} = -2\varkappa_{ab}$

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Some Important Features of Ultra-Light Particles, Induced Cosmological Constant and Massive Gravitons in Modern Cosmology Theories

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Abstract: Some important features of ultra-light masses and induced cosmological constant implemented in Einstein gravity theory from supergravities arguments and non-minimal coupling effects are presented and discussed in some details in modern cosmology where massive gravitons are taken into account.

Keywords: Ultra-Light Particles, Induced Cosmological Constant, Gravitons, Accelerated Expansion, Relativistic Theory of Gravity, Brane, Cyclic Universe
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1. Introduction

The recent discovery of dark energy as a theoretical explanation of the accelerated expansion of the Universe is a big unattended surprise in cosmology. In fact, the recent observations of the Type Ia supernovae have led to the idea that our universe undergoes accelerated expansion at the present epoch tending to a flat de-Sitter space-time as predicted by inflation theory [1,2,3]. Theoretical physics explain this fact by the presence of a non-luminous or dark matter known as "quintessence or Λ-field" with negative pressure obeying the state equation $w = p/\rho < -1/3$ and accounting for the missing energy if one really believes inflation theory in all its aspects predicting $\Omega = 1$ [4,5,6]. In reality, this strange matter present in the early phase of the Universe adds to the cosmological constant problem (the lambda problem) another ones. Several theoretical models, theories and possible solutions have been proposed, including a cosmological constant, a time-varying energy density, real and complex scalar fields, dilaton from string theory, supersymmetric exotic particles, massive neutrinos, holographic dark energy, $N = 8$ and $N = 2$ supergravity, M/string theory, etc [7,8,9,10,11]. In most of these theories, we still find difficulties to solve the cosmological constant problem, to find a suitable dynamical theoretical scenario generating a small "lambda" and explain at the same time the recent astrophysical observed parameters. In reality, there exists observational evidence where $w < -1$ (tachyonic or phantom dark energy) corresponding to a very fast acceleration or superacceleration [12,13]. It is highly important to know precisely which principles and observations restrictions we need to follow and obey. We believe that minimal coupling theories between the scalar field the scalar Ricci curvature cannot achieve such a superacceleration regime, especially with time-dependent state equation, but can be achieved in models containing only a real (or complex scalar field), non-minimally coupled to the Ricci curvature and with positive definite kinetic energy density [14,15,16]. Note that non-minimal couplings can in general be constrained by classical tests of gravity theories [17]. Recently, we have investigated a particular cosmological model with complex scalar self-interacting inflation field non-minimally coupled to gravity, based on supergravities argument [18]. It was shown that in the case of non-minimal coupling between the scalar curvature and the density of the scalar field such as $L = -(\xi/2)\sqrt{-g}R\phi\phi^*$ ($R$is the scalar curvature or the Ricci scalar,ξ is the non-minimal coupling constant and g is the scalar metric) and for a particular scalar complex potential field $V(\phi\phi^*) = (3m^2/4)(\zeta\phi^2\phi^{*2} - 1)$,($\zeta = O(1)$), inspired from supergravity inflation theories, ultra-light masses m ($|m^2| \approx H^2$) are implemented naturally in the Einstein field equations (EFE), leading to a cosmological constant Λ in accord with observations. The metric tensor of the spacetime is treated as a background and the Ricci scalar in the non-minimal coupling term, regarded as an external parameter, was found to be $R = 4\Lambda - 3m^2 = 4\bar{\Lambda}$where $\bar{\Lambda} = \Lambda - (3m^2/4)$ is the effective cosmological constant. In fact, the property $|m^2| \sim H^2$ is required in many theoretical quintessence models based on extended supergravity. These ultra-light masses occur in supergravity quintessence model with de-Sitter minimum and have important additional features in astrophysics, cosmology, the early universe and the standard electroweak model [19,20,21,22,23,24]. Adopting this result, we explored in a recent paper some of the consequences concerning their presence in the field equations. Briefly talking, when we ignore all gauge coupling, the Klein-Gordon equation (KGE) with no mass term associated with this choice reads $\Box\phi - (R/6)\,\phi - \partial V/\partial\phi^* = 0$. In fact, a possible mass term for the scalar field and the cosmological constant are embodied in our quartic complex potential $V(\phi\phi^*)$ and our scalar curvature$R = 4\Lambda - 3m^2$. By substituting into the last equation, we obtain $\Box\phi - M^2\phi = 0$ where $M^2 = \tilde{M}^2 + 2h\phi\phi^*$, $\tilde{M}^2 = (2\Lambda/3) - (m^2/2) = \xi R$ and $h = (3/4)\,\zeta m^2 > 0$. If we treat the Ricci scalar as a parameter, the complex potential takes the following form $\tilde{V}(\phi\phi^*) = (2\Lambda\xi - (3m^2/2)\,\xi)\,\phi\phi^* + (h/2)\,(\phi\phi^*)^2$ which can be written in a real-form as $V(\varphi) = (1/2)\,\mu^2\varphi^2 + (\lambda/4)\,\varphi^4 \equiv (\lambda/2)\left(\varphi^2 - v^2\right)^2 + V_0$ assuming $\phi(t) = \varphi(t)\exp(i\theta(t))$ where $\mu^2 = (4\Lambda - 3m^2)\,\xi$, $\lambda = (3/2)\,\zeta m^2 > 0$, $\mu^2 = -\lambda v^2$ and $V_0 = -\mu^4/4\lambda \equiv -\left(4\Lambda - 3m^2\right)^2\xi^2\Big/4\lambda$. The point $\varphi = \mathrm{v}$ corresponds to the minimum of the negative potential with of course a spontaneous symmetry breaking (SSB). In our model, the presence of ultra-light particles is responsible for inducing the SSB provided that $4\Lambda < 3m^2$ and $\xi > 0$. Note that both constraints have no effects on the sign of V_0. For the particular case $4\Lambda = 9m^2$ and $\zeta = 0$ we have$R = 6m^2 > 0$, and as a result we can obtain $(\Box - R/6)\,\phi = 0$. The Higgs field is then supplanted by a massless KGE conformally coupled to the scalar curvature with tiny real rest mass. Consequently, for $h = \Lambda = 0$, the effective cosmological constant is negative, SSB still occurs and is controlled by the presence of the term m^2. The major feature of our Mexican potential is its dependence on the scalar curvature and on the non-minimal coupling parameter. When the scalar curvature is zero or $4\Lambda = 3m^2$, the potential vanishes whatever is the sign of ξ. That is the universe is flat without inflation. But in reality, SSB occurs if $4\Lambda < 3m^2$, that is we are dealing with a negative potential unless $\xi < 0$. We are in fact interested in the following two cases: ($\xi > 0, 4\Lambda < 3m^2$) and ($\xi < 0, 4\Lambda < 3m^2$) corresponding for SSB in the first evolution stage of the universe. This would imply that we live in an open Friedmann-Robertson-Walker (FRW) accelerating universe with negative spatial curvature and positive cosmological constant[25]. The resulting field equations for a matter free universe but dominated with dark energy scalar field give the Ricci scalar for zero index curvature as $R = 6\left(\dot{H} + H^2\right) = 4\Lambda - 3m^2 = 4\bar{\Lambda}_-$ or in the following form $\dot{H} + H^2 = 2\bar{\Lambda}_-\big/3 \equiv \tilde{V}_-$. This is a Riccatti equation indicating the possibility of a time or Hubble-dependent effective cosmological constant[26-32]. Another interesting case corresponds to $\tilde{V} > 0$ and positive scalar curvature. This is achieved when we deal with negative potential and negative cosmological constant, e.g. $V(\phi\phi^*) = -\left(3m^2/4\right)\left(\zeta\phi^2\phi^{*2} - 1\right)$ and $\Lambda < 0$. This case is favoured by string theorists. In this way $R = 6\left(\dot{H} + H^2\right) = -4\hat{\Lambda} + 3m^2, \hat{\Lambda} = -\Lambda$ and $\dot{H} + H^2 \equiv \tilde{V}_+$. ($X_\pm$ corresponds to whether X is positive or negative respectively). In fact, letting $r^2 = \tilde{V}_+$, the solution is given by $H(t) = r(e^{rt} - ke^{-rt})/(e^{rt} + ke^{rt})$, $k$is a constant[33]. Let us observe that as time becomes infinite, the Hubble parameter approaches the limiting value r, that is $H = \sqrt{\tilde{V}_+}$ implying $3H_\infty^2 = 2\left(3m^2 - 4\hat{\Lambda}\right), \Lambda < 0, 3m^2 > -4\Lambda$. In this way, $m \approx H_\infty$

as expected from sueprgravities theories[5]. The new potential $\tilde{V}$is negative and this will lead to a contraction[34]. Therefore, we can write $H\left(t\right) = H_{\infty}(e^{rt} - ke^{-rt})/(e^{rt} + ke^{rt})$. If at the origin of time, $H\left(t\right) \equiv H_0$, then $k = (1-\theta)/(1+\theta)$, $\theta \equiv H_0/H_{\infty}$. In this case, the solution can be written in the following form $H\left(t\right) = H_{\infty}(\theta + k \tanh rt)/(1 + \theta \tanh rt)$. The corresponding scale factor $a\left(t\right)$ evolves as $a\left(t\right) \propto \left(\cosh rt + \theta \sinh rt\right)^{H_{\infty}/r}$assuming that at the origin of time $a = a_0$, e.g. a non-singular Universe.[2]* A negative cosmological constant may be consistent with recent astronomical observations if the present accelerating is generated by another dark energy component [35].

Motivated by these results, we present in this paper some additional important cosmological and astrophysical features and implications of the ultra-light masses and the induced cosmological constant. The paper is organized in independent sections as follows: in section **2**, we will explore the evolution of a homogeneous flat universe but with positive scalar curvature dominated by ultra-light matter and the cosmological constant and we investigate about the necessary condition for eternal accelerated expansion with no need of any form of tachyonic matter. In section **3**, some important aspects and features of a cosmological model with phenomenological decay of the ultra-light masses, the vacuum cosmological constant and the matter density are discussed. In section **4**, we use extended supergravities theories to show the important role playing by the decaying gravitons masses and decaying quantized ultra-light masses in the evolution Universe within the framework of the Relativistic Theory of Gravitation (RTG). In section **5**, we proposed a modification of the Standard Hot Big Bang Cosmology (*SHBBC*), in which the Universe is flat with a total energy density taken to be the sum of the contributions from vacuum and ultra-light mass term responsible of the dominant driver of expansion at a late epoch of the Universe. In section **6** we illustrate the modification of the gravitational wave propagation through a special vacuum medium with a total density being the sum of the true vacuum density and the ultra-light particle supergravity density and finally conclusions are given in section **7**.

2. Supergravity, Ultra-light Masses and New Condition for Eternal Accelerated Expansion

As we mentioned in the previous paragraph, it is a matter of fact that the discovery of the accelerated expansion of the universe plays an important and leading role in modern cosmological theories [1,2]. After a lot of investigations, the dark energy seems for the majorities the responsible. Of course, one can deal with dark energy in a polite and soft way including the celebrated cosmological constant and quintessence [35]. Within the same context and within the framework of standard Friedmann-Roberston-Walker (**FRW**) cosmology, the resulting dynamical equations for an homogeneous flat universe dominated

[2] **The possibility that we are living in a non-singular accelerating Universe with positive scalar Ricci curvature generated from negative extended supergravity potential is an interesting idea, in particular for string theorists. It may have additional interesting features that are under progress.*

by the cosmological constant and the ultra-light masses are[18,19,20,21,23,24,36,37]:

$$\frac{\dot{a}^2}{a^2} = \frac{\Lambda}{3}\left(1 + \frac{3m^2}{\Lambda}\right) \tag{1}$$

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3}\left(1 - \frac{3m^2}{2\Lambda}\right) \tag{2}$$

We know from inflation scalar theory that $m^2/\Lambda \approx \dot{\phi}$, $\dot{\phi} \equiv d\phi/dt$ being the derivative with respect to time of the inflaton scalar field [34]. For this, we feel motivated to define $\dot{F}^2 = \alpha m^2/\Lambda, F$ represents our new field and α is a positive constant. That is

$$\frac{\dot{a}^2}{a^2} = \frac{\Lambda}{3}\left(1 + 3\frac{\dot{F}^2}{\alpha}\right) \tag{3}$$

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3}\left(1 - \frac{3}{2}\frac{\dot{F}^2}{\alpha}\right) \tag{4}$$

The expansion is accelerated if $\ddot{a}/a > 0$, that is $m^2/\Lambda \equiv \dot{F}^2\big/\alpha < 2/3$. As a result, the universe is flat but with positive scalar curvature. In fact, one can recover equations (3) and (4) from another alternative by assuming that the cosmos is dominated by a special fluid with density and pressure behaving like:

$$\rho = \frac{\Lambda}{8\pi G}\left(1 + 3\frac{\dot{F}^2}{\alpha}\right) \tag{5}$$

$$\rho + 3p = -\frac{\Lambda}{4\pi G}\left(1 - \frac{3}{2}\frac{\dot{F}^2}{\alpha}\right) \tag{6}$$

The negative fluid pressure is easily found to be independent of $\dot{F}$, that is $p = -\Lambda/8\pi G$ $(\Lambda > 0)$ and from equation (5) $p = -\rho\Big/\left(1 + 3\dot{F}^2\right)$ or:

$$p = -\frac{\rho}{1 + 3\frac{m^2}{\Lambda}} \equiv \omega\rho \tag{7}$$

where $\omega = -1/(1 + 3m^2/\Lambda) = -1/(5 - R/\Lambda) \approx -1/(5 - R/\gamma H^2)$ (*where* $R = 4\Lambda - 3m^2$ *and* $\Lambda \equiv \gamma H^2$, $0 < \gamma \leq 3$). The dependence of ω on the scalar curvature is a remarkable feature. Equation (7) tells us that for $m^2 = 0$ or $\dot{F} = 0$, $p = -\rho$ or $\omega = -1$. From equation (6), $\dot{F}$ reaches its limiting value at $\dot{F}^2 = 2\alpha/3$. In this case, $2\Lambda = 3m^2$ and $p = -\rho/3$ or $w = -1/3$. That is $-1 \leq w \leq -1/3$ or in other words, we have eternal accelerated expansion. That is the flat universe with positive scalar curvature filled with fluid having negative pressure, positive density and positive kinetic energy is accelerating with time with no need of tachyon matter. If for our massive field $m^2 << \Lambda \sim H^2$, than $p \approx -\rho\left(1 - 3m^2/\Lambda\right)$. In this way, $w \approx 3m^2/\Lambda - 1 > -1$. A remarkable feature of equations (3) and (4) is the presence of some similarity with Sen tachyon matter cosmology [36,37]. In fact, letting

$$V_{\text{effective}}\,(F) \equiv \tilde{V}\,(F) = \frac{\Lambda_{\text{effective}}}{8\pi G} \tag{8}$$

$$\begin{aligned}
\Lambda_{\text{effective}} &= \Lambda\left(1+3\frac{\dot{F}^2}{\alpha}\right)\sqrt{1-\frac{\dot{F}^2}{\alpha}} && (9)\\
&= \Lambda\left(1+3\frac{m^2}{\Lambda}\right)\sqrt{1-\frac{m^2}{\Lambda}} && (10)\\
&= \Lambda\left(1+3\mu H^2\right)\sqrt{1-\mu H^2} && (11)\\
&= \left(3H^2+2\dot{H}\right)\left(1+3\mu H^2\right)\sqrt{1-\mu H^2} && (12)
\end{aligned}$$

where $\tilde{V}\,(F)$ is the effective potential of the field F and μ is assumed to be a positive constant, equations (3) and (4) takes the form:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\frac{\tilde{V}\,(F)}{\sqrt{1-\frac{\dot{F}^2}{\alpha}}} \tag{13}$$

$$\frac{\ddot{a}}{a} = \frac{8\pi G}{3}\frac{\tilde{V}\,(F)}{\left(1+3\frac{\dot{F}^2}{\alpha}\right)\sqrt{1-\frac{\dot{F}^2}{\alpha}}}\left(1-\frac{3}{2}\frac{\dot{F}^2}{\alpha}\right) \tag{14}$$

The cosmological constant in contrast to Sen's model is not set equal to zero. Equation (13) is the Friedman equation in the standard form within the framework of Sen tachyon field (α=1) and equation (14) is the Raychaudhuri equation within the same framework but with the presence of the term $1+\left(3\dot{F}^2\big/\alpha\right)$. In this way, the accelerated cosmos expansion ($m^2/\Lambda \equiv \dot{F}^2\big/\alpha < 2/3$, $\Lambda > 0$) is dominated by a special fluid with density and pressure as:

$$\rho = \frac{\tilde{V}\,(F)}{\sqrt{1-\frac{\dot{F}^2}{\alpha}}} \tag{15}$$

$$p = -\tilde{V}\,(F)\,\frac{1}{\left(1+3\frac{\dot{F}^2}{\alpha}\right)\sqrt{1-\frac{\dot{F}^2}{\alpha}}} = -\frac{\rho}{1+3\frac{\dot{F}^2}{\alpha}} \equiv w\rho \tag{16}$$

with

$$\rho + p = \frac{\tilde{V}\,(F)}{\sqrt{1-\frac{\dot{F}^2}{\alpha}}}\left(\frac{3\frac{\dot{F}^2}{\alpha}}{1+3\frac{\dot{F}^2}{\alpha}}\right) \tag{17}$$

where again $w = -1/(1+3m^2/\Lambda)$. The entropy conservation of our special fluid will be:

$$\frac{d}{dt}\left(\frac{\tilde{V}\,(F)}{\sqrt{1-\frac{\dot{F}^2}{\alpha}}}\right) + 3\frac{\dot{a}}{a}\left(\frac{\tilde{V}\,(F)}{\sqrt{1-\frac{\dot{F}^2}{\alpha}}}\left(\frac{3\frac{\dot{F}^2}{\alpha}}{1+3\frac{\dot{F}^2}{\alpha}}\right)\right) = 0 \tag{18}$$

and the field energy decreases with time as it is expected due to the universe accelerated expansion. In summary, we have investigated the possibility of having accelerated eternal expansion without implementing any form of tachyonic matter as long as the following condition $m^2/\Lambda < 2/3$ holds. As a result, the Ricci scalar curvature is positive. The cosmos is assumed to be filled with a special fluid with positive cosmological constant, positive energy density and negative energy pressure obeying the particular state equation $p = w\rho$, where $-1 \leq w = -1/(1 + (3m^2/\Lambda)) \leq -1/3$. This indicates how importance is the presence of the ultra-light masses implemented in our theory from supergravity and non-minimal coupling arguments as necessary ingredients in modern cosmology.[3*]

3. Cosmic Acceleration with Two Decaying Cosmological Constants

The unified modern gauge field theories inform us that it is impossible to neglect the Einstein cosmological constant "Λ" introduced by Albert Einstein in the search for static cosmological solutions to his General Relativity Theory. This famous Lambda is so tiny but in the context of high energy physics, it is not really the case. When quantum corrections are used, that is, within the context of quantum gravity, we get at Planck scale, a cosmological constant cancelled at the electroweak scale energy scale and below to one part in 10^{119} [38-44]. As a result, the problem still persists and it seems that the "*lambda*" problem is a difficult physical problem at very "*low energies scales*". There exist in literature several attempts to reduce the vacuum density to a very small value over cosmological timescales, at least from dynamical point of view. It has also been noted that spontaneous symmetry breaking and phase transitions can be induced by curvature via the non-minimal coupling with the external gravitational field [45-49]. It was shown for the simple case of a scalar field with a potential depending on the Ricci scalar or spacetime curvature that if the field starts near the origin, it will slows down and the vacuum density will decrease as inverse of the square of time ($\rho_{vac} \propto t^{-2}$). These suggest the beautiful possibility that the universe evolves to a state with time decaying effective cosmological constant. Such scenarios are interesting because *a red-shifting vacuum energy can have effects over many expansion times, while a constant vacuum density could only have become dynamically important at very recent epochs....and dynamical models of decaying vacuum energy of a rather general variety are consistent with observational cosmology; however, the deviation from the standard model must be small*[44]. Particle creation of light nonminimally coupled scalar fields due to the changing geometry of a spacetime at an early inflationary phase was also treated and discussed in details by several authors leading to a total density parameter $\Omega_T \approx 1$ [50-55]. Inspired from the relation between the cosmological constant, the ultra-light masses and the Ricci scalar, we assume that in

[3] * *An important issue to treat in a future work is the impact of all these in string theory, the connection to tachyon field theory as well as the implication of the effective cosmological constant represented by equation (12) in standard* ***FRW*** *cosmology.*

addition to the ordinary matter and radiation, there is a time-dependent energy density associated with the ultra-light masses, say $\rho_m(t) \equiv 3m^2(t)/8\pi G$ and $\Lambda = \Lambda(t)$ where $m(t)$ represents the ultra-light masses, assuming at first that the gravitational constant "G" is constant and that the ultra-light pressure is zero. This additional or extra energy density must certainly appear in the Friedman dynamical equation (*solution of the Einstein's field equations*) describing the "*flat*" FRW standard cosmology as follows:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\left(\rho + \frac{3m^2}{8\pi G}\right) + \frac{\Lambda}{3} = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} + m^2 \tag{19}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3m^2}{8\pi G} + 3p\right) + \frac{\Lambda}{3} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} - \frac{m^2}{2} \tag{20}$$

"p" and "ρ" are the pressure and density of the perfect fluid with equation of state assuming through this work to be $p = (\tilde{\gamma} - 1)\rho$, "$\tilde{\gamma} \equiv w + 1$" being a constant. Equations (19) and (20) give:

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -8\pi G p + \Lambda \tag{21}$$

$$\frac{d(\rho a^3)}{dt} + p\frac{d}{dt}\left(a^3\right) + \frac{a^3}{8\pi G}\frac{d}{dt}\left(\Lambda + 3m^2\right) = 0 \tag{22}$$

The goal of this work is to study the effect of the presence of phenomenological decaying ultra-light masses on cosmic acceleration. For this we follow Arbab, Ozer, Lima, etc., and we consider the following three phenomenological laws: $m^2(t) = \varepsilon\dot{a}^2/a^2$, $\Lambda(t) = \beta\ddot{a}/a$ and $\rho(t) = (3\delta/8\pi G)\,\dot{a}^2/a^2$, ε,β and δ are constants[56-66]. Remember that the FRW Ricci scalar contains both terms $\dot{a}^2/a^2$ and $\ddot{a}/a$[67]. Equation (21) yields:

$$(2-\beta)\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\left(1 + 3\delta(\tilde{\gamma} - 1)\right) = 0 \tag{23}$$

with the general solution:

$$a(t) = \left(C\frac{\beta - 3 - 3\delta(\tilde{\gamma} - 1)}{\beta - 2}t\right)^{\frac{\beta-2}{\beta-3-3\delta(\tilde{\gamma}-1)}} \tag{24}$$

C is a constant, $\beta \neq 2$. Note that in this scenario, the scale factor is independent of the ultra-light masses parameter ε. It follows that:

$$\Lambda(t) = \frac{(\beta-2)(\beta+6\delta(\tilde{\gamma}-1))}{(\beta-3-3\delta(\tilde{\gamma}-1))^2}\frac{1}{t^2} \tag{25}$$

$$m^2(t) = \frac{\varepsilon(\beta-2)^2}{(\beta-3-3\delta(\tilde{\gamma}-1))^2}\frac{1}{t^2} \tag{26}$$

$$\rho(t) = \frac{3\delta}{8\pi G}\frac{(\beta-2)^2}{(\beta-3-3\delta(\tilde{\gamma}-1))^2}\frac{1}{t^2} \tag{27}$$

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2} = \frac{1+3\delta(\tilde{\gamma}-1)}{2-\beta}, \beta \neq 2 \tag{28}$$

$$H \equiv \frac{\dot{R}}{R} = \frac{\beta - 2}{\beta - 3 - 3\delta\left(\tilde{\gamma} - 1\right)}\frac{1}{t} \tag{29}$$

Equation (28) represents the deceleration parameter.

- For the matter dominated universe, $\tilde{\gamma} = 1$, than for $\beta > 2$, $q < 0$, $\Lambda > 0$, $m^2 = \left(\varepsilon\left(\beta - 2\right)/\beta\right)\Lambda > 0$, the density parameter of the universe $\Omega_M = \rho/\rho_c = \delta$ ($\rho_c = 3H^2/8\pi G, H \equiv \dot{a}/a$ *are respectively the critical density and the Hubble parameter*), the density parameter of the Λ-vacuum $\Omega_\Lambda = \Lambda/3H^2 = \left(\beta/3\left(\beta - 2\right)\right)$, the density parameter of the m^2-vacuum defined as $\Omega_m = m^2/H^2 = \varepsilon$. The total density parameter defined as $\Omega_T = \Omega_M + \Omega_\Lambda + \Omega_m$. In our scenario, the ultra-light masses vary as $m^2\left(t\right) = \varepsilon(\beta - 2)/(\beta - 3)t^2$. In order to have $m^2 > 0$, we need to have $\varepsilon > 0$ and $\beta > 3$. As a result for:

$\beta = 3.5, q = -2/3, \Omega_M^0 = \delta, \Omega_\Lambda^0 = 7/9, \Omega_m^0 = \alpha, \Omega_T^0 = \delta + 7/9 + \varepsilon \Rightarrow \delta + \varepsilon \approx 2/9$so that $\Omega_T^0 \approx 1$
$\beta = 4, q = -0.5, \quad \Omega_M^0 = \delta, \Omega_\Lambda^0 = 2/3, \Omega_m^0 = \alpha, \Omega_T^0 = \delta + 2/3 + \varepsilon \Rightarrow \delta + \varepsilon \approx 1/3$for $\Omega_T^0 \approx 1$
$\beta = 5, q = -0.33, \quad \Omega_M^0 = \delta, \Omega_\Lambda^0 = 5/9, \quad \Omega_m^0 = \alpha, \Omega_T^0 = \delta + 5/9 + \varepsilon \Rightarrow \delta + \varepsilon \approx 4/9$ for $\Omega_T^0 \approx 1$
$\beta = 6, q = -0.25, \Omega_M^0 = \delta, \Omega_\Lambda^0 = 1/2, \quad \Omega_m^0 = \alpha, \Omega_T^0 = \delta + 1/2 + \varepsilon \Rightarrow \delta + \varepsilon \approx 1/2$ for $\Omega_T^0 \approx 1$

(*The subscript* 0 *denotes the present value of the quantity*). Arbab suggests the value $\beta = 5$ as the best fit-value with the age of the universe [57]. In our scenario, for this value of "β", the universe is accelerating with time with a positive decaying cosmological constant, positive decaying ultra-light masses and positive decaying matter density accompanied with $\Omega_T^0 \approx 1$, $\delta \in (0.3 \pm 0.1)$ and $\varepsilon \approx 4/9 - \delta > 0$. The age of the universe is found to be from equation (29) to be larger than the standard model. If $\beta = 3$, we fall into the inflationary regime.

- For the radiation dominated universe, $\tilde{\gamma} = 4/3$ yielding from the above equations and requirements:

$$\Lambda\left(t\right) = \frac{\left(\beta - 2\right)\left(\beta + 2\delta\right)}{\left(\beta - 3 - \delta\right)^2}\frac{1}{t^2} \tag{30}$$

$$m^2\left(t\right) = \frac{\varepsilon\left(\beta - 2\right)^2}{\left(\beta - 3 - \delta\right)^2}\frac{1}{t^2} \tag{31}$$

$$\rho\left(t\right) = \frac{3\delta}{8\pi G}\frac{\left(\beta - 2\right)^2}{\left(\beta - 3 - \delta\right)^2}\frac{1}{t^2} \tag{32}$$

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2} = \frac{1 + \delta}{2 - \beta}, \beta \neq 2 \tag{33}$$

As a result, the density parameter of the universe at the radiation epoch $\Omega_M = \rho/\rho_c = \delta$, the density parameter of the Λ-vacuum $\Omega_\Lambda = \Lambda/3H^2 = \left((\beta + 2\delta)/3\left(\beta - 2\right)\right)$, the density parameter of the m^2-vacuum defined as $\Omega_m = m^2/H^2 = \varepsilon$. The total density parameter defined as $\Omega_T = \Omega_M + \Omega_\Lambda + \Omega_m > 1$.

If the value of $\beta = 5$ still holds in radiation dominated era, $\Omega_\Lambda^0 = (5 + 2\delta)/9$, $\Omega_T^0\left(radiation\right) = \varepsilon + \delta + (5 + 2\delta)/9 \Rightarrow \varepsilon + 11\delta/9 \approx 4/9$so that $\Omega_T^0\left(radiation\right) \approx 1$,

$q<0$, $\Lambda>0$ and $m^2>0$. Again this implies $\varepsilon\approx(4-11\delta)/9>0$ if $\delta<0.36$. For $\beta=3$, $a(t)\propto t^{-1/\delta}$. For the same value of β, but for $\tilde{\gamma}=0$ ($p=-\rho$), we find $a(t)\propto t^{1/3\delta}$, $q=3\delta-1<0$ for $\delta<1/3$. For this particular case, the vacuum universe is accelerating with time. Astronomical observations predict $\delta=0.3\pm0.1$, that is to conclude that for the particular value $\beta=3$, the universe will not belong to an inflationary regime as predicted by inflation cosmology. For $\beta=5$ and $\tilde{\gamma}=0$, $a(t)\propto t^{3/(2+3\delta)}$ with the deceleration parameter $q=(3\delta-1)/(3)<0$ for $\delta<1/3$. In summary for $\beta=5$ and $\delta<1/3$ with $(\Lambda(t), m^2(t), \rho(t)\propto t^{-2})$:

Vacuum state: $p=-\rho$, $a(t)\propto t^{3/(2+3\delta)}$, $q<0$
Radiation dominated epoch: $p=\rho/3$, $a(t)\propto t^{3/(2-\delta)}$, $q<0$
Matter dominated epoch: $p=0, a(t)\propto t^{3/2}$, $q<0$

For the particular case $\delta=0.25$ with $(\Lambda(t), m^2(t), \rho(t)\propto t^{-2})$:

Vacuum state: $p=-\rho, a(t)\propto t^{1.09}, q<0$
Radiation dominated epoch:$p=\rho/3, a(t)\propto t^{1.7}, q<0$
Matter dominated epoch: $p=0, a(t)\propto t^{1.5}, q<0$

For this particular case, the universe will undergo an accelerating regime when passing from the vacuum state to the radiation dominated epoch, continue its acceleration with time but with a tiny decrease of its accelerating expansion. If the gravitational constant is assumed to vary with time, than for the case of matter dominated era, one can easily prove that $G(t)\propto t^{\beta/(\beta-3)}$ while in the radiation dominated era, $G(t)\propto t^{(\beta+2\delta)/(\beta-3-\delta)}$ and in the vacuum state, $G(t)\propto t^{(\beta-6\delta)/(\beta-3+3\delta)}$[57]. Again, for $\beta=5$ and $\delta=0.25$ with $(\Lambda(t), m^2(t), \rho(t)\propto t^{-2})$:

Vacuum state: $p=-\rho, a(t)\propto t^{1.09}, q<0, G(t)\propto t^{2.5}$
Radiation dominated epoch: $p=\rho/3, a(t)\propto t^{1.7}, q<0, G(t)\propto t^{3.14}$
Matter dominated epoch: $p=0, a(t)\propto t^{1.5}, q<0, G(t)\propto t^{1.27}$

In this case, the gravitational constant passes through different passes, it increases when passing from the vacuum state to the radiation dominated epoch, continue decreasing but with a reducing factor when passing to the matter dominated era[57,67,68,69,70]. In conclusion, for the special parameters $\beta=5$ and $\delta=0.25$, the universe will passes through different phases. The transition from the vacuum state to the radiation one is followed by a increasing in the gravitational constant, an increasing of the cosmic acceleration while the ultra-light masses, the cosmological constant and the density decreases as $1/t^2$. The transition from the radiation dominated epoch to the matter dominated one is also followed by increasing of the gravitational constant but with a reduced term and this reducing follows also the scale factor as well as $m^2(t), \Lambda(t)$ and $\rho(t)$. Certainly the exact choice and exact values of the parameters within the theory described need more astrophysical data.

4. Extended Supergravities and Accelerated Universe from Relativistic Theory of Gravitation with Decaying Gravitons and Quantized Ultra-Light Masses

In **RTG**, Minkowski space is the fundamental space for all physical fields. The Special Theory of Relativity (**STR**) was assumed to play the major role in the construction of **RTG** [71,72]. The Faraday-Maxwell physical gravitational field in Minkowski space is introduced and allows the notion of an energy-momentum tensor of the gravitational field to be used. This means that the gravitational field could be localize. The gravitational field in **RTG** is characterized by the curvature tensor alone. While in Einstein General Relativity (**EGR**), it is characterized by both the curvature and the 4-vector of force. The **RTG** denies the total geometrization. The gravitational field is described by a real physical field with energy-momentum density and the rest mass m and a polarization states corresponding to spin two and zero, that is a symmetric second rank tensor $\phi^{\mu\nu}$constraints to the condition $D_\mu \phi^{\mu\nu} = 0$ where D_μ is the covariant derivative in the Minkowski space. The field $\phi^{\mu\nu}$is related to the Minkowski space metric $\gamma^{\mu\nu}$via the Lagrangian density of matter according to the following rule

$$L_M \left(\tilde{\gamma}^{\mu\nu}, \phi_{matter}\right) \rightarrow L_M \left(\tilde{g}^{\mu\nu}, \phi_{matter}\right) \tag{34}$$

where $\tilde{g}^{\mu\nu} = \tilde{\gamma}^{\mu\nu} + \tilde{\phi}^{\mu\nu}$ and $\tilde{a}^{\mu\nu} = \sqrt{-g}a^{\mu\nu}, a = \{g, \gamma, \phi\}$.The effective Riemann space is produced by $\phi^{\mu\nu}$ in the Minkowski space. This means that complicated topology is excluded from RTG. The total Lagrangian density of matter and gravitational field is then given by

$$L = \frac{1}{16\pi}\tilde{g}^{\mu\nu}\left(G^\lambda_{\mu\nu}G^\sigma_{\lambda\sigma} - G^\lambda_{\mu\rho}G^\sigma_{\nu\lambda}\right) - \frac{m^{*2}}{16\pi}\left(\frac{1}{2}\gamma_{\mu\nu}\tilde{g}^{\mu\nu} - \sqrt{-g} - \sqrt{-\gamma}\right) + L_{matter}\left(\tilde{g}^{\mu\nu}, \phi_{matter}\right) \tag{35}$$

where $G^\lambda_{\mu\nu} = 1/2\tilde{g}^{\mu\nu}\left(D_\mu g_{\sigma\nu} + D_\nu g_{\sigma\mu} - D_\sigma g_{\mu\nu}\right)$ and $m^*$is the graviton mass. Using the last action principle $\delta L/\delta\tilde{g}_{\mu\nu} = \delta L_{matter}/\delta\phi_{\mu\nu}$ =0 and the fact that $\nabla_\lambda T^{\lambda\mu} = 0$ where ∇_λ is the covariant derivative, the complete system of equations is ($G = \hbar = c = 1$):

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \frac{m^{*2}}{2}\left[g^{\mu\nu} + \left(g^{\mu\nu}g^{\nu\beta} - \frac{1}{2}g^{\nu\beta}g^{\alpha\beta}\right)\gamma^{\alpha\beta}\right] = \frac{8\pi}{\sqrt{-g}}T^{\mu\nu} \tag{36}$$

with $D_\mu\tilde{g}^{\mu\nu} = 0$. In **RTG**, the insertion of the cosmological term into the field equations (36) kills its physical and mathematical logic and structure, because the additional repulsing physical field is not affected by the presence of matter. So, we will reject the possibility of adding a cosmological constant to equation (36). It was proved that the evolution of the isotropic and homogeneous **FRW** Universe in the context of the **RTG** could be a flat $D = 4$ manifold with density greater than the critical density $\rho_c = 3H_0^2/8\pi G$

where H_0 is Hubble constant and as a result, some missing dark energy must exist. One important feature of the **RTG** is $m^* \neq 0$ and due to this, the Universe is free from singularity, evolves cyclically and with deceleration parameter greater than unity, e.g. in contrast with recent observations of the Type Ia supernovae suggesting that our universe undergoes accelerated expansion at the present epoch tending to a flat de-Sitter space-time as predicted by inflation theory. In our ultra-light masses theory, the total density can be interpreted as the sum of the ordinary matter and an ultra-light one (ρ_m) where$\rho_m = \pm 3m^2/8\pi G$, $m \approx H$ (+ for positive potential and - for negative potential). In fact, in the absence of matter, $\rho_m = 3m^2/8\pi G \equiv \rho_c = 3H_0^2/8\pi G$ for $m = H_0$. In this way, ρ_m can be interpreted as dark energy density. In de-Sitter regime where $\rho_{matter} = 0$, the time-dependent Hubble constant is $H(t) = \rho_m/3$. In this way, we have an inflation or deflation regime corresponding to whether ρ_m is positive or negative. Note that the positive sign corresponds to a positive scalar curvature unless $m^2 < 0$(tachyons; we note that negative energy density occurs also in wormholes physics). In reality a large class of supersymmetric Calabi-Yau string compactifications, have classical configurations with negative energy density, e.g. from a four dimensional perspective, there can be constrained finite regions of space-time manifolds with negative energy and positive scalar curvature on a compact Ricci flat manifold admitting a covariant constant spinor leads to negative energy density [73]. The role of positive Ricci scalar curvature is negligible at late times, but can played an important and crucial role in the early stages of the Universe [74-82]. In order to reconcile the **RTG** with recent observations, we would like to use another alternative:$D = 4$ *extended supergravities having a de Sitter solution corresponding to the extrema of the negative effective potentials* $V(\phi)$ *for some scalar fields* ϕ. An interesting results of these solutions is that the squared mass of these scalars in all theories with $N = 2$ (*extended supergravities with unstable de-Sitter (dS) vacua*) is quantized in units of the Hubble constant H_0. That is $m^2 = nH_0^2$, n are integers (in units of unity Planck Mass). It was also proved that the accelerated expansion of the Universe in the framework of **RTG** can be achieved by the introduction of weak-coupling light quintessence in the energy momentum tensor generating gauge coupling corrections. Motivated by these results, we will restudy the **RTG** with negative effective supergravity power-law potentials with maximum $V_0 = 3m^2/4$ at $\phi = 0$ and zero cosmological constant and where the field equations are:

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \frac{m^{*2}}{2}\left[g^{\mu\nu} + \left(g^{\mu\nu}g^{\nu\beta} - \frac{1}{2}g^{\nu\beta}g^{\alpha\beta}\right)\gamma^{\alpha\beta}\right] = \frac{8\pi}{\sqrt{-g}}\left[\left(p + \rho - \frac{3m^2}{8\pi G}\right)u_\mu u_\nu + pg_{\mu\nu}\right] \tag{37}$$

ρ being the density and p the pressure. In other words, we will take into considerations the contribution of $\rho_m = -3m^2/8\pi G$.As we mentioned below, this density can be viewed positive if $m^2 < 0$ and negative if $m^2 > 0$. By applying this field equation to the flat FRW flat metric $ds^2 = -dt^2 + a^2(t)\,\delta_{ij}dx^i dx^j =, i, j = 1, 2, 3$, the dynamical equations in matter dominated epoch are found to be($P = 0$):[70,71]

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\tilde{\rho}}{3} - \frac{m^{*2}}{6}\left(1 - \frac{1}{a^2}\right)^2\left(1 + \frac{1}{2a^2}\right) \tag{38}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G\tilde{\rho}}{3} - \frac{m^{*2}}{6}\left(1 - \frac{1}{a^6}\right) \tag{39}$$

where $\tilde{\rho} = \rho - 3m^2/8\pi G$. As a result, it is easy to show that the density of the gravitational matter is

$$\rho_{m^*} = \rho - \rho_c = \frac{m^{*2}}{16\pi G}\left(1 + 6\frac{m^2}{m^{*2}}\right) \approx 2.5\rho_c\left(1 + 4\frac{nH_0^2}{\pi^2}\right) \tag{40}$$

where $\rho_c = 3H_0^2/8\pi G$, $H_0 = \dot{a}/a$ and $m^{*2} \approx (3/2)\,\pi^2 H_0^2$as deduced from the cyclic **RTG**. In reality, for the dust dominant epoch of the Universe ($\rho_{total} \propto a^{-3}(t)\,, t >> t_0$), $a(t) \propto \sin^{2/3}(\lambda t/2)$, $\lambda = 3m^*/\sqrt{6}$ with a good accuracy. Making use of equation (38), one can deduce the actual value of the deceleration parameter:

$$q_0 = -\left(\frac{\ddot{a}}{a}\right)_0 \frac{1}{H_0^2} \approx 4.2 - \frac{n}{2} \tag{41}$$

In order to have negative deceleration parameter, we need to have $n > 9$ with n integer. In this way, we have a cyclic expansion of the Universe but with $q_0 < 0$. If instead we use a positive effective supergravity potential then $q_0 = 4.2 + n/2$, not in agreement with recent astrophysical observations unless $n < 0$. In the absence of macroscopic matter, $q_0 = 2.4 - n/2$ and accelerated expansion occurs also for $n > 9$. Note that in $N = 2$supergravity having de-Sitter vacua without tachyons, near **dS** extremum all masses of scalar particles are quantized in terms of the Hubble parameter with $n = 12, -2, -6$[83]. This new cyclic cosmology provides a surprisingly new picture of the Universe history in which inflation exists. The Universe undergoes a periodic eternal sequence of Big Bangs and Big Crunches and where each bang is followed by a stage with $q_0 < 0$. This is a mysterious feature of the model explanation has to be found and need more detailed investigations. From phenomenological point of view, one may be inspired from the interesting property of scalar masses quantized in terms of cosmological constant in **dS** vacuum and explore the **RTG** with our supergravity potential (positive/negative) assuming that the ultra-light masses decays as $m^2 = \pm\omega\dot{a}^2/a^2, \omega$ is a positive/negative parameter respectively (the negative (positive) sign corresponds to the positive (negative) potential). That is:
Negative Supergravity Potential: $\tilde{\rho} = \rho - 3m^2/8\pi G, m^2 = \omega\dot{a}^2/a^2, \omega < 0$
Positive Supergravity Potential: $\tilde{\rho} = \rho + 3m^2/8\pi G, m^2 = \omega\dot{a}^2/a^2, \omega > 0$

In reality, the case where $\omega < 0$ corresponds to negative (mass)^2 and in gauged supergravities it corresponds to AdS_4 and plays an important role in string theory, string black hole and branes [84,85]. In this way we have two possible scenarios and equations (38) and (39) are modified as:

(1) ***Negative Supergravity Potential*:**

$$(1+\beta)\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} - \frac{m^{*2}}{6}\left(1 - \frac{1}{a^2}\right)^2\left(1 + \frac{1}{2a^2}\right) \tag{42}$$

$$\frac{\ddot{a}}{a} - \frac{\beta}{2}\frac{\dot{a}^2}{a^2} = -\frac{4\pi G\rho}{3} - \frac{m^{*2}}{6}\left(1 - \frac{1}{a^6}\right) \tag{43}$$

Equation (42) can be written as:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi\bar{G}\rho}{3} - \frac{\bar{m}^{*2}}{6}\left(1 - \frac{1}{a^2}\right)^2\left(1 + \frac{1}{2a^2}\right) \tag{44}$$

where $\bar{G} = G/(1+\omega)$ and $\bar{m}^{*2} = m^{*2}/(1+\omega)$with $-1 < \omega < 0$are the modified gravitational constant and gravitons mass. Assuming again $\rho(t) = \chi(\gamma)/a^3(t)$ with the state equation$p(t) = (\gamma - 1)\rho(t)$, $\chi(\gamma)$ being integration constant depending on the evolution state of the cosmic fluid, the solution is given by:

$$a(t) \approx \sqrt[3]{\frac{32\pi G\chi(1)}{3m^{*2}}}\sin^{2/3}\left(\frac{m^* t\sqrt{3}}{2\sqrt{2(1+\omega)}}\right) \tag{45}$$

The resulting critical mass of the gravitons is then $m^{*2} \approx (3(1+\omega)/2)\pi^2 H_0^2$ and the density of the gravitational field is $\rho_{m^*} \approx \bar{m}^{*2}/16\pi\bar{G} \approx 2.5\rho_c$. The second Friedmann equation (43) gives:

$$\begin{aligned}\frac{\ddot{a}}{a} &= \frac{\omega}{2}\left(\frac{8\pi\bar{G}\rho}{3} - \frac{\bar{m}^{*2}}{6}\left(1 - \frac{1}{a^2}\right)^2\left(1 + \frac{1}{2a^2}\right)\right) - \frac{4\pi G\rho}{3} - \frac{m^{*2}}{6}\left(1 - \frac{1}{a^6}\right) \\ &\approx \frac{\omega}{2}\left(\frac{8\pi\bar{G}\rho}{3} - \frac{\bar{m}^{*2}}{6}\right) - \frac{4\pi G\rho}{3} - \frac{m^{*2}}{6} \quad = -\frac{4\pi G}{3}\left(\frac{3(1+\omega)\rho_{m^*} + \rho_c}{1 \pm \omega}\right) \\ &= -\frac{4\pi G\rho_c(8.5 + 7.5\omega)}{3(1+\omega)}\end{aligned} \tag{46}$$

and the resulting deceleration parameter is $q_0 \approx 4.2 + (7.5\omega/2)$. For $-1 < \omega < 0$, $0.45 < q_0 < 4.5$.

(2) ***Positive Supergravity Potential*****:**

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi\bar{G}\rho}{3} - \frac{\bar{m}^{*2}}{6}\left(1 - \frac{1}{a^2}\right)^2\left(1 + \frac{1}{2a^2}\right) \tag{47}$$

$$\frac{\ddot{a}}{a} + \frac{\omega}{2}\frac{\dot{a}^2}{a^2} = -\frac{4\pi G\rho}{3} - \frac{m^{*2}}{6}\left(1 - \frac{1}{a^6}\right) \tag{48}$$

where $\bar{G} = G/(1-\omega)$ and $\bar{m}^{*2} = m^{*2}/(1-\omega)$are the modified gravitational constant and gravitons mass corresponding to this special case with $\omega < 1$ and the resulting deceleration parameter is $q_0 \approx 4.2 - (7.5\omega/2)$. For $0 < \omega < 1$, $0.45 < q_0 < 4.5$ as in the previous case but with some amelioration. One may in both cases generates negative deceleration parameter if we require the presence of tachyon mass and negative gravitational constant, a special case not favored by observations but favored with cyclic string and brane theorists [84-92]. One can also interpret this scenario by stating that the negative gravity force generated in the presence of tachyons prevent the Universe from being completely crushed into a singularity [93,94]. One can reconcile these situations by assuming a time-decreasing gravitons mass as $m^{*2} \equiv \gamma\dot{a}^2/a^2 + \eta\ddot{a}/a \propto 1/t^2 \to 0$ as $t \to \infty$, γ and η are free parameters of the model. In this way, for negative and positive supergravity potentials, we find a power-law evolution of the scale factor as $a \propto t^p$

where $p = (4+\eta)/(6+\eta+\gamma)$independent of ω. In this way, for $\alpha < -2$and$\delta > 0$, $p > 0$ and accelerated expansion may occur. The gravitons mass evolves as $m^{*2} = ((\gamma+\eta)\,p^2 - \eta p)/t^2 > 0$for $(\gamma+\eta)\,p > \eta, p > 1, \gamma < -2$.That is there is a critical scale factor where the gravitons mass tends to zero, e.g. $a\,(t) = (C\,((\gamma+\eta)/\eta)\,t)^{(\eta+\gamma)/\eta}$. For $\gamma < -2$ and$\eta > 0$, $(\eta+\gamma)/\gamma < 1$ and the universe is not yet in the stage of accelerated expansion. From the other sight, one can have accelerated expansion if during the decaying of the gravitons masses $\gamma, \eta > 0$. In this way, the decaying of the graviton masses is responsible of the cosmic acceleration. At very later times, when all the gravitons disappear, the Universe enters a slow dying epoch where he scale factor evolves as $a \propto t^p$ with $p < 1$.

5. Accelerating Flat-Brane cosmology in the presence of Ultra-Light Masses

As we mentioned in the previous sections, the first evidence for the accelerating universe came from observations of distant supernovae. However, the data were also consistent with an open universe because the total energy density was less than the so-called critical density with a low mass density and no cosmological constant. The energy density of the universe is composed of matter (*both ordinary visible matter and invisible or dark matter*) and the energy density of the vacuum. The size of the latter, which is sometimes called quintessence or "*dark energy*" defines the cosmological constant. This new accelerating energy has a larger energy density than the mass density of the Universe. Many cosmologists and high energy physicists have explored a cosmological constant, a decaying vacuum energy and quintessence as possible explanations for such an acceleration [91,92]. In the context of quantum field theories the notion of empty space has been replaced with that of a vacuum state, defined to be the ground (*lowest energy density*) state of a collection of quantum fields. A peculiar and truly quantum mechanical feature of the quantum fields is that they exhibit zero-point fluctuations everywhere in space, even in regions, which are otherwise "empty". These zero-point fluctuations of the quantum fields, as well as other "vacuum phenomena" of quantum field theory, give rise to an enormous vacuum energy density [95]. Zero-point energies of particle physics theories cannot be ignored when gravitation is taken into account and densities are given by$\rho_{vac} \approx m^4c^3/\hbar^3$, where "$m$" is the ultra-violet cut-off. In this section, we would like to study a radiative flat universe consisting only of vacuum energy and ultra-light densities. In others words, we take the total energy density to be the sum of two terms: the contributions from vacuum with density$\rho_{vac} \approx m^4c^3/\hbar^3$ and the contribution from ultra-light masses with densities$\rho_m = 3m^2c^4/8\pi G\hbar^2, m \leq \hbar H/c^2,$"$H$" is the Hubble constant. The ultra-light term may arise from self-interactions between light particles. The nature of this force is unclear, but could be interpreted as a long-range quantum fifth force. From equilibrium and statistical mechanics considerations based on the scaling of the partition function, one finds the equation of state $p = -a\rho/3$ where$F\,(r) \propto r^{a-1}$("a" is a positive number). The new model has then the attractive characteristic that quantum matter alone is sufficient to provide a flat geometry. As a first approximation, we will suppose that is of the same nature in both densities ρ_{vac} and ρ_m. The Friedman-Robertson-Walker (FRW) equation is modified by the addition of this new term and takes one of the forms:

$$H^2 = A\rho_{vac} + B\rho_{vac}^{1/2} \tag{49}$$

$$H^2 = A\rho_m + B\rho_m^2 \tag{50}$$

$A \equiv 8\pi G/3$ and B is a constant. Equation (50) appears in the general in the context of *brane cosmology* [96.97,98]. In the brane world, all matter fields and forces except gravity are localized on the 3−brane in a higher dimensional spacetime. In models where gravity is confined on the brane the Einstein's gravitational equations on the 3 brane are given by:

$$G_{\mu\nu}^{(4)} + \Lambda^{(4)} g_{\mu\nu} + C_{bcd}^{a} n_a n^c q_\mu^b q_\nu^d = \kappa_4^2 T_{\mu\nu} + \kappa_5^4 - \frac{1}{4}\tau_{\mu\alpha}\tau_\nu^\alpha + \frac{1}{12}\tau\tau_{\mu\nu} + \frac{1}{8}q_{\mu\nu}\left(\tau_{\alpha\beta}\tau^{\alpha\beta} - \frac{\tau^2}{3}\right) \tag{51}$$

where $G_{\mu\nu}^{(4)}$ is the Einstein tensor with respect to the intrinsic metric $g_{\mu\nu}$, $\Lambda^{(4)}$ is the 4-dimensional cosmological constant given by $\Lambda^{(4)} = \kappa_5^2/2(\Lambda^{(5)} + \kappa_5^2\sigma^2/6)$, $T_{\mu\nu}$ is the energy-momentum tensor of matter fields confined on the brane defined by $T_{\mu\nu} = -\sigma q_{\mu\nu} + \tau_{\mu\nu}$, σ being the brane tension and $\tau_{\mu\nu}$ is the matter-energy momentum, $E_{\mu\nu}$ is part of the 5-dimensional Weyl tensor and carries some information about bulk geometry with $E_\mu^\mu = 0, n_a = (1,0,0,0,0)$, $\kappa_4^2 = 8\pi G_4 = \kappa_5^4\sigma/6$ and $\kappa_5^2 = 8\pi G_5$ are 4-dimensional and 5-dimensional gravitational constants. Assuming the Friedmann-Robertson-Walker metric, equation (51) yields the Friedmann equation was found to be:

$$H^2 + \frac{k}{a^2} = \frac{\rho}{3m_4^2} + \frac{\rho^2}{36m_5^6} + \frac{C}{R^4} \tag{52}$$

where $m_4 = \kappa_4^{-1} = 2.4 \times 10^{18} GeV$, $m_5 = \kappa_5^{-2/3}$,"k"is a curvature constant, "ρ" is the total energy density of matter, C is a constant coming from $E_{\mu\nu}$. $\Lambda^{(4)}$ is set equal to zero. The important future of equation (4) is the presence of the last two terms. When $C = k = 0$, equation (52) looks like equation (50). In principle, equation (52) can be written in the following form:

$$H^2 + \frac{k}{R^2} = \frac{8\pi G_4\rho}{3} + \frac{4\pi G_4\rho^2}{3\sigma} - \frac{E_{00}}{3} \tag{53}$$

In fact, one can consider the vacuum state of a particle field as containing itself virtual pairs of particles with mass assuming here to be of the same order of ultra-light particles (m)and with an effective density $n \propto 1/\lambda_c$ where $\lambda_c = \hbar/mc$ is the Compton wavelength. Then one can considers the gravitational interaction energy of these virtual pairs, which is $G\lambda_c/m^2$¸ for one pair, thus leading to a contribution to an effective energy density in the vacuum of order of magnitude$\rho_c = Gm^6c^4/\hbar^4$. If we take now the energy density to be the sum of three terms$\rho_m, \rho_{vac}, \rho_c$, then the following equations holds:

$$H^2 = A\rho_{vac}^{1/2} + B\rho_{vac} + C\rho_{vac}^{3/2} \tag{54}$$

$$H^2 = A\rho_m + B\rho_m^2 + C\rho_m^3 \tag{55}$$

$$H^2 = A\rho_c^{1/3} + B\rho_c^{2/3} + C\rho_c \tag{56}$$

In fact, such equations may arise from fundamental theories of gravity in higher dimensions or from an extra contribution to the energy-momentum tensor on the right hand side of (*ordinary four dimensional*) Einstein's equations as our cosmological model [99,100,101]. Equation (54) (or (55)) is identical to the one obtained not only in brane but also in Cardassian models with ρ replaced byρ_m [102,103,104]. Only when $m = \hbar H/c^2$, the two densities are equal, then we fall into the classical and standard model. From equation (50), equation (55) could be viewed as brane perturbations. Replacing the density in equation (5) by$\rho_m = 3m^2/8\pi G_4$, we get:

$$H^2 + \frac{k}{R^2} = \frac{8\pi G_4 \rho_m}{3} + \frac{4\pi G_4 \rho_m^2}{3\sigma} - \frac{E_{00}}{3} \tag{57}$$

or

$$H^2 + \frac{k}{R^2} = A\rho_m + B\rho_m^2 - \frac{E_{00}}{3} \tag{58}$$

This equation is identical to equation (55) if $E_{00} \propto \rho_m^3 \approx \rho_c \approx \rho_{vac}^{3/2} \approx m^6 \approx H^6 \quad (\hbar = c = 1)$, that is$H^6 \equiv \dot{a}^6/a^6 \propto 1/a^4$, or $a\,(t) \propto t^{3/2}$ (*accelerating expansion*) and $k = 0$ (*flat geometry*). In this way, the radiation energy varies with time as $E_{00} \propto t^{-6}$which is a too fast decreasing function. Finally, we study and discuss the phenomenology of the ansatz in equation (49). We first mention that the ghost contribution ρ_m has a negative pressure (*simple calculations gives* $p_m = -\rho_m/2$), which is responsible for the Universe's acceleration. In this way, the quantum fifth force varies as $F\,(r) \propto a^{0.5}$. From equation (49), it is clear that this kind of behavior is qualitatively very different from the standard braneworld cosmology (*as equation* (50)) because it implies a modification of gravity at very low energy scales rather than very high ones. We suppose that the new density in equation (48) is considered to be initially negligible. It only comes to dominate recently, at the redshift $z_{eq} \approx O\,(1)$ indicated by the supernovae observations. Once the second term dominates, it causes the universe to accelerate even if there is no macroscopic matter contribution. We take the vacuum density to scale with the redshift as$\rho_{vac} \propto a^{-3}$. That is, when the second term in equation (1) dominates, it causes the Universe to accelerate as the scale factor grows as $a \propto t^{4/3}$so that$\ddot{a} > 0$. The second term starts to dominate at a redshift z_{eq} such that $A\rho\,(z_{eq}) = B\rho^{0.5}\,(z_{eq})$. In this way[105,106]:

$$A = \frac{H_0^2}{\rho_o^{vac}} - \frac{B}{\sqrt{\rho_0^{vac}}} \tag{59}$$

$$B = \frac{H_0^2\,(1 + z_{eq})^{3/2}}{\rho_0^{vac}\left[1 + (1 + z_{eq})^{3/2}\right]} \tag{60}$$

where ρ_0^{vac} is the present vacuum density. We have one parameter in this model, z_{eq} or B. Observations of the cosmic background radiation show that we live in a flat Universe.

We defines the critical quantum density as

$$\rho_c = \frac{3m_0^2c^4}{8\pi G\hbar^2} \times \frac{1}{\left[1 + (1 + z_{eq})^{3/2}\right]} \tag{61}$$

where $m_0 = \hbar H_0/c^2$. As a result:

$$\begin{aligned} \rho_c^{vac} &\approx 0.35 \frac{3H_0^2}{8\pi G} \\ &\approx 0.35 \times 1.88 \times 10^{-29} h_0^2 gm \times cm^{-2} \end{aligned} \tag{62}$$

where h_0 is the Hubble constant today in unit of 100km/s/Mpc. The critical density is much lower than previously estimated and satisfy most of the observational constraints. It became much lower if we reformulate the problem with equation (53), but than the acceleration is constant. Several characteristics of this model are in progress.

6. Massive Gravitons, Massive Ultra-Light Particles and the Cosmological Constant in General Relativity

In the previous sections, we mention that one can describe dark energy in some $D = 4$ extended supergravities that have a de Sitter solutions. These **dS** solutions correspond to the extrema of the effective potentials $V(\phi)$ for some scalar fields "ϕ". An interesting result of these solutions is that the squared mass of these scalars in all theories with $N = 2$ (*extended supergravities with unstable dS vacua*) is quantized in units of the Hubble constant "H_0". That is $m^2 = nH_0^2$ where "n"are integers of order of unity (*in units* $M_P = 1, M_P$ *being the Planck mass*). In extended supergravities with a positive cosmological constant, one always has $3m^2 = n\Lambda$, "Λ" is the cosmological constant having dimension two in energy unit. Along this, reducing the vacuum density (*using the context of non-minimal coupling of scalar fields to gravity*) to a very small value over cosmological timescales was discussed by several authors [18,19,20,21,22,107-111]. In this work, we will be interested on extended supergravities with positive cosmological constant and positive "m^2". For that we adopt the chaotic inflaton complex potential-like $V(\phi\phi^*) = am^2(\zeta\phi^2\phi^{*2} + 1)$, $a \approx 3/4$. In this particular case and for $\zeta << 1$, the Einstein field equations $G_{\mu\nu} - \Lambda g_{\mu\nu} + \sum T_{\mu\nu} = 0$ ($G_{\mu\nu}$ *is the Einstein curvature tensor and* $\sum T_{\mu\nu}$ *is the sum of all the stress-energy tensor implemented in the theory*) yields the following Ricci scalar $R = -4\Lambda + 3m^2$ [18]. As a result, a possible candidate static field equations corresponding to this scalar curvature are:

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = -\left(\frac{3m^2}{4} - \Lambda\right)g_{\alpha\beta} \tag{63}$$

"m" is the ultra-light masses and "$g_{\alpha\beta}$" is material interior metric tensor. In general, this equation is identical to the general Einstein field equations

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = -8\pi G\left[(p + \rho)\, u_\alpha u_\beta - p g_{\alpha\beta}\right] \tag{64}$$

if we assume $p = -\rho$ and $32\pi G\rho = 3m^2 - 4\Lambda$, "$G$" being the gravitational constant. As long as $3m^2 > 4\Lambda$, the density remains positive. It follows that if the mass of the ultra-light particle vanishes, the cosmological constant must be assumed negative in order to get positive vacuum density and a corresponding negative pressure. That is to say also that the total vacuum density is the sum of the true vacuum energy represented by the cosmological constant and the false vacuum energy represented by the ultra-light masses. At the same time, there are some theoretical arguments that the graviton mass can have a non-zero rest mass as a result of the interaction with matter [112,113,114]. The gravitational waves were showed to be absorbed by the false vacuum having the equation of state $p = -\rho$. Thus we expect that there must exit a special relation between the graviton mass, the ultra-light mass and the cosmological constant. In fact, conformal relativity is a theory of mass having zero Weyl conformal curvature tensor and a special metric tensor$g_{\alpha\beta} = e^{\psi}\eta_{\alpha\beta}$, $\eta_{\alpha\beta}$ is the Minkowski metric and "ψ" is some function of the coordinates in the theory described. The resulting Einstein field equations[115]:

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = -8\pi G\left[(p+\rho)\,\delta^0_\alpha\delta^0_\beta - pg_{\alpha\beta}\right] \tag{65}$$

becomes adopting that the density of the background cosmic fluid is $\rho = (3m^2 - 4\Lambda)/32\pi G$:

$$\frac{\partial^2\psi}{\partial x^\alpha \partial x^\beta} + \frac{1}{2}\eta_{\alpha\beta}\eta^{\mu\nu}\frac{\partial^2\psi}{\partial x^\mu \partial x^\nu} - \frac{3}{2}g_{\alpha\beta}\psi = 8\pi G\rho g_{\alpha\beta} \tag{66}$$

or in the following form and in natural units:

$$\left(\Box - \frac{64\pi G}{3c^2}\left(\frac{3m^2c^4}{32\pi G\hbar^2} - \frac{\Lambda c^2}{8\pi G}\right)\right)\psi = \frac{32\pi G}{3c^2}\left(\frac{3m^2c^4}{32\pi G\hbar^2} - \frac{\Lambda c^2}{8\pi G}\right) \tag{67}$$

Here $\Box \equiv \partial^2/\partial x^\mu \partial x_\mu$. Comparing to the Proca-Yukawa massive gravitoelectomagnetics static theory describing a universe as a mode collective behavior of dust and graviton particles [116,117]:

$$\left(\Box - \frac{m_g^2c^2}{\hbar^2}\right)\psi = \frac{1}{2}\frac{m_g^2c^2}{\hbar^2} \tag{68}$$

where "m_g" is the graviton mass, we get from equations (68) and (66):

$$\Lambda = \frac{3}{8}\frac{c^2}{\hbar^2}\left(2m^2 - m_g^2\right) \equiv \frac{3}{8}\frac{m^2c^2}{\hbar^2}\left(2 - \frac{m_g^2}{m^2}\right) \equiv \frac{3}{8}\frac{m^2c^2}{\hbar^2}\left(2 - f^2\right) \tag{69}$$

where $f \equiv m_g/m$. This equation is of interest: *it relates for the first time the cosmological constant to both the ultra-light mass and the graviton mass.* As long as $f < \sqrt{2}$ or $m_g < \sqrt{2}m$, the cosmological constant is positive. If "m" is set equal to the "*Hubble mass*", that is $m \equiv m_H^0 = \hbar H_0/c^2 \approx 3.8\cdot 10^{-66}h\,(g)\,, h = 0.71\pm 0.07$, then we get $m_g < \sqrt{2}\hbar H_0/c^2$ [118]. For $m_g = \sqrt{2}\hbar H_0/c^2 \approx 1.4m_H^0$, the cosmological constant vanishes. Equation (69) takes then the simple form:

$$\Lambda = \frac{3}{8}H_0^2\left(2 - f^2\right) \tag{70}$$

If $m_g = m_{H_0}$, than $\Lambda = 3m_g^2 c^2/8\hbar^2 = 3/8H_0^2 = 0.375H_0^2$. This could explain the weakness of the cosmological constant. It is worth noting that equations (69) and (70) are in agreement with MAXIMA-1 recent constraints on cosmological parameters suggest that $0 < \Lambda < 2.28H_0^2$ [119]. Within the framework of Relativistic Theory of Gravity, some authors claimed that $0.24 \leq f^2 \leq 1.08$ [71,72,120,121]. As a result, the Ricci scalar as well as the cosmic fluid densities $\rho = 3m^2c^4f^2/64\pi G\hbar^2$ determined from the above requirements are both positive. For $m = m_g = m_{H_0}$, $\rho = 3H_0^2 f^2/64\pi G < \rho_c \equiv 3H_0^2/8\pi G$ in agreement with observations. In conclusion, a gravitational wave propagating through a special medium with a total density which is the sum of the true vacuum density and the ultra-light particle false density is influenced by these latter. This is manifested by the production of massive gravitons, by reducing the value of the cosmological constant and the density of matter $\rho < \rho_{critical} = 3H_0^2/8\pi G$, relating the ultra-light masse, the graviton mass and the cosmological constant in one simple relation and finally by generating a positive curvature scalar $R = 3m^2 - 4\Lambda > 0$.

7. Conclusions

In this work, we illustrated some important aspects, implications and features of ultralight masses in modern cosmology. We showed the importance of the presence of the ultra-light masses as necessary ingredients in modern cosmology. The evolution of a homogeneous flat universe with positive scalar curvature dominated by ultra-light matter and the cosmological constant are discussed and we investigated about the necessary condition for eternal accelerated expansion with no need of any form of tachyonic matter. We discussed some important aspects and features of a cosmological model with phenomenological decay of the ultra-light masses, the vacuum cosmological constant and the matter density as $m^2(t) = \varepsilon\dot{a}^2/a^2$, $\Lambda(t) = \beta\ddot{a}/a$ and $\rho(t) = (3\delta/8\pi G)\,\dot{a}^2/a^2$ respectively, α,β and δ are constants. We showed that, for the special parameters $\beta = 5$ and $\delta = 0.25$, the universe will passes through different phases. The transition from the vacuum state to the radiation one is followed in our scenario by a increasing in the gravitational constant, an increasing of the cosmic acceleration while the ultra-light masses, the cosmological constant and the density decreases as $1/t^2$. Within the same context, the transition from the radiation dominated epoch to the matter dominated one is also followed by increasing of the gravitational constant but with a reduced term and this reducing follows also the scale factor as well as $m^2(t), \Lambda(t)$ and $\rho(t)$. We discussed also the role of decaying ultralight masses and gravitons masses within the framework of RTG. We showed the important role playing by the decaying gravitons masses and decaying quantized ultra-light masses from extended supergravities in the evolution of the (RTG) Universe for positive and negative supergravities potentials. The decaying of the graviton masses is responsible of the cosmic acceleration. At very later times, when all the gravitons disappear, the Universe enters a slow dying epoch where he scale factor evolves as $a \propto t^p$ with $p < 1$. We proposed also a simple modification of the Standard Hot Big Bang Cosmology (*SHBBC*), in which the Universe is flat with a total energy density taken to be the sum of the contributions from vacuum and ultra-light mass term responsible of the dominant driver of expansion at a late epoch of the Universe. When the new term dominates, the universe was found to be accelerated with time. The ultra-light quantum energy density required to close the Universe was found to be much smaller than in *SHBBC*, so that quantum matter can be sufficient to provide a flat geometry. It was also found that quantum ultra-light matter particles interactions are interpreted as a quantum fifth force and it is found to vary as$F(r) \propto r^{0.5}$. Finally, we illustrated the modification of the gravitational wave propagation through a special vacuum medium with a total density being the sum of the true vacuum density and the ultra-light particle supergravity density by reducing the value of the cosmological constant up to $\Lambda = 3H_0^2(2-f^2)/8$ ($f \equiv m_g/m_{H_0}$) and the density of matter $\rho < \rho_{critical} = 3H_0^2/8\pi G$, by relating the ultra-light masse, the graviton mass and the cosmological constant in one simple relation and finally by generating a positive Ricci scalar curvature. The gravitational wave propagating through a special medium with a total density which is the sum of the true vacuum density and the ultra-light particle false density was shown to be influenced by these latter. This is manifested by the production of massive gravitons, by reducing the value of the cosmological constant and the density of matter $\rho < \rho_{critical} = 3H_0^2/8\pi G$, relating the ultra-light masse, the graviton mass and the cosmological constant in one simple relation and finally by generating a positive curvature scalar $R = 3m^2 - 4\Lambda > 0$. We hope that this work will lead to some important cosmological, astrophysical and theoretical consequences (*the cosmological constant problem, the quantization of General Relativity, the dark matter problem, the effects of the presence of ultra-light masses and gravitons on black holes physics, etc*) [122-131]. For this, further consequences and details are in process.

Appendix

The parameters used in this paper are summarized as fellows:

(1) ξ is the non-minimal coupling constant appearing in our supergravity inflationary potential $V(\phi\phi^*) = (3m^2/4)(\zeta\phi^2\phi^{*2} - 1)$ (Section **1**)

(2) $\zeta = O(1)$ is a supergravity inflaton scalar field parameter (Section **1**)

(3) $\theta \equiv H_0/H_\infty$ introduced in attempt to describe non-singular universe accelerating Universe with positive scalar Ricci curvature generated from negative extended supergravity potential (Section **1**)

(4) $\dot{F}^2 = \alpha m^2/\Lambda, F$ represents the new inflaton field with α is a positive constant introduced in attempt to investigate about the necessary condition for eternal accelerated expansion with no need of any form of tachyonic matter (Section **2**)

(5) $\omega = -1/(1+3m^2/\Lambda) \equiv p/\rho$ is the state equation coefficient (Section **2**)

(6) $w \equiv \omega = p/\rho$(Section 3)

(7) $\tilde{\gamma} \equiv w + 1$(Section 3)

(8) $m^2(t) = \varepsilon\dot{a}^2/a^2$, $\Lambda(t) = \beta\ddot{a}/a$ and $\rho(t) = (3\delta/8\pi G)\dot{a}^2/a^2$ are the three phenomenological laws for time-varying ultra-light masses, cosmological constant and matter density (Section **4**)

(9) $m^2 = \pm\omega\dot{a}^2/a^2$is the phenomenological time-varying law for the ultra-light masses within the framework of **RTG** (Section **4**)

(10) $m^{*2} \equiv \gamma\dot{a}^2/a^2 + \eta\ddot{a}/a$ is the phenomenological time-varying law for the gravitons masses within the framework of **RTG** (Section **4**)

(11) $a \propto t^p, \quad p = (4+\eta)/(6+\eta+\gamma)$ is the corresponding scale factor parameter (notes 9 and 10) (Section **4**)

(12) $p = -a\rho/3, a$ is a positive number (Section **5**)

(13) $A \equiv 8\pi G/3, B$ is a constant used in the brane world scenario (Section **5**)

(14) $m^2 = nH_0^2$ where "n" are integers of order of unity (Mostly in all sections)

(15) $a \approx 3/4$ used in the positive supergravity inflationary potential $V(\phi\phi^*) = am^2(\zeta\phi^2\phi^{*2} + 1)$ (Section **6**)

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Building of Heat Kernel on Non-Compact Homogeneous Spaces

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Abstract: Method of the solution of the main problem of homogeneous spaces thermodynamics on non-compact spaces in the case of non-compact homogeneous spaces is presented in the article. The method is based on the formalism of coadjoint orbits. In that article we present algorithm that allows efficiently evaluate heat kernel on non-compact homogeneous spaces. The method is illustrated with non-trivial example.

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1. Introduction

The goal to be achieved in present work is to demonstrate how method of orbits can be applied to the problems of quantum statistic mechanics (thermodynamics of homogeneous spaces) on non-compact homogeneous space [2]. Method of coadjoint orbits appeared to be quite powerful tool in theory of representations, Fourier analysis on homogeneous spaces, geometric quantization and integration of PDE's. As it was shown in [1] the use of the orbits method is the most fruitful if not the ultimate way to solve the main problem of homogeneous spaces thermodynamics for non-compact unimodular Lie groups with left-invariant riemannian metric. Non-compact Lie groups were chosen there as the object for the investigation because they give

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us simple but transparent enough example of spaces for which heat kernel and and statistic sum hardly can be found in the framework of methods existed before, for instance widely used separation of variables. Below we will consider much wider class of spaces then Lie groups with riemannian metric and we will describe the algorithm for solution of the main problem of quantum statistic mechanics on arbitrary non-compact homogeneous space.

The methods traditionally used for solution of PDE's are hardly applicable for integration of heat-kernel equation on homogeneous spaces because of the same reasons as for Lie groups. We do not tend to describe them here since we discussed all of necessary details of the problem in [1].

Problem to be discussed in that article is interesting and important because for non-compact space is not yet possible way to represent statistic sum as series with factorized volume of the manifold for n-dimensional compact space

$$Z_\beta = \frac{Vol_M}{(4\pi\beta)^{n/2}} \sum a_i \beta^i, \tag{1}$$

were coefficients a_i represent spectral invariants, which may be expressed through functions on the manifold, symbol of operator H and it's derivatives. All existing results in that field were related to the compact manifolds and non-compact manifolds of finite volume ([3],[4],[5]). Below we shall show the algorithm to build heat kernel on arbitrary non-compact homogeneous space. Solution of wave and heat kernel equation for different classes of manifolds is the problem of great interest in modern mathematical physics. In works by Chalykh and Veselov ([6],[7]) that problem was solved for some compact spaces and non-compact spaces as well and authors obtained explicit formulas for heat kernel on these spaces. That became possible because of special structure of groups of motions and since that spaces themselves (most part of considered spaces were symmetric with simple or semisimple group of motion).

In general our task here is evaluation of the statistic sum (distribution function) Z_β as a sum over the spectrum of energy operator $H = -\Delta$ (Laplace-Beltrami operator) on homogeneous space

$$Z_\beta = \sum_n d_n \exp(-\beta E_n), \tag{2}$$

where d_n is a degeneration of E_n, β - inverse temperature. In case of elliptic operator H on compact space series (2) is always convergent [2].

Statistic sum (2) on homogeneous space can be expressed as a trace of density matrix (heat kernel) $\rho_\beta(x, x')$

$$Z_\beta = \int \rho_\beta(x, x) d\mu(x), \quad d\mu(x) = \sqrt{|g|} dx, \tag{3}$$

being a solution of Bloch equation (heat kernel equation) on corresponding manifold

$$\frac{\partial \rho_\beta(x, x')}{\partial \beta} + H(x)\rho_\beta(x, x') = 0, \quad \rho_\beta(x, x')|_{\beta=0} = \delta(x - x')/\sqrt{|g|}. \tag{4}$$

In case of arbitrary homogeneous space integration of heat kernel equation (3) is not that obvious. Below we shall develop the method that would allow us to get global solution of that equation.

2. Harmonic Analysis on Homogeneous Spaces

Size required for the article can not allow us to give detailed description of harmonic analysis on homogeneous spaces based on method of orbits. Since that we will mention here only the most necessary of it's principal constructions. Details of the method interested reader may find in [8].

Let real group G acts on smooth manifold M, i.e. $M = G/H$ — homogeneous space, where H — isotropy subgroup of marked point x_0, $\{e_A\}$ — basic vectors of Lie algebra $\mathfrak{g}$ of group G. With use of local coordinates (x) action of transformation group on manifold M can be expressed through the action of vector fields X_A, which form basis of Lie algebra $\mathfrak{g}$:

$$X_A = X_A^a(x)\frac{\partial}{\partial x^a}, \quad \operatorname{rank} X_A^a(x_0) = \dim M; \quad [X_A, X_B] = C_{AB}^C X_C. \tag{5}$$

Any algebra $\mathfrak{g}$ of generators X_A of transformation group can be put into correspondence associate algebra $\mathbf{D}(M)$ of invariant and pseudo-invariant operators Y on M with condition

$$[Y, X_A] = 0, \quad A = 1, \ldots, \dim \mathfrak{g}.$$

Algebra $\mathbf{D}(M)$ is finitely generated (has finite number of generators). Let's note it's basis through $\{Y_\mu\}$ — set of functionally independent operators which means that any of invariant operators can be expressed as a function of operators Y_μ. Obviously commutator of any pair of invariant operators will be invariant operator. Since that there are such symmetrized operator functions $\Omega_{\mu\nu}(Y_1, Y_2, \ldots)$, that

$$[Y_\mu, Y_\nu] = \Omega_{\mu\nu}(Y). \tag{6}$$

Associate algebra with basis commutation rules (6) is called *functional* algebra or shortly $\mathcal{F}$–algebra. In particular, if $\Omega_{\mu\nu}$ are quadratic polynomes such algebra is called quadratic algebra. *Index* ($\operatorname{ind}\mathcal{F}$) of $\mathcal{F}$–algebra is called number of elements that generate it's center.

Dimension and index of $\mathcal{F}$–algebra of invariant operators are easy to count knowing structural constants of algebra $\mathfrak{g}$ and isotropy subalgebra $\mathfrak{h}$:

$$\dim \mathcal{F} = \dim \mathfrak{g} + \dim \mathfrak{h}^\lambda - 2\dim \mathfrak{h}, \quad \lambda \in \mathfrak{h}^\perp = \{f \in \mathfrak{g}^* \mid \langle f, \mathfrak{h}\rangle = 0\}; \tag{7}$$

$$\operatorname{ind} \mathcal{F} = \dim \mathfrak{g}^\lambda/\mathfrak{h}^\lambda, \quad \mathfrak{g}^\lambda = \{X \in \mathfrak{g} \mid \langle \lambda, [X, \mathfrak{g}]\rangle = 0\}, \quad \mathfrak{h}^\lambda = \mathfrak{g}^\lambda \cap \mathfrak{h}. \tag{8}$$

Here and below λ is generally situated linear functional that belongs to the subspace $\mathfrak{h}^\perp$. The way formulas (7), (8) were derived and and algoritm to find algebra of invariant operators is describedin work [9].

Lets introduce symbols of operators as functions on cotangent bundle T^*M:

$$X_A(x, \partial_x) \to X_A(x, p) = X_A^a(x)p_a, \quad Y_\mu(x, \partial_x) \to Y_\mu^{cl}(x, p).$$

One can show that any invariant operator $Y(x, \partial_x)$ can be corresponded with invariant function $Y^{cl}(x, p)$ cotangent bundle T^*M and vice versa .

Symbols of operators satisfy following relations in respect to Poisson bracket defined using canonic symplectic 2-form $\omega = dp \wedge dx$:

$$\{X_A(x,p), X_B(x,p)\} = C_{AB}^C X_C(x,p), \quad \{Y_\mu^{cl}(x,p), Y_\nu^{cl}(x,p)\} = \Omega_{\mu\nu}(Y^{cl}(x,p)),$$
$$\{X_A(x,p), Y_\mu^{cl}(x,p)\} = 0.$$

Moments mappings

$$\mu: \ T^*M \to \mathfrak{g}^*, \ X(x,p) = f \in \mathfrak{g}^*; \quad \tilde{\mu}: \ T^*M \to \mathcal{F}^*, \ Y^{cl}(x,p) = g \in \mathcal{F}^*$$

are Poisson mappings of Poisson algebra of functions on T^*M at Poisson algebras on $\mathfrak{g}^*$ and $\mathcal{F}^*$ with brackets:

$$\{\varphi, \psi\}^{\mathcal{F}}(g) = \Omega_{\mu\nu}(g) \frac{\partial \varphi(g)}{\partial g_\mu} \frac{\partial \psi(g)}{\partial g_\nu}; \quad g = g_\mu E^\mu \in \mathcal{F}^*; \quad \varphi, \psi \in C^\infty(\mathcal{F}^*);$$
$$\{\varphi, \psi\}(f) = C_{AB}^C f_C \frac{\partial \varphi(f)}{\partial f_A} \frac{\partial \psi(f)}{\partial f_B}; \quad f = f_A e^A \in \mathfrak{g}^*; \quad \varphi, \psi \in C^\infty(\mathfrak{g}^*).$$

Even more, symplectic sheets $\Omega \subset \mathfrak{g}^*$ and $\tilde{\Omega} \subset \mathcal{F}^*$ are in mutual correspondence [10]:

$$\Omega = \mu(\tilde{\mu}^{-1}(\tilde{\Omega})), \quad \tilde{\Omega} = \tilde{\mu}(\mu^{-1}(\Omega)); \quad \operatorname{codim} \Omega = \operatorname{codim} \tilde{\Omega}. \tag{9}$$

Formula (9) means that centers of Poisson algebras coincide in the sense that basises of Casimir functions in $C^\infty(\mathfrak{g}^*)$ and $C^\infty(\mathcal{F}^*)$ may be chosen following way:

$$K_\alpha(X(x,p)) = \tilde{K}_\alpha(Y^{cl}(x,p)), \quad \alpha = 1, \ldots, \operatorname{ind} \mathcal{F}.$$

Last equation is satisfied even if we change Casimir functions for symmetrized functions of operators:

$$K_\alpha(iX(x,\partial_x)) = \tilde{K}_\alpha(Y(x,\partial_x)), \quad \alpha = 1, \ldots, \operatorname{ind} \mathcal{F}. \tag{10}$$

Let's note as U Lagrange submanifold to the symplectic sheet $\tilde{\Omega}$. *Defect* $d(M)$ of homogeneous space M is defined as dimension of Lagrange submanifold to the symplectic sheet on coalgebra of invariant operators [9]:

$$d(M) = \dim U = \frac{1}{2} \dim \tilde{\Omega}.$$

Following equality definitely takes place

$$\dim \mathcal{F} - \operatorname{ind} \mathcal{F} = \dim \tilde{\Omega} = 2d(M).$$

If we substitute in it (7), (8) for dimension and index of $\mathcal{F}$–algebra we shall aquire following expression

$$d(M) = \frac{1}{2} \dim \mathfrak{g}/\mathfrak{g}^\lambda - \dim \mathfrak{h}/\mathfrak{h}^\lambda, \quad \lambda \in \mathfrak{h}^\perp. \tag{11}$$

To make calculation procedure more convenient we rewrite (11) as follows

$$d(M) = \frac{1}{2}\operatorname{rank}\langle\lambda, [\mathfrak{g}, \mathfrak{g}]\rangle - \operatorname{rank}\langle\lambda, [\mathfrak{g}, \mathfrak{h}]\rangle, \quad \lambda \in \mathfrak{h}^{\perp}. \tag{12}$$

Homogeneous spaces with $d(M) = 0$ are called *commutative.* For commutative spaces $\dim\mathcal{F} = \operatorname{ind}\mathcal{F}$, i.e. algebra of invariant operators is commutative and since that (10) is generated by Cazimir operators of algebra $\mathfrak{g}$. For instance all symmetric and weakly symmetric spaces are commutative [12].

Let Q be Lagrange submanifold to the symplectic sheet Ω (coadjoint orbit) in $\mathfrak{g}^*$. Manifold Q is of dimension

$$\dim Q = \frac{1}{2}\dim\mathfrak{g}/\mathfrak{g}^{\lambda}, \quad \lambda \in \mathfrak{h}^{\perp}. \tag{13}$$

Let's represent algebra $\mathfrak{g}$ as algebra of differential operators $l(q, \partial_q, J)$, that give exact irreducible representation of algebra $\mathfrak{g}$ in space of functions on Q (λ–representation [11]):

$$[l_A(q, \partial_q; J), l_B(q, \partial_q; J)] = C^C_{AB} l_C(q, \partial_q; J), \quad K_\alpha(-il(q, \partial_q, J)) = \kappa_\alpha(J), \quad \det\frac{\partial\kappa_\alpha(J)}{\partial J_\beta} \neq 0. \tag{14}$$

Here q are local coordinates on Q, J are parameters enumerating orbits in $\mathfrak{g}$ which are integer.

Let's build irreducible representation of algebra $\mathcal{F}$ by differential operators $\zeta(u, \partial_u; J)$ on Lagrange submanifold U of the symplectic sheet $\tilde{\Omega}$ in $\mathcal{F}^*$ being in agreement with λ–representation.

$$[\zeta_\mu(u, \partial_u; J), \zeta_\nu(u, \partial_u; J)] = -\Omega_{\mu\nu}(\zeta(u, \partial_u; J)), \quad \tilde{K}_\alpha(\zeta(u, \partial_u, J)) = \kappa_\alpha(J). \tag{15}$$

Set of generalized functions $D^J_{qu}(x)$ is to be defined from the equations

$$(X_A(x, \partial_x) + l_A(q, \partial_q; J))D^J_{qu}(x) = 0, \quad (Y_\mu(x, \partial_x) - \zeta_\mu(u, \partial_u; J))D^J_{qu}(x) = 0 \tag{16}$$

and is full and orthogonal $C^\infty(M)$ [8]:

$$\int\limits_M D^{\tilde{J}}_{\tilde{q}\tilde{u}}(x)\overline{D^J_{qu}(x)}\, d\mu(x) = \delta(J, \tilde{J})\delta(q, \tilde{q})\delta(u, \tilde{u}); \tag{17}$$

$$\int D^J_{qu}(x)\overline{D^J_{qu}(\tilde{x})}\, d\mu(J)d\mu(q)d\mu(u) = \delta(x, \tilde{x}). \tag{18}$$

Here $d\mu(J)$ is spectral measure of Casimir operators, $\delta(J, \tilde{J})$ is δ–function in respect to that measure, $d\mu(x)$ $d\mu(q)$, $d\mu(u)$ are quasi-invariant measures on homogeneous space M and on Lagrange manifolds Q and U, $\delta(x, \tilde{x})$, $\delta(q, \tilde{q})$, $\delta(u, \tilde{u})$ are δ–function in respect to corresponding measures.

Set of functions $D^J_{qu}(x)$ is full and orthogonal and allows direct and inverse Fourier transform to be performed:

$$\varphi(x) = \int \overline{\psi(q, u, J)} D^J_{qu}(x)\, d\mu(J)d\mu(q)d\mu(u), \quad \psi(q, u, J) = \int \overline{\varphi(x)} D^J_{qu}(x)\, d\mu(x). \tag{19}$$

If functions $\varphi(x)$ and $\psi(q,u,J)$ are connected by the relations (19) ($\varphi \sim \psi$) one easily can prove that the same relations connect functions:

$$X_A(x,\partial_x)\varphi(x) \sim l_A(q,\partial_q,J)\psi(q,u,J), \quad Y_\mu(x,\partial_x)\varphi(x) \sim \zeta_\mu(u,\partial_u,J)\psi(q,u,J). \qquad (20)$$

We assume here that the measures are chosen the way that operators $X_A,\ l_A$ are anti-hermitian and operators $Y_\mu,\ \zeta_\mu$ are hermitian. That assumption was done to make the text of that article easier and even more, that case is quite widespread. No obstacles stop us from consideration of the general case when such measures do not exist.

Let operator $H(x,\partial_x)$ is invariant under action of group G. Then it can be presented as a function of invariant operators $H = H(Y(x,\partial_x))$ and after fourier transform is done (19) it goes into $\tilde{H}(\zeta(u,\partial_u,J))$ which depends of $d(M)$ (see formula (11)) independent variables u. If operator $H(x,\partial_x)$ is element of enveloping algebra $U(\mathfrak{g})$, i.e. $H(x,\partial_x) = H(X(x,\partial_x))$ then after fourier transform is done it will take form $\tilde{H}(l(q,\partial_q,J))$ and it will depend on $\dim Q$ (see formula (13)) independent variables q.

So the variant of harmonic analysis on homogeneous spaces based on method of orbits, presented above, allows us to effectively integrate linear differential equations with non-commuting symmetries [13]. We shall apply that formalism to find heat kernel on homogeneous riemannian spaces.

3. Integration of Bloch Equation on Homogeneous Spaces

Below we shall consider Bloch equation (4) on homogeneous space and apply the formalism of harmonic analysis described above to integrate it.

Let group G acts on homogeneous space $M = G/H$. We supply that space with the structure of rimannian manifold introducing metric tensor on the manifold according to the rule

$$g^{ij} = G^{ab}X_a^iX_b^j, \qquad (21)$$

where G^{ab} is symmetric matrix. Here X_a^i are generators of group action given by (5).That metric is so-called central metric and integration of geodesic flows on homogeneous spaces with such metrics was discussed in [14].

Firstly we must evaluate defect $d(M)$ of homogeneous space according to (11) or (12) and dimension and index of $\mathcal{F}$—algebra. Then one must find explicit formulas for operators $X_i(x)$ — generators of action of transformation group on homogeneous space, which are obtained as left-invariant vector fields restricted on the space M. The method to build operators $X_i(x)$ on $M = G/H$ quite easily knowing structural constants of Lie algebras of Lie groups G and H was described in [15] and may be realized as a computer program.

Operator $H(x)$ in Bloch equation (4) on homogeneous space M we obtain as a quadratic function of operators $X_i(x)$:

$$H(-i\hbar X) = -\hbar^2 G^{ab}X_aX_b = -\hbar^2\Delta, \qquad (22)$$

where Δ is Laplace-Beltrami operator on manifold with riemannian metric (21).

The main idea of presented method is to make number of independent variables in Bloch equation smaller using non-commuting symmetry operators. That can be done using formalism of harmonic analysis when solution of Bloch equation is presented as Fourier decomposition according to (19). That gives us possibility to represent heat kernel $\rho_\beta(x, x')$ on entire homogeneous space by expression

$$\rho_\beta(x,x') = \int \mathcal{R}_\beta(q,u,q',u',J,J')\overline{D^J_{qu}}(x')D^{J'}_{q'u'}(x)d\mu(q)d\mu(q')d\mu(u)d\mu(u')d\mu(J)d\mu(J'), \quad (23)$$

where (23) function $\mathcal{R}_\beta(q, u, q'u', J, J')$ can be treated as a heat kernel on a Lagrange submanifold to the symplectic sheet Q (coadjoint orbit).

Bloch equation on homogeneous space after Fourier transform is done goes into Bloch equation on corresponding orbit

$$\frac{\partial \mathcal{R}_\beta(q,u,q'u',J,J')}{\partial \beta} + H(-i\hbar l)\mathcal{R}_\beta(q,u,q',u',J,J') = 0 \quad (24)$$

with initial condition

$$\mathcal{R}_\beta(q,u,q',u',J,J')|_{\beta=0} = \delta(q,q')\delta(u,u')\delta(J,J'), \quad (25)$$

where number of independent variables q is sufficiently reduced and equals $n' = \dim Q$. We must point that in one interesting case when $n' = 1$ reduced Bloch equation appears to be ordinary differential equation and since that integrable in quadratures. Although mentioned situation is not common the method allows to sufficiently simplify Bloch equation and as a result to integrate it.

Since operators of λ – representation depend only of variables q, we can represent heat kernel on Lagrange submanifold $\mathcal{R}_\beta(q, u, q'u', J, J')$ following way

$$\mathcal{R}_\beta(q,u,q',u',J,J') = \mathcal{R}_\beta(q,q')\delta(u,u')\delta(J,J'), \quad \mathcal{R}_\beta(q,q')|_{\beta=0} = \delta(q,q'), \quad (26)$$

where $\mathcal{R}_\beta(q, q')$ satisfies the same Bloch equation (24) with operator $H(-i\hbar l)$, which follows from (20). The way we considered function $\mathcal{R}_\beta(q, u, q'u', J, J')$ in (26) makes possible to do integrations in (23) over $d\mu(u')$ and $d\mu(J')$. After integration we will obtain following simplified formula for $\rho_\beta(x, x')$:

$$\rho_\beta(x,x') = \int \mathcal{R}_\beta(q,q')\overline{D^J_{qu}}(x')D^J_{q'u}(x)d\mu(q)d\mu(q')d\mu(u)d\mu(J), \quad (27)$$

So we finally have to solve the equation for function $\mathcal{R}_\beta(q, q')$

$$\frac{\partial \mathcal{R}_\beta(q,q')}{\partial \beta} + H(-i\hbar l)\mathcal{R}_\beta(q,q') = 0, \mathcal{R}_\beta(q,q')|_{\beta=0} = \delta(q,q') \quad (28)$$

After solution of equation (28) and substituting it in (26) we obtain a solution of reduced Bloch equation (24) on Lagrange submanifold of a symplectic sheet to the orbit that corresponds to the homogeneous space. The transition to the solution on the entire space $\rho_\beta(x, x')$ can be obtained using transformation (27).

4. Example

As an example of presented method we chose the solution of the main problem of thermodynamics tree-dimensional homogeneous space with four-dimensional Lie group acting on it. The algebra of Lie group G is determined by following commutation rules:

$$[e_2, e_3] = e_1, [e_2, e_4] = e_3, [e_3, e_4] = -e_2. \tag{29}$$

Casimir functions of that algebra are $K_1 = f_1, \quad K_2 = f_2^2 + f_3^2 - 2f_1 f_4$.

Let's consider 3-dimensional homogeneous space $M = G/H$, where Lie algebra of one-dimensional subgroup H is $\{e_3\}$. We choose linear functional $\lambda(J) = \{-2j^2, 0, 0, n\}, n \in N$.

It's easy to calculate that the defect of that space $d(M) = 0$ and since that considered space is symmetric and we don't have here generators of $\mathcal{F}$ – algebra except for Casimir operators.

Generators X_a of group action on M look as follows

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = -x_2 \sin(x_4)\frac{\partial}{\partial x_1} + \cos(x_4)\frac{\partial}{\partial x_2},$$
$$X_3 = x_2 \cos(x_4)\frac{\partial}{\partial x_1} + \sin(x_4)\frac{\partial}{\partial x_2}, \quad X_4 = \frac{\partial}{\partial x_4}. \tag{30}$$

Operators of λ-representation are

$$l_1 = -2ij^2, \quad l_2 = j(\partial_q - q), \quad l_3 = -ij(\partial_q + q), \quad l_4 = -i(\partial_q + n), \tag{31}$$

with polarization chosen as $\{e_1, e_2 - ie_3, e_4\}$.

Functions $D^{\lambda}_{qu}(x)$ can easily be found from equations (16)

$$D^J_{qu}(x) = D^J_q(x) = \exp(2jqx_2e^{ix_3} - x_2^2j^2 + 2ijx_1 - \frac{q^2}{2}e^{2ix_2} - inx_3), \tag{32}$$

and since defect $d(M)$ of homogeneous space M is zero we have here only dependence on variable q. Measure on q-space is $d\mu(q) = \exp(-q\overline{q})d^2q/\pi$ and δ-function in respect to that measure takes form $\delta(\overline{q'}, q) = \exp(q\overline{q'})$.

Matrix G was chosen as

$$G^{ab} = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & B & -C \\ 0 & 0 & C & 0 \end{pmatrix}$$

with condition $A, B, C > 0, B > 2C$.

Operator of Bloch equation on considered homogeneous space (22) looks as follows

$$H(x) = (A + Bx_2^2)\frac{\partial^2}{\partial x_1^2} + B\frac{\partial^2}{\partial x_2^2} + C(x_2 \sin(x_4)\frac{\partial}{\partial x_1} - \cos(x_4)\frac{\partial}{\partial x_2}). \tag{33}$$

That operator coincides with Laplace-Beltrami operator built according to the well known rule for riemannian manifold using metric (21).

Since that Bloch equation on that space is

$$\frac{\partial \rho_\beta(x,x')}{\partial \beta} - \hbar^2((A+Bx_2^2)\frac{\partial^2}{\partial x_1^2} + B\frac{\partial^2}{\partial x_2^2} + C(x_2\sin(x_4)\frac{\partial}{\partial x_1} - \cos(x_4)\frac{\partial}{\partial x_2}))\rho_\beta(x,x') = 0. \quad (34)$$

Bloch equation on corresponding Lagrange submanifold of a symplectic sheet to the orbit contains only one independent variable and takes following form

$$\frac{\partial \mathcal{R}_\beta(q,q')}{\partial \beta} - (-4j^4A - 2j^2B + Cjq)\mathcal{R}_\beta(q,q') + (-4Bj^2q - Cj)\frac{\partial}{\partial q}\mathcal{R}_\beta(q,q') = 0. \quad (35)$$

Solution of (35) which represents density matrix on orbit and satisfies special initial condition is

$$\mathcal{R}_\beta(q,q') = \exp(-4j^2B\beta(\frac{C^2}{B^2j^2} + \frac{j^2A}{B} + \frac{1}{2}))\exp(\frac{Cq}{4Bj} - \frac{C((4Bqj+C)\exp(-4Bj^2\beta) - C)}{16B^2j^2}) \times$$
$$\times \exp(\frac{q'((4Bqj+C)\exp(-4Bj^2\beta) - C)}{4Bj}). \quad (36)$$

Finally solution of Bloch equation on entire homogeneous space we obtain as integral over variable j

$$\mathcal{R}_\beta(x,x') = \int \frac{j}{2\pi^2}\delta(x_3 - x'_3)\exp(-4j^4\beta A - \frac{\beta C^2}{4B} - 2i(x_1 - x_1')j - (2B\beta + x_2^2 + x_2'^2) -$$
$$- \frac{Cx_2e^{-ix_3}}{2B}(e^{-4Bj^2\beta} - 1) - \frac{C^2e^{-2ix_3}(e^{4Bj^2\beta} - 1)}{32B^2j^2}(e^{4Bj^2\beta}(2e^{2ix_3} - 1) + 1))dj. \quad (37)$$

5. Conclusion

Above we have shown clear algorithm that partly allows to solve the main problem of homogeneous spaces thermodynamics for arbitrary homogeneous space. Solution of Bloch equation (4) and building of density matrix are just the first but important step to the solution of much more complex task — to find statistic sum for arbitrary non-compact homogeneous space, the problem that was stated many years ago but yet to be solved. We must point out that in mentioned works were obtained exact explicit formulas for heat kernel. Method presented above does not give answer immediately at the moment we define the structure of the manifold. Our method is an algorithm that must be applied to a given homogeneous space to find the heat kernel on it.

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Radiating Shell Supported by a Phantom Energy

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Abstract: I describe the evolution of a thin spherically symmetric self-gravitating phantom shell around the radiating shell. The general equations describing the motion of shell with a general form of equation of state are derived. The stability analysis of this phantom shell to linearized spherically symmetric perturbation about static equilibrium solution is carried out.

Keywords: General Relativity, Spherically Symmetric Thin Shell, Phantom Energy
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1. Introduction

Recent astrophysical observations [1,2] related to distant supernovas, cosmic microwave background and galaxy clustering all together essentially changed our view on the evolution of the universe. Now it is generally accepted that the universe at present is expanding with acceleration. The explanation of such cosmological behavior in the framework of general relativity (GR) requires the supposition that a considerable part of the universe consists of a hypothetical dark energy: the exotic matter with a positive energy density $\rho > 0$ and a negative pressure $p = \omega\rho$ with $\omega < -\frac{1}{3}$. In the last few years intensive efforts with a variety of theoretical ideas and models concerning dark gravity and scalar tensor theories, bran world models, dark energy models with negative potentials, tachyon scalar field, scalar field with a negative kinetic energy were discussed (cf [3,4,5]).

The most exotic form of dark energy is a phantom energy with $\omega < -\frac{1}{3}$ [6], for which the weak energy conditions is violated. The exotic nature of phantom energy reveals itself in a number of unusual cosmological consequence (cf. [7]). Therefore, one can consider the phantom energy as a possible candidate for exotic matter. It is possible to extend the

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motion of phantom energy on the case of spherically symmetric space time configurations. Suppose that it is characterized by the equation of state $p = \omega\rho$ with $\omega < -\frac{1}{3}$, where p is the negative radial pressure.

Sushkov [8] discussed a model of static spherically symmetric wormholes with phantom energy. The aim of this paper is to describe the thin spherically symmetric phantom shell around a radiating body.

The paper is organized as follows. In Section 2 the general concepts of the dynamics spherically symmetric thin shell are outlined with special attention to Schwarzschild space time. In Section 3 the evolution of thin shell with phantom equation of state is analyzed. Outline of a general linearized stability analysis procedure is given in Section 4. A general conclusion is given in Section 5.

2. Dynamics of spherically symmetric thin shell

The line element of any spherically symmetric space time can be written in the form

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = Adt^2 + 2Hdtdq + Bdq^2 + r^2\left(t,q\right)d\Omega^2 \tag{1}$$

Here t and q are the timelike and spacelike coordinates, $A, H$and $B$are functions of t and q only, and $r(t,q)$ is the radius of a two dimensional sphere (in the sense that the area of the sphere is $4\pi r^2$), and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, is the line element of the unit sphere.

For the given space time the coefficients$A, H$and$B$are not uniquely defined. One can transform the line element (1) to the new coordinate system which converses explicitly the spherically symmetric form of the metric

$$\tilde{t} = \tilde{\tau}\left(t,q\right) \quad , \quad \tilde{q} = \tilde{q}\left(t,q\right).$$

The radius $r$is invariant under this transformation. The other very important invariant is

$$\Delta = \gamma^{\alpha\beta}r_{,\alpha}r_{,\beta}$$

where $\gamma^{\alpha\beta}$ is inverse to the two-dimensional metric tensor $\gamma_{\alpha\beta}$.

In the flat Minkowskian space time $\Delta = -1$, all the surfaces $r = const$ are time like and r can be chosen as spatial coordinate $q = r$. In the curved space time, Δ can be positive and negative: (1) The region with $\Delta < 0$ is called $R - region$ and the radius can be chosen as a radial coordinate q. (2) The region with $\Delta > 0$ is called $T - region$ and the surfaces $r = const$ are spacelike (the normal vector is timelike) and the radius can be chosen as a time coordinate t. In $T - region$ there is no $\dot{r} = 0$ (where "dot" means a time derivative), hence it must be either $\dot{r} > 0$ (such region of expansion is called $T_+ - region$) or $\dot{r} < 0$ (such region of contraction is called $T_- - region$). The same holds for $R - regions$. They are divided into two classes which are called $R_+ - region$ with $r' > 0$ and $R_- - region$ with $r' < 0$ (where prime stands for a spatial derivative). These R and T regions are separated by the surfaces $\Delta = 0$ which are called the apparent horizons, which can be null, timelike or spacelike apparent horizon.

The metric (1) in a Schwarzschild space time has the form

$$ds^2 = f\,(r)\,dt^2 - f^{-1}\,(r)\,dr^2 - r^2 d\Omega^2 \tag{2}$$

where

$f\,(r) = 1 - \frac{2m}{r}$ with $m > 0$.

One of the most important features of GR is that the equations of motion of matter fields are incorporated into the Einstein equations. The Einstein equations of GR are nonlinear partial differential equations. This means that the motion of test particles or fields on the given background will in general be different from that of the matter for the self-consistent solutions. It makes analysis very complicated. To obtain some definite results, choose the simplest possible model, like self-gravitating thin shell. In this section I give a brief history on the equation of motion of a thin shell. When dealing with the time like spherically symmetric thin shell, adjust the covariant formalism derived by Israel [9] to the case of interest.

Let hypersurface Σ divided the whole space time into two parts, "in" and "out", and can connect some special coordinate system called Gauss normal coordinates to this hypersurface Σ. The line element in these coordinates takes the form

$$ds^2 = d\tau^2 - dn^2 - r^2\,(\tau, n)\,d\Omega^2 \tag{3}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, is the line element of the unit sphere, τ is the proper time of the observer sitting fixed on Σ. The coordinate n grows from the "in" to the "out" region in the outer normal direction to the hypersurface Σ and $r(\tau, n)$ is the radius of the sphere. The hypersurface is situated at $n = 0$ and the intrinsic metric to Σ is $ds^2_\Sigma = d\tau^2 - R^2\,(\tau)\,d\Omega^2$, where $R\,(\tau) = r\,(\tau, 0)$.

Keeping in mind that the metric itself is continuous but some of its derivatives make a jump across the shell. The jump of the extrinsic curvature K_{ab} is $[K_{ab}] = K^{out}_{ab} - K^{in}_{ab}$, where a quantity in square brackets stands for the difference of that quantity evaluated on the outer side, say the "out", minus the quantity evaluated on the inner side, the "in" side.

The hypersurface Σ represents the history of a surface layer (a singular hypersurface of order one) if $K^{out}_{ab} \neq K^{in}_{ab}$. This hypersurface Σ is called the singular shell if some energy momentum tensor is concentrated on it, namely: $T^k_i = t^k_i \delta(n) + ...$ Otherwise the hypersurface is nonsingular. The Einstein equation determines the relation between the extrinsic curvature K_{ab} and the three dimensional intrinsic energy momentum tensor t_{ab} is given by the Lanczos equation

$$[K_{ab}] = -8\pi \left(t_{ab} - \frac{1}{2} t g_{ab} \right)$$

where $t = t^a_a$. This relation can be written in the form

$$t_{ab} = \frac{-1}{8\pi}\left([K_{ab}] - g_{ab}\,[K]\right) \tag{4}$$

where $[K] = g^{ab}[K_{ab}]$ is the trace of the extrinsic curvature. In this case due to spherical symmetry the only nonzero components of t_a^b are t_0^0 and $t_2^2 = t_3^3$ [10]. The invariant function Δ equals

$$\Delta = R_{,r}^2 - R_{,n}^2$$

and

$$R_{,n}|_\Sigma = \varepsilon\sqrt{\dot{R}^2 - \Delta}$$

where $\varepsilon = +1$ if radii increase in the direction of the outer normal, and $\varepsilon = -1$ if radii decrease. Evidently, $\varepsilon = +1$ in $R_+ - region$ and $\varepsilon = -1$ in $R_- - region$. From (4) the components are:

$$\frac{2\varepsilon_{in}}{R}\sqrt{\dot{R}^2 - \Delta_{in}} - \frac{2\varepsilon_{out}}{R}\sqrt{\dot{R}^2 - \Delta_{out}} = 8\pi R t_0^0 \tag{5}$$

$$\begin{gathered}\tfrac{2\varepsilon_{in}}{R}\sqrt{\dot{R}^2 - \Delta_{in}} - \tfrac{2\varepsilon_{out}}{R}\sqrt{\dot{R}^2 - \Delta_{out}} + \tfrac{\varepsilon_{in}}{\sqrt{\dot{R}^2-\Delta_{in}}}\ddot{R} - \tfrac{\varepsilon_{out}}{\sqrt{\dot{R}^2-\Delta_{out}}}\ddot{R} + \\ \tfrac{\varepsilon_{in}}{2R\sqrt{\dot{R}^2-\Delta_{in}}}(1+\Delta_{in}) - \tfrac{\varepsilon_{out}}{2R\sqrt{\dot{R}^2-\Delta_{out}}}(1+\Delta_{out}) + 4\pi R\left({}^{out}T_n^n - {}^{in}\mathrm{T}_n^n\right) = 8\pi t_2^2\end{gathered} \tag{6}$$

The continuity equation for the energy-momentum tensor is transformed to

$$\frac{dt_0^0}{d\tau} + \frac{2\dot{R}}{R}\left(t_0^0 - t_2^2\right) + \left({}^{out}\mathrm{T}_0^n - {}^{in}\mathrm{T}_0^n\right) = 0 \tag{7}$$

This equation is a differential consequence of the first two ones.

In this paper the space-time inside the shell will be the Vaydia metric and outside the shell is the Schwarzschild metric, then the equations (5), (6) and (7) becomes

$$\varepsilon_{in}\sqrt{\dot{R}^2 + F_{in}} - \varepsilon_{out}\sqrt{\dot{R}^2 + F_{out}} = 4\pi R t_0^0 \tag{8}$$

$$\ddot{R} = -4\pi^2 R\left(t_0^0\right)^2 + 8\pi^2 R t_0^0 t_2^2 - \frac{(m_{in} + m_{out})}{2R^2} - \frac{\Delta m^2 t_2^2}{8\pi^2 R^5\left(t_0^0\right)^3} \tag{9}$$

$$\dot{t}_0^0 + \frac{2\dot{R}}{R}\left(t_0^0 - t_2^2\right) = 0 \tag{10}$$

where

$$\Delta_{out} = -F_{out} = -1 + \frac{2m_{out}}{R}$$

$$\Delta_{in} = -F_{in} = -1 + \frac{2m_{in}(\upsilon)}{R}$$

and $m_{in}(\nu)$ is the mass function in the interior space and depending whether the null coordinate υ is advanced or retarded.

3. Dynamics of phantom shell

The equation of state describing phantom energy in cosmology is usually taken as $p = \omega\rho$ where $\omega < -1$ and $p$is a negative spatially homogeneous pressure. By analogy, it is possible to use the same equation of state for a spherically symmetric distribution of phantom energy but with $p$is the negative radial pressure and written in the form $p = -k\rho$ with $k > 1$. Consider a simple linear equation of state by relation

$$t_0^0 = kt_2^2 \tag{11}$$

In the phantom case $k > 1$. The solution of (10) is

$$t_0^0 = CR^{2(k-1)} \tag{12}$$

where $C$is a constant denoted to the shell power.

Using equation (8) and taking into account ($\varepsilon_{in} = -1, \varepsilon_{out} = +1$) one gets the following two equations:

$$\dot{R}^2 = -1 + \frac{1}{R}((m_{in} + m_{out}) + \frac{\delta m^2}{4x} + x) \quad , \tag{13}$$

$$\ddot{R} = -\frac{1}{2R^2}[(m_{in} + m_{out}) + k\frac{\delta m^2}{x} - 2(2k-1)x] \tag{14}$$

where

$$x \equiv 4\pi^2 C^2 R^{4k-1} \tag{15}$$

The sign conditions of the equation of motion of shell will be

$$\varepsilon_{in} = sign\left[\delta m + 8\pi^2 R^3 \left(t_0^0\right)\right] \tag{16}$$

$$\varepsilon_{out} = sign\left[\delta m - 8\pi^2 R^3 \left(t_0^0\right)\right] \tag{17}$$

where $\delta m = m_{out} - m_{in}$. Using (15) to get a more convenient form of the sign conditions

$$\varepsilon_{in} = sign\left(\delta m + 2x\right) \tag{18}$$

$$\varepsilon_{out} = sign\left(\delta m - 2x\right) \tag{19}$$

When the function $\dot{R}^2$ has roots, it is possible to represent both the finite and infinite motion. The change of sign of the acceleration $\ddot{R}$ in (14) occurs when $\ddot{R} = 0$. This corresponds to the quadratic equation whose positive root is

$$x_0 = \frac{m_{in} + m_{out} + \sqrt{(4k-1)^2\delta m^2 + 4m_{in}}m_{out}}{4(2k-1)} \tag{20}$$

It is convenient to define the parameter space of the problem using $m_{in}, m_{out}, k > \frac{1}{4}$ and R as free parameters.

Consider at first the case of $\delta m > 0$. According to (18), ε_{in} must be positive $\varepsilon_{in} = +1$, the ε_{out} changes sign at $x = x_1 \equiv \frac{\delta m}{2}$ and $\varepsilon_{out} = -1$ if $x > x_1$. Hence, in the case $x \to \infty$, then $\varepsilon_{out} = -1$ and if $x \to 0$ then $\varepsilon_{out} = +1$. The value of R corresponding to x_1 is denoted by R_1. Then equation (14) will be $\ddot{R}\left(x_1\right) = -\frac{m_{out}}{R_1^2}\langle 0$. Therefore $R_1 < R_0$ which corresponds the value of R at x_0 in (15). From (13) one obtains $\dot{R}^2\left(x_1\right) = -1 + \frac{2m_{out}}{R_1}$. For the shell moving from infinity to infinity. Consider now the case of $\delta m < 0$. The density t_0^0 is assumed to be always positive. So the negativity of δm is caused by the gravitational mass defect. The ε_{in} changes sign at $x = x_2 \equiv \frac{-\delta m}{2}$ and $\varepsilon_{in} = +1$ if $x > x_2$, $\varepsilon_{out} = -1$ at the same time. In the case of $x \to \infty$ then $\varepsilon_{in} = +1$ and $\varepsilon_{in} = -1$ if $x \to 0$. The value of R corresponding to x_2 is

$\ddot{R}\,(x_2) = -\frac{m_{in}}{R_2^2}\langle 0$ and $\dot{R}^2\,(x_2) = -1 + \frac{2m_{in}}{R_2}$.

In the case of $\delta m < 0$ the shell evolves under horizons and cannot reach a distant observer living in $R_+ - region$. But in the case of $\delta m > 0$, the shell can show itself in $R_+ - region$.

4. Linearized stability analysis

Rearranging equation (13) into the form

$$\dot{R}^2 = -1 + \frac{1}{R}((m_{in} + m_{out}) + \frac{(m_{out} - m_{in})^2}{4x} + x) \tag{21}$$

where $x \equiv 4\pi^2 C^2 R^{4k-1}$, C is a power constant and $k > 1$. This equation can be written in the dynamical form

$$\dot{R}^2 + V(R) = 0 \tag{22}$$

with the potential given by

$$V(R) = 1 - \frac{1}{R}((m_{in} + m_{out}) + \frac{(m_{out} - m_{in})^2}{4x} + x)$$

The factor $F(R)$and $G(R)$ introduced for computational convenience, are defined by

$$F(R) = 1 - \frac{1}{R}(m_{in} + m_{out})$$

$$G(R) = \frac{(m_{out} - m_{in})}{R}$$

So that the potential $V(R)$ takes the form

$$V(R) = F(R) - 4\pi^2 C^2 R^{4k-2} - \frac{RG^2}{4x} \tag{23}$$

From equation (15) and $M = 4\pi R^2 t_0^0 = 4\pi C R^{2k}$, equation (23) takes the form

$$V(R) = F(R) - (\frac{M}{2R})^2 - (\frac{RG}{M})^2 \tag{24}$$

where M is the surface mass of the shell. Linearized around the stable solution at $R = R_0$, consider a Taylor expansion of $V(R)$ around R_0 to second order, provides

$$V(R) = V(R_0) + V'(R_0)(R - R_0) + \frac{1}{2}V''(R_0)(R - R_0)^2 + O\left[(R - R_0)^3\right] \tag{25}$$

Where the prime denotes a derivative with respect to R. The first and second derivatives of $V(R)$are given by

$$V'(R) = F'(R) - 2(\frac{M}{2R})(\frac{M}{2R})' - 2(\frac{RG}{M})(\frac{RG}{M})' \tag{26}$$

$$V''(R) = F''(R) - 2[(\frac{M}{2R})']^2 - 2(\frac{M}{2R})(\frac{M}{2R})'' - 2[(\frac{RG}{M})']^2 - 2(\frac{RG}{M})(\frac{RG}{M})'' \quad (27)$$

Evaluated at the static solution ($R = R_0$) and through a long calculation, I find that $V(R_0) = 0$ and $V'(R_0) = 0$. From the condition $V'(R_0) = 0$, one extracts the following useful equilibrium relationship

$$(\frac{M}{2R_0})' \equiv \Gamma = (\frac{R_0}{M})[F'(R) - -2(\frac{R_0G}{M})(\frac{R_0G}{M})'] \quad (28)$$

So that,

$$\dot{R}^2 = -\frac{1}{2}V''(R_0)(R - R_0)^2 + O[(R - R_0)^3].$$

If $V''(R_0) < 0$is verified, then the potential $V(R)$has a local maximum at R_0, where a small perturbation in the surface radius will produce an irreversible contraction or expansion of the shell. Therefore, the solution is stable if and only if $V(R)$ has a local minimum at R_0 and

$V''(R_0) > 0$ is verified. The latter stability condition takes the form

$$(\frac{M}{2R})(\frac{M}{2R})'' < \Psi - \Gamma^2 \quad (29)$$

where $\Psi$is defined as

$\Psi = \frac{F''}{2} - [(\frac{RG}{M})']^2 - (\frac{RG}{M})(\frac{RG}{M})''$ and $\Gamma = (\frac{M}{2R_0})'$.

5. Conclusion

The motivation of this work is the fact that in many physically interesting situations in cosmology and astrophysics the essential role was played the full account for gravitational back reaction. In this case of phantom shell such a back reaction may appear crucial for formation of the space time.

The matter is that in GR any type of energy is gravitating, not only energy density but also the tension and pressure are gravitating. The pressure plays a twofold role. The positive pressure causes both repulsion and attraction, the attraction is due to its contribution to the gravitating source. The negative pressure leads to the gravitational repulsion. Hence, the phantom shell is even more repulsive.

In the case of $\delta m > 0$ the distance observer may see the shell but can not register the energy flux of the shell. The stability analysis of this phantom shell to linearized spherically symmetric perturbation about static equilibrium solution is carried out.

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Radial Matrix Elements for the Hydrogen Atom

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Abstract: It is known that the hydrogenlike atom can be studied as a Morse oscillator, then here we show that these fact leads to an interesting method to obtain the matrix elements for the Coulomb potential.

Keywords: Coulomb and Morse potentials; Langer transformation; matrix elements.
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1. Introduction

For the hydrogenic atom its radial wave function $\frac{1}{r}g_{nl}$ depends of the total (n) and orbital (l) quantum numbers, which are associated to eigenvalues for energy and angular momentum, respectively. Lee [1] showed that the Langer transformation [2] permits to study a non-relativistic hydrogenlike system as a vibrational Morse oscillator (MO) [3], such that n gives the parameters of the Morse well and l determines an energy level in these well. In Sec.2 we exhibit this result of Lee.

In according with [1] the function g_{nl} is proportional to the corresponding (MO) wave function, which means that the matrix elements $\left\langle nl_2 \left| r^k \right| nl_1 \right\rangle$ of the hydrogenic atom are equivalent to $\langle N_2 \left| e^{-\gamma u} \right| N_1 \rangle$,$\gamma = k+2$, of its MO. Thus the knowledge on Morse matrix elements can be used to determine $\left\langle r^k \right\rangle$ for the Coulomb potential. In Sec.3 we apply this approach to obtain $\left\langle nl_2 \left| r^k \right| nl_2 \right\rangle$, $k =$*integer* ≥ -2, without factorization techniques [4, 5] as in [6]; we reproduce, as particular cases, the elements $\left\langle nl \left| r^k \right| nl \right\rangle$, $k = \pm 1, \pm 2$, deduced analytically by Landau-Lifshitz [7].

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2. Hydrogenlike Atom As A Morse Oscillator

Here we exhibit the result of Lee [1]: The motion of an electron into the Coulomb field generated by a nucleus with charge Ze, is equivalent to the vibrational dynamics of a MO.

It is very well known [7] that the radial wave function $\frac{1}{r}g_{nl}$ satisfies the Schrödinger equation (in natural units $\hbar = m = 1$):

$$-\frac{1}{2}\left[\frac{d^2}{dr^2} - \frac{l\,(l+1)}{r^2}\right] g_{nl} - \frac{Ze^2}{4\pi\varepsilon_0 r} g_{nl} = -\frac{Z^2e^2}{32\pi^2\varepsilon_0^2 n^2} g_{nl} \tag{1}$$

where $n = 1,\ 2, ...$ and $l = 0,\ 1, ..., n-1$. Now the quantities r, g_{nl} are changed to u, ψ_N via the Langer transformation [1, 2]:

$$g_{nl} = \frac{e^{-\frac{u}{2}}}{\left[bn\left(l+\frac{1}{2}\right)\right]^{\frac{1}{2}}} \Psi_N\,(u) \tag{2}$$

with $r = bn^2 e^{-u}$, and $b = \frac{4\pi\varepsilon_0}{Ze^2}$, then (1) adopts the form:

$$-\frac{1}{2}\frac{d^2}{du^2}\Psi_N + D\left(e^{-2u} - 2e^{-u}\right)\ \Psi_N = E\,\Psi_N \tag{3}$$

which is the Schrödinger equation for a MO [3, 5] with parameters:

$$\tilde{k} = \frac{2}{a}\sqrt{2D} = 2n, D = \frac{n^2}{2}, a = 1, \tag{4}$$

$$E = -\frac{1}{8}(\tilde{k} - 2N - 1)^2 = -\frac{1}{2}\left(l + \frac{1}{2}\right)^2,$$

$$N = n - l - 1,$$

thus each n generates one MO with width $a = 1$, depth $D = \frac{n^2}{2}$ and vibrational

frequency $\frac{a}{2\pi}\sqrt{2D} = \frac{n}{2\pi}$. Finally, the value of l determines the eigenstate ψ_N, $N = n - l - 1$, with energy $E = -\frac{1}{2}(l + \frac{1}{2})^2$.

3. Matrix Elements for the Coulomb Potential

The principal aim of our work is the calculation of the matrix elements:

$$\langle nl_2|\, r^k \,|nl_1\rangle = \int_0^\infty g_{nl_2} r^k g_{nl_1} dr, \quad k = \text{ integer } \geq -2 \tag{5}$$

The factorization method [4, 6] calculates (5) using ladder operators for the proper states g_{nl}; the analytical approach [7] employs the explicit expression of g_{nl} and determines directly the integral (5). Here we apply the Langer transformation [1, 2] to obtain (5) via the relationship between the Coulomb and Morse interactions.

In fact, if we put (2) into (5):

$$\langle nl_2|\, r^k\, |nl_1\rangle = n^{2k+1} b^k \left[\left(l_1+\frac{1}{2}\right)\left(l_2+\frac{1}{2}\right)\right]^{-\frac{1}{2}} \langle N_2|\, e^{-\gamma u}\, |N_1\rangle \tag{6}$$

with $N_j = n - l_j - 1$, $j = 1, 2$ and $\gamma = k + 2 = 0, 1, 2, ...$, which means that any $\langle r^k \rangle$ for the Coulomb potential is proportional to a matrix element of the corresponding MO. The elements $\langle e^{-\gamma u}\rangle$ are determined in [8, 9]:

$$\begin{aligned} \langle N_2|\, e^{-\gamma u}\, |N_1\rangle &= \tfrac{(-1)^{N_1+N_2}}{\tilde{k}^{\gamma}} \left[\tfrac{Q_1 Q_2 N_2!\, \Gamma\left(\tilde{k}-N_2\right)}{N_1!\, \Gamma\left(\tilde{k}-N_1\right)}\right]^{\frac{1}{2}} \bullet \\ \bullet \sum_{j=0}^{N_2} & \tfrac{(-1)^j \Gamma\left(N_1+\gamma-j\right) \Gamma\left(\tilde{k}-N_1-1+\gamma-j\right)}{j!\ \left(N_2-j\right)!\, \Gamma\left(\tilde{k}-N_2-j\right) \Gamma\left(\gamma-j\right)} \end{aligned} \tag{7}$$

where Γ denotes the gamma function, $Q_c = \tilde{k} - 2N_c - 1$, $c = 1,\ 2$, and without loss of generality we have accepted $N_1 \geq N_2$(that is, $l_2 \geq l_1$). Then (6) and (7) with $\tilde{k} = 2n$ imply the exact expression:

$$\begin{aligned} \langle nl_2|\, r^k\, |nl_1\rangle &= \tfrac{(-1)^{l_1+l_2}}{2n} \left(\tfrac{bn}{2}\right)^k \left[\tfrac{(n-l_2-1)!\, (n+l_2)!}{(n-l_1-1)!\, (n+l_1)!}\right]^{\frac{1}{2}} \bullet \\ \bullet \sum_{j=0}^{n-l_2-1} & \tfrac{(-1)^j (n+k-l_1-j)!\, (n+k+l_1-j+1)!}{j!(n-l_2-1-j)!\, (n+l_2-j)!\, (k+1-j)!} \end{aligned} \tag{8}$$

which is not explicitly in the literature, and it is more simple than the corresponding relation deduced in [6] using factorization techniques. Special applications of (6) and (8) are:

a) $k = -2$.

In this case$\gamma = 0$, then from (6) it is immediate that:

$$\langle nl_2|\, r^{-2}\, |nl_1\rangle \propto \langle N_2 \mid N_1\rangle = \delta_{N_1 N_2} \tag{9}$$

therefore only if $l_1 = l_2$ we have$\langle r^2\rangle \neq 0$, which is the result of Pasternack-Sternheimer mentioned in [6].

b) $l_1 = l_2 = l, \quad k = \pm 1,\ \pm 2$

The general expression (8) reproduces easily the following particular examples of Landau-Lifshitz [7]:

$$\begin{gathered} \left\langle r^{-2}\right\rangle = \frac{b^{-2}}{n^3\left(l+\frac{1}{2}\right)}, \\ \langle r\rangle = \frac{b}{2}\left[3n^2 - l\left(l+1\right)\right], \\ \left\langle r^{-1}\right\rangle = \frac{b^{-1}}{n^2}, \\ \left\langle r^2\right\rangle = \frac{b^2 n^2}{2}\left[5n^2 + 1 - 3l\left(l+1\right)\right]. \end{gathered} \tag{10}$$

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A Simply Regularized Derivation of the Casimir Force

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Abstract: We want to calculate the Casimir force between two parallel, uncharged, perfectly conducting plates by a simple automatically regularized approach. Although in the well-known methods one should explicitly subtract the energy term due to the empty space to regularize the calculation, here, the regularization is simply/implicitly achieved by considering only the energy per unit area of each plate.

Keywords: Quantum Theory of Electromagnetic Fields, The Casimir Effect
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1. Introduction

There are three well-known technical types of derivation of the Casimir force for different geometries including the simplest geometry of two parallel, uncharged, perfectly conducting plates firstly explored by Casimir [1]. One modern method is the quantum field theoretical approach based on the appropriate Green's function of the geometry of problem [2]. The other technical type is the dimensional regularization method that involves the mathematical complications of the Riemann zeta function and the analytical continuation [2]. The last (the most elementary/the simplest) method is based on modes summation by using the Euler-Maclurian integral formula [3-5].

The problem of finding the Casimir force, not only for the simplest geometry of two plates that we want to study here but also for other more complicated geometries, indispensably/automatically involves some infinities/irregularities; thus, one should regularize the calculation for arriving at the desired finite physical result(s). In the Green' function method, one uses the subtraction of two terms (two Green's functions) to do the required

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regularization. In the dimensional regularization method, although there isn't an explicit subtraction for the regularization of the problem, as is clear from its name, the calculation is regularized dimensionally by going to a complex plane with a mathematically complicated/ambiguous approach. In the simplest method in which the Euler-Maclurian formula is used, the regularization is performed by the subtraction of the zero-point energy of the free space (no plates) from the energy expression under consideration/calculation (e.g. summation of the interior and exterior zero-point energies of the two parallel plates).

Here, as an almost simple method and in an approach near to the methods using the Euler-Maclurian integral formula but of course without explicitly regularizing the calculation (i.e. without subtracting the energy term due to the empty space), we want to derive the Casimir force for the simplest geometry of two parallel, uncharged, perfectly conducting plates. This does not mean that we can omit the regularization by short cutting the problem! In fact, the calculation is automatically/implicitly regularized.

2. The Zero-Point Energy of the Electromagnetic (Em) Fields

The quantization of the operator form of the Hamiltonian

$$H_{EM} = \frac{1}{2}\int (\varepsilon_0 E^2 + \mu_0 B^2) d^3x \tag{1}$$

by considering the electric and magnetic fields as dynamical operators of the system leads to the result [3]

$$H = \sum_m \hbar\omega_m (a_m^+ a_m + \frac{1}{2} I) \tag{2}$$

where ω_m is the angular frequency corresponding to the eigenmodes expansion of the fields operators and the raising and lowering operators, a_m^+ and a_m, satisfy the commutation relation

$$[a_m, a_n^+] = I\delta_{mn} \tag{3}$$

Therefore, the Hamiltonian operator for the EM fields is equivalent to the Hamiltonian operator for a system of infinite number of independent oscillators.

The lowest energy, the zero-point energy (quantum field theoretically: the vacuum energy), for one mode is $\frac{1}{2}\hbar\omega = \frac{1}{2}\hbar ck$; thus, since there are infinitely many modes of arbitrary high frequency in any finite volume, it follows that there should be an infinite zero-point (vacuum) energy in any (finite) volume of space?! Although in many situations it is stated that this zero-point energy is not observable and for this reason the theory should be defined by normal ordering [6], Casimir recognized that such a conclusion is incorrect when he showed that zero-point fluctuations in electromagnetic fields gave rise to an attractive force between parallel, perfectly conducting plates [1].

3. The Casimir Force Between Parallel, Uncharged, Perfectly Conducting Plates

Consider two infinitely large, parallel, uncharged, perfectly conducting plates at a distance (a) from each other (assume each plate is in the xy plane). In order to having the appropriate boundary conditions for the electromagnetic fields, the components of the wave-vectors corresponding to their Fourier transforms should satisfy [3]

$$k_i = \frac{n_i \pi}{x_i} \quad (i = 1, 2, 3) \tag{4}$$

with n_i's being nonnegative integers and x_is are the Cartesian dimensions of the problem (here we work with $x_1 = x_2 = 1$ and $x_3 = a$).

Since the dimensions of the plates in the xy plane is infinitely large, the allowed values of k_x, k_y approximate a continuum, the density of modes in the $k_x k_y$ plane is $\frac{1_{area}}{\pi^3}$ and therefore

$$\frac{1}{(area)} \sum_{k_x, k_y} \frac{1}{2} \hbar\omega \rightarrow \frac{1}{2} \hbar c \int \frac{d^2k}{(2\pi)^2} \sqrt{(k_x^2 + k_y^2)} \quad (\omega = ck) \tag{5}$$

Thus, by means of (2), (4) and (5), we have the following expression for the energy per unit transverse area of each plate [1]

$$\mathrm{E}_{area} = 2 \times \hbar c [\frac{1}{2} \sum_{n=1}^{\infty} \int \frac{d^2k}{(2\pi)^2} \sqrt{(k_x^2 + k_y^2) + (\frac{n\pi}{a})^2}] \tag{6}$$

where the pre-factor 2 has come from summing on two polarization degrees of freedom of the electromagnetic fields.

Changing the (k_x, k_y) system of variables to a system of two dimensional polar variables (θ, k_θ), makes (6) take the form

$$\mathrm{E}_{area} = 2 \times \hbar c \{\frac{1}{2} \sum_{n=1}^{\infty} [\int \frac{k_\theta dk_\theta d\theta}{(2\pi)^2} \sqrt{k_\theta^2 + (\frac{n\pi}{a})^2}]\} = \hbar c \{\sum_{n=1}^{\infty} [\int \frac{k_\theta dk_\theta}{2\pi} \sqrt{k_\theta^2 + (\frac{n\pi}{a})^2}]\} \tag{7}$$

where the integration on k_θ is from 0 to ∞.

With another change of variable $u = [k_\theta^2 + (\frac{n\pi}{a})^2]$, we find

$$\begin{aligned} \mathrm{E}_{area} &= \frac{1}{2\pi} \hbar c \{\sum_{n=1}^{\infty} [\frac{1}{2} \int_{(\frac{n\pi}{a})^2}^{\infty} du \sqrt{u}]\} = \frac{1}{6\pi} \hbar c \{\lim_{(U \to \infty)} \sum_{n=1}^{\infty} [U^{\frac{3}{2}} - (\frac{n\pi}{a})^3]\} \\ &= \frac{1}{6\pi} \hbar c \{\lim_{(U, N \to \infty)} [U^{\frac{3}{2}} N - \sum_{n=1}^{N} (\frac{n\pi}{a})^3]\} \end{aligned} \tag{8}$$

[1] We should mention that just from here (from the equation (6) to the next), the approach we have used here differs from other well-known calculations (e.g. the calculations in [3-5]). Indeed, to the best of our knowledge, the authors who use the Euler-Maclurian integral formula (particularly L. E. Ballentine [3], C. Itzykson and J. B. Zuber [4], and K. Huang [5]), all work with additional energy terms (the energy terms corresponding to the exterior region of the plates and/or the empty space). Here, simply, it is enough to consider only the energy per unit transverse area of each plate.

By changing $(\frac{a}{\pi})^3 U^{\frac{3}{2}} N = \frac{1}{4}X^4 = \int\limits_0^X y^3 dy$ and considering the limiting process on U, N and thus X, we obtain

$$\mathrm{E}_{area} = (\frac{\pi^2 \hbar c}{6a^3})(\int\limits_0^\infty y^3 dy - \sum_{n=1}^{\infty} n^3) \tag{9}$$

Using Euler-Maclurian integral formula (see **Section A**), we arrive at the result

$$\mathrm{E}_{area} = \frac{-\pi^2 \hbar c}{720 a^3} \tag{10}$$

The desired force per unit area between the plates is obtained by taking the negative derivative of E_{area} with respect to a

$$F_{area} = -\frac{d}{da}\mathrm{E}_{area} = \frac{-\pi^2 \hbar c}{240 a^4} \tag{11}$$

This is just the force per unit area between parallel, uncharged, perfectly conducting plates firstly found by Casimir [1].

4. Remark

The method used here for finding the Casimir force may mislead one that there is no need to regularize the calculation. In fact, in the present/standard quantum theory of fields, there are some indispensable infinities/irregularities and we always try to regularize our calculations to find out the physical finite/regular results. But, what has happened here? Is there really no regularization in this work? The answer is clearly negative. Clearly, if we observe the right-hand side of the equation (8) we can see that the two terms in the bracket are individually infinite/irregular and only their subtraction is finite/regular. Thus, we have simply regularized the calculation through the mathematical trick introduced in the equation (8). Of course, one point is important here and should be considered as an advantage. The energy per unit area of each plate (E_{area}) with which we have worked here is a physical/finite/regular quantity (in spite of those absolute energy terms that are infinite/irregular).

5. Section A

The Euler-Maclurian formula is usually used for estimating sums with integrals. The complete version with enough explanation can be found in mathematical physics texts as [7]. A number of examples can be found in [8].

What we need (have used) here is the following formula

$$\sum_{n=1}^{\infty} f(n) - \int\limits_0^\infty f(n) dn = \frac{-1}{2} f(0) + \frac{-1}{6\times 2!} f'(0) + \frac{1}{30\times 4!} f'''(0) + \cdots \quad \text{(A-1)}$$

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