Gödel's Geometry: Embedding and Lanczos Spintensor

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Abstract: We exhibit an open problem: To investigate if the Gödel's metric accepts local and isometric embedding into E_6. Besides, we show that in this metric there is a symmetric tensor which generates algebraically to Riemann tensor and differentially to Weyl tensor.

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Gödel [1] proposed the geometry [2,3] (signature +2):

$$ds^2 = -\left(dx^1\right)^2 - 2e^{x^4} dx^1 dx^2 - \frac{1}{2} e^{2x^4} \left(dx^2\right)^2 + \left(dx^3\right)^2 + \left(dx^4\right)^2, \tag{1}$$

to represent a rotational universe, under the hypothesis that the cosmos is composed by an incoherent perfect fluid. The line element (1) can be embedded locally and isometrically into E_n, $n \geq 5$, if [3-6] there exist a set of functions $z^r(x^j)$, $r = 1, ..., n$ such that (1) adopts the form:

$$ds^2 = \sum_{r=1}^{n} \varepsilon_r \left(dx^r\right)^2, \; \varepsilon_j = \pm 1. \tag{2}$$

It is worth to recall [3] that any R_4 allows embedding into E_{10}, thus we will restrict ourselves to the range $5 \leq n \leq 10$.

The $n = 9$ and $n = 10$ cases were already solved in [7] and [4], respectively; so for

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instance, with the functions:

$$\begin{array}{ll} z^1 = \frac{1}{\sqrt{2}} e^{x^4} \cos x^2, & z^2 = \frac{1}{\sqrt{2}} e^{x^4} \sin x^2, \\ z^3 = \sqrt{2} e^{\frac{x^4}{2}} \cos \frac{1}{2} \left(x^1 + x^2 \right), & z^4 = \sqrt{2} e^{\frac{x^4}{2}} \sin \frac{1}{2} \left(x^1 + x^2 \right), \\ z^5 = \sqrt{2} e^{\frac{x^4}{2}} \cos \frac{1}{2} \left(x^1 - x^2 \right), & z^6 = \sqrt{2} e^{\frac{x^4}{2}} \sin \frac{1}{2} \left(x^1 - x^2 \right), \\ z^7 = x^1, & z^8 = x^3, \\ z^9 = \rho + \frac{1}{2} Ln \left(\frac{\rho - 1}{\rho + 1} \right), & \rho = \left(1 + \frac{1}{2} e^{2x^4} \right)^{1/2}, \end{array} \tag{3}$$

the metric (1) adopts the form (2):

$$ds^2 = -\left(dz^1\right)^2 - \left(dz^2\right)^2 - \left(dz^3\right)^2 - \left(dz^4\right)^2 + \left(dz^5\right)^2 + \left(dz^6\right)^2 - \left(dz^7\right)^2 + \left(dz^8\right)^2 + \left(dz^9\right)^2 . \tag{4}$$

On the other hand, it is known that by employing indirect methods [5], the geometry (1) accepts embedding into E_7, and hence into E_8 too; but in the literature nobody has published the corresponding z^r for such cases, and therefore this turns out an open problem. Besides, the model (1) can not be [7-10] embedded into E_5. At the present time it is ignored yet [5,6] if this Gödel solution accepts local and isometric embedding into E_6.

Is the Gödel space of class two?

One says that a given R_4 has class two if it can be embedded into E_6 [3,6]; our interest is to know if (1) can be represented as a surface of a pseudo-Euclidian 6-space. As a first possibility, one can initiate the quest (if they exist) of the z^r, $r = 1, ..., 6$ which reduce (1) to the structure (2), and in case of success the corresponding explicit embedding will be obtained.

Another option consists in solving the differential equations of Gauss–Codazzi-Ricci (GCR) [3,6] for (1), next trying to construct the second fundamental forms ${}_{(j)}b_{ac}$, $j = 1, 2$ and the Ricci vector A_c, which determine the extrinsic geometry of R_4 with respect to E_6. If somebody were able to show that this GCR system has no solution, then the Gödel metric will not be of class two.

Yakupov [6,11-13] has shown that in every R_4 embedded into E_6, the following algebraic necessary condition has to be fulfilled:

$$Y \equiv {}^{*} R^{*tjkc} \, {}^{*}R_{arkc} \, R^{ar}_{tj} = 0, \tag{5}$$

where ${}^{*}R_{ijkc}$ and ${}^{*}R^{*}_{ijkc}$ are the duals [2,7,9,10,14-16] of the Riemann tensor. This means that if (1) had $Y \neq 0$ then would be impossible its embedding into E_6; but after a long calculation it can be checked the truth of (5) for the Gödel geometry, which do not help to decide if (1) is of class two because (5) is a necessary but not sufficient requisite for the embedding. Hence, it will be convenient to find new necessary algebraic/differential conditions alternatives to (5), rendering more information about the possibility of embedding (1) into E_6.

Lanczos spintensor

Now we shall show that in this geometry there is a symmetric tensor which generates differentially to conformal tensor and algebraically to curvature tensor. In fact, for the Gödel spacetime (1) we have [17, 18] the Lanczos potential [15]:

$$K_{aij} = \frac{1}{9}\left(R_{ji\,;\,a} - R_{ja\,;\,i}\right) \quad , \tag{6}$$

where $R_{ij} = R^{c}{}_{ijc}$ is the Ricci tensor, being R_{acij} the curvature tensor, besides ; r means covariant derivative. On the other hand, the metric tensor satisfies $g_{ac\,;\,r} = 0$, and (1) has the constant vector $(B_r) = (0, 0, 1, 0)$,that is, $B_{r\,;\,c} = 0$. Then it is evident that (6) is equivalent to:

$$K_{aij} = \frac{\sqrt{2}}{18}\left(b_{ji\,;\,a} - b_{ja\,;\,i}\right) \ , \tag{7}$$

with

$$b_{ij} = \sqrt{2}\ \left[R_{ij} + \frac{1}{2}\left(B_i B_j - g_{ij}\right)\right] \ . \tag{8}$$

Thus we have that (8) is a differential generator, via the Lanczos potential, for the Weyl tensor:

$$C_{aijr} = K_{aij\,;r} - K_{air;j} + K_{jra\,;\,i} - K_{jri\,;\,a} + g_{ar}K_{ji} - g_{aj}K_{ri} + g_{ij}K_{ra} - g_{ir}K_{ja} \quad , \tag{9}$$

where $Kij = K^{r}_{ij;r}$ But b_{ij} also is an algebraical generator for the Riemann tensor because it verifies the Gauss equation [3]:

$$R_{acij} = b_{ai}b_{cj} - b_{aj}b_{ci} \ . \tag{10}$$

We must note that (8) violates the Coddazi relation [3] $b_{ac\,;\,r} - b_{ar\,;\,c} = 0$ because $b_{12\,;\,4} \neq b_{14\,;\,2}$, that is, it is impossible the embedding of (1) into E_5 [5,8].

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