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Modified Moyal-Weyl Star product in a Curved Non Commutative space-time

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Abstract: To generate gravitational terms in a curved noncommutative space-time, new Moyal-Weyl star product as well as Weyl ordering are defined. As an example, a complex scalar mass term action is considered.

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1. Introduction

During the last two decades many efforts have been made to solve or at least to understand the reaming unsolved outstanding problems of theoretical physics by using new ideas like quantum groups, deformation theory, noncommutative geometry etc.. [1] − [9]. This may shed a light on the real microscopic geometry and structure of our universe.

One approach, is to consider a noncommutative space-time where the dynamical variables become operators and therefore, the formalism of the quantum fields theories constructions must be modified. It turns out that in a flat space-time geometry, this amounts basically to replace the ordinary products by a Moyal-Weyl star products and taking into account the Weyl ordering [1] − [10]. The goal of this paper is to consider a curved space-time (presence of a gravitational background) and define the corresponding new Moyal-Weyl star product and Weyl ordering. In section 2, we present our mathematical formalism and consider an example a mass term of a complex scalar field. Finally, in section 3, we draw our conclusions.

2. Formalism

2.1 *The ordinary Moyal − Weyl ∗ −product*

The Moyal-Weyl ∗-product of any two smooth functions f and g such as [12]

$$f(x) = (2\pi)^{-\frac{3}{2}} \int d^4k \, e^{ikx} \, \widetilde{f}(k) \tag{1}$$

$$g(x) = (2\pi)^{-\frac{3}{2}} \int d^4k \, e^{ikx} \, \widetilde{g}(k) \tag{2}$$

can be defined as follows:

First we associate to f and g the Weyl operators $W(f)$ and $W(g)$ defined by

$$\begin{aligned} W(f) &= (2\pi)^{-\frac{3}{2}} \int d^4k \, e^{ik\widehat{x}} \, \widetilde{f}(k) \\ W(g) &= (2\pi)^{-\frac{3}{2}} \int d^4k \, e^{ik\widehat{x}} \, \widetilde{g}(k) \end{aligned} \tag{3}$$

where $\widehat{x}^{\mu}$ are non commuting operators satisfying

$$[\widehat{x}^{\mu} \, , \, \widehat{x}^{\nu}] = i\theta^{\mu\nu} \tag{4}$$

Next we define the product $W(f)\,W(g)$ as

$$W(f)\,W(g) = (2\pi)^{-\frac{3}{2}} (2\pi)^{-\frac{3}{2}} \int d^4k \, d^4p \, e^{ik\widehat{x}} \, e^{ip\widehat{x}} \, \widetilde{f}(k) \, \widetilde{g}(p) \tag{5}$$

Using the C-B-H formula, the Weyl product $W(f)\,W(g)$ reads :

$$W(f)\,W(g) = (2\pi)^{-\frac{3}{2}} (2\pi)^{-\frac{3}{2}} \int d^4k \, d^4p \, e^{ik\widehat{x}+ip\widehat{x}-\frac{i}{2}k_{\mu}p_{\nu}\theta^{\mu\nu}} \, \widetilde{f}(k) \, \widetilde{g}(p) = W(f * g) \tag{6}$$

Where $f * g$ is a new classical function defined by:

$$(f * g)(x) = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial y^{\nu}}} f(x)\,g(y) \tag{7}$$

This is the ordinary Moyal ∗-product. To the second ordre in θ The Moyal ∗-product reads:

$$\begin{aligned} (f * g)(x) = {} & f(x)\,g(x) + \frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^{\mu}} f(x) \frac{\partial}{\partial x^{\nu}} g(x) + \\ & + \frac{i}{2}\theta^{\mu\nu}\frac{i}{2}\theta^{\alpha\beta}\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial x^{\alpha}} f(x) \frac{\partial}{\partial x^{\nu}}\frac{\partial}{\partial x^{\beta}} g(x) + ... \end{aligned} \tag{8}$$

Notice here, that the operators $\widehat{x}^{\mu}$ are only defined modulo terms which vanish at the classical limit, for example x^{μ} and $x^{\mu}+\Sigma^{\mu}_{\alpha\beta}x^{\alpha}x^{\beta}-\Sigma^{\mu}_{\alpha\beta}x^{\beta}x^{\alpha}$ are equal but the corresponding non commutating operators are not except if $\Sigma^{\mu}_{\alpha\beta}$ is symmetric

$$\widehat{x}^{\mu}\neq\widehat{x}^{\mu}+\Sigma^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}-\Sigma^{\mu}_{\alpha\beta}\widehat{x}^{\beta}\widehat{x}^{\alpha}=\widehat{x}^{\mu}+i\theta^{\alpha\beta}\Sigma^{\mu}_{\alpha\beta} \tag{9}$$

2.2 *The deformed Moyal − Weyl ∗ −product*

Here, by deforming the ordinary Moyal ∗-product, we propose a new Moyal-Weyl ⊛-product which take in consideration the missing terms cited above and which generate gravitational terms to the order θ^2. To any smooth function f we associate the Weyl operator $W(f)$

$$f(x)=(2\pi)^{-\frac{3}{2}}\int d^4x\; e^{ikx}\;\widetilde{f}(k)\;\rightarrow W(f)=(2\pi)^{-\frac{3}{2}}\int d^4k\; e^{ik\widehat{X}}\;\widetilde{f}(k) \tag{10}$$

where $\widehat{X}^{\mu}$ are non commuting operators associated to the following classical variables

$$X^{\mu}=x^{\mu}+\Gamma^{\mu}_{\alpha\beta}x^{\alpha}x^{\beta}-\frac{1}{2}\Gamma^{\mu}_{\rho\lambda}\Gamma^{\rho}_{\alpha\beta}x^{\alpha}x^{\beta}x^{\lambda} \tag{11}$$

where $\Gamma^{\mu}_{\alpha\beta}(x)=\Gamma^{\mu}_{\beta\alpha}(x)$ is the symmetric affine connection . The non commuting operators $\widehat{X}^{\mu}$ are defined by a symmetrization procedure:

$$\widehat{X}^{\mu}=\widehat{x}^{\mu}+\left(\widehat{\Gamma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\right)_{w}-\frac{1}{2}\left(\widehat{\Gamma}^{\mu}_{\rho\lambda}\widehat{\Gamma}^{\rho}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\widehat{x}^{\lambda}\right)_{w} \tag{12}$$

Where the Weyl ordering is defined by:

$$\left(\widehat{\Sigma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\right)_{w}=(\widehat{\Sigma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}-2\widehat{x}^{\alpha}\widehat{\Sigma}^{\mu}_{\alpha\beta}\widehat{x}^{\beta}+\widehat{x}^{\alpha}\widehat{x}^{\beta}\widehat{\Sigma}^{\mu}_{\alpha\beta})/\sqrt{-g} \tag{13}$$

with

$$g=\det g_{\mu\nu} \tag{14}$$

and direct simplifications (see Appendix) give:

$$\left(\widehat{\Sigma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\right)_{w}=i\theta^{\beta\lambda}i\theta^{\alpha\sigma}\partial_{\sigma}\partial_{\lambda}\widehat{\Sigma}^{\mu}_{\alpha\beta}/\sqrt{-g} \tag{15}$$

Where we have used the fact that $\widehat{\Sigma}^{\mu}_{\alpha\beta}$ is symmetric, and:

$$[\widehat{x}^{\mu}\;,\;\widehat{x}^{\nu}]=i\theta^{\mu\nu}$$

and

$$\left[\widehat{x}^{\mu}\;,\;\widehat{f}(x)\right]=i\theta^{\mu\nu}\partial_{\nu}\widehat{f}(x) \tag{16}$$

Thus the second term in eq.(15) reads:

$$\left(\widehat{\Gamma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\right)_{w} = -\theta^{\beta\lambda}\ \theta^{\alpha\sigma}\partial_{\sigma}\partial_{\lambda}\widehat{\Gamma}^{\mu}_{\alpha\beta}/\sqrt{-g} \tag{17}$$

It is worth to mention that we can get the same result if we define the Weyl ordering as:

$$\left(\widehat{\Gamma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\right)_{w} = (\overleftrightarrow{\widehat{\Gamma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}}\widehat{x}^{\beta} - \widehat{x}^{\alpha}\overleftrightarrow{\widehat{\Gamma}^{\mu}_{\alpha\beta}\widehat{x}^{\beta}} + \widehat{\Gamma}^{\mu}_{\alpha\beta}\overleftrightarrow{\widehat{x}^{\alpha}\widehat{x}^{\beta}})/\sqrt{-g} \tag{18}$$

Where

$$\overleftrightarrow{\widehat{x}^{\alpha}\widehat{x}^{\beta}} \equiv \left[\widehat{x}^{\alpha},\widehat{x}^{\beta}\right] \tag{19}$$

and

$$\overleftrightarrow{\widehat{x}^{\mu}\widehat{f}\,(x)} \equiv \left[\widehat{x}^{\mu}\ ,\ \widehat{f}\,(x)\right] \tag{20}$$

Now using these relations and the fact that

$$\widehat{\Gamma}^{\mu}_{\alpha\beta}\overleftrightarrow{\widehat{x}^{\alpha}\widehat{x}^{\beta}} = \widehat{\Gamma}^{\mu}_{\alpha\beta}\left[\widehat{x}^{\alpha},\widehat{x}^{\beta}\right] = \ i\theta^{\alpha\beta}\widehat{\Gamma}^{\mu}_{\alpha\beta} \tag{21}$$

eq.(18) can be rewritten as

$$\left(\widehat{\Gamma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\right)_{w} = -\theta^{\beta\lambda}\theta^{\alpha\sigma}\partial_{\sigma}\partial_{\lambda}\widehat{\Gamma}^{\mu}_{\alpha\beta}/\sqrt{-g}$$

Which is the same result as above.

Similarly, one can define the Weyl ordering $\left(\widehat{\Sigma}^{\mu}_{\alpha\beta\lambda}\widehat{x}^{\alpha}\widehat{x}^{\beta}\widehat{x}^{\lambda}\right)_{w}$ as:

$$\left(\widehat{\Sigma}^{\mu}_{\alpha\beta\lambda}\widehat{x}^{\alpha}\widehat{x}^{\beta}\widehat{x}^{\lambda}\right)_{w} = (\overleftrightarrow{\widehat{\Sigma}^{\mu}_{\alpha\beta\lambda}\widehat{x}^{\alpha}}\overleftrightarrow{\widehat{x}^{\beta}\widehat{x}^{\lambda}} + \overleftrightarrow{\widehat{\Sigma}^{\mu}_{\alpha\beta\lambda}\widehat{x}^{\beta}}\overleftrightarrow{\widehat{x}^{\alpha}\widehat{x}^{\lambda}} + \overleftrightarrow{\widehat{\Sigma}^{\mu}_{\alpha\beta\lambda}\widehat{x}^{\lambda}}\overleftrightarrow{\widehat{x}^{\alpha}\widehat{x}^{\beta}})/\sqrt{-g} \tag{22}$$

and straightforward simplifications by using the fact that $\widehat{\Sigma}^{\mu}_{\alpha\beta\lambda}$ is symmetric with respect to $\alpha\beta$ give:

$$\left(\widehat{\Sigma}^{\mu}_{\alpha\beta\lambda}\widehat{x}^{\alpha}\widehat{x}^{\beta}\widehat{x}^{\lambda}\right)_{w} = -2i\theta^{\beta\lambda}i\theta^{\alpha\sigma}\partial_{\sigma}\widehat{\Sigma}^{\mu}_{\alpha\beta\lambda}/\sqrt{-g} \tag{23}$$

and one can deduce that:

$$\left(\widehat{\Gamma}^{\mu}_{\rho\lambda}\widehat{\Gamma}^{\rho}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\widehat{x}^{\lambda}\right)_{w} = -2i\theta^{\beta\lambda}i\theta^{\alpha\sigma}\partial_{\sigma}\left(\widehat{\Gamma}^{\mu}_{\rho\lambda}\widehat{\Gamma}^{\rho}_{\alpha\beta}\right)/\sqrt{-g} \tag{24}$$

and the noncommuting variable $\widehat{X}^{\mu}$ can be rewritten as:

$$\widehat{X}^{\mu} = \widehat{x}^{\mu} + \left(\widehat{\Gamma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\right)_{w} - \frac{1}{2}\left(\widehat{\Gamma}^{\mu}_{\rho\lambda}\widehat{\Gamma}^{\rho}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\widehat{x}^{\lambda}\right)_{w} \tag{25}$$

where after the corresponding expressions substitutions reads:

$$\widehat{X}^{\mu} = \widehat{x}^{\mu} + \frac{1}{2}i\theta^{\beta\lambda}i\theta^{\alpha\sigma}\partial_{\sigma}\widehat{R}^{\mu}{}_{\alpha\beta\lambda}\left(x\right)/\sqrt{-g} \tag{26}$$

Here $\widehat{R}^{\mu}{}_{\alpha\beta\lambda}$ stands for the Riemann curvature tensor defined as:

$$\widehat{R}^{\mu}{}_{\alpha\beta\lambda}(x) = \partial_\beta \widehat{\Gamma}^{\mu}_{\alpha\lambda} - \partial_\lambda \widehat{\Gamma}^{\mu}_{\alpha\beta} + \widehat{\Gamma}^{\mu}_{\rho\beta}\widehat{\Gamma}^{\rho}_{\alpha\lambda} - \widehat{\Gamma}^{\mu}_{\rho\lambda}\widehat{\Gamma}^{\rho}_{\alpha\beta} \tag{27}$$

Using the C-B-H formula, one can write:

$$e^{ik\widehat{X}}e^{ip\widehat{X}} = e^{ik\widehat{X}+ip\widehat{X}+\frac{1}{2}\left[ik\widehat{X}\ ,\ ip\widehat{X}\right]+...} = e^{ik\widehat{x}+ip\widehat{x}+ik_\mu\Delta\widehat{x}^\mu+ip_\nu\Delta\widehat{x}^\nu-\frac{i}{2}\theta^{\mu\nu}k_\mu p_\nu...} \tag{28}$$

with

$$\Delta\widehat{x}^\mu = \frac{1}{2} i\theta^{\beta\lambda} i\theta^{\alpha\sigma}\partial_\sigma \widehat{R}^{\mu}{}_{\alpha\beta\lambda}/\sqrt{-g} \tag{29}$$

Thus, The Moyal-Weyl $\circledast$-product reads:

$$W(f)\,W(g) = (2\pi)^{-\frac{3}{2}}(2\pi)^{-\frac{3}{2}}\int d^4k d^4p e^{ik\widehat{x}+ip\widehat{x}+ik_\mu\Delta\widehat{x}^\mu+ip_\nu\Delta\widehat{x}^\nu-\frac{i}{2}\theta^{\mu\nu}k_\mu p_\nu}\widetilde{f}(k)\,\widetilde{g}(p) = W(f\circledast g) \tag{30}$$

where

$$(f\circledast g)(x) = e^{\Delta x^\mu\frac{\partial}{\partial x^\mu}+\Delta x^\nu\frac{\partial}{\partial y^\nu}+\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}}\left[f(x)\,g(y)\right]_{x=y} \tag{31}$$

and to the second order in θ^2 , one obtains:

$$\begin{aligned}(f\circledast g)(x) = {} & f(x)\,g(x) + \Delta x^\mu\partial_\mu\left(f(x)\,g(x)\right) + \frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}f(x)\,\frac{\partial}{\partial x^\nu}g(x) \\ & + \frac{i}{2}\theta^{\mu\nu}\frac{i}{2}\theta^{\alpha\beta}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\alpha}f(x)\,\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\beta}g(x) + ...\end{aligned} \tag{32}$$

which can be rewritten as

$$\begin{aligned}(f\circledast g)(x) = {} & f(\overline{x})\,g(\overline{x}) + \frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}f(\overline{x})\,\frac{\partial}{\partial x^\nu}g(\overline{x}) + \\ & \frac{i}{2}\theta^{\mu\nu}\frac{i}{2}\theta^{\alpha\beta}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\alpha}f(\overline{x})\,\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\beta}g(\overline{x}) + ..\end{aligned} \tag{33}$$

where

$$\overline{x}^\mu = x^\mu + \Delta x^\mu = x^\mu + \frac{1}{2}i\theta^{\beta\lambda}i\theta^{\alpha\sigma}\partial_\sigma R^{\mu}{}_{\alpha\beta\lambda}/\sqrt{-g}. \tag{34}$$

Notice that one can add to the expression of $\widehat{X}^\mu$ the term $\widehat{\Gamma}^{\mu}_{\alpha\beta}\widehat{\Gamma}^{\rho}_{\rho\lambda}\widehat{x}^\alpha\widehat{x}^\beta\widehat{x}^\lambda$. However, if we require that the Weyl ordering of a product of non coupled terms like $\widehat{\Sigma}^{\mu}_{\alpha\beta\lambda...\varrho}\widehat{x}^\alpha\widehat{x}^\beta\widehat{x}^\lambda...\widehat{x}^\varrho$ and $\widehat{\Lambda}^{\nu}_{\pi\sigma\tau...\kappa}\widehat{x}^\pi\widehat{x}^\sigma\widehat{x}^\tau...\widehat{x}^\kappa$ factorizes i.e.:

$$W\left(\widehat{\Sigma}^{\mu}_{\alpha\beta\lambda...\varrho}\widehat{x}^\alpha\widehat{x}^\beta\widehat{x}^\lambda...\widehat{x}^\varrho\widehat{\Lambda}^{\nu}_{\pi\sigma\tau...\kappa}\widehat{x}^\pi\widehat{x}^\sigma\widehat{x}^\tau...\widehat{x}^\kappa\right) = W\left(\widehat{\Sigma}^{\mu}_{\alpha\beta\lambda...\varrho}\widehat{x}^\alpha\widehat{x}^\beta\widehat{x}^\lambda...\widehat{x}^\varrho\right)W\left(\widehat{\Lambda}^{\nu}_{\pi\sigma\tau...\kappa}\widehat{x}^\pi\widehat{x}^\sigma\widehat{x}^\tau...\widehat{x}^\kappa\right) \tag{35}$$

Then, this term does not contribute since

$$\left(\widehat{\Gamma}^{\rho}_{\rho\lambda}\widehat{x}^{\lambda}\right)_{w} = \overleftrightarrow{\widehat{\Gamma}^{\rho}_{\rho\lambda}\widehat{x}^{\lambda}}/\sqrt{-g} = -i\theta^{\lambda\sigma}\partial_{\sigma}\widehat{\Gamma}^{\rho}_{\rho\lambda}/\sqrt{-g} \tag{36}$$

Now, using the fact that

$$\Gamma^{\rho}_{\rho\lambda} = \partial_{\lambda}\log\left(g\right) \tag{37}$$

one deduce that:

$$\left(\widehat{\Gamma}^{\rho}_{\rho\lambda}\widehat{x}^{\lambda}\right)_{w} = -i\theta^{\lambda\sigma}\partial_{\sigma}\widehat{\Gamma}^{\rho}_{\rho\lambda}/\sqrt{-g} = -i\theta^{\lambda\sigma}\partial_{\sigma}\partial_{\lambda}\log\left(g\right)/\sqrt{-g} = 0 \tag{38}$$

2.3 *The Noncommutative Action*

Let us calculate the mass term $\Phi^{+} \circledast \Phi$ where Φ is a complex scalar field in this new noncommuting space-time:

$$\left(\Phi^{+} \circledast \Phi\right)(x) = \Phi^{+}\left(\overline{x}\right)\Phi\left(\overline{x}\right) + \text{total derivative} \tag{39}$$

Eq.(39) can be rewritten as:

$$\left(\Phi^{+} \circledast \Phi\right)(x) = \Phi^{+}\left(x+\Delta x\right)\Phi\left(x+\Delta x\right) = \Phi^{+}\left(x\right)\Phi\left(x\right) + \Delta x^{\mu}\partial_{\mu}\left[\Phi^{+}\left(x\right)\Phi\left(x\right)\right] + \text{total derivative} \tag{40}$$

which after direct simplifications, the action reads:

$$I = \int d^{4}x\left(\Phi^{+} \circledast \Phi\right)(x) = \int d^{4}x\Phi^{+}\left(x\right)\Phi\left(x\right) + \frac{1}{2}\int d^{4}x\theta^{\beta\lambda}\theta^{\alpha\sigma}R^{\mu}_{\alpha\beta\lambda}\partial_{\sigma}\partial_{\mu}\left[\Phi^{+}\left(x\right)\Phi\left(x\right)\right]\sqrt{-g} \tag{41}$$

2.4 $2-d$ *Gravity coupled to a scalar field*

Let us calculate this action in two dimensions, choosing $\theta^{01} = +\eta$, with $\eta \ll 1$, The gravitational term reads

$$\theta^{\beta\lambda}\theta^{\alpha\sigma}R^{\mu}_{\alpha\beta\lambda}\partial_{\sigma}\partial_{\mu}\left[\Phi^{+}\left(x\right)\Phi\left(x\right)\right]/\sqrt{-g} = \theta^{\beta\lambda}\theta^{\alpha\sigma}R_{\mu\alpha\beta\lambda}\partial^{\mu}\partial_{\sigma}\left[\Phi^{+}\left(x\right)\Phi\left(x\right)\right]/\sqrt{-g} \tag{42}$$

which can be simplified as:

$$\theta^{\beta\lambda}\theta^{\alpha\sigma}R^{\mu}{}_{\alpha\beta\lambda}\partial_{\sigma}\partial_{\mu}\left[\Phi^{+}\left(x\right)\Phi\left(x\right)\right]/\sqrt{-g} = -2\eta^{2}R_{0101}g^{\mu\nu}\partial_{\mu}\partial_{\nu}\left[\Phi^{+}\left(x\right)\Phi\left(x\right)\right]/\sqrt{-g} \tag{43}$$

In two dimensions, the scalar curvature:

$$R = g^{\mu\nu} R_{\mu\nu} \tag{44}$$

is related to the component R_{0101} by the relation :

$$R_{0101} = \frac{1}{2} g R \tag{45}$$

Straightforward simplifications lead to:

$$R_{00} = g^{11} R_{0101} \tag{46}$$

$$R_{11} = g^{00} R_{0101} \tag{47}$$

and

$$R_{01} = R_{10} = -g^{01} R_{0101}$$

and consequently, the scalar curvature R reads

$$R = 2g^{-1} R_{0101} \tag{48}$$

Finally the action gets the form:

$$I = \int d^2x \Phi^+ (x) \Phi (x) - \frac{1}{2}\eta^2 \int d^2x \sqrt{-g} R g^{\mu\nu} \partial_\mu \partial_\nu \left[\Phi^+ (x) \Phi (x)\right] \tag{49}$$

3. Conclusion

Thought this work we have considered a noncommutative curved space-time in a gravitational background and define new Moyal-Weyl star product and Weyl ordering at the order of θ^2 (θ is the order parameter of the noncommutative of the space time)where the geometric structure is included. As an example, we have considered the mass term of a complex field and show explicitly the gravitation effect on the noncommutative-space time. (More studies are under investigations).

Appendix

we have

$$\left(\widehat{\Sigma}^{\mu}_{\alpha\beta} \widehat{x}^{\alpha} \widehat{x}^{\beta}\right)_w = (\left(\widehat{\Sigma}^{\mu}_{\alpha\beta} \widehat{x}^{\alpha} - \widehat{x}^{\alpha} \widehat{\Sigma}^{\mu}_{\alpha\beta}\right) \widehat{x}^{\beta} - \widehat{x}^{\alpha} \left(\widehat{\Sigma}^{\mu}_{\alpha\beta} \widehat{x}^{\beta} - \widehat{x}^{\beta} \widehat{\Sigma}^{\mu}_{\alpha\beta}\right))/\sqrt{-g} \tag{A-1}$$

which can be rewritten as:

$$\left(\widehat{\Sigma}^{\mu}_{\alpha\beta} \widehat{x}^{\alpha} \widehat{x}^{\beta}\right)_w == (\widehat{x}^{\alpha} \left[\widehat{x}^{\beta}, \widehat{\Sigma}^{\mu}_{\alpha\beta}\right] - \left[\widehat{x}^{\alpha}, \widehat{\Sigma}^{\mu}_{\alpha\beta}\right] \widehat{x}^{\beta})/\sqrt{-g} \tag{A-2}$$

using the relations eq.(16) we obtain:

$$\left(\widehat{\Sigma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\right)_w = i\theta^{\beta\lambda}\left(\widehat{x}^{\alpha}\partial_\lambda\widehat{\Sigma}^{\mu}_{\alpha\beta} - \partial_\lambda\widehat{\Sigma}^{\mu}_{\beta\alpha}\widehat{x}^{\alpha}\right)/\sqrt{-g} = i\theta^{\beta\lambda}\left(\widehat{x}^{\alpha}\partial_\lambda\widehat{\Sigma}^{\mu}_{\alpha\beta} - \partial_\lambda\widehat{\Sigma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\right)/\sqrt{-g} \tag{A-3}$$

thus,

$$\left(\widehat{\Sigma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\right)_w = i\theta^{\beta\lambda}\left[\widehat{x}^{\alpha},\partial_\lambda\widehat{\Sigma}^{\mu}_{\alpha\beta}\right]/\sqrt{-g} = i\theta^{\beta\lambda}i\theta^{\alpha\sigma}\partial_\sigma\partial_\lambda\widehat{\Sigma}^{\mu}_{\alpha\beta}/\sqrt{-g} \tag{A-4}$$

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