Volume 3 Number 12

Electronic Journal of Theoretical Physics

ISSN 1729-5254

EJTP

Editors

http://www.ejtp.com September, 2006 E-mail:info@ejtp.com

Volume 3 Number 12

Electronic Journal of Theoretical Physics

ISSN 1729-5254

EJTP

Editors

Ammar Sakaji **Ignazio Licata**

http://www.ejtp.com September, 2006 E-mail:info@ejtp.com

Table of Contents

EJTP **3, No. 12** (2006) 1–18 *Electronic Journal of Theoretical Physics*

Duality and a Renormalization Scheme for Einsteinian Gravity as a Fix Point Within a Gravitational Gauge Framework

Eckehard W. Mielke *

Departamento de Física, Universidad Autónoma Metropolitana–Iztapalapa, Apartado Postal 55-534, C.P. 09340, México, D.F., MEXICO

Received 11 Augest 2006, Accepted 25 August 2006, Published 20 September 2006

Abstract: A general scheme for a *field redefinition* (FR) of the coframe and the connection is developed. Within a Yang–Mills type gauge dynamics of gravity, configurations with double dual curvature induced by a θ-type Chern-Simons terms as generating function reside on an effective Einsteinian background. The effect of the FR on the renormalization and the relation of gravity to effective string models is studied. One encounters a *duality of weak and strong couplings* of Einsteinian and renormalizable Yang–Mills type gravity as well as an induced cosmological constant of the Anti–de Sitter space.

Keywords: Duality, gravity, renormalization
PACS (2006): 04.60.+m

1. Introduction

The *duality* of electric and magnetic fields in Maxwell's theory was already known to Von Laue [91] and Silberstein [89]. Later it was generalized to the symmetry of *duality rotations* by Rainich [87] and developed further in *geometrodynamics* by Misner and Wheeler [74], cf. [61]. More recently, Motonen and Olive [76, 82] noted that then also a duality of the strong-weak coupling regime of gauge fields is generated, the so-called *S–duality.* In the context of magnetic monopoles it plays nowadays a predominant role in M–theory [18, 96].

We are going to apply this to a Yang–Mills–type formulation of gravitational interactions, regarding it as a *field redefinition* [38, 39, 66]. In general, not only the energy–momentum content of matter, but also its spin couples to a dynamical geometry with

* ekke@xanum.uam.mx

translational and Lorentz-rotational curvature [32, 34]. This general framework encompasses the Einstein–Cartan (EC) theory as an important (but from the variational point of view degenerate) subcase which is macroscopically indistinguishable from Einstein's theory of general relativity (GR).

2. Field Redefinition Scheme

In perturbative quantum gravity [1], there arise counterterms ΔV of higher order in the curvature. According to 't Hooft [38, 39], these terms can be simulated, already on the classical level, by a *field redefinition* (FR)

$$g_{ij} \to \widetilde{g}_{ij} = g_{ij} + aRic_{ij} + bg_{ij}\, Ric_k{}^k \tag{2.1}$$

of the metric in the first order approximation. In exterior form notation, the symmetric Ricci tensor is the holonomic version of the zero-form $Ric_{\alpha\beta} := {}^*(R_{(\alpha}{}^\delta \wedge \eta_{\delta|\beta)})$. In a gauge framework based on the Poincaré group and summarized in the Appendix, the independent variables are the one-forms ϑ^α and $\Gamma^{\alpha\beta}$. Then a generally nonlinear FR of these basic variables are dictated by the appropriate form degree and the correct physical dimension:

$$\vartheta^\alpha \quad \to \quad \widetilde{\vartheta}^\alpha := \vartheta^\alpha + \ell^2\, e_\beta \rfloor {}^*H^{\alpha\beta}\,, \tag{2.2}$$

$$\Gamma_\alpha{}^\beta \quad \to \quad \widetilde{\Gamma}_\alpha{}^\beta := \Gamma_\alpha{}^\beta + \ell^2\, e_\alpha \rfloor {}^*H^\beta\,. \tag{2.3}$$

Here the field momenta H_α and $H_{\alpha\beta}$ are understood as arising from a generating n–form G as part of some effective gauge Lagrangian $V_{\rm eff}$ which includes the counterterms from the searched-for renormalization. Observe also that a *fundamental length* ℓ squared necessarily occurs for dimensional reasons. These *deformations* of the gauge potentials can also be viewed as *nonlinear prolongations* [6].

As a consequence, the spacetime metric

$$g = g_{ij}\, dx^i \otimes dx^j = o_{\alpha\beta}\, \vartheta^\alpha \otimes \vartheta^\beta\,. \tag{2.4}$$

gets redefined by

$$g \quad \to \quad \widetilde{g} = g + 2\ell^2 o_{\alpha\beta}\vartheta^\alpha \otimes e_\mu \rfloor {}^*H^{\beta\mu} + \ell^4 o_{\alpha\beta} e_\mu \rfloor {}^*H^{\alpha\mu} \otimes e_\nu \rfloor {}^*H^{\beta\nu}\,, \tag{2.5}$$

i.e. by curvature excitations[1] up to quadratic order. This generalizes 't Hooft's ansatz (2.1) for the metric, used there in an attempt at perturbative renormalization of GR on a Riemannian background. For a general counterterm ΔV in the effective gauge Lagrangian $V_{\rm eff}$, our geometrical variables become redefined according to the "intertwining relations" (2.2,2.3) via the corresponding $H_{\rm eff}$.

[1] For nonvanishing non-metricity $Q_{\alpha\beta} := -Dg_{\alpha\beta}$, there would be an additional FR of the anholonomic metric via $g_{\alpha\beta} \quad \to \quad \widetilde{g}_{\alpha\beta} = g_{\alpha\beta} + {}^*m_{\alpha\beta}$ where $m_{\alpha\beta} := -\partial G/\partial g_{\alpha\beta}$ is an n–form, cf. Eq. (3.11.13) of Ref. [34].

The dual two-form and the volume four–form are deformed according to

$$\eta_{\alpha\beta} \quad \to \quad \widetilde{\eta}_{\alpha\beta} = \eta_{\alpha\beta} + \ell^2 \eta_{\alpha\beta\gamma} \wedge e_\mu \rfloor {}^*H^{\mu\gamma} + \frac{\ell^4}{2} \eta_{\alpha\beta\gamma\delta} (e_\mu \rfloor {}^*H^{\gamma\mu}) \wedge (e_\nu \rfloor {}^*H^{\delta\nu}) \,, \tag{2.6}$$

and

$$\begin{aligned} \eta \quad \to \quad \widetilde{\eta} = \eta &+ \ell^2 \eta_{\alpha\beta} \wedge {}^*H^{\alpha\beta} + \frac{\ell^4}{2} \eta_{\alpha\beta} (e_\mu \rfloor {}^*H^{\alpha\mu}) \wedge (e_\nu \rfloor {}^*H^{\beta\nu}) + \cdots \\ &+ \frac{\ell^8}{4!} \eta_{\alpha\beta\gamma\delta} (e_\mu \rfloor {}^*H^{\alpha\mu}) \wedge (e_\nu \rfloor {}^*H^{\beta\nu}) \wedge (e_\rho \rfloor {}^*H^{\gamma\rho}) \wedge (e_\omega \rfloor {}^*H^{\delta\omega}) \,, \end{aligned} \tag{2.7}$$

respectively.

In our dynamical approach, the $(n-2)$–forms H_α and $H_{\alpha\beta}$ will be gauge field momenta canonically conjugated to the coframe and the Lorentz connection, respectively. Due to the *semidirect product* structure of the Poincaré group $P := R^4 \rtimes SO(1,3)$, the gauge field momenta contribute to the gauge potentials via $H^{\alpha\beta} \to \vartheta^\alpha$ and $H^\alpha \to \Gamma^{\alpha\beta}$ in the field redefinition (2.2,2.3) just in an *intertwined* manner.

In the four–dimensional gauge theory, a Hodge star for gauge field momenta H can be dismissed as it is done in Ref. [66], but in dimension $n \neq 4$ it is necessary to use ${}^*H$ in order obtain the correct form degree. In the FR (2.3) of the connection we could have included, similarly as for Yang–Mills fields, the term $\ell^2\, {}^*DH_\alpha{}^\beta$. However, 'on shell', i.e. when the vacuum field equation (5.18) is satisfied, this is equivalent to ${}^*\left(H^\beta \wedge \vartheta_\alpha\right) \equiv e_\alpha \rfloor {}^*H^\beta$ due to the identity (3.7.13) of Ref. [34]. In the FR (2.2) of the coframe, the same situation arises, with the modification that the 'on shell' term ${}^*DH^\alpha \cong {}^*E^\alpha$ is second order in the field strength and therefore equivalent to a higher order generation functional G. When coupling to matter, FRs have to be taken with care because they may induce violations of the macroscopic principle of equivalence, cf. Brans [11].

2.1 Legendre transformation

For exhibiting physically equivalent gauge field Lagrangians via a Legendre transformation, let us proceed from the Hilbert–Einstein of general relativity (GR) or the Einstein–Cartan (EC) Lagrangian

$$V_{\rm EC} = -\frac{1}{2\ell^2} R^{\alpha\beta} \wedge \eta_{\alpha\beta} \tag{2.8}$$

as an example [95].

If we compare this with the more general Lagrangian

$$\widetilde{V} = -\sum_{k=0}^{K} (1/2k)\, R^{\alpha\beta} \wedge \overset{(2k)}{H}{}_{\alpha\beta} + V_\theta \,, \tag{2.9}$$

which is quadratic, quartic, etc. in the curvature. The first term in this expansion corresponds to the Stephenson-Kilmister-Yang (SKY) Lagrangian quadratic in the curvature [94, 36, 46, 98, 58, 97]. The *gauge field momentum* $\widetilde{H}_{\alpha\beta} := -\partial \widetilde{V}/\partial R^{\alpha\beta}$ can be expanded as $\widetilde{H}_{\alpha\beta} = \overset{(2)}{H}{}_{\alpha\beta} + \overset{(4)}{H}{}_{\alpha\beta} + \cdots$. For the time being, the field momentum $\widetilde{H}_\alpha := -\partial \widetilde{V}/\partial T^\alpha$

conjugated to the torsion $T^\alpha := D\vartheta^\alpha$ is put to zero. From the field equations (5.17, 5.18) of PG theory, we can then infer the resulting *Yang–Mills type field equations*

$$-\widetilde{E}_\alpha := -e_\alpha\rfloor\widetilde{V} - (e_\alpha\rfloor R^{\beta\gamma}) \wedge \widetilde{H}_{\beta\gamma} = \Sigma_\alpha\,, \tag{2.10}$$

$$D\widetilde{H}_{\alpha\beta} = \tau_{\alpha\beta}\,. \tag{2.11}$$

However, the *Legendre transformation* [42, 66]

$$\widetilde{V} \;\rightarrow\; V = -\frac{1}{2}\left(R^{\alpha\beta} \wedge \widetilde{H}_{\alpha\beta} - \widetilde{V}\right) \tag{2.12}$$

provides a physically equivalent gravitational Lagrangian V. (The overall factor 1/2 is chosen so as to render the EC Lagrangian invariant.) The new rotational gauge field momenta

$$H_{\alpha\beta} := -\frac{\partial V}{\partial R^{\alpha\beta}} = \widetilde{H}_{\alpha\beta} + \frac{1}{2}R^{\mu\nu} \wedge (\partial\widetilde{H}_{\mu\nu}/\partial R^{\alpha\beta})\,, \tag{2.13}$$

will depend on the *Hessian*

$$\widetilde{H}_{\alpha\beta\mu\nu} := \frac{\partial^2\widetilde{V}}{\partial R^{\mu\nu}\partial R^{\alpha\beta}} = -\frac{\partial\widetilde{H}_{\mu\nu}}{\partial R^{\alpha\beta}} \tag{2.14}$$

of the Lagrangian $\widetilde{V}$ we started with.[2]

2.2 Constant Hessian and S–duality

As the first illustrative case let us first consider a *constant* Hessian, i.e.

$$\widetilde{H}_{\alpha\beta\mu\nu} = 2\theta_{\rm L} g_{[\alpha|\mu}g_{\beta]\nu} + \theta^\star_{\rm L}\,\eta_{\alpha\beta\mu\nu} \tag{2.15}$$

arising from the curvature type θ terms in the topological term V_θ. Then from (2.13) the following form for the new rotational field momenta can be inferred

$$H_{\alpha\beta} = \widetilde{H}_{\alpha\beta} - \theta_{\rm L}R_{\alpha\beta} - \theta^\star_{\rm L}\overset{(\star)}{R}_{\alpha\beta}\,, \tag{2.16}$$

where we disregard for the moment a kernel discussed below in (2.19). Note that the θ-terms can be regarded as induced via the boundary terms (5.15) of Pontrjagin and

[2] Capovilla et al. [12] presented a reformulation of GR in which a gauge potential and an arbitrary scalar density $\sigma = \sqrt{|\det g_{\mu\nu}|}$ (but no metric) occur as dynamical variables. Then the Hodge dual is constructed via $\eta_{\alpha\beta\gamma\delta} := \sigma\,\epsilon_{\alpha\beta\gamma\delta}$, i.e. from the Levi–Civita symbol multiplied by this scalar density. Following Jakubiec and Kijowski [42], we have pointed out [66] that this reformulation as well as Ashtekar's complex variables [2, 3] can be interpreted as a FR applied to the EC Lagrangian. By translating the $SO(3,C)$ formulation of Ref. [12] into our formalism, the equivalent of $\overset{(4)}{H}_{\alpha\beta} = R^{\gamma\delta} \wedge {}^*(R_{\alpha\gamma} \wedge R_{\beta\delta})$ is the only nonvanishing term. By varying $\widetilde{V}$ with respect to $\Gamma^{\alpha\beta}$ and σ, the *vacuum* field equation $D\widetilde{H}_{\alpha\beta} = 0$ and the energy–momentum trace $(\vartheta^\alpha \wedge \widetilde{E}_\alpha)/\sigma = (4\widetilde{V} + 2R^{\alpha\beta} \wedge \widetilde{H}_{\alpha\beta})/\sigma = 0$ are found. If we insert this $\overset{(4)}{H}_{\alpha\beta}$ into the metric, the curvature plays the role of a 'cubic root' of the deformed metric (2.5), similarly as in the chiral alternative of 't Hooft [40]. In our scheme. however, also order six terms will arise.

Euler, respectively. Using the decomposition (5.12) into self– or antiselfdual fields with respect to the Lie dual, this can be resolved for the curvature as

$$R^{(\pm)}_{\alpha\beta} = \frac{1}{(\theta_{\rm L} \pm \theta^{\star}_{\rm L})} \left(\widetilde{H}^{(\pm)}_{\alpha\beta} - H^{(\pm)}_{\alpha\beta} \right) \tag{2.17}$$

with a coupling constant which is *inverse* to those of the θ terms. Then upon the Legendre transformation (2.12), the new Lagrangian can be rewritten as

$$V = -\frac{1}{2(\theta_{\rm L} \pm \theta^{\star}_{\rm L})} \left(\widetilde{H}^{(\pm)}_{\alpha\beta} - H^{(\pm)}_{\alpha\beta} \right) \wedge \widetilde{H}^{\alpha\beta} + \frac{1}{2}\widetilde{V}\,. \tag{2.18}$$

Thereby the originally *weak* coupling to the θ terms is converted to a *strong* coupling regime $1/\theta$ for the field momenta and vice versa. This so-called *S–duality* of strong and weak coupling was first noted by Motonen and Olive [76] in the contex of magnetic monopoles and plays nowadays a prominant role in M-theory [18, 96]. In the context of non-Abelian Yang–Mills theories, a related equivalence with respect to S–duality was first encountered in Refs. [24, 75] and then, in a particular case, applied [26] to the MacDowell–Mansouri gauge theory of gravity .

2.3 Vanishing Hessian: GR as a stable fix point

Since EC theory may arise from different higher order models, we have an infinite ambiguity in such a "renormalization" program, cf. Kaku [44], p. 210. However, we can improve this by showing that EC theory is a *stable* fix point of the quadratic SKY gravity, e.g. For a *fixed point* of the transformation (2.12), the Hessian $\widetilde{H}_{\alpha\beta\mu\nu}$ obviously has to vanish. This condition, i.e. $\partial\widetilde{H}_{\mu\nu}/\partial R^{\alpha\beta} = 0$, can be readily solved. If parity violating terms such as $\theta_{\rm T}\, R^{\alpha\beta} \wedge \vartheta_\alpha \wedge \vartheta_\beta$ arising from the Nieh-Yan term (5.7) are admitted, then we obtain the relation

$$\widetilde{\eta}_{\alpha\beta} := \frac{\theta^*_{\rm T}}{2}\eta_{\alpha\beta} - \frac{\theta_{\rm T}}{2}\,\vartheta_\alpha \wedge \vartheta_\beta - \ell^2\, \widetilde{H}_{\alpha\beta} = 0 \tag{2.19}$$

which can be regarded as a singular FR derived from (2.2), but arising from a different effective[3] Lagrangian $\widetilde{V}$.

Accordingly $\eta_{\alpha\beta}$ and $\widetilde{H}_{\alpha\beta}$ interchange their role as generalized coordinates and momenta, respectively. If we had started from $\widetilde{V} = V_{\rm EC}$ then we would be led back to $V = V_{\rm EC}$ for the choice $\theta^*_{\rm T} = 1$ and $\theta_{\rm T} = 0$. In the case $\theta^*_{\rm T} = 1$ and $\theta_{\rm T} = i$ this leads to a *chiral* formulation of gravity [65]. Thus, the EC Lagrangian or its chiral version remains as a *"stable"* Lagrangian under FR, provide we embbed it in a class of gravity Lagrangians for which $V_{\rm EC}$ is located at some local minimum.

Our gauge framework clearly exhibits the coupling to fundamental matter, such as to the Dirac field. If we reinsert (2.19) into (2.11), we recover the algebraic Cartan type equation $\eta_{\alpha\beta\gamma} \wedge T^\gamma = 2\ell^2\, \tau_{\alpha\beta}$. In the Dirac case, this implies a nonvanishing *axial*

[3] In D=11 supergravity, a similar relation holds after compactification for the 7-volume form on S^7, i.e. $\eta_7 \approx H = -\partial V/\partial dB$, where B is the Kalb–Ramond three–form, cf. [18].

torsion. However, as already stressed by 't Hooft [39], a FR of the *coframe* may ruin the nice features of the Dirac Lagrangian which, in GR and its RC extensions, has to be formulated in terms of ϑ^α in a *multiplicative* way: Dangerous derivative couplings must be avoided in the Dirac equation and the positivity of energy needs to asured during this procedure [90].

In the transformation to Ashtekar's complex variables, the coframe is kept fixed, whereas the connection is subjected to the *complex* FR $\Gamma_\alpha{}^\beta \quad\rightarrow\quad \overset{(\pm)}{\Gamma}{}_\alpha{}^\beta := \Gamma_\alpha{}^\beta \mp (i\ell^2/2)\, e_\alpha \rfloor \overset{(\mp)}{H}{}^\beta$, induced by the translational Chern–Simons term $idC_{\rm TT}$, cf. [62, 63]. The resulting Sen type connection still couples *minimally* to the Dirac field, but poses the issue of *reality conditions*, cf. [69].

2.4 GR from effective strings

A more general situation arises, when we consider the tree–level effective action, corresponding to the lowest order in the string loop expansion, in the physical Einstein frame: According to Damour and Vilenkin [14], the following nonlinear terms arise

$$\begin{aligned} V_{\rm Stringeff} &= \left(\frac{1}{\alpha'}\hat{R} + \hat{R}^2 + (\alpha')\hat{R}^3 + (\alpha')^2\hat{R}^4 + \cdots\right)\eta \\ &= \frac{1}{(\alpha')^2}\sum_{n=1}^{\infty}(\alpha'\hat{R})^n\eta = \frac{\hat{R}/\alpha'}{1-\alpha'\hat{R}}\eta\,, \end{aligned} \tag{2.20}$$

where α' is the slope parameter and $\hat{R}^n$ stands in for generic higher–order curvature invariants.

For $|\alpha'\hat{R}| < 1$, the formal summation to a *geometric series* is inspired by the *nonlinear graviton* construction of Penrose [85]. To be justified, we need, for simplicity, to identify $\hat{R} = R = {}^*(R^{\alpha\beta}\wedge\eta_{\alpha\beta})$ with the curvature scalar. Then the Lorentz-rotational field momentum (5.19) reduces to $\widetilde{H}_{\alpha\beta} := -(\partial\widetilde{V}/\partial R)(\partial R/\partial R^{\alpha\beta}) = -\eta_{\alpha\beta}(\partial V/\partial R)$, and the Legendre transformation (2.12) simplifies to

$$\begin{aligned} \widetilde{V} \;\rightarrow\; V = \frac{1}{2}R\frac{\partial\widetilde{V}}{\partial R} + \frac{1}{2}\widetilde{V} &= \frac{R/\alpha'}{1-\alpha' R}\eta + \frac{R^2}{2(1-\alpha' R)^2}\eta \\ &\simeq \frac{1}{\alpha'}R\eta + \frac{1}{2}R^2\,. \end{aligned} \tag{2.21}$$

Again the Lagrangian truncated at $\alpha' \rightarrow 0$ corresponds to the perturbatively renormalizable quadratic model of Stelle [93].

3. Field Redefinition Induced by Theta Terms

In the gauge framework with torsion, the most general *quadratic* Lagrangian[35, 32, 81] reads

$$V_{\rm QPG} = \frac{\Lambda}{\ell^2}\,\eta + \frac{a_0}{4\ell^2}R^{\alpha\beta}\wedge\eta_{\alpha\beta} - \frac{1}{2}T^\alpha\wedge H_\alpha - \frac{1}{2}R^{\alpha\beta}\wedge H_{\alpha\beta} + V_\theta$$
$$H_\alpha := -\frac{1}{\ell^2}\,{}^*\left(\sum_{M=1}^{3} a_{(M)}\,{}^{(M)}T_\alpha\right), \quad H_{\alpha\beta} := -\frac{a_0}{2\ell^2}\,\eta_{\alpha\beta} - \frac{1}{g^2}\,{}^*\left(\sum_{N=1}^{6} b_{(N)}\,{}^{(N)}R_{\alpha\beta}\right) \quad (3.1)$$

The dimensionless coupling constant g and the fundamental length ℓ fix the relative strength of the rotational and translational interaction parts of the gravitational Lagrangian V. In the field momenta, each of the three irreducible torsion and six irreducible curvature pieces contribute to the Lagrangian[4] with an individual weight $a_{(M)}$ and $b_{(N)}$, respectively.

If we resolve the condition of *constant Hessian* for the allowed form of the rotational field momenta , we obtain the *generalized double duality* ansatz (DD)

$$H_{\alpha\beta}(**) = \theta_{\rm L}\,R_{\alpha\beta} + \theta^\star_{\rm L}\,R^{(\star)}_{\alpha\beta} + \frac{\theta^*_{\rm T}}{2\ell^2}\,\eta_{\alpha\beta} - \frac{\theta_{\rm T}}{2\ell^2}\vartheta_\alpha\wedge\vartheta_\beta\,, \quad (3.2)$$

where $\theta_{\rm T}$, $\theta^*_{\rm T}$, $\theta_{\rm L}$, and $\theta^\star_{\rm L}$ are dimensionless constants which can be related to the individual coupling constants in the θ–type boundary term (5.15). It has been demonstrated in much detail elsewhere [58, 59, 61, 72, 99] that the DD ansatz maps the second field equation (5.18) into the second Bianchi identity (5.6), provided the translational gauge field momenta fulfill certain algebraic conditions. By inserting the duality ansatz[5] into the second field equation (5.18), these may be derived from

$$\frac{\theta^*_{\rm T}}{2\ell^2}\eta_{\alpha\beta\gamma}T^\gamma - \vartheta_{[\beta}\wedge H_{\alpha]}(**) + \frac{\theta_{\rm T}}{\ell^2}\vartheta_{[\beta}\wedge T_{\alpha]} = \tau_{\alpha\beta}\,. \quad (3.3)$$

---

[4] The propagating modes and particle content of this Lagrangian were investigated by Sezgin and van Nieuwenhuizen[88]. A subclass of Lagrangians survived their selection criteria motivated by quantum field theoretical considerations, such as ghost-freeness and positive energy of the physical modes. By performing a mode decomposition based on a *flat* Minkowskian background, Kuhfuß and Nitsch [49] found a three–parameter class of unitary PG Lagrangians (see [32] and further References). therein. However, there may arise problems with the Cauchy formulation, shock waves [51], and positivity of the gravitational energy, see Hecht et al. [30, 31] as well as [27] for a review. A more general class has been employed [19] in a FR, where axial torsion gets identified with a dynamical axion field.

[5] Instanton solutions of the Stepenson-Kilmister-Yang (SKY) theory of gravity were already 1981 classified [58] with such an ansatz simplified by the choice $\theta_{\rm T} = \theta^*_{\rm T} = \theta_{\rm L} = \theta^\star_{\rm L} = 0$; much earlier than, e.g., García-Compeán et al. [25] in the context of S-duality. (For an extension to metric-affine theory, see [97].) In the wider framework of quadratic gravitational gauge models this 1982 ansatz of Baekler et al. [7] for $\theta_{\rm L} = 0$ induces for purely imaginary $Im(\theta_{\rm T}) \neq 0$ *complex* dual variables similar to those found later by Ashtekar [2] and Minkowski [73]. Recently, Soo [92] 'recovered' a more specialized version of our earlier DD ansatz. The modified self-dual action of Barbero [9] corresponds to a real $\theta_{\rm T}$, but necessarily faces the problem of CP violation, cf. [71]. The same problem arises in the so-called 'Immirzi ambiguity' [41], which is generated [37] by a part of the Nieh-Yan term (5.7). Contrary to the statement of Gambini et al. [23], this translational $\theta_{\rm T}$ is a total divergence in RC spacetime.

For spinless matter (cf. [59] for the general case), the translational momentum $H_\alpha(**)$ subject to the constraint (3.2) takes the form

$$H_\alpha(**) = -\frac{\theta_{\rm T}^*}{\ell^2}\,{}^*\left[T_\alpha - \vartheta_\alpha \wedge (e_\beta\rfloor T^\beta) - \frac{1}{2}\,e_\alpha\rfloor(T^\beta \wedge \vartheta_\beta)\right] + \frac{\theta_{\rm T}}{\ell^2}T_\alpha$$
$$= -\frac{\theta_{\rm T}^*}{2\ell^2}\,K^{\beta\gamma}\wedge\eta_{\alpha\beta\gamma} + \frac{\theta_{\rm T}}{\ell^2}T_\alpha = \frac{\theta_{\rm T}^*}{\ell^2}\,K_\alpha^{(\star)} + \frac{\theta_{\rm T}}{\ell^2}T_\alpha. \tag{3.4}$$

As a further consequence, the first field equation (5.17) reduces to the Einstein equation

$$\frac{\theta_{\rm T}^*}{2}R^{\{\}\beta\gamma}\wedge\eta_{\alpha\beta\gamma} - \Lambda_{\rm eff}\,\eta_\alpha = \ell^2\,\widehat{\Sigma}_\alpha \tag{3.5}$$

for the Riemannian background with an *effective* "cosmological" constant

$$\Lambda_{\rm eff} = \Lambda - \Lambda_\theta\,, \qquad \Lambda_\theta = \frac{3(\theta_{\rm T}^* + a_0)^2}{2\ell^2(\theta_{\rm L} + \theta_{\rm L}^\star + b_6/g^2)} \tag{3.6}$$

of microscopic origin [59, 61, 8, 54, 55]. Observe that the 'bare' cosmological constant Λ gets subtractively *renormalized* by a term induced by the Lorentz rotational boundary terms. For $b_6 = 0$, this persists even in the weak coupling limit $g \to 0$.

By inserting (3.2) and (3.4) into (3.1), the same can be obtained on the Lagrangian level. Since the torsion terms drop out due to the Nieh-Yan relation (5.7) and the teleparallelism identity (5.14), we are left with an *effective* Hilbert–Einstein Lagrangian

$$V_{\rm eff} = -\frac{\theta_{\rm T}^*}{2\ell^2}R^{\{\}\alpha\beta}\wedge\eta_{\alpha\beta} + \Lambda_{\rm eff}\,\eta\,. \tag{3.7}$$

This result complies with that of Refs.[59, 8], because the topological boundary terms have already been included in the quadratic Lagrangian (3.1) we started with. In order to attain macroscopic correspondence, the coupling of the effective Einstein equation (3.5) to the symmetrized [67] energy–momentum current $\widehat{\Sigma}_\alpha$ of matter requires $\theta^*_{\rm Tphys.} = 1$ for consistency.

What are the consequences of the duality ansatz for a *general* gravitational gauge model in terms of the field redefinition of the basic fields, the coframe and connection? Since $e_\beta\rfloor\eta^{\alpha\beta} = 0$, we find

$$\widetilde{\vartheta}^\alpha = (1 + \frac{3}{2}\theta_{\rm T}^*)\vartheta^\alpha - \theta_{\rm L}\ell^2\,e_\beta\rfloor{}^*R^{\alpha\beta} - \theta_{\rm L}^\star\ell^2\,e_\beta\rfloor{}^*R^{\alpha\beta(\star)}\,, \tag{3.8}$$

$$\widetilde{\Gamma}_\alpha{}^\beta = \Gamma_\alpha{}^\beta + \frac{\theta_{\rm T}^*}{2}e_\alpha\rfloor{}^*K^{\beta(\star)} + \theta_{\rm T}\,e_\alpha\rfloor{}^*T^\beta = \Gamma_\alpha^{\{\}\beta} - K_\alpha{}^\beta + \frac{\theta_{\rm T}^*}{2}e_\alpha\rfloor{}^*K^{\beta(\star)} + \theta_{\rm T}\,e_\alpha\rfloor{}^*T^\beta\,. \tag{3.9}$$

Besides a different normalization, a curvature piece and a double dual one related to the Euler four–form (5.13) deformes the coframe. In the special case $\theta_{\rm T}^* = -2/3$, the coframe (or holonomic metric) arises as a concept derived from the curvature, thus leading to an Eddington type theory [20]. On the other hand, in the deformed connection occurs only a reshuffling of the contortional pieces such that the Riemannian connection stays invariant. It is interesting to note that then the volume four–form (2.7), will change for the simplifying choice $\theta_{\rm L} = 0$ to

$$\widetilde{\eta} = (\theta_{\rm L}^\star)^4 \frac{\ell^8}{4!}\, \eta_{\alpha\beta\gamma\delta} e_\mu \rfloor {}^*R^{\alpha\mu(\star)} \wedge \cdots \wedge e_\mu \rfloor {}^*R^{\delta\mu(\star)}\,, \tag{3.10}$$

which for $\ell = \ell_{\rm Planck}$ is of the order 10^{-256} times $\theta_{\rm L}^{\star 4} |R|^4$. Do we have here some clue for a macroscopically tiny effective cosmological term $\Lambda\widetilde{\eta}$ induced by the Lorentz-rotational $\theta_{\rm L}^\star$–term? Or does this correspond to the sought-for "dark energy" induced by Chern-Simons terms?

4. Duality of Weak and Strong Coupling Limit

Can we use this FR scheme in order to start from a general PG Lagrangian which avoids Cauchy, ghost mode, and renormalization problems and end up after renormalization with an effective Einstein equation at the *macroscopic* level?

In order to probe the virtue of our construction on a more simple model, let us compare the Hilbert–Einstein or EC Lagrangian with the Lagrangian

$$\widetilde{V}_{\rm SKY} = -\frac{1}{2g^2} R^{\alpha\beta} \wedge {}^*R_{\alpha\beta} + \frac{1}{2}\theta_{\rm L}^\star\, R^{\alpha\beta} \wedge R^{(\star)}_{\alpha\beta} \tag{4.1}$$

of SKY gravity supplemented by the Euler term, where $\widetilde{H}_{\alpha\beta} = {}^*R_{\alpha\beta}/g^2 - \theta_{\rm L}^\star R^{(\star)}_{\alpha\beta}$. From the work of Stelle [93] we know that this curvature squared gravity is perturbatively *renormalizable* but plagued with ghost [50].

Via the singular FR

$$\eta_{\alpha\beta} \quad \rightarrow \quad \widetilde{\eta}_{\alpha\beta} := \eta_{\alpha\beta} - \frac{\ell^2}{g^2}\, {}^*R_{\alpha\beta} + \theta_{\rm L}^\star \ell^2\, R^{(\star)}_{\alpha\beta} = 0\,. \tag{4.2}$$

we retain from the Legendre transformation (2.12) the EC Lagrangian (2.8) plus an induced cosmological constant.[6] A similar reduction happens for the duality ansatz (3.2) as we have seen.

Since $R^{(\pm)}_{\alpha\beta} = \pm g^2/(g^2\theta_{\rm L}^\star \mp 1)\eta_{\alpha\beta}/\ell^2$, the weak coupling limit $g \rightarrow 0$ implies vanishing (chiral) curvature, cf. [83]. On the other hand, the weak coupling regime $\ell \rightarrow \ell_{\rm P} = 10^{-33}{\rm cm} \approx 0$ of macroscopic Einstein gravity implies for $g = 1$ a strong curvature scalar $R = 12(g/\ell)^2 = 48/\alpha' \rightarrow \infty$, i.e. the strong coupling regime of SKY gravity as part of the effective string or curvature-saturated Lagrangian, cf. Ref. [47]. Moreover, for the Taub–NUT metric this induces a rotation in the plane spanned by mass M and *dual mass* N (angular momentum), as felt by chiral fermions [60]. This duality of strong and weak coupling resembles that found by Montonen and Olive [76] in the contex of magnetic monopoles.

[6] In a rather ad hoc fashion, such a FR was applied already in Ref. [59] and later by Obukhov and Hehl [80] to Euler and Pontrjagin type terms. However, such deformations change the latter four-forms from being anymore boundary terms, thus preventing a topological interpretation in the spirit of S–duality.

5. Chromogravity and the Anti-De Sitter Model of Quark Confinement

The occurrence of an 'induced' cosmological constant (3.6) and its metrical Anti-de Sitter solution (AdS) has led to many speculation for its meaning in particle physics. It was already Einstein [21] who pointed that that ".. the scalar curvature plays the role of a negative pressure, ... whose gradient balances the electrodynamic force." Later on a similar idea was taken up by Salam et al. in his *strong gravity* model of quark confinement, cf. [56, 57, 77].

Independent of the physical interpretation, a 4D Anti-de Sitter background provides an *geometrodynamical mechanism* for quark confinement already on the semi-classical level. Then the *generally covariant* Klein–Gordon equation for a tortoise type radial coordinate ρ^* reduces to an effective Schrödinger equation with a Pöschl–Teller type effective potential $U_{\text{eff}} \sim 1 + \tan^2(\rho^*)$ familiar from oscillations of diatomic molecules. The energy spectrum is the same as that of a non-relativistic harmonic oscillator, see [61] for details. Since this potential 'wall' is infinitely rising, there exist equally spaced excited states but NO disintegration of the constituents can occur. Thus our AdS model is a fully relativistic model of an harmonic oscillator.

Today it is advocated to use the effective D=11 supergravity resulting from M-theory after compactification to $\text{AdS}_4 \otimes S^7$ space [96], as a calculational means ('analog computer') [78] for the strong coupling regime of *quark confinement* in QCD.

Acknowledgments

We would like to thank Friedrich W. Hehl and Yuri N. Obukhov for helpful discussions and comments. Moreover, thanks are due to Noelia, Markus Gérard Erik and Miryam Sophie Naomi for encouragement. This article is dedicated to the memory of Yuval Ne'eman.

Appendix A: Redefined Maxwell Fields

Let us consider as a guiding example Maxwell's or Yang–Mills theory in n dimensions, where $A := A^I_\mu dx^\mu \lambda_I$ is the Lie-algebra valued gauge potential one–form and $F := dA + A \wedge A$ the field strength. A rather general deformation or *field redefinition* (FR) of the gauge connection A which respects gauge-invariance and the form degree reads

$$A \quad \rightarrow \quad \widetilde{A} = A + \xi\rfloor {}^*H - {}^*DH + {}^*j\,, \tag{5.1}$$

where $H := -\partial G/\partial F$ is the excitation $(n-2)$–form of some *generating* n–form G. Quite generally, such a FR does not depend on a dimensionfull parameter and assures in the Abelian case that the new field strength $\widetilde{F} := d\widetilde{A} = F + d(\xi\rfloor {}^*H + {}^*DH - {}^*j)$ is derivable from a vector potential. In the case of a topological boundary term $G_\infty = \theta dC = -(\theta/2)Tr(F \wedge F)$ derived from a Chern–Simons term C, there will arise a θ induced FR with $H = \theta F$ and $DH = \theta DF \equiv 0$ due to the gauge-covariant Bianchi identity $DF \equiv 0$. Our construction generalizes the rather formal field redefinition $\overline{A} = A + \delta G/\delta j$ of Dixon [15] for non–Abelian gauge theories, where $j := \delta G/\delta A \cong -DH$ is a gauge current $(n-1)$–form. 'On shell', only the term depending on the vector field ξ will remain, due to $DH \cong j$. A related one-form $\hat{A} := {}^*j - {}^*dH^{\pm} = A + {}^*(A \wedge H^{\pm})$ has been recently used by Ganor and Sonnenschein [24] in order to replace A by $\hat{A}$ in a classical duality of gauge theories, cf. Eq.(11) of Ref. [75]. Also Born–Infeld type Lagrangians may be generated, cf. [28]. For Maxwell's theory the corresponding *canonical transformation* was given by Lozano [53] within a Hamiltonian formulation, whereas in the non-Abelian case the theory turns out to be of the Freedman–Townsend type [22] where the new two-forms are not derivable from a covector potential, or one-form. In our case the generating functional reads $G^{(\pm)} = H^{(\pm)}_{\alpha\beta} \wedge \widetilde{H}^{\alpha\beta}$.

If the vector field ξ is *normal* to a spacelike hypersurface, i.e. $\xi \rfloor dt = 1$, the normal part $A_\perp := \xi \rfloor A$ remains invariant under this deformation. The ansatz of Ashtekar and Rovelli [4] used for projecting out positive frequency fields in the Hamiltonian formulation can be viewed as such a FR for a special choice of G.

Experimentally accessible appears [16] to be the duality rotation of the Aharanov–Bohm/Casher topological phases in Maxwell's theory.

In effective string theory, the roles of the 'theta angle' in front of the topological boundary term $G_\infty = \theta dC$ and the gauge coupling constant g in the Yang–Mills Lagrangian

$$L_{\rm YM} = -\frac{1}{2g^2} Tr\,(F \wedge {}^*F) \tag{5.2}$$

are related to the vacuum expectation value (VEV) of the *axion a* and the dilaton field φ, respectively, via $\theta = 2\pi\langle 0|a|0\rangle$ and $g^2 = 4\pi\langle 0|e^{\varphi}|0\rangle = 32\pi G/\alpha' = 4\ell^2_{\rm Planck}\alpha'$, where $2\pi\alpha'$ is the string tension.

Appendix B: Geometry of a Riemann–Cartan Spacetime

Our geometrical arena consists of a four–dimensional manifold which is equipped with a local Riemannian metric (2.4) of Lorentz signature $(o_{\alpha\beta}) = \mathrm{diag}(-1, 1, 1, 1)$. For the representation of spinors in a curved spacetime, it is necessary to have the anholonomic formalism available on par. Therefore, we introduce an orthonormal local frame and coframe field of dimension [1/length] and [length], respectively

$$e_\alpha = e^i{}_\alpha\, \partial_i\,, \qquad \vartheta^\alpha = e_j{}^\alpha\, dx^j\,. \tag{5.3}$$

According to our conventions, $\alpha, \beta, ... = 0, 1, ..., 3$ are anholonomic frame indices, $i, j, k, ... = 0, 1, ..., 3$ are holononic or world indices, and $\wedge$ denotes the exterior product. The *coframe* field of basis one–forms are reciprocal to the frame e_α with respect to the *interior product* $\rfloor$, i.e., $e_\alpha \rfloor \vartheta^\beta = e^i{}_\alpha\, e_i{}^\beta = \delta^\beta_\alpha$.

In a Yang–Mills type gauge theory of gravity, the coframe ϑ^α of dimension [length] and the dimensionless connection one-form $\Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha} = \Gamma_i{}^{\alpha\beta}\, dx^i$ are regarded as gauge potentials of non–linearly represented *local translations* and *local Lorentz transformations*, respectively, cf. Ref. [61, 70, 52, 43]. The corresponding translational field strength is the *torsion* two–form

$$T^\alpha := D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta = \frac{1}{2}\, T_{ij}{}^\alpha\, dx^i \wedge dx^j, \tag{5.4}$$

of dimension [length] and the dimensionless Riemann–Cartan (RC) *curvature* two–form [13]

$$R^{\alpha\beta} := d\Gamma^{\alpha\beta} - \Gamma^{\alpha\gamma} \wedge \Gamma_\gamma{}^\beta = \frac{1}{2}\, R_{ij}{}^{\alpha\beta}\, dx^i \wedge dx^j. \tag{5.5}$$

These field strengths obey the *first* and *second Bianchi identities*

$$DT^\alpha \equiv R_\gamma{}^\alpha \wedge \vartheta^\gamma, \qquad \text{and} \qquad DR^{\alpha\beta} \equiv 0\,. \tag{5.6}$$

The corresponding Lagrangians [33] are the Chern–Simons type boundary terms

$$dC_{\rm TT} = \frac{1}{2\ell^2}\left(T^\alpha \wedge T_\alpha + R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta\right) =: V_{\rm NY}\,, \qquad dC_{\rm RR} = -\frac{1}{2} R_\alpha{}^\beta \wedge R_\beta{}^\alpha =: V_{\rm Pontrjagin}\,, \tag{5.7}$$

where ℓ is a fundamental length. Up to normalizations, they are also known as Nieh–Yan four-form [79] and Pontrjagin term, respectively. Both Chern–Simons terms arise as part of the CS term $\hat{C} = C_{\rm RR} - 2C_{\rm TT}$ of a gauge theory with $SL(5,R)$ as structure group, containing the de Sitter groups $SO(1,4)$ or $SO(2,3)$ as subgroups, see footnote 31 of Ref. [34] and Pagels [84] for a dynamical scheme.

The Riemannian content of our geometrical framework can be brought out by splitting the RC connection according to $\Gamma^{\alpha\beta} = \Gamma^{\{\}\,\alpha\beta} - K^{\alpha\beta}$ into the unique Levi–Civita connection $\Gamma^{\{\}\,\alpha\beta}$ of Riemannian geometry and into the *contortion*

$$K_{\alpha\beta} = -K^{\beta\alpha} = e_{[\alpha}\rfloor T_{\beta]} - \frac{1}{2}(e_\alpha\rfloor e_\beta\rfloor T_\gamma)\vartheta^\gamma\,. \tag{5.8}$$

It follows from (5.4) that the latter is related to torsion implicitly via $T^\alpha = K^\alpha{}_\beta \wedge \vartheta^\beta$. In turn, the RC curvature two–form (5.5) decomposes as follows

$$R^{\alpha\beta} = R^{\{\}\,\alpha\beta} + D^{\{\}}\, K^{\alpha\beta} + K^\alpha{}_\mu \wedge K^{\mu\beta}. \tag{5.9}$$

Appendix C: Dual Forms

On an n–dimensional manifold with metric index s, the Hodge dual of p–forms is almost involutive: ${}^{**}\alpha = (-1)^{p(n-p)+s}\alpha$. For spacetimes where $s = 1$ holds, it induces an *almost complex structure*, cf. [10]. In four dimensions, the *Hodge dual* applied to two–forms is *conformally invariant* [5]. Vice versa, an initially metric-free *involutive* star operation ${}^\#$ on arbitrary two–forms allows to *reconstruct* [17, 29] a metric h which is conformally related to g. Our *Hodge dual* * of exterior forms is defined such that the normalization

$${}^*(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta) = \eta^{\alpha\beta\gamma\delta}, \qquad \text{where} \quad \eta_{\alpha\beta\gamma\delta} := +\delta^{0123}_{\alpha\beta\gamma\delta} \quad \text{and} \quad D\eta_{\alpha\beta\gamma\delta} = 0 \tag{5.10}$$

holds.

From the volume four–form $\eta = \frac{1}{4!}\eta_{\alpha\beta\gamma\delta}\,\vartheta^\alpha\wedge\vartheta^\beta\wedge\vartheta^\gamma\wedge\vartheta^\delta$, the so-called η- or dual basis $\{\eta,\, \eta_\alpha,\, \eta_{\alpha\beta},\, \eta_{\alpha\beta\gamma}\, \eta_{\alpha\beta\gamma\delta}\}$ of exterior forms can now be generated by consecutive interior products: $\eta_\alpha := e_\alpha\rfloor\eta = {}^*\vartheta_\alpha$, $\eta_{\alpha\beta} := e_\beta\rfloor\eta_\alpha = \eta_{\alpha\beta\gamma}\,\vartheta^\gamma = e_\beta\rfloor e_\alpha\rfloor\eta = {}^*(\vartheta_\alpha\wedge\vartheta_\beta) = \frac{1}{2}\eta_{\alpha\beta\gamma\delta}\,\vartheta^\gamma\wedge\vartheta^\delta$, and $\eta_{\alpha\beta\gamma} := e_\gamma\rfloor\eta_{\alpha\beta} = {}^*(\vartheta_\alpha\wedge\vartheta_\beta\wedge\vartheta_\gamma)$. Anholonomic indices are lowered by $o_{\alpha\beta} = e^i{}_\alpha\, e^j{}_\beta\, g_{ij}$, where $(o_{\alpha\beta}) = \mathrm{diag}(-1,\,1,\,1,\,1)$ denotes the signature of spacetime.

The *Lie dual* of Lorentz algebra–valued forms such as contortion and curvature is defined by

$$K_\alpha^{(\star)} := \frac{1}{2}\eta_{\alpha\beta\gamma}\wedge K^{\beta\gamma}\,, \qquad R_{\alpha\beta}^{(\star)} := \frac{1}{2}\eta_{\alpha\beta\gamma\delta}R^{\gamma\delta}\,. \tag{5.11}$$

and satisfies $DR_{\alpha\beta}^{(\star)} \equiv 0$.

We will also employ the self– or anti-selfdual torsion and curvature two forms

$$T_\alpha^{\pm} := \frac{1}{2}\left(T_\alpha \pm {}^*T_\alpha\right)\,, \qquad R_{\alpha\beta}^{\pm} := \frac{1}{2}\left(R_{\alpha\beta} \pm {}^*R_{\alpha\beta}\right)\,, \qquad R_{\alpha\beta}^{(\pm)} := \frac{1}{2}\left(R_{\alpha\beta} \pm R_{\alpha\beta}^{(\star)}\right) \tag{5.12}$$

in terms of the Hodge or Lie dual, respectively. In view of this, the teleparallel boundary term and the topological Euler terms can be written as

$$dC_{\mathrm{TT}^*} := \frac{1}{2\ell^2}d(\vartheta^\alpha\wedge{}^*T_\alpha) = \frac{1}{2\ell^2}\left(T^\alpha\wedge{}^*T_\alpha - D\,{}^*T_\alpha\right)\,, \qquad dC_{\mathrm{RR}^\star} := -\frac{1}{2}\,R^{\alpha\beta}\wedge R_{\beta\alpha}^{(\star)} = V_{\mathrm{Euler}}\,. \tag{5.13}$$

From (5.9) results the geometric identity [34]

$$\begin{aligned} R^{\{\}\alpha\beta}\wedge\eta_{\alpha\beta} &\equiv R^{\alpha\beta}\wedge\eta_{\alpha\beta} - K^{\alpha\mu}\wedge K_\mu{}^\beta\wedge\eta_{\alpha\beta} + K^{\alpha\beta}\wedge T^\gamma\wedge\eta_{\alpha\beta\gamma} + d(K^{\alpha\beta}\wedge\eta_{\alpha\beta}) \\ &= R^{\alpha\beta}\wedge\eta_{\alpha\beta} + T^\alpha\wedge{}^*\!\left(-\,{}^{(1)}T_\alpha + 2\,{}^{(2)}T_\alpha + \frac{1}{2}\,{}^{(3)}T_\alpha\right) + 4\ell^2\,dC_{\mathrm{TT}^*} \end{aligned} \tag{5.14}$$

which relates the Hilbert–Einstein Lagrangian to the proper *teleparallelism* model, with the Lagrangian constraint $R^{\alpha\beta} = 0$. These topological terms have been proposed, for instance in Ref. [63], in the combination

$$V_\theta := \theta_{\mathrm{T}}\,dC_{\mathrm{TT}} + \theta_{\mathrm{T}}^*\,dC_{\mathrm{TT}^*} + \theta_{\mathrm{L}}\,dC_{\mathrm{RR}} + \theta_{\mathrm{L}}^\star dC_{\mathrm{RR}^\star} \tag{5.15}$$

as *generating function* for obtaining more general Ashtekar type variables in the Hamiltonian formulation for $\theta_{\mathrm{L}}^\star = 0$. With exception of the Euler terms, these θ terms *violate parity*, but for purely *imaginary* θ parameters they would conserved the combined charge conjugation C and parity transformation P, i.e. CP, cf. [71]. Some of these terms arise also in the chiral anomaly [68, 64].

Appendix D: Framework of Gravitational Gauge Dynamics

The total action of interacting matter and gravitational gauge fields

$$W = \int\left[L(\vartheta^\alpha,\Psi,D\Psi) - V(\vartheta^\alpha,T^\alpha,R^{\alpha\beta})\right] \tag{5.16}$$

is assumed to be a functional of suitable matter fields Ψ and of the geometrical variables ϑ^α and $\Gamma^{\alpha\beta}$. Besides the Euler-Lagrange equation $\delta L/\delta\Psi = 0$ for matter, their *independent* variations yield the following two, general *nonlinear* field equations:

$$DH_\alpha - E_\alpha = \Sigma_\alpha, \tag{5.17}$$

$$DH_{\alpha\beta} + \vartheta_{[\alpha} \wedge H_{\beta]} = \tau_{\alpha\beta}. \tag{5.18}$$

Here the *gauge field momenta* are defined by the $(n-2)$–forms:

$$H_\alpha := -\frac{\partial V}{\partial T^\alpha}, \qquad \text{and} \qquad H_{\alpha\beta} := -\frac{\partial V}{\partial R^{\alpha\beta}}. \tag{5.19}$$

Note that in $n = 4$ dimensions H_α has dimension [length]. In addition to the material currents of energy–momentum $\Sigma_\alpha := \partial L/\partial\vartheta^\alpha$ and dynamical spin $\tau_{\alpha\beta} := \partial L/\partial\Gamma^{\alpha\beta}$, there occur the three–forms of *energy–momentum* $E_\alpha := \partial V/\partial\vartheta^\alpha = e_\alpha \rfloor V + (e_\alpha \rfloor T^\beta) \wedge H_\beta + (e_\alpha \rfloor R^{\beta\gamma}) \wedge H_{\beta\gamma}$ and the *translational spin current* $E_{\alpha\beta} = -\vartheta_{[\alpha} \wedge H_{\beta]}$ of the gravitational gauge fields themselves [35, 32]. This is due to the universality of gravitational interactions.

References

[1] E. Alvarez, Rev. Mod. Phys. **61** (1989) 561.

[2] A. Ashtekar, Phys. Rev. Lett. **57** (1986) 2244; *New Perspectives in Canonical Gravity* (Bibliopolis, Napoli 1988).

[3] A. Ashtekar: *Lectures on Non-perturbative Canonical Gravity* (World Scientific, Singapore 1991).

[4] A. Ashtekar and C. Rovelli, Class. Quantum Grav. **9** (1992) 1121.

[5] M.F. Atiyah, N.J. Hitchin, and I.M. Singer, Proc. R. Soc. (London) **A 362** (1978) 425.

[6] P. Baekler, Class. Quantum Grav. **8** (1991) 1023.

[7] P. Baekler, F. W. Hehl, and E. W. Mielke, in: *Proceedings of the 2nd Marcel Grossmann Meeting on the Recent Progress of the Fundamentals of General Relativity*, R. Ruffini, ed. (North–Holland, Amsterdam 1982), p.413.

[8] P. Baekler and E.W. Mielke, Phys. Lett. **113A**, 471 (1986); Fortschr. Phys. **36** (1988) 549.

[9] J. Barbero, Phys. Rev. **D51** (1995) 5507.

[10] C.H. Brans, J. Math. Phys. **15**, 1559 (1974); **16**, 1008 (1975); see also R. Debever, Cahiers Phys. **168**, 303 (1964).

[11] C. Brans, Class. Quantum Grav. **5** (1988) L197.

[12] R. Capovilla, T. Jacobson, and J. Dell, Phys. Rev. Lett **63**, 2325 (1989); Class. Quantum Grav. **8**, 41, 59 (1990).

[13] É. Cartan: *On Manifolds with an Affine Connection and the Theory of General Relativity*, English translation of the French original (Bibliopolis, Napoli 1986); for a book review, see F.W. Hehl, Gen. Rel. Grav. **21** (1989) 315.

[14] T. Damour and A. Vilenkin, Phys. Rev. **D** (1996).

[15] J.A. Dixon, Nucl. Phys. **B99** (1975) 420.

[16] J.P. Dowling, C.P. Williams, and J.D. Franson, Phys. Rev. Lett. **83** (1999) 2486.

[17] T. Dray, R. Kulkarni, and J. Samuel, J. Math. Phys. **30** (1989) 1306.

[18] M.J. Duff, Nucl. Phys. **B442** (1995) 47; Int. J. Mod. Phys. **A11** (1996) 4031; 5623.

[19] M.J. Duncan, N. Kaloper, and K.A. Olive, Nucl. Phys. **B 387** (1992) 215.

[20] A.S. Eddington: *The Mathematical Theory of Relativity*, 2nd Ed. (Cambridge University Press, Cambridge 1924).

[21] A. Einstein, Sitzungsber. Preuss. Akad. Wiss. (Berlin) **20** (1919) 349.

[22] D.Z. Freedman and P.K. Townsend, Nucl. Phys. **B177** (1981) 282.

[23] R. Gambini, O. Obregón, and J. Pullin, Phys. Rev. **D 59** (1999) 047505.

[24] O. Ganor and J. Sonnenschein, Int. J. Mod. Phys. **A11** (1996) 5701.

[25] H. García-Compeán, O. Obregón, J.F. Plebañski. and J. Socorro, Phys. Rev. **D 57** (1998) 7501.

[26] H. García-Compeán, O. Obregón, and C. Ramírez, Phys. Rev. **D 58** (1998) 104012-1.

[27] M.S. Gladchenko and V.V. Zhytnikov, Phys. Rev. **D50** (1994) 5060.

[28] M. Halsuda, K. Kamimura, and S. Sekiya: "Electric-magnetic duality invariant Lagrangians", KEK-TH 631 (1999).

[29] G. Harnett, J. Math. Phys. **32** (1991) 84; J. Phys. **A25** (1992) 5649; Yu.N. Obukhoff and F.W. Hehl, Phys. Lett **B458** (1999) 466.

[30] R.D. Hecht, J. Lemke, and R.P. Wallner, Phys. Rev. **D44** (1991) 2442.

[31] R.D. Hecht, J.M. Nester, and V.V. Zhytnikov, Phys. Lett. **A 222** (1996) 37.

[32] F.W. Hehl, in: *Proceedings of the 6th Course on Spin, Torsion. Rotation, and Supergravity*, held at Erice, Italy, May 1979, P. G. Bergmann, V. de Sabbata, eds. (Plenum, New York 1980), p. 5.

[33] F.W. Hehl, W. Kopczyński, J.D. McCrea, and E.W. Mielke, J. Math. Phys. **32** (1991) 2169.

[34] F.W. Hehl, J.D. McCrea, E.W. Mielke, and Y. Ne'eman, Found. Phys. **19** (1989) 1075; Phys. Rep. **258** (1995) 1.

[35] F.W. Hehl, P. von der Heyde, G.D. Kerlick, and J.M. Nester, Rev. Mod. Phys. **48** (1976) 393.

[36] P.W. Higgs, Nuovo Cimento **11** (1959) 816.

[37] S. Holst, Phys. Rev. **D53** (1996) 5966.

[38] G . 't Hooft, Nucl. Phys. **B79** (1974) 276.

[39] G. 't Hooft, in: *Trends in Elementary Particle Theory*, H. Rollnik and K. Dietz, eds. (Springer, Berlin 1975), p.92.

[40] G. 't Hooft, Nucl. Phys. **B357** (1991) 211.

[41] G. Immirzi, Class. Quantum Grav. **14** (1997) L177.

[42] A. Jakubiec and J. Kijowski, *Phys. Rev.* **D37**, 1406 (1988).

[43] J. Julve, A. López–Pinto, A. Tiemblo and R. Tresguerres, Gen. Rel. Grav. **28** (1996) 759.

[44] M. Kaku: *Quantum Field Theory* (Oxford University Press 1993).

[45] D.M. Kerrick, Phys. Rev. Lett. **75** (1995) 2074.

[46] C.W. Kilmister and D.L. Newman, Proc. Cambridge Phil. Soc. (Math.Phys. Sci) **57** (1961) 851.

[47] H. Kleinert and H. J. Schmidt, "Cosmology with curvature-saturated gravitational Lagrangian $R/\sqrt{1+l^4R^2}$," Gen. Rel. Grav. **34**, 1295 (2002).

[48] D. Kreimer and E.W. Mielke: "Comment on: Topological invariants, instantons, and the chiral anomaly on spaces with torsion", Phys. Rev. **D63** 048501 (2001) 1–4.

[49] R. Kuhfuß and J. Nitsch, Gen. Rel. Grav. **18**, 1207 (1986).

[50] C.-Y. Lee and Y. Ne'eman, Phys. Lett **B242** (1990) 59.

[51] J. Lemke, Phys. Lett. **A143** (1990) 13.

[52] A. López–Pinto, A. Tiemblo and R. Tresguerres, Class. Quantum. Grav. **12** (1995) 1503.

[53] Y. Lozano, Phys. Lett. **B364** (1995) 19; Hep-th/9610024.

[54] J. D. McCrea, in: *Proceedings of the 14th Int. Conference on* Differential Geometric Methods in Mathematical Physics *Salamanca 1985, P.L. García and A. Pérez-Rendón, eds., Lecture Notes in Mathematics, Vol. 1251* (Springer, Berlin 1987), p.222.

[55] J.D. McCrea, in: *Algebraic Computing in General Relativity*, lecture notes from the First Brazilian School on Computer Algebra, Vol. 2, M. J. Reboucas and W. L. Roque, eds. (Oxford University Press, Oxford 1991), p. 173 – 263.

[56] E.W. Mielke, Phys. Rev. Lett. **39** (1977) 530 – 533; 851 (E).

[57] E.W. Mielke, Int. J. Theor. Phys. **19**, (1980) 189 – 209.

[58] E.W. Mielke, Gen. Rel. Grav. **13** (1981) 175 – 187.

[59] E. W. Mielke, J. Math. Phys. **25**, 663 (1984); Fortschr. Phys. **32**, 639 (1984).

[60] E. W. Mielke, *Phys. Lett.* **A110** (1985) 87.

[61] E.W. Mielke: *Geometrodynamics of Gauge Fields — On the geometry of Yang–Mills and gravitational gauge theories* (Akademie–Verlag, Berlin 1987).

[62] E.W. Mielke, Phys. Lett. **A149** (1990) 345: **A 151** (1990) 567 (E); Phys. Rev. **D42** (1990) 3388.

[63] E.W. Mielke, Ann. Phys. (N.Y.) **219** (1992) 78.

[64] E.W. Mielke, Acta Phys. Polonica **29B** (1998) 871 – 880; Phys. Lett. **A 251** (1999) 349.

[65] E.W. Mielke, P. Baekler, F.W. Hehl, A. Macías and H.A. Morales–Técotl, in: *Gravity, Particles and Space–Time*, ed. by P. Pronin and G. Sardanashvily (World Scientific, Singapore, 1996), pp. 217–254.

[66] E.W. Mielke and F.W. Hehl, Phys. Rev. Lett. **67** (1991) 1370.

[67] E.W. Mielke, F.W. Hehl and J.D. McCrea, Phys Lett. **A140** (1989) 368.

[68] E.W. Mielke and D. Kreimer, Int. J. Mod. Phys. **D7** (1998) 535 – 548; Gen. Rel. Gravitation **31**, 701–712.

[69] E.W. Mielke and A. Macías, Annalen der Physik (Leipzig) **8** (1999) 301- 317.

[70] E.W. Mielke, J.D. McCrea, Y. Ne'eman, and F.W. Hehl, Phys. Rev. **D48** (1993) 673.

[71] E.W.Mielke, Macías, A. , and Y. Ne'eman: "CP–symmetry in chiral gravity", *Proc. of the Eigth Marcel Grossman Meeting on General Relativity*, Jerusalem, 1997, T. Piran and R. Ruffini, eds. (World Scientific, Singapore, 1999) pp. 901–903.

[72] E.W. Mielke and R.P. Wallner, Nuovo Cimento **101 B** (1988) 607; **102 B** (1988) 555.

[73] P. Minkowski, Phys. Lett. **B174** (1986) 191.

[74] C.W. Misner and J.A. Wheeler, Ann. Phys. (N.Y.) **2** (1957) 525.

[75] N. Mohammedi: "Classical duality in gauge theories", hep-th/9507040.

[76] C. Montonen and D. Olive , Phys. Lett. **B 72** (1997) 117.

[77] Y. Ne'eman and Dj. Šijački: "Chromogravity explains 'strong gravity'", in *Salamfestschrift.* A collection of talks from the Conference on Highlights of Particle and Condensed Matter Physics, A. Ali, J. Ellis and S. Randjbar-Daemi, eds. (World Scientific, Singapore 1994), pp. 540–551.

[78] Y. Ne'eman: "QCD confinement parameters from dual supergravity", in: *Quark Confinement and the Hadron Spectrum III*, Proc. of the 3rd International Conference held at Jefferson Lab., Virginia 1998, N. Isgur ed. (World Scientific, Singapore), in print.

[79] H.T. Nieh and M.L. Yan, J. Math. Phys. **23** (1982) 373–374.

[80] Yu.N. Obukhov and F.W. Hehl, Acta Phys. Polonica **B27** (1996) 2685.

[81] Yu.N. Obukhov, V.N. Ponomariev, and V.V. Zhytnikov, Gen. Relat. Grav. **21** (1989) 1107–1142.

[82] D. I. Olive:"Exact electromagnetic duality," Nucl. Phys. Proc. Suppl. **45A**, 88 (1996).

[83] J.-F. Pascual-Sánchez, Phys. Lett. **A108** (1985) 387; Lett. Nuovo Cimento **44** (1985) 326.

[84] H.R. Pagels, Phys. Rev. **D 29** (1984) 1690.

[85] R. Penrose, Gen. Rel. Grav. **7** (1976) 31.

[86] J. Polchinski, Phys. Rev. Lett **75** (1995) 4724.

[87] G.Y. Rainich, Trans. Am. Math. Soc. **27** (1925) 106.

[88] E. Sezgin and P. van Nieuwenhuizen, Phys. Rev. **D21** (1980) 3269–3280.

[89] L. Silberstein, Ann. Physik **24** (1907) 783 -784.

[90] L.M. Sokolowski, Class. Quantum Grav. **6**, 2045 (1989).

[91] A. Sommerfeld, Ann. Phys. (Leipzig) **32** (1910) 753, footnote p. 768.

[92] C. Soo, Phys. Rev. **D52** (1995) 3484.

[93] K.S. Stelle, Phys. Rev. **D16** (1977) 953.

[94] G. Stephenson, Nuovo Cimento **9** (1958) 263.

[95] A. Trautman, in: *Differential Geometry*, Symposia Mathematica Vol. **12** (Academic Press, London 1973) p. 139–162.

[96] E. Witten, Adv. Theor. Math. Phys. **2** (1998) 505.

[97] D. Vassiliev, Gen. Rel. Grav. **34**, 1239 (2002).

[98] C.N. Yang, Phys. Rev. Lett **33** (1974) 445.

[99] V.V. Zhytnikov, J. Math. Phys. **35** (1994) 6001–6017.

EJTP **3, No. 12** (2006) 19–35 *Electronic Journal of Theoretical Physics*

High-Dimensional Dynamics in the Delayed Hénon Map

J. C. Sprott*

Department of Physics, University of Wisconsin, Madison, WI 53706, USA

Received 3 April 2006, Accepted 16 August 2006, Published 20 September 2006

Abstract: A variant of the Hénon map is described in which the linear term is replaced by one that involves a much earlier iterate of the map. By varying the time delay, this map can be used to explore the transition from low-dimensional to high-dimensional dynamics in a chaotic system with minimal algebraic complexity, including a detailed comparison of the Kaplan-Yorke and correlation dimensions. The high-dimensional limit exhibits universal features that may characterize a wide range of complex systems including the spawning of multiple coexisting attractors near the onset of chaos.

Keywords: Delayed Hénon Map, Dynamical systems, Chaos, Bifurcations
PACS (2006): 05.45.-a, 05.45.Pq , 05.45.Gg, 05.45.Jn, 05.45.Ac, 05.45.Pq,45.30.+s, 05.45.-a

1. Introduction

The behavior of low-dimensional chaotic maps and flows has been extensively studied and characterized [1]. Hence, much of the interest in nonlinear dynamics is now turning to an understanding of the high-dimensional complex systems that characterize most of the real world. The intuition that has arisen from the study of low-dimensional systems does not necessarily extend to high-dimensional systems whose behavior is often quite different and in some ways simpler.

The goal of this paper is to explore the transition from low-dimensional to high-dimensional dynamics in a particularly simple example of an iterated map that is a variant of the familiar Hénon map [2]. The dynamics will be governed by a single parameter whose value determines the dimension of the system and hence its complexity. The system is algebraically minimal in that it has a single (quadratic) nonlinearity and a single linearity.

* Tel: +1-608-263-4449; fax: +1-608-262-7205
E-mail: sprott@physics.wisc.edu.

2. Delayed Hénon map

The system considered here is the time-delayed Hénon map given by

$$x_n = 1 - ax_{n-1}^2 + bx_{n-d} \tag{1}$$

For $d = 2$, this map is the familiar two-dimensional dissipative Hénon map whose solutions are chaotic for typical values of $a = 1.4$ and $b = 0.3$. This system can also be viewed as a quadratic map with time-delayed linear feedback in which d is a measure of the delay time, such as might be encountered in a chaos control scheme [3].

For $d = 1$, the map is equivalent to the one-dimensional logistic map [4]

$$y_n = Ay_{n-1}(1 - y_{n-1})$$

as is evident from the transformation

$$y = ax/A + 1/2 - b/2A$$

with the condition

$$A^2 - 2A + (2b - b^2 - 4a) = 0$$

from which it follows that a given choice of the parameters a and b maps into A according to

$$A = 1 \pm \sqrt{1 - 2b + b^2 + 4a} \tag{2}$$

3. Fixed Points

Equation (1) has fixed point solutions at

$$x_{\pm} = \frac{b-1}{2a} \pm \frac{1}{2a}\sqrt{(b-1)^2 + 4a}$$

These fixed points are born simultaneously in a saddle-node (blue-sky) bifurcation [5] at $a = -(b-1)^2/4$ with the smaller one (x_-) initially unstable and the larger one (x_+) initially stable. Thus for fixed b, the quantity a can be used as a bifurcation parameter to take the system from a stable fixed point into chaos, analogous to the A in the logistic map as suggested by Eq. (2).

4. Regions of Various Dynamical Behaviors

Figure 1 shows the regions of various dynamical behaviors for Eq. (1) in the ab-plane as determined numerically for various values of d. For this purpose, the initial conditions were taken as the mean of the two fixed points $x_0 = (b-1)/2a$, and the regions of chaos were identified by calculating the largest Lyapunov exponent using a variant of the Wolf algorithm [6]. There is no guarantee that a different initial condition would not produce a different dynamic in the various regions, but a full study of the basins of attractions for every (a, b, d) combination is computationally infeasible and unessential for the purposes of this paper (however, see Section 7 below where it is shown that multiple coexisting attractors are common only near the onset of chaos).

From plots such as Fig. 1, it was determined that the values $a = 1.6$ and $b = 0.1$, as shown by the dotted lines in the figure, give chaotic solutions for all $d \geq 1$. These values are close to the largest value of b for which chaos exists for fixed a (the actual value is closer to $a = 1.5933$ and $b = 0.10834$). For much of what follows, these values will be assumed except where describing the routes to chaos, in which case b will be fixed at 0.1 and a varied over the range 0 to 2. For $d = 1$, according to Eq. (2), this choice corresponds to varying A in the logistic map from 1.9 to 3.968164416..., with the value of $a = 1.6$ corresponding to $A = 3.685144316...$, which coincidentally is very near the Misiurewicz point [7] at $A = 3.678573510...$ where two chaotic bands coalesce into one.

5. Attractors

Figure 2 shows the attractors for the system in Eq. (1) with $a = 1.6$ and $b = 0.1$ for several values of d. The global structure is dominated by the quadratic map as expected for the small value of b [8], but the dimension of the attractor clearly increases with increasing d. This increase can be quantified by calculating the Kaplan-Yorke dimension D_{KY} [9] from the spectrum of Lyapunov exponents as shown in Fig. 3 along with three of the Lyapunov exponents. Linear least-squares fits to these results over the range $1 \leq d \leq 100$ give

$$D_{KY} \cong 0.192d + 0.699$$

$$\lambda_1 \cong 0.354 - 2.3 \times 10^{-5} d$$

For these values of a and b, there is a single positive Lyapunov exponent (no hyperchaos), and the sum of the exponents is $\Sigma\lambda_i = \log|b| = -2.302585093...$ for all $d \geq 2$. Consequently, the other Lyapunov exponents tend to cluster at small negative values ($\lambda_i \sim -2.65/d$ for $2 \leq i \leq d$) in the limit of large d. The actual mean at $d = 100$ is -0.0268 with a standard deviation of ± 0.0049.

The metric entropy, which by Pesin's identity [10] is the sum of the positive Lyapunov exponents, is one measure of the complexity and is identical to λ_1 and nearly independent of d [11]. The single positive Lyapunov exponent is presumably a consequence of the fact that all the stretching and folding occur along a single direction in the d-dimensional state space. Note that all the exponents in this paper are base-e.

Because of the very smooth and predictable near linear variation of the attractor dimension with d, this system provides a perfect opportunity for a critical comparison of the Kaplan-Yorke dimension with the correlation dimension. Figure 3 includes data with error bars for the correlation dimension D_C [12] determined by the extrapolation method of Sprott and Rowlands [13]. Such an extrapolation is crucial for accurately calculating these high correlation dimensions. A linear least-squares fit to the correlation dimension data over the range $1 \leq d \leq 35$ gives

$$D_C \cong 0.189d + 0.560$$

which suggests that $D_C \cong 0.981 D_{KY} - 0.126$ in keeping with the theoretical expectation [14-16] of

$$D_C \leq D_I = D_{KY}$$

where D_I is the information dimension. Similar behavior has been reported in an extensive survey of 3-dimensional chaotic systems by Chlouverakis and Sprott [17], and the results here can be viewed as an extension of that work up to $d = 35$. Such a careful comparison of these two dimensions over such a wide range is apparently a new result.

6. Routes to Chaos

It is well known that the logistic map and the Hénon map exhibit a period-doubling route to chaos as shown in the left panel of Fig. 4 for $d = 2$ and $b = 0.1$. Also typical of low-dimensional maps is the existence of dense stable periodic windows in the chaotic regime as evidenced by the negative value of λ_1 and a Kaplan-Yorke dimension of zero.

By contrast, the high-dimensional case in the right panel of Fig. 4 with $d = 100$ and $b = 0.1$ shows a much smoother transition into chaos and a complete absence of periodic windows, although there are several period doublings before the onset of chaos. A curious feature is the oscillation between simple chaos (with $\lambda_2 < 0$) and hyperchaos (with $\lambda_2 > 0$) with increasing a in the chaotic ($\lambda_1 > 0$) regime. The high-dimensional case also shows a lack of superstable orbits where λ_1 is infinitely negative. The absence of periodic windows for large d thus relates more to the dimension of the attractor than to the presence of hyperchaos in contrast to the conjecture by Thomas, *et al.* [18]. Similar behavior has been observed in delay differential equations [19], convection models governed by partial differential equations [20], lattices of coupled logistic maps [21, 22], artificial neural networks [23], and competitive Lotka-Volterra models [24].

The detailed behavior of this case near the onset of chaos is shown in Fig. 5. At a value of $a = 1.10$, the system has already period-doubled twice and exhibits a 4-cycle. When a reaches about 1.10893 without further period doubling, a Neimark-Sacker bifurcation [25, 26] occurs, leading to the appearance of a drift ring (a 2-torus in the corresponding flow) as evidenced by $\lambda_1 = 0$ and $D_{KY} = 1$. The drift ring undergoes successive period doublings, followed by a finite region of chaos with $\lambda_1 > \lambda_2 = 0$ beginning about $a = 1.13577$ before the onset of hyperchaos with $\lambda_1 > \lambda_2 > 0$. One of the Lyapunov exponents tends to remain at zero even after the onset of chaos, suggesting the existence of a neutrally stable global manifold even in the presence of chaos and hyperchaos.

The route to chaos is exhibited differently by the attractors in Fig. 6. The plot of x_n versus x_{n-50} in the upper left is a zoom into the vicinity of one of the points in the four-cycle just after the Neimark-Sacker bifurcation occurs. The succession of images shows a circular drift ring growing to a rectangular shape and then period-doubling before the onset of chaos. The bifurcations of the drift ring to periods 2, 4, and 8 occur at $a \cong 1.12991$, 1.134524, and 1.1357040, respectively, implying a Feigenbaum number of 3.91 ± 0.01, which is similar to but possibly different from the value of $\delta = 4.669201609\ldots$ for unimodal maps with a quadratic maximum [27]. Note that the plots indicate a homoclinic tangle [28] near the corners of the rectangle as the onset of chaos is approached.

7. Global Bifurcations

The study of global bifurcations and multiple attractors in high-dimensional systems is still in its infancy [29, 30]. The system described here provides an opportunity to study such bifurcations in a particularly simple mapping. For this purpose, we characterize an attractor by a single scalar value

$$\langle r^2 \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (x_n - x_{ref})^2 \tag{3}$$

which is the mean square deviation (the variance) of the attractor from the reference point x_{ref} projected onto one axis of the time-delay embedding space. Except for a set of measure zero, any choice of x_{ref} will give a unique and different value of $\langle r^2 \rangle$ for each attractor. For fixed parameters and many different initial conditions, multiple coexisting attractors will be indicated by values of $\langle r^2 \rangle$ that cluster around distinct values. With two different reference points, even the small potential degeneracy could be resolved by plotting the respective values in a plane, with each attractor having values that cluster near a point in the plane. Abrupt changes in the value or slope of $\langle r^2 \rangle$ as a parameter is varied will indicate a discontinuous (catastrophic or subcritical) or continuous (subtle or supercritical) bifurcation, respectively. For the study here, the reference point was taken as $x_{ref} = b/2a$ (the minimum of the parabola in Eq. (1)) and initial conditions were chosen from a normal random distribution with mean x_{ref} and variance 1.0, although other choices give similar results.

Figure 7 shows $\langle r^2 \rangle$ versus a for $b = 0.1$ and $b = 0.3$ with $d = 100$. For the smaller value of b, there is a single attractor for all values of a, but for $b = 0.3$, there is a range of a from about 0.4 to 0.8 where multiple attractors coexist, and they are most numerous near the onset of chaos. Figure 8 shows the relative probability for 8000 cases with various values of $\langle r^2 \rangle$ for $0.6 < \langle r^2 \rangle < 0.73$ with $a = 0.7$, $b = 0.3$, and $d = 100$, indicating the presence of at least seven distinct attractors. A closer examination indicates that each of these seven cases is really a cluster of distinct but similar attractors, totaling at least sixteen cases. Some of these attractors have very small basins of attraction since they occur infrequently or have very similar values of $\langle r^2 \rangle$, which makes it difficult to be confident that they are distinct.

Figure 9 shows four of the most common of these coexisting attractors. Three of them are weakly chaotic ($\lambda_1 \sim 0.002$), but the one in the upper right is a period-4 drift ring with $\lambda_1 < 10^{-8}$ (and presumably zero). Some of the attractors, such as the two at the bottom of Fig. 9, look almost identical but are clearly distinct, as evident by the very different values of $\langle r^2 \rangle$ and the largest Lyapunov exponent.

This behavior may represent a new route to chaos through "attractor spawning." Figure 7 indicates that these attractors appear gradually as their basins of attraction grow slowly or as they gradually separate from one another. The attractors coalesce into a single strange attractor once the chaos is fully developed, and all bounded orbits then have initial conditions that lie within its basin of attraction.

Of course Eq. (3) is only one of many possible ways to characterize an attractor. The largest Lyapunov exponent λ_1 could serve as another, and its value for each of the four attractors is shown in Fig. 9. However, it is more difficult to calculate, and it tends to converge more slowly. Furthermore, it would be useless for distinguishing quasiperiodic attractors (tori) since they all have $\lambda_1 = 0$. The full spectrum of Lyapunov exponents could also be used to characterize an attractor, but that would be even more computationally demanding.

8. Summary

Simple systems such as the one described here are useful for exploring the transition from low-dimensional and high-dimensional chaotic systems. The characteristic intricate bifurcation structure and periodic windows in the midst of chaos gives way to a smoother variation and more robust behavior as the dimension increases, especially in the chaotic regime. The Kaplan-Yorke dimension increases linearly with system dimension, but the largest Laypunov exponent and metric entropy remain relatively constant. The correlation dimension is a relatively constant fraction of about 98% of the Kaplan-Yorke dimension. The period-doubling route to chaos that is common at low dimension transitions to a quasiperiodic route as the dimension increases. A method for identifying global bifurcations is described, and it shows the existence of a large number of coexisting attractors near the onset of chaos, suggesting a new route to chaos in high-dimensional systems.

Acknowledgments

Discussions with David Albers, Konstantinos Chlouverakis, and Morris Hirsch are gratefully acknowledged.

References

[1] J. C. Sprott, Chaos and Time-Series Analysis, Oxford University Press, Oxford (2003).

[2] M. Hénon, A two-dimensional mapping with a strange attractor, Comm. Math. Phys. 50 (1976) 69-77.

[3] T. Buchner and J. J. Zebrowski, Logistic map with a delayed feedback: Stability of a discrete time-delay control of chaos, Phys. Rev. E 63 (2001) 016210 [7 pages].

[4] R. May, Simple mathematical models with very complicated dynamics, Nature 261 (1976) 45-67.

[5] R. H. Abraham and C. D. Shaw, Dynamics: the geometry of behavior, Part 4: bifurcation behavior, Aerial Press, Santa Cruz, CA, (1988).

[6] A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, Determining Lyapunov exponents from a time series, Physica D 16 (1985) 285-317.

[7] M. Misiurewicz, Absolutely continuous measures for certain maps of an interval, Publ. Math. I.H.E.S. 53 (1981) 17-51.

[8] G. Rowlands, Non-linear Phenomena in Science and Engineering, Horwood, Chichester (1990).

[9] J. Kaplan and J. Yorke, Chaotic behavior of multidimensional difference equations, In Functional differential equations and approximations of fixed points, Lecture Notes in Mathematics, Vol. 730 (ed. H. –O. Peitgen and H. –O. Walther), pp. 228-237, Springer, Berlin (1979).

[10] Ya. B. Pesin, Characteristic Lyapunov exponents and smooth ergodic theory, Russian Mathematical Surveys 32 (1977) 55-114.

[11] E. F. Manffra, H. Kant, and W. Just, Periodic orbits and topological entropy of delayed maps, Phys. Rev. E. (2001) 046203 [6 pages]

[12] P. Grassberger and I. Procaccia, Characterization of strange attractors, Phys. Rev. Lett. 50 (1983) 346-349.

[13] J. C. Sprott and G. Rowlands, Improved correlation dimension calculation, International Journal of Bifurcation and Chaos 11 (2001) 1861-1880.

[14] J. D. Farmer, E. Ott, and J. A. Yorke, The dimension of chaotic attractors, Physica D 7 (1983) 153-180.

[15] L. -S. Young, Dimension, entropy, and Lyapunov exponents in differentiable dynamical systems, Physica A 124 (1984) 639-646.

[16] F. Ledrappier and L. -S. Young, The metric entropy of diffeomorphisms, Annals of Mathematics 2 (1985) 509-574.

[17] K. E. Chlouverakis and J. C. Sprott, A comparison of correlation and Lyapunov dimensions, Physica D 200 (2004) 156-164.

[18] R. Thomas, V. Basios, M. Eiswirth, T. Kruel, and O. E. Rössler, Hyperchaos of arbitrary order generated by a single feedback circuit, and the emergence of chaotic walks, Chaos 14 (2004) 669-674.

[19] J. D. Farmer, Chaotic attractors of an infinite-dimensional dynamical system, Physica D 4 (1982) 366-393.

[20] H. Chaté and P. Manneville, Transition to turbulence via spatiotemporal intermittency, Phys. Rev. Lett. 58 (1987) 112-115.

[21] P. Grassberger, Information content and predictability of lumped and distributed dynamical systems, Physica Scripta 40 (1989) 346-353.

[22] K. Kaneko, Supertransients, spatiotemporal intermittency and stability of fully developed spatiotemporal chaos, Phys. Lett. A 149 (1990) 105-112.

[23] D. J. Albers, J. C. Sprott, and W. D. Dechert, Routes to chaos in neural networks with random weights, International Journal of Bifurcation and Chaos 8 (1998) 1463-1478.

[24] J. C. Sprott, J. C. Wildenberg, and Y. Azizi, A simple spatiotemporal chaotic Lotka-Volterra model, Chaos, Solitons and Fractals 26 (2005) 1035-1043.

[25] J. Neimark, On some cases of periodic motions depending on parameters, Doklady Akademii nauk Souiza Sovetskikh Skotsialisticheskikh Respuplik 129 (1959) 736-739.

[26] R. S. Sacker, A new approach to the perturbation theory of invariant surfaces, Communications on Pure and Applied Mathematics 18 (1965) 717-732.

[27] M. J. Feigenbaum, Quantitative universality for a class of nonlinear transformations, J. Stat. Phys. 19 (1978) 24-52.

[28] H. Poincaré, Analyse des travaux scientifiques de Henri Poincaré faites par lui-même, Acta Mathematica 38 (1921) 1-135.

[29] S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer, New York (1990).

[30] Y. A. Kuznetsov, Elements of Applied Bifurcation Theory (2^{nd} ed.), Springer, New York (1995).

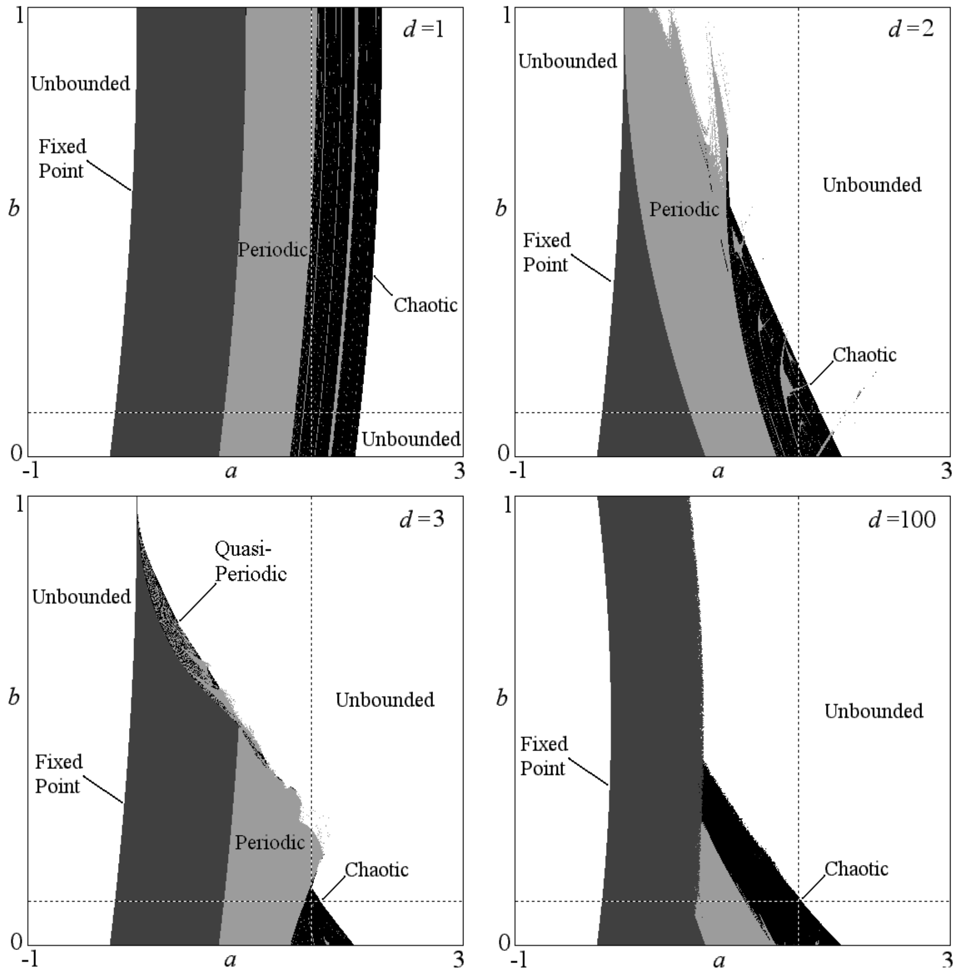

Fig. 1 Regions of dynamical behaviors for Eq. (1) for various values of the time delay.

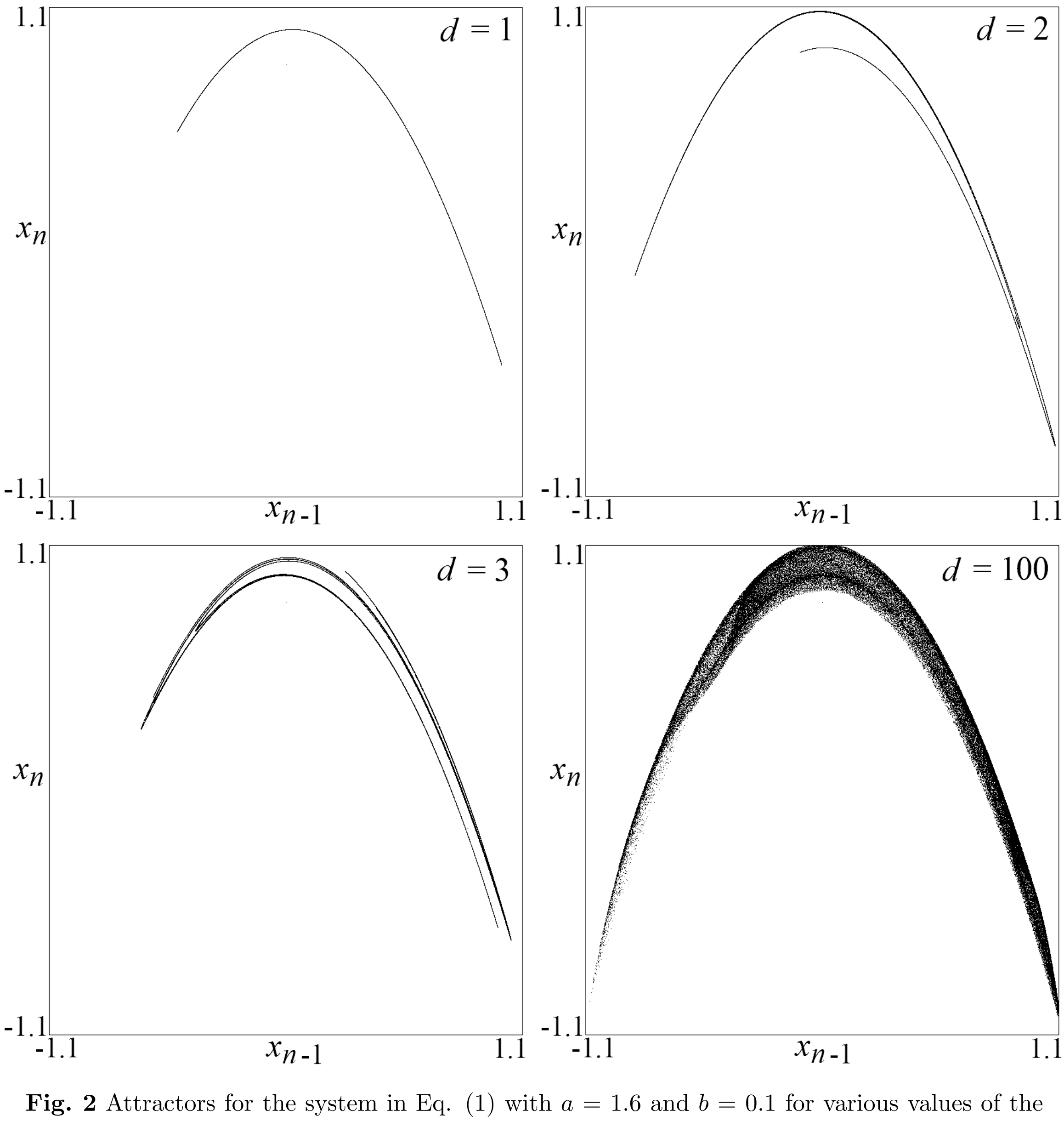

Fig. 2 Attractors for the system in Eq. (1) with $a = 1.6$ and $b = 0.1$ for various values of the time delay.

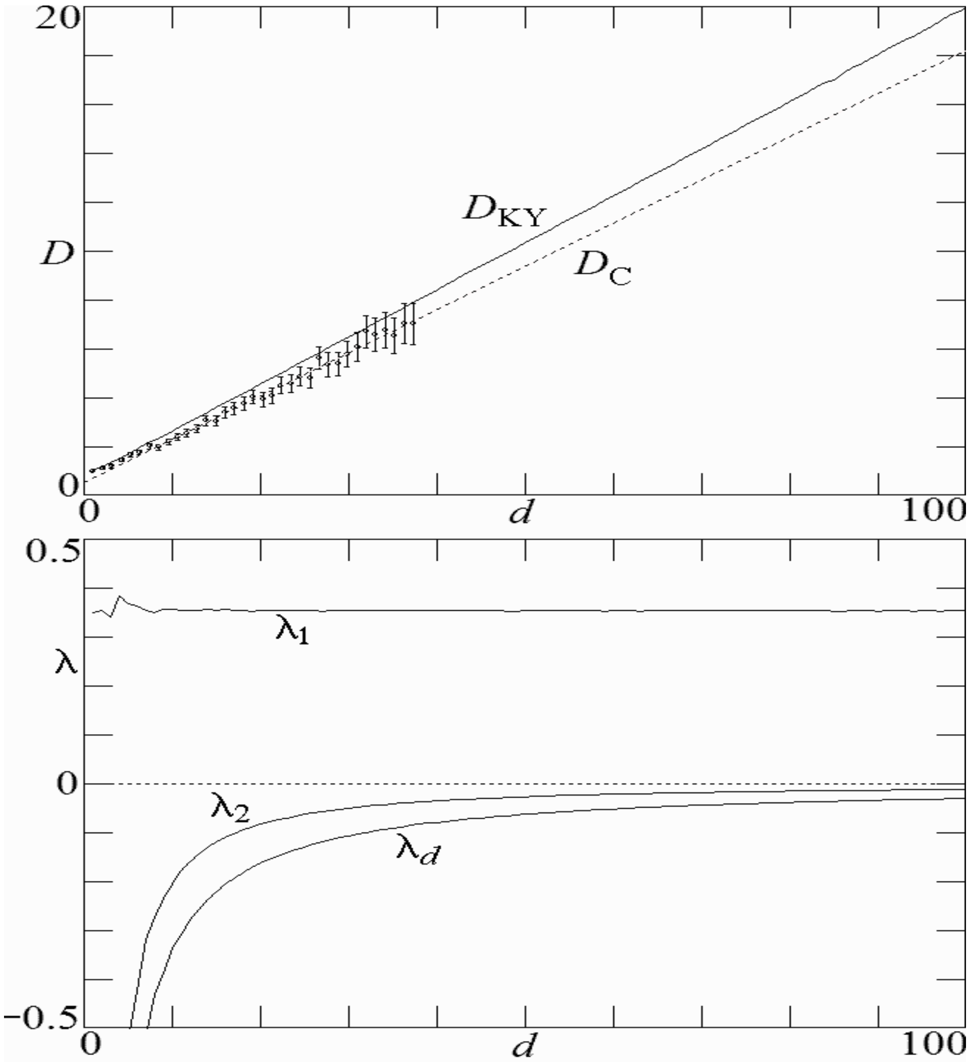

Fig. 3 Kaplan-Yorke dimension and Lyapunov exponents for the system in Eq. (1) with $a = 1.6$ and $b = 0.1$ versus time delay.

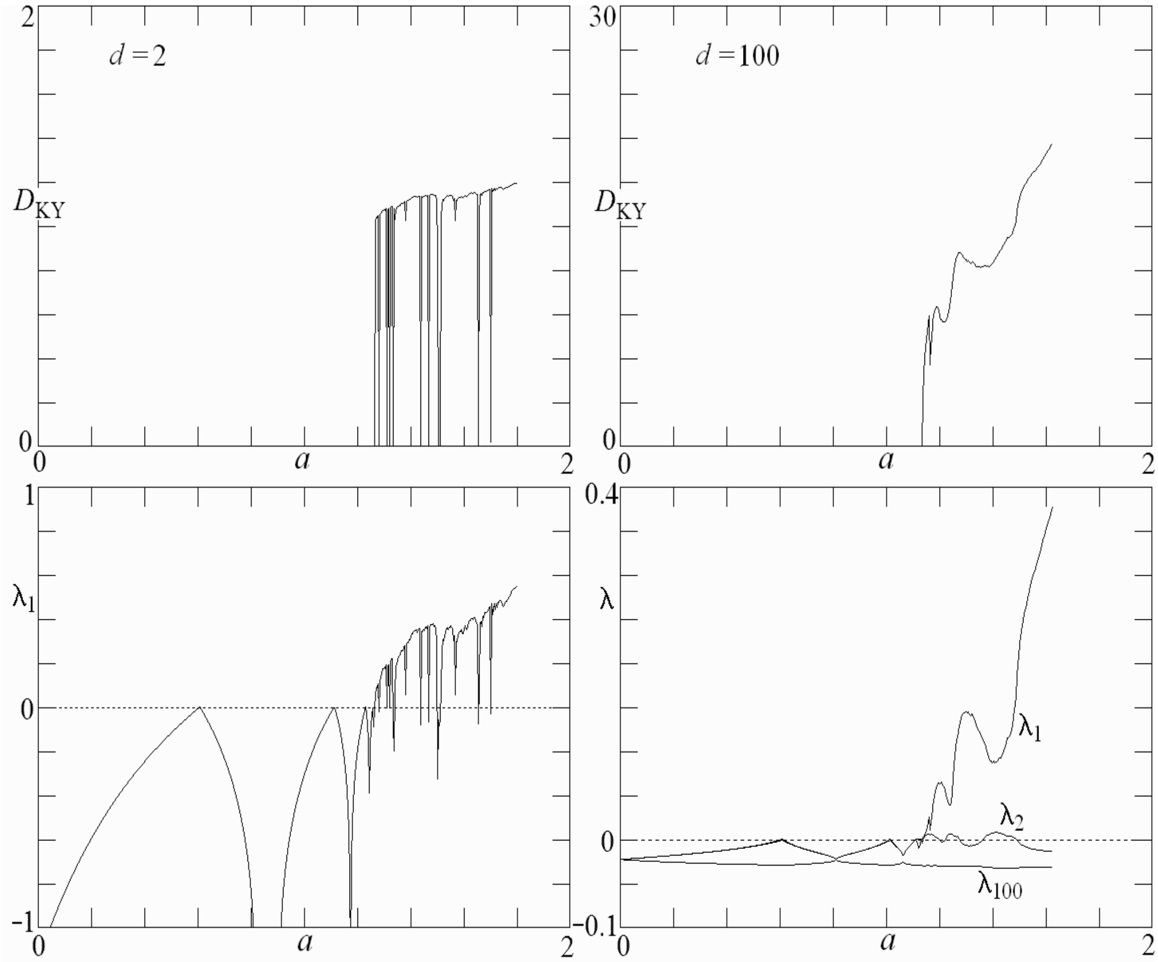

Fig. 4 Kaplan-Yorke dimension and Lyapunov exponents for the system in Eq. (1) with $b = 0.1$ showing the route to chaos at low dimension (d= 2) and high dimension (d = 100).

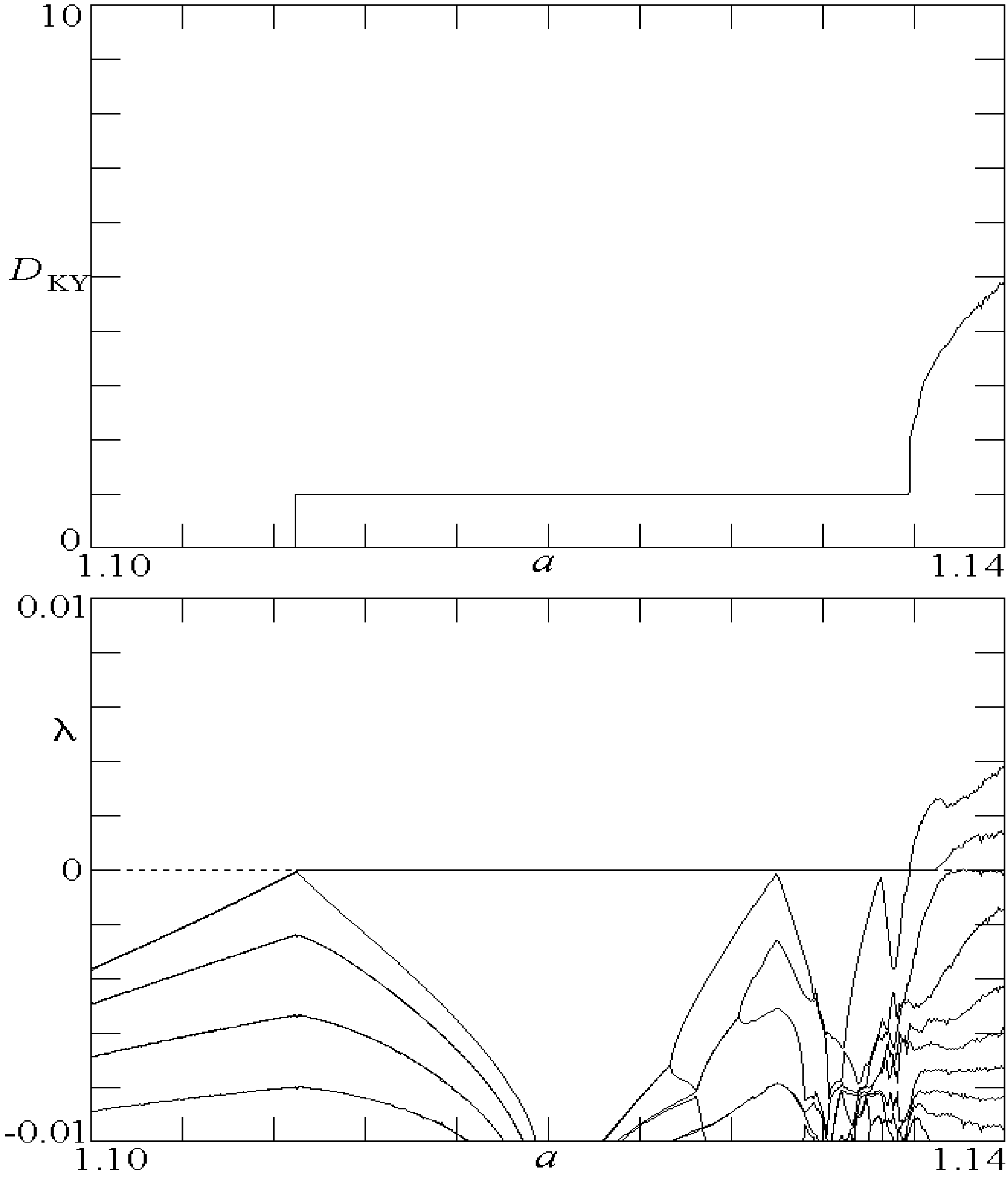

Fig. 5 Kaplan-Yorke dimension and a few of the largest Lyapunov exponents for the system in Eq. (1) with $b = 0.1$ and $d = 100$ showing in more detail the onset of chaos.

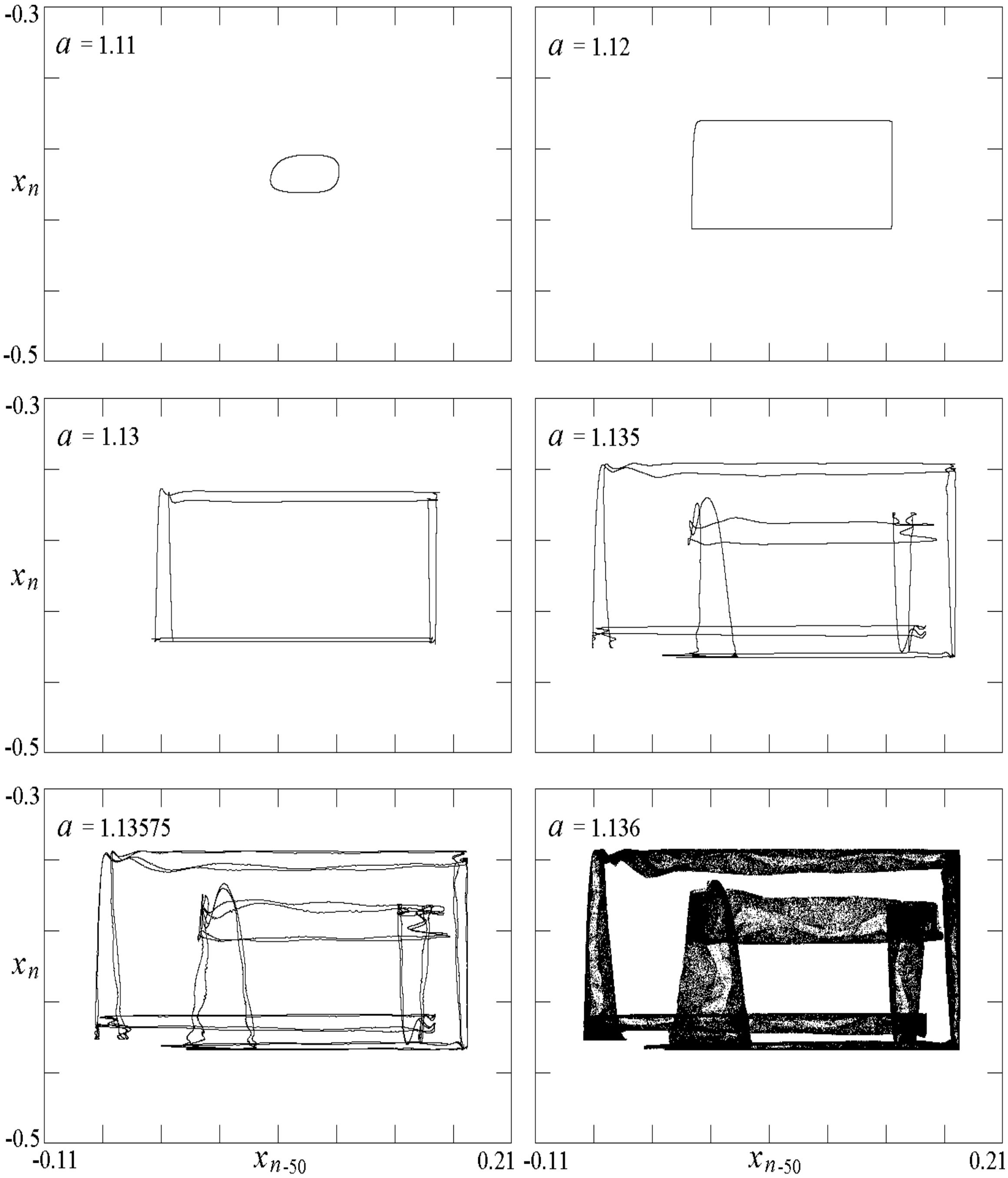

Fig. 6 Attractors for the system in Eq. (1) with $b = 0.1$ and $d = 100$ showing period doubling of a drift ring approaching the onset of chaos.

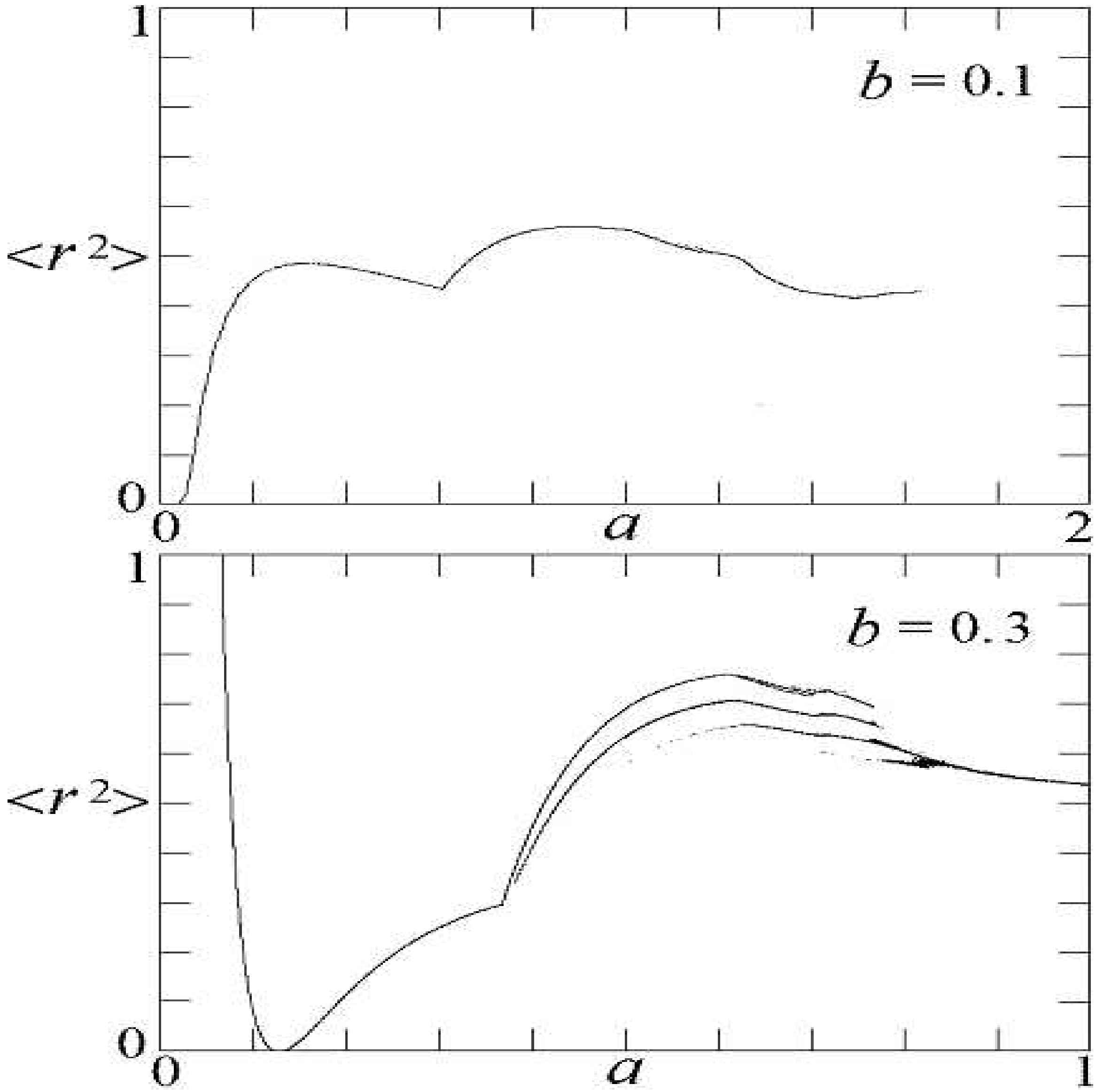

Fig. 7 Global bifurcations and multiple attractors for two values of b with $d = 100$.

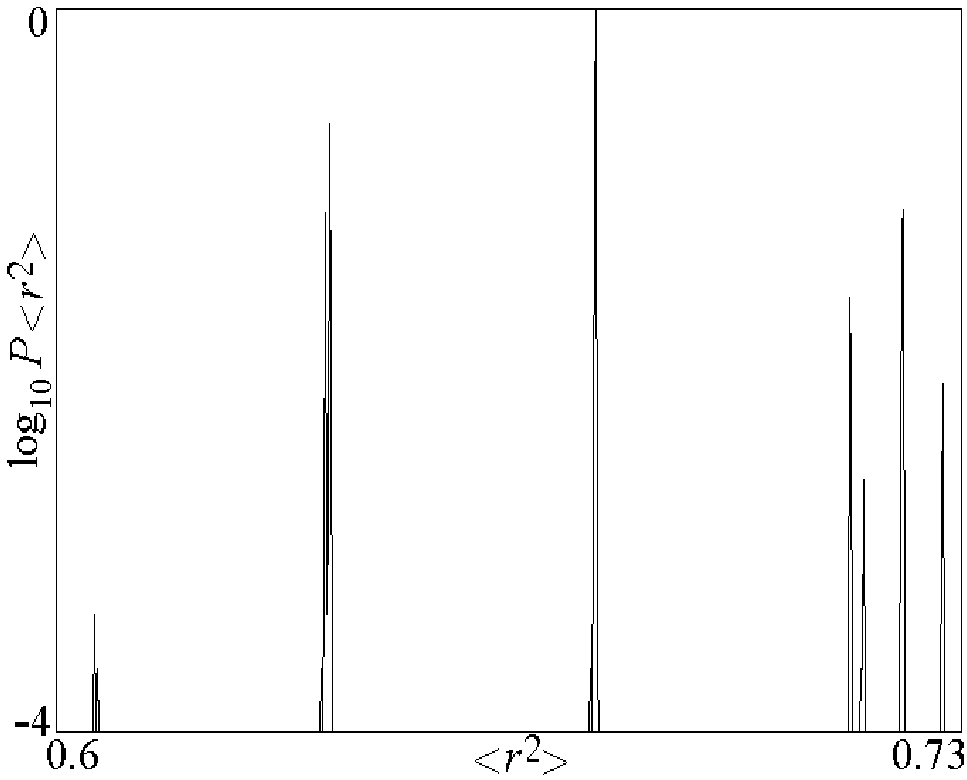

Fig. 8 Relative probability of different values of $\langle r^2 \rangle$ for $a = 0.7$, $b = 0.3$, and $d = 100$, indicating the existence of at least seven distinct attractors.

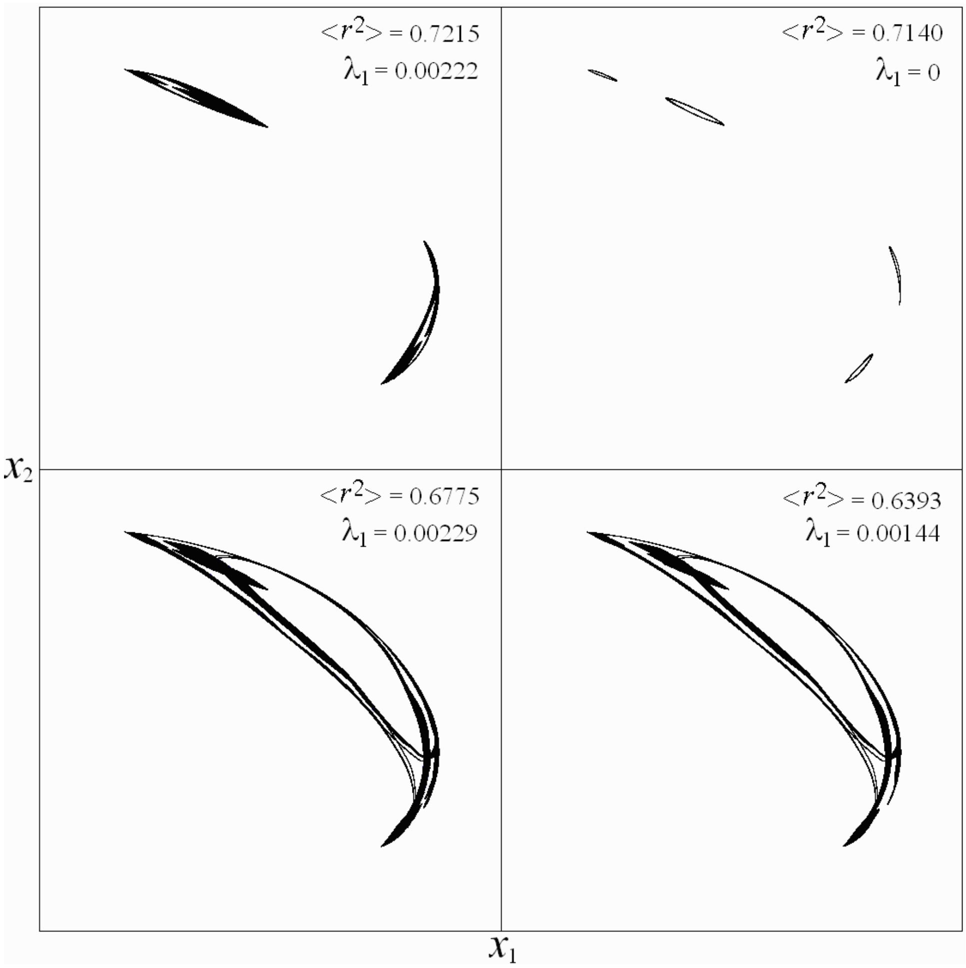

Fig. 9 Four coexisting attractors for $a = 0.7$, $b = 0.3$, and $d = 100$ near the onset of chaos.

EJTP **3, No. 12** (2006) 37–45 *Electronic Journal of Theoretical Physics*

Modified Moyal-Weyl Star product in a Curved Non Commutative space-time

N.Mebarki,F.Khallili , M.Boussahel, and M.Haouchine

Laboratoire de Physique Mathématique et Subatomique.
Mentouri University,Constantine,ALGERIA

Received 6 July 2006, Accepted 16 August 2006, Published 20 September 2006

Abstract: To generate gravitational terms in a curved noncommutative space-time, new Moyal-Weyl star product as well as Weyl ordering are defined. As an example, a complex scalar mass term action is considered.

Keywords: Quantum Groups, Non Commutative Geometry, Moyal-Weyl Star Product
PACS (2006): 02.20.Uw, 11.10.Nx, 11.15.?q, 02.40.Gh

1. Introduction

During the last two decades many efforts have been made to solve or at least to understand the reaming unsolved outstanding problems of theoretical physics by using new ideas like quantum groups, deformation theory, noncommutative geometry etc.. [1] − [9]. This may shed a light on the real microscopic geometry and structure of our universe.

One approach, is to consider a noncommutative space-time where the dynamical variables become operators and therefore, the formalism of the quantum fields theories constructions must be modified. It turns out that in a flat space-time geometry, this amounts basically to replace the ordinary products by a Moyal-Weyl star products and taking into account the Weyl ordering [1] − [10]. The goal of this paper is to consider a curved space-time (presence of a gravitational background) and define the corresponding new Moyal-Weyl star product and Weyl ordering. In section 2, we present our mathematical formalism and consider an example a mass term of a complex scalar field. Finally, in section 3, we draw our conclusions.

2. Formalism

2.1 *The ordinary Moyal − Weyl ∗ −product*

The Moyal-Weyl ∗-product of any two smooth functions f and g such as [12]

$$f(x) = (2\pi)^{-\frac{3}{2}} \int d^4k\, e^{ikx}\, \widetilde{f}(k) \tag{1}$$

$$g(x) = (2\pi)^{-\frac{3}{2}} \int d^4k\, e^{ikx}\, \widetilde{g}(k) \tag{2}$$

can be defined as follows:

First we associate to f and g the Weyl operators $W(f)$ and $W(g)$ defined by

$$W(f) = (2\pi)^{-\frac{3}{2}} \int d^4k\, e^{ik\widehat{x}}\, \widetilde{f}(k) \tag{3}$$
$$W(g) = (2\pi)^{-\frac{3}{2}} \int d^4k\, e^{ik\widehat{x}}\, \widetilde{g}(k)$$

where $\widehat{x}^{\mu}$ are non commuting operators satisfying

$$[\widehat{x}^{\mu}\,,\ \widehat{x}^{\nu}] = i\theta^{\mu\nu} \tag{4}$$

Next we define the product $W(f)\,W(g)$ as

$$W(f)\,W(g) = (2\pi)^{-\frac{3}{2}}\,(2\pi)^{-\frac{3}{2}} \int d^4k\, d^4p\, e^{ik\widehat{x}}\, e^{ip\widehat{x}}\, \widetilde{f}(k)\ \widetilde{g}(p) \tag{5}$$

Using the C-B-H formula, the Weyl product $W(f)\,W(g)$ reads :

$$W(f)\,W(g) = (2\pi)^{-\frac{3}{2}}\,(2\pi)^{-\frac{3}{2}} \int d^4k\, d^4p\, e^{ik\widehat{x}+ip\widehat{x}-\frac{i}{2}k_{\mu}p_{\nu}\theta^{\mu\nu}}\, \widetilde{f}(k)\ \widetilde{g}(p) = W(f * g) \tag{6}$$

Where $f * g$ is a new classical function defined by:

$$(f * g)(x) = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial y^{\nu}}} f(x)\, g(y) \tag{7}$$

This is the ordinary Moyal ∗-product. To the second ordre in θ The Moyal ∗-product reads:

$$(f * g)(x) = f(x)\,g(x) + \frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^{\mu}}f(x)\,\frac{\partial}{\partial x^{\nu}}g(x) + \tag{8}$$
$$+\frac{i}{2}\theta^{\mu\nu}\frac{i}{2}\theta^{\alpha\beta}\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial x^{\alpha}}f(x)\,\frac{\partial}{\partial x^{\nu}}\frac{\partial}{\partial x^{\beta}}g(x) + ...$$

Notice here, that the operators $\widehat{x}^{\mu}$ are only defined modulo terms which vanish at the classical limit, for example x^{μ} and $x^{\mu}+\Sigma^{\mu}_{\alpha\beta}x^{\alpha}x^{\beta}-\Sigma^{\mu}_{\alpha\beta}x^{\beta}x^{\alpha}$ are equal but the corresponding non commutating operators are not except if $\Sigma^{\mu}_{\alpha\beta}$ is symmetric

$$\widehat{x}^{\mu}\neq\widehat{x}^{\mu}+\Sigma^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}-\Sigma^{\mu}_{\alpha\beta}\widehat{x}^{\beta}\widehat{x}^{\alpha}=\widehat{x}^{\mu}+i\theta^{\alpha\beta}\Sigma^{\mu}_{\alpha\beta} \tag{9}$$

2.2 *The deformed Moyal − Weyl ∗ −product*

Here, by deforming the ordinary Moyal ∗-product, we propose a new Moyal-Weyl ⊛-product which take in consideration the missing terms cited above and which generate gravitational terms to the order θ^2. To any smooth function f we associate the Weyl operator $W(f)$

$$f(x)=(2\pi)^{-\frac{3}{2}}\int d^4x\, e^{ikx}\,\widetilde{f}(k)\ \to W(f)=(2\pi)^{-\frac{3}{2}}\int d^4k\, e^{ik\widehat{X}}\,\widetilde{f}(k) \tag{10}$$

where $\widehat{X}^{\mu}$ are non commuting operators associated to the following classical variables

$$X^{\mu}=x^{\mu}+\Gamma^{\mu}_{\alpha\beta}x^{\alpha}x^{\beta}-\frac{1}{2}\Gamma^{\mu}_{\rho\lambda}\Gamma^{\rho}_{\alpha\beta}x^{\alpha}x^{\beta}x^{\lambda} \tag{11}$$

where $\Gamma^{\mu}_{\alpha\beta}(x)=\Gamma^{\mu}_{\beta\alpha}(x)$ is the symmetric affine connection . The non commuting operators $\widehat{X}^{\mu}$ are defined by a symmetrization procedure:

$$\widehat{X}^{\mu}=\widehat{x}^{\mu}+\left(\widehat{\Gamma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\right)_w-\frac{1}{2}\left(\widehat{\Gamma}^{\mu}_{\rho\lambda}\widehat{\Gamma}^{\rho}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\widehat{x}^{\lambda}\right)_w \tag{12}$$

Where the Weyl ordering is defined by:

$$\left(\widehat{\Sigma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\right)_w=(\widehat{\Sigma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}-2\widehat{x}^{\alpha}\widehat{\Sigma}^{\mu}_{\alpha\beta}\widehat{x}^{\beta}+\widehat{x}^{\alpha}\widehat{x}^{\beta}\widehat{\Sigma}^{\mu}_{\alpha\beta})/\sqrt{-g} \tag{13}$$

with

$$g=\det g_{\mu\nu} \tag{14}$$

and direct simplifications (see Appendix) give:

$$\left(\widehat{\Sigma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\right)_w=i\theta^{\beta\lambda}i\theta^{\alpha\sigma}\partial_{\sigma}\partial_{\lambda}\widehat{\Sigma}^{\mu}_{\alpha\beta}/\sqrt{-g} \tag{15}$$

Where we have used the fact that $\widehat{\Sigma}^{\mu}_{\alpha\beta}$ is symmetric, and:

$$[\widehat{x}^{\mu}\ ,\ \widehat{x}^{\nu}]=i\theta^{\mu\nu}$$

and

$$\left[\widehat{x}^{\mu}\ ,\ \widehat{f}(x)\right]=i\theta^{\mu\nu}\partial_{\nu}\widehat{f}(x) \tag{16}$$

Thus the second term in eq.(15) reads:

$$\left(\widehat{\Gamma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\right)_w = -\theta^{\beta\lambda}\ \theta^{\alpha\sigma}\partial_\sigma\partial_\lambda\widehat{\Gamma}^{\mu}_{\alpha\beta}/\sqrt{-g} \tag{17}$$

It is worth to mention that we can get the same result if we define the Weyl ordering as:

$$\left(\widehat{\Gamma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\right)_w = (\overleftrightarrow{\widehat{\Gamma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}}\widehat{x}^{\beta} - \widehat{x}^{\alpha}\overleftrightarrow{\widehat{\Gamma}^{\mu}_{\alpha\beta}\widehat{x}^{\beta}} + \widehat{\Gamma}^{\mu}_{\alpha\beta}\overleftrightarrow{\widehat{x}^{\alpha}\widehat{x}^{\beta}})/\sqrt{-g} \tag{18}$$

Where

$$\overleftrightarrow{\widehat{x}^{\alpha}\widehat{x}^{\beta}} \equiv \left[\widehat{x}^{\alpha},\widehat{x}^{\beta}\right] \tag{19}$$

and

$$\overleftrightarrow{\widehat{x}^{\mu}\widehat{f}(x)} \equiv \left[\widehat{x}^{\mu}\ ,\ \widehat{f}(x)\right] \tag{20}$$

Now using these relations and the fact that

$$\widehat{\Gamma}^{\mu}_{\alpha\beta}\overleftrightarrow{\widehat{x}^{\alpha}\widehat{x}^{\beta}} = \widehat{\Gamma}^{\mu}_{\alpha\beta}\left[\widehat{x}^{\alpha},\widehat{x}^{\beta}\right] = \ i\theta^{\alpha\beta}\widehat{\Gamma}^{\mu}_{\alpha\beta} \tag{21}$$

eq.(18) can be rewritten as

$$\left(\widehat{\Gamma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\right)_w = -\theta^{\beta\lambda}\theta^{\alpha\sigma}\partial_\sigma\partial_\lambda\widehat{\Gamma}^{\mu}_{\alpha\beta}/\sqrt{-g}$$

Which is the same result as above.

Similarly, one can define the Weyl ordering $\left(\widehat{\Sigma}^{\mu}_{\alpha\beta\lambda}\widehat{x}^{\alpha}\widehat{x}^{\beta}\widehat{x}^{\lambda}\right)_w$ as:

$$\left(\widehat{\Sigma}^{\mu}_{\alpha\beta\lambda}\widehat{x}^{\alpha}\widehat{x}^{\beta}\widehat{x}^{\lambda}\right)_w = (\overleftrightarrow{\widehat{\Sigma}^{\mu}_{\alpha\beta\lambda}\widehat{x}^{\alpha}}\overleftrightarrow{\widehat{x}^{\beta}\widehat{x}^{\lambda}} + \overleftrightarrow{\widehat{\Sigma}^{\mu}_{\alpha\beta\lambda}\widehat{x}^{\beta}}\overleftrightarrow{\widehat{x}^{\alpha}\widehat{x}^{\lambda}} + \overleftrightarrow{\widehat{\Sigma}^{\mu}_{\alpha\beta\lambda}\widehat{x}^{\lambda}}\overleftrightarrow{\widehat{x}^{\alpha}\widehat{x}^{\beta}})/\sqrt{-g} \tag{22}$$

and straightforward simplifications by using the fact that $\widehat{\Sigma}^{\mu}_{\alpha\beta\lambda}$ is symmetric with respect to $\alpha\beta$ give:

$$\left(\widehat{\Sigma}^{\mu}_{\alpha\beta\lambda}\widehat{x}^{\alpha}\widehat{x}^{\beta}\widehat{x}^{\lambda}\right)_w = -2i\theta^{\beta\lambda}i\theta^{\alpha\sigma}\partial_\sigma\widehat{\Sigma}^{\mu}_{\alpha\beta\lambda}/\sqrt{-g} \tag{23}$$

and one can deduce that:

$$\left(\widehat{\Gamma}^{\mu}_{\rho\lambda}\widehat{\Gamma}^{\rho}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\widehat{x}^{\lambda}\right)_w = -2i\theta^{\beta\lambda}i\theta^{\alpha\sigma}\partial_\sigma\left(\widehat{\Gamma}^{\mu}_{\rho\lambda}\widehat{\Gamma}^{\rho}_{\alpha\beta}\right)/\sqrt{-g} \tag{24}$$

and the noncommuting variable $\widehat{X}^{\mu}$ can be rewritten as:

$$\widehat{X}^{\mu} = \widehat{x}^{\mu} + \left(\widehat{\Gamma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\right)_w - \frac{1}{2}\left(\widehat{\Gamma}^{\mu}_{\rho\lambda}\widehat{\Gamma}^{\rho}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\widehat{x}^{\lambda}\right)_w \tag{25}$$

where after the corresponding expressions substitutions reads:

$$\widehat{X}^{\mu} = \widehat{x}^{\mu} + \frac{1}{2}i\theta^{\beta\lambda}i\theta^{\alpha\sigma}\partial_\sigma\widehat{R}^{\mu}{}_{\alpha\beta\lambda}(x)/\sqrt{-g} \tag{26}$$

Here $\widehat{R}^{\mu}{}_{\alpha\beta\lambda}$ stands for the Riemann curvature tensor defined as:

$$\widehat{R}^{\mu}{}_{\alpha\beta\lambda}(x) = \partial_{\beta}\widehat{\Gamma}^{\mu}_{\alpha\lambda} - \partial_{\lambda}\widehat{\Gamma}^{\mu}_{\alpha\beta} + \widehat{\Gamma}^{\mu}_{\rho\beta}\widehat{\Gamma}^{\rho}_{\alpha\lambda} - \widehat{\Gamma}^{\mu}_{\rho\lambda}\widehat{\Gamma}^{\rho}_{\alpha\beta} \tag{27}$$

Using the C-B-H formula, one can write:

$$e^{ik\widehat{X}}e^{ip\widehat{X}} = e^{ik\widehat{X}+ip\widehat{X}+\frac{1}{2}\left[ik\widehat{X}\ ,\ ip\widehat{X}\right]+\dots} = e^{ik\widehat{x}+ip\widehat{x}+ik_{\mu}\Delta\widehat{x}^{\mu}+ip_{\nu}\Delta\widehat{x}^{\nu}-\frac{i}{2}\theta^{\mu\nu}k_{\mu}p_{\nu}\dots} \tag{28}$$

with

$$\Delta\widehat{x}^{\mu} = \frac{1}{2}i\theta^{\beta\lambda}i\theta^{\alpha\sigma}\partial_{\sigma}\widehat{R}^{\mu}{}_{\alpha\beta\lambda}/\sqrt{-g} \tag{29}$$

Thus, The Moyal-Weyl $\circledast$-product reads:

$$W(f)\,W(g) = (2\pi)^{-\frac{3}{2}}(2\pi)^{-\frac{3}{2}}\int d^4k d^4p e^{ik\widehat{x}+ip\widehat{x}+ik_{\mu}\Delta\widehat{x}^{\mu}+ip_{\nu}\Delta\widehat{x}^{\nu}-\frac{i}{2}\theta^{\mu\nu}k_{\mu}p_{\nu}}\widetilde{f}(k)\,\widetilde{g}(p) = W(f\circledast g) \tag{30}$$

where

$$(f\circledast g)(x) = e^{\Delta x^{\mu}\frac{\partial}{\partial x^{\mu}}+\Delta x^{\nu}\frac{\partial}{\partial y^{\nu}}+\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial y^{\nu}}}\left[f(x)\,g(y)\right]_{x=y} \tag{31}$$

and to the second order in θ^2 , one obtains:

$$\begin{aligned}(f\circledast g)(x) = {} & f(x)\,g(x) + \Delta x^{\mu}\partial_{\mu}\left(f(x)\,g(x)\right) + \frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^{\mu}}f(x)\frac{\partial}{\partial x^{\nu}}g(x) \\ & + \frac{i}{2}\theta^{\mu\nu}\frac{i}{2}\theta^{\alpha\beta}\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial x^{\alpha}}f(x)\frac{\partial}{\partial x^{\nu}}\frac{\partial}{\partial x^{\beta}}g(x) + \dots\end{aligned} \tag{32}$$

which can be rewritten as

$$\begin{aligned}(f\circledast g)(x) = {} & f(\overline{x})\,g(\overline{x}) + \frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^{\mu}}f(\overline{x})\frac{\partial}{\partial x^{\nu}}g(\overline{x}) + \\ & \frac{i}{2}\theta^{\mu\nu}\frac{i}{2}\theta^{\alpha\beta}\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial x^{\alpha}}f(\overline{x})\frac{\partial}{\partial x^{\nu}}\frac{\partial}{\partial x^{\beta}}g(\overline{x}) + ..\end{aligned} \tag{33}$$

where

$$\overline{x}^{\mu} = x^{\mu} + \Delta x^{\mu} = x^{\mu} + \frac{1}{2}i\theta^{\beta\lambda}i\theta^{\alpha\sigma}\partial_{\sigma}R^{\mu}{}_{\alpha\beta\lambda}/\sqrt{-g}. \tag{34}$$

Notice that one can add to the expression of $\widehat{X}^{\mu}$ the term $\widehat{\Gamma}^{\mu}_{\alpha\beta}\widehat{\Gamma}^{\rho}_{\rho\lambda}\widehat{x}^{\alpha}\widehat{x}^{\beta}\widehat{x}^{\lambda}$. However, if we require that the Weyl ordering of a product of non coupled terms like $\widehat{\Sigma}^{\mu}_{\alpha\beta\lambda\dots\varrho}\widehat{x}^{\alpha}\widehat{x}^{\beta}\widehat{x}^{\lambda}\dots\widehat{x}^{\varrho}$ and $\widehat{\Lambda}^{\nu}_{\pi\sigma\tau\dots\kappa}\widehat{x}^{\pi}\widehat{x}^{\sigma}\widehat{x}^{\tau}\dots\widehat{x}^{\kappa}$ factorizes i.e.:

$$W\left(\widehat{\Sigma}^{\mu}_{\alpha\beta\lambda\dots\varrho}\widehat{x}^{\alpha}\widehat{x}^{\beta}\widehat{x}^{\lambda}\dots\widehat{x}^{\varrho}\widehat{\Lambda}^{\nu}_{\pi\sigma\tau\dots\kappa}\widehat{x}^{\pi}\widehat{x}^{\sigma}\widehat{x}^{\tau}\dots\widehat{x}^{\kappa}\right) = W\left(\widehat{\Sigma}^{\mu}_{\alpha\beta\lambda\dots\varrho}\widehat{x}^{\alpha}\widehat{x}^{\beta}\widehat{x}^{\lambda}\dots\widehat{x}^{\varrho}\right)W\left(\widehat{\Lambda}^{\nu}_{\pi\sigma\tau\dots\kappa}\widehat{x}^{\pi}\widehat{x}^{\sigma}\widehat{x}^{\tau}\dots\widehat{x}^{\kappa}\right) \tag{35}$$

Then, this term does not contribute since

$$\left(\widehat{\Gamma}^{\rho}_{\rho\lambda}\widehat{x}^{\lambda}\right)_w = \overleftrightarrow{\widehat{\Gamma}^{\rho}_{\rho\lambda}\widehat{x}^{\lambda}}/\sqrt{-g} = -i\theta^{\lambda\sigma}\partial_\sigma\widehat{\Gamma}^{\rho}_{\rho\lambda}/\sqrt{-g} \quad (36)$$

Now, using the fact that

$$\Gamma^{\rho}_{\rho\lambda} = \partial_\lambda \log(g) \quad (37)$$

one deduce that:

$$\left(\widehat{\Gamma}^{\rho}_{\rho\lambda}\widehat{x}^{\lambda}\right)_w = -i\theta^{\lambda\sigma}\partial_\sigma\widehat{\Gamma}^{\rho}_{\rho\lambda}/\sqrt{-g} = -i\theta^{\lambda\sigma}\partial_\sigma\partial_\lambda \log(g)/\sqrt{-g} = 0 \quad (38)$$

2.3 *The Noncommutative Action*

Let us calculate the mass term $\Phi^+ \circledast \Phi$ where Φ is a complex scalar field in this new noncommuting space-time:

$$\left(\Phi^+ \circledast \Phi\right)(x) = \Phi^+(\overline{x})\,\Phi(\overline{x}) + \text{total derivative} \quad (39)$$

Eq.(39) can be rewritten as:

$$\left(\Phi^+ \circledast \Phi\right)(x) = \Phi^+(x+\Delta x)\,\Phi(x+\Delta x) = \Phi^+(x)\,\Phi(x) + \Delta x^\mu \partial_\mu\left[\Phi^+(x)\,\Phi(x)\right] + \text{total derivative} \quad (40)$$

which after direct simplifications, the action reads:

$$I = \int d^4x\left(\Phi^+ \circledast \Phi\right)(x) = \int d^4x \Phi^+(x)\,\Phi(x) + \frac{1}{2}\int d^4x \theta^{\beta\lambda}\theta^{\alpha\sigma}R^{\mu}_{\alpha\beta\lambda}\partial_\sigma\partial_\mu\left[\Phi^+(x)\,\Phi(x)\right]\sqrt{-g} \quad (41)$$

2.4 $2-d$ *Gravity coupled to a scalar field*

Let us calculate this action in two dimensions, choosing $\theta^{01} = +\eta$, with $\eta \ll 1$, The gravitational term reads

$$\theta^{\beta\lambda}\theta^{\alpha\sigma}R^{\mu}_{\alpha\beta\lambda}\partial_\sigma\partial_\mu\left[\Phi^+(x)\,\Phi(x)\right]/\sqrt{-g} = \theta^{\beta\lambda}\theta^{\alpha\sigma}R_{\mu\alpha\beta\lambda}\partial^\mu\partial_\sigma\left[\Phi^+(x)\,\Phi(x)\right]/\sqrt{-g} \quad (42)$$

which can be simplified as:

$$\theta^{\beta\lambda}\theta^{\alpha\sigma}R^{\mu}{}_{\alpha\beta\lambda}\partial_\sigma\partial_\mu\left[\Phi^+(x)\,\Phi(x)\right]/\sqrt{-g} = -2\eta^2 R_{0101}g^{\mu\nu}\partial_\mu\partial_\nu\left[\Phi^+(x)\,\Phi(x)\right]/\sqrt{-g} \quad (43)$$

In two dimensions, the scalar curvature:

$$R = g^{\mu\nu} R_{\mu\nu} \tag{44}$$

is related to the component R_{0101} by the relation :

$$R_{0101} = \frac{1}{2} g R \tag{45}$$

Straightforward simplifications lead to:

$$R_{00} = g^{11} R_{0101} \tag{46}$$

$$R_{11} = g^{00} R_{0101} \tag{47}$$

and

$$R_{01} = R_{10} = -g^{01} R_{0101}$$

and consequently, the scalar curvature R reads

$$R = 2g^{-1} R_{0101} \tag{48}$$

Finally the action gets the form:

$$I = \int d^2x \Phi^+ (x) \Phi (x) - \frac{1}{2}\eta^2 \int d^2x \sqrt{-g} R g^{\mu\nu} \partial_\mu \partial_\nu \left[\Phi^+ (x) \Phi (x)\right] \tag{49}$$

3. Conclusion

Thought this work we have considered a noncommutative curved space-time in a gravitational background and define new Moyal-Weyl star product and Weyl ordering at the order of θ^2 (θ is the order parameter of the noncommutative of the space time)where the geometric structure is included. As an example, we have considered the mass term of a complex field and show explicitly the gravitation effect on the noncommutative-space time. (More studies are under investigations).

Appendix

we have

$$\left(\widehat{\Sigma}^{\mu}_{\alpha\beta} \widehat{x}^{\alpha} \widehat{x}^{\beta}\right)_w = (\left(\widehat{\Sigma}^{\mu}_{\alpha\beta} \widehat{x}^{\alpha} - \widehat{x}^{\alpha} \widehat{\Sigma}^{\mu}_{\alpha\beta}\right) \widehat{x}^{\beta} - \widehat{x}^{\alpha} \left(\widehat{\Sigma}^{\mu}_{\alpha\beta} \widehat{x}^{\beta} - \widehat{x}^{\beta} \widehat{\Sigma}^{\mu}_{\alpha\beta}\right))/\sqrt{-g} \tag{A-1}$$

which can be rewritten as:

$$\left(\widehat{\Sigma}^{\mu}_{\alpha\beta} \widehat{x}^{\alpha} \widehat{x}^{\beta}\right)_w == (\widehat{x}^{\alpha} \left[\widehat{x}^{\beta}, \widehat{\Sigma}^{\mu}_{\alpha\beta}\right] - \left[\widehat{x}^{\alpha}, \widehat{\Sigma}^{\mu}_{\alpha\beta}\right] \widehat{x}^{\beta})/\sqrt{-g} \tag{A-2}$$

using the relations eq.(16) we obtain:

$$\left(\widehat{\Sigma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\right)_{w} = i\theta^{\beta\lambda}\left(\widehat{x}^{\alpha}\partial_{\lambda}\widehat{\Sigma}^{\mu}_{\alpha\beta} - \partial_{\lambda}\widehat{\Sigma}^{\mu}_{\beta\alpha}\widehat{x}^{\alpha}\right)/\sqrt{-g} = i\theta^{\beta\lambda}\left(\widehat{x}^{\alpha}\partial_{\lambda}\widehat{\Sigma}^{\mu}_{\alpha\beta} - \partial_{\lambda}\widehat{\Sigma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\right)/\sqrt{-g} \tag{A-3}$$

thus,

$$\left(\widehat{\Sigma}^{\mu}_{\alpha\beta}\widehat{x}^{\alpha}\widehat{x}^{\beta}\right)_{w} = i\theta^{\beta\lambda}\left[\widehat{x}^{\alpha}, \partial_{\lambda}\widehat{\Sigma}^{\mu}_{\alpha\beta}\right]/\sqrt{-g} = i\theta^{\beta\lambda}i\theta^{\alpha\sigma}\partial_{\sigma}\partial_{\lambda}\widehat{\Sigma}^{\mu}_{\alpha\beta}/\sqrt{-g} \tag{A-4}$$

Acknowledgement

One of us (N. Mebarki) is very grateful to Profs. P. Aurenche and F. Boudjemaa for the hospitality during the visit to LAPTH d'Annecy.This work is supported by the research grant N°D2005/13/04

References

[1] J.Madore, S.Schraml,P.Schupp and J.Wess, Eur.Phys.Jour.C16(2000)161

[2] M.R.Douglas and N.A.Nekrasov,Rev.Mod.Phys.73,(2001)977.

[3] J.K.Glikman and S.Nowak, Int.Journ.Mod.Phys.A12,(2003)299

[4] X.Calmet,B.Jurco,J.Wess and M.Wohlgenannt, Eur.Phys.Jour.C23,(2002)363.

[5] X.Calmet,and M.Wohlgenannt,Phys.Rev.D68(2003)025016.

[6] J.Madore, An introduction to Noncommutetive geometry and its physical applications, Cambridge University Press1999.

[7] F.Ardalan, H.Arafi and M.M.Sheikh-Jabbari JHEP(1999)9902.

[8] N.Seiberg and E.Witten, JHEP(1999)032.

[9] M.Chaichan, A.Demitchev, P.Presnajder, M.M.Sheick-jabbari and A.Tureanu, Nucl.Phys.B611(2001)383

[10] M.M.Sheick-Jabbari,JHEP(1999)015.

[11] J.E.Moyal, Proc.Cambridge Phil. Soc.45,(1949)99.

EJTP **3, No. 12** (2006) 47–53 *Electronic Journal of Theoretical Physics*

Light Scattering Studies on the Orientational Behavior of Macromolecular Solutions in a Shear Flow

J.A. Kupriyanova* and A.V. Zatovsky

Department of Theoretical Physics, Odessa National I.I. Mechnikov University, 2 Dvoryanskaya str., 65026 Odessa, Ukraine

Received 19 April 2006, Accepted 20 July 2006, Published 20 September 2006

Abstract: Theoretical investigation of Rayleigh light scattering by a suspension of anisotropic ellipsoidal particles subjected to a shear flow is carried out. Some properties of the suspension of such particles caused by Brownian rotation of these particles are studied. It is shown that the action of a shear flow induces deformations in the shape of scattering line and results into the non-monotonic frequency dependence of depolarized scattering spectral lines with additional local maxima in the spectra.

This work is being dedicated to the memory of Alexandr V. Zatovsky who passed away on August 18, 2006. His field of interest was on the Dynamic Properties of Complex Fluids and Disperse Systems.

Keywords: Shear flow, correlation functions, Brownian motion, light scattering.
PACS (2006): 61.25.Hq; 47.15Gf; 05.40+j, 47.15.gp

1. Introduction

The intensity, polarization and spectral distribution of the scattered light convey significant information about processes that occur in the liquid. Further information on the processes can be obtained by studying the liquid which is subjected to different fields controllable under laboratory conditions. Such fields cause splittings, shifts and spectrum line deformations to appear. The applying of external fields changes the feature of Brownian motion and provides control over the changes in experimental spectra. We investigate theoretically Rayleigh light scattering by a suspension of anisotropic ellipsoidal particles placed into a laminar flow of constant velocity gradient. Such systems can be

* j_kupriyanova@rambler.ru

used to model macromolecular solutions, solutions of anisotropic bacteria or colloidal particles. There are various experimental data showing the shear flow influence on the dynamic characteristics of liquids or the admixtures in liquids [1-3]. The wing of the Rayleigh scattering line by dilute solutions of anisotropic particles can be ascribed to rotational motion of the particles. The changes in scattering spectra are caused by the orientation of the particles in a shear flow. In order to describe the spectroscopy study results, the knowledge of the time correlation functions (CF) for the spherical harmonics of the Euler angles, determining the orientation of the molecule's own reference frame with respect to the laboratory one, is necessary:

$$\Psi^{(2)}_{MK}(t) = \left\langle D^{(2)}_{MK}(t) D^{(2)^*}_{MK}(0) \right\rangle \tag{1}$$

If the shape of a large macromolecule is approximated, for example, with an ellipsoid, then in the case of equilibrium liquids, these correlations functions are known from a well-developed theory of rotational Brownian motion [4]. But in the case of even simplest Couette flows in liquids, the study CF (1) becomes a very difficult task. Macromolecules in a shear flow are in rotation motion, which leads to significant changes in the scattering spectra [1-3].

2. Analysis

Let the shape of an impurity particle is represented by an ellipsoid. Any change in the orientation of such a particle is only due to its rotation, i. e. its angular velocity. The latter can be represented as a sum of two terms. One is the regular contribution caused by the orienting action of the heterogeneous shear flow, and the other is the random contribution caused by the disorientation Brownian motion of the particle. We restrict ourselves to the case of a flow with a single velocity component $V_x = \Gamma y$. The regular component of the ellipsoid's angular velocity in the ellipsoid's own reference frame then has the Cartesian components [5, 6]

$$\vec{\Omega}^0 = \frac{1}{2}\Gamma \left(\lambda n_1 n_2,\ -\lambda n_2 n_3,\ 1 + \lambda \left(n_2^2 - n_1^2 \right) \right) \ , \quad \lambda = \left(b_{||}^2 - b_{\perp}^2 \right) / \left(b_{||}^2 + b_{\perp}^2 \right), \tag{2}$$

where (n_1, n_2, n_3) - are the unit vectors in this frame along the principal axes of inertia of the ellipsoid, and the parameter λ characterizes the anisotropy of the particle with semi-axes $b_{||}, b_{\perp}$. Using the complex spherical components of the angular velocity vector, we can represent its components through the Vigner functions of the Euler angles:

$$\Omega^0_\mu = \frac{\Gamma}{2}\delta_{\mu 0} + \sum_{\alpha=-2}^{2} a_{\mu\alpha} D^{(2)}_{\alpha 0},\ \mu = -1,\ 0,\ 1\ . \tag{3}$$

Here, the matrix is

$$a_{\mu\alpha} = -\frac{\lambda\Gamma}{2\sqrt{3}} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} .$$

As an external field that determines the heterogeneous conditions in which the impurity macromolecules move, we can choose a constant electric field. In this case, the nonpolar macromolecules are caused by the field to gain an angular momentum, and the regular part of the angular velocity has the components [5]

$$\vec{\Omega} = \left(-\sigma E^2 n_2 n_3,\ \sigma E^2 n_1 n_3\,,\ 0\right).$$

The coefficient σ bears the information of the geometrical parameters of the macromolecules, the polarizability anisotropy, and the properties of the surroundings. A detailed study of the orientation motion of the ellipsoid with an induced dipole moment in an external electric field was initiated by one of us in [7, 8]. Here we take advantage of those results to analyze the orientation motion of the impurity macromolecules in a Couette flow.

We proceed from Langevin's equations, describing the changes in the Vigner functions for the ellipsoid's orientation variables caused by its rotation with constant angular velocity [9]

$$\frac{d}{dt} D^{(l)}_{MK}(t) = -i \sum_{\lambda,\rho} \left[\Omega^0_\lambda(t) + \Omega^r_\lambda(t)\right] D^{(l)}_{M\rho}(t)\ \langle l\rho\,|I_\lambda|\,lK\rangle\,. \tag{4}$$

Here $\Omega^0_\lambda(t)$ and $\Omega^r_\lambda(t)$ are respectively the components of the regular and random components of the angular velocity in a moving reference frame tied up to the object , and $\langle \ldots |I| \ldots \rangle$ are the matrix elements of the projection of the rotation operator on the coordinate axes of the molecular frame in units of h.

Using the method and approach of works [4, 7] and the properties of the matrix elements of the rotation operator, we can find an integro-differential equation for CF (1) of impurity axisymmetric particles in a liquid with a shear flow:

$$\begin{aligned}\frac{d}{dt}\Psi^{(l)}_{MK}(t) = &-\tfrac{l(l+1)-K^2}{3\tau_\Gamma^2} \int_0^t dt' \Psi^{(2)}_{10}(t-t')\Psi^{(l)}_{MK}(t') - (\delta_{1,M}+\delta_{-1,M})\delta_{2,l}\delta_{0,K}\tfrac{l(l+1)-K^2}{6\tau_\Gamma^2} \\ &\times \left[\int_0^t dt' \Psi^{(l)}_{10}(t-t')\Psi^{(l)}_{1K}(t') - t\Psi^{(l)}_{1K}(t)\Psi^{(l)}_{1K}(0)\right] - \tfrac{1}{\tau_{lK}}\Psi^{(l)}_{MK}(t),\end{aligned} \tag{5}$$

The expressions for the characteristic time τ_Γ of the orientation change in a shear flow and the relaxation time τ_{lK} of the particle's orientation due to the thermal motion of the particle are of the form

$$\frac{1}{\tau_\Gamma} = \frac{|\lambda|\ \Gamma}{2\sqrt{3}}, \quad \frac{1}{\tau_{lK}} = l(l+1)\Theta_1 + K^2(\Theta_3 - \Theta_1). \tag{6}$$

Here, Θ_i are the principal values of the rotational diffusion tensor.

Note that equation (5) is obtained under assumption $\tau_\Omega << \tau_{lK}$, where τ_Ω is the characteristic time of the angular velocity relaxation. This assumption allows us to consider (4) as a stochastic differential equation with quickly changing disturbance $\Omega^r_\lambda(t)$, so that equation (5) for CF (1) is correct in the first order in magnitude in the small parameter τ_Ω/τ_{lK}.

Hereinafter, we limit the study to the spectral characteristics of CF with the indices $l = 2$,$K = 0$ and $M = 0, \pm 1, \pm 2$. The solution of the integro-differential equation is convenient to search with use of the unilateral Fourier transform for CF:

$$\tilde{\Psi}^{(l)}_{MK}(\omega) = \int_0^\infty dt \Psi^{(l)}_{MK}(t) \exp(i\omega t). \tag{7}$$

In this case, for the CF spectra$\tilde{\Psi}^{(2)}_{00}(\omega)$, $\tilde{\Psi}^{(2)}_{10}(\omega)$ $\tilde{\Psi}^{(2)}_{20}(\omega)$ we have the following closed system of equations

$$\frac{\Psi^{(2)}_{10}(0)}{\tau_\Gamma^2} \frac{\partial \tilde{\Psi}^{(2)}_{10}(\omega)}{\partial(i\omega)} + \left(-i\omega + \frac{1}{\tau_{20}}\right) \tilde{\Psi}^{(2)}_{10}(\omega) + \frac{3}{\tau_\Gamma^2} \left[\tilde{\Psi}^{(2)}_{10}(\omega)\right]^2 = \Psi^{(2)}_{10}(0). \tag{8}$$

$$\tilde{\Psi}^{(2)}_{00}(\omega) = \frac{\Psi^{(2)}_{00}(0)}{-i\omega + \frac{1}{\tau_{20}} + \frac{2}{\tau_\Gamma^2} \tilde{\Psi}^{(2)}_{10}(\omega)}, \quad \tilde{\Psi}^{(2)}_{20}(\omega) = \frac{\Psi^{(2)}_{20}(0)}{-i\omega + \frac{1}{\tau_{20}} + \frac{2}{\tau_\Gamma^2} \tilde{\Psi}^{(2)}_{10}(\omega)} \tag{9}$$

The solution of this system reduces to the integration of nonlinear differential first-order equation (8). The latter can be integrated numerically or analytically. Let us introduce the following unitless quantities values

$$x = \left(-i\omega + \frac{1}{\tau_{20}}\right) \frac{\tau_\Gamma}{\sqrt{\Psi^{(2)}_{10}(0)}} = a\ (-i\omega\tau_{20} + 1)\,, \quad z(x) = -\frac{3}{\tau_\Gamma \sqrt{\Psi^{(2)}_{10}(0)}} \tilde{\Psi}^{(2)}_{10}(\omega), \tag{10}$$

where a is a unitless parameter depending of the velocity gradient of the shear flow:

$$a^{-2} = (\,\tau_{20}/\tau_\Gamma)^2\, \Psi^{(2)}_{10}(0). \tag{11}$$

In the variables x and $z(x)$, differential equation (8) is a special case of the Riccati equation

$$\frac{d}{dx} z(x) + z^2(x) = x z(x) + 3 \tag{12}$$

with additional condition

$$z(0) = -\frac{3}{\tau_E \sqrt{\Psi^{(2)}_{10}(0)}} \int_0^\infty dt \Psi^{(2)}_{10}(t) = -\frac{3}{\sqrt{\Psi^{(2)}_{10}(0)}} \frac{\tau_\vartheta}{\tau_E}. \tag{13}$$

Here $\tau_\vartheta(\Gamma \neq 0)$ is the relaxation time for the CF of the orientation variables $\Psi^{(2)}_{10}(t)$. The solution of the Riccati equation (8) is expressed through algebraic functions and a definite integral (which reduces the error integral by integration by parts; we omit the integration result because of its awkwardness)

$$z(x) = x + \frac{2x}{1+x^2} + \frac{1}{(1+x^2)^2 A(x)}\ ,$$

$$A(x) = \int_0^1 du \left[\frac{xe^{\frac{1}{2}x^2(1-u^2)}}{(1+x^2u^2)^2} - \frac{ae^{\frac{1}{2}(x^2-a^2u^2)}}{(1+a^2u^2)^2} \right] + \frac{e^{\frac{1}{2}(x^2-a^2)}}{(1+a^2)^2} \left[z(0) - a - \frac{2a}{1+a^2} \right]^{-1} . \quad (14)$$

It follows from here and from equation (8) that the function $z(\omega)$ is not monotone. It has a maximum at the frequency value ω_e determined from the relation

$$2Rez(\omega_e) = a - 2Re\left(a^2(1 + i\omega_e\tau_{20})^2 + 4\right)^{1/2}. \quad (15)$$

A non-vanishing solution of (15) occurs if the condition $\tau_\vartheta > 2\tau_{20}/(1 + \sqrt{1 + 12a^{-2}})$ is met. Thus, the spectral density of CF for the orientation variables of the ellipsoid in a shear flow is in general of non-Lorenz form and can have a local maximum that strongly depend on the velocity gradient of the flow.

The complete analysis of the obtained solution of (14) is difficult because of its awkwardness. For the sake of convenience, we calculated the CF spectra by numerical integrating of the equation system (8) for the real and imaginary parts in unitless variables $\tilde{\omega} = \omega\tau_{20}$, $\Psi' + i\Psi'' = \tilde{\Psi}_{00}^{(2)}/\tau_{20}$ taking the initial conditions importance's $\Psi' = \tau_\vartheta(\Gamma \neq 0)/\tau_{20}(\Gamma = 0) \approx 1$, $\Psi''(0) = 0$.

The relaxation times for the orientation motion in absence of the flow and in its presence are not identical, but their explicit dependences of remained undetermined. The figure represents the dependence of the real part of the spectra of $\Psi_{00}^{(2)}$ on the unitless frequency for different values of the shear flow gradient squared $G = a^{-2} \propto \Gamma^2$.

At the presence of a shear flow the contour of a wing of a Rayleigh line is essentially not Lorentzian and can have one or several local maxima. Both shape of a spectrum and peak intensity of these additional maxima varies with an alternation of parameter G value. A position of maxima on the frequencies axis in a short-wave portion of the spectrum also is moved when parameter G increases.

3. Conclusions

Our method makes possible to receive spectra of a Rayleigh light scattering by weak solutions of ellipsoidal particles placed in a shear flow. The presence of a shear flow changes the nature of ellipsoids Brownian rotation that has had an effect on the shape and spectral content of scattered lines. In particular, the contour of a Rayleigh scattering line wing is not Lorentzian and has a fine structure with local maxima. A position on frequencies axis and peak intensity of these additional maxima in Rayleigh scattered line depends both on a shear flow value and the particles sizes. This fact enables to estimate the fragments sizes on changes of light scattering spectra if a shear flow presence.

References

[1] Y. Doppke, W. Heller, J. Phys. Chem. 83 (1989) 1717.

[2] A. V. Lomakin, V. A. Noskin, Lett. JETP 28 (1978) 592.

[3] D. Rusu, P. Van Puyvelde, E. Peuvrel-Disdier, Polymer 40 (1999) 1353.

[4] K. A. Valiev, E. N. Ivanov, UFN 109 (1973) 31.

[5] A.Peterlin, H.A. Stuart, Zeitschrift für Physik 112 (1938) 129.

[6] E. Hinch, L. Leal, J. Fluid Mech. 76 (1976) 187.

[7] A. V. Kyrylyuk, A.V. Zatovsky, Ukr. J. Phys. 49 (2004) 570.

[8] A. V. Kyrylyuk, A.V. Zatovsky, J. Mol. Liq. 120 (2005) 55.

[9] A. S. Davydov, Excited States of Atomic Nuclei, Atomizdat, Moscow, 1967.

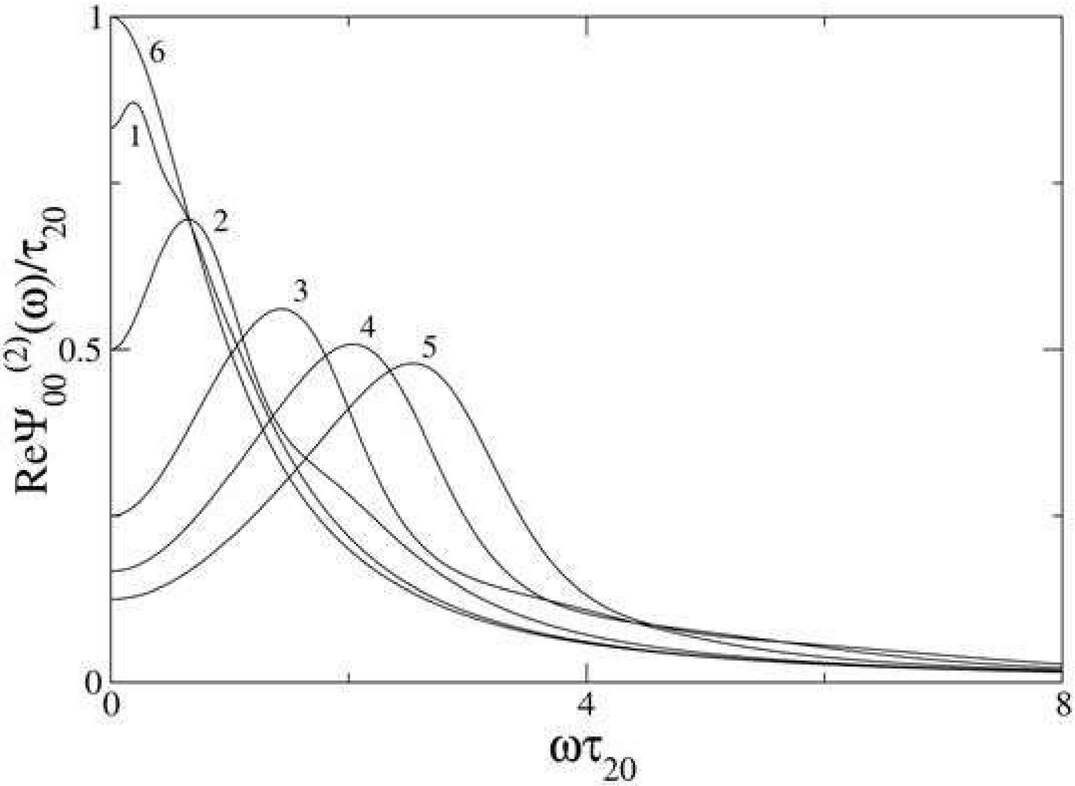

Fig. 1 The spectra of the real part of the correlation function $\Psi_{00}^{(2)}(\omega)/\tau_{20}$ versus the unitless frequency $\tilde{\omega} = \omega\tau_{20}$ for different values of the flow velocity gradient (1 - G=0.1, 2 - 0.5, 3 - 1.5, 4 - 2.5, 5 - 3.5) and in the absence of the flow(6 - G=0).

EJTP **3, No. 12** (2006) 55–58 *Electronic Journal of Theoretical Physics*

Gödel's Geometry: Embedding and Lanczos Spintensor

R. García-Olivo , J. López-Bonilla*, S. Vidal-Beltrán ,
SEPI-ESIME-Zacatenco

Instituto Politécnico Nacional
Edif.Z-4, 3er. Piso, Col. Lindavista, 07738 México DF

Received 21 December 2005, Accepted 2 February 2005, Published 25 June 2006

Abstract: We exhibit an open problem: To investigate if the Gödel's metric accepts local and isometric embedding into E_6. Besides, we show that in this metric there is a symmetric tensor which generates algebraically to Riemann tensor and differentially to Weyl tensor.

Keywords: Embedding of Riemannian spaces; Gödel solution; Lanczos potential; Gauss-Coddazi equations
PACS (2006): 02.40.Ky, 04.20.-q, 04.20.Cv, 04.90.+e,02.10.-v, 02.40.-k

Gödel [1] proposed the geometry [2,3] (signature +2):

$$ds^2 = -\left(dx^1\right)^2 - 2e^{x^4}dx^1dx^2 - \frac{1}{2}e^{2x^4}\left(dx^2\right)^2 + \left(dx^3\right)^2 + \left(dx^4\right)^2, \tag{1}$$

to represent a rotational universe, under the hypothesis that the cosmos is composed by an incoherent perfect fluid. The line element (1) can be embedded locally and isometrically into E_n, $n \geq 5$, if [3-6] there exist a set of functions $z^r(x^j)$, $r = 1, ..., n$ such that (1) adopts the form:

$$ds^2 = \sum_{r=1}^{n} \varepsilon_r \left(dx^r\right)^2,\ \varepsilon_j = \pm 1. \tag{2}$$

It is worth to recall [3] that any R_4 allows embedding into E_{10}, thus we will restrict ourselves to the range $5 \leq n \leq 10$.

The $n = 9$ and $n = 10$ cases were already solved in [7] and [4], respectively; so for

* jlopezb@ipn,mx, lopezbjl@hotmail.com

instance, with the functions:

$$\begin{aligned}
&z^1 = \tfrac{1}{\sqrt{2}} e^{x^4} \cos x^2, && z^2 = \tfrac{1}{\sqrt{2}} e^{x^4} \sin x^2, \\
&z^3 = \sqrt{2} e^{\frac{x^4}{2}} \cos \tfrac{1}{2}\left(x^1 + x^2\right), && z^4 = \sqrt{2} e^{\frac{x^4}{2}} \sin \tfrac{1}{2}\left(x^1 + x^2\right), \\
&z^5 = \sqrt{2} e^{\frac{x^4}{2}} \cos \tfrac{1}{2}\left(x^1 - x^2\right), && z^6 = \sqrt{2} e^{\frac{x^4}{2}} \sin \tfrac{1}{2}\left(x^1 - x^2\right), \\
&z^7 = x^1, && z^8 = x^3, \\
&z^9 = \rho + \tfrac{1}{2} Ln\left(\tfrac{\rho-1}{\rho+1}\right), && \rho = \left(1 + \tfrac{1}{2} e^{2x^4}\right)^{1/2},
\end{aligned} \tag{3}$$

the metric (1) adopts the form (2):

$$ds^2 = -\left(dz^1\right)^2 - \left(dz^2\right)^2 - \left(dz^3\right)^2 - \left(dz^4\right)^2 + \left(dz^5\right)^2 + \left(dz^6\right)^2 - \left(dz^7\right)^2 + \left(dz^8\right)^2 + \left(dz^9\right)^2 . \tag{4}$$

On the other hand, it is known that by employing indirect methods [5], the geometry (1) accepts embedding into E_7, and hence into E_8 too; but in the literature nobody has published the corresponding z^r for such cases, and therefore this turns out an open problem. Besides, the model (1) can not be [7-10] embedded into E_5. At the present time it is ignored yet [5,6] if this Gödel solution accepts local and isometric embedding into E_6.

Is the Gödel space of class two?

One says that a given R_4 has class two if it can be embedded into E_6 [3,6]; our interest is to know if (1) can be represented as a surface of a pseudo-Euclidian 6-space. As a first possibility, one can initiate the quest (if they exist) of the z^r, $r = 1, ..., 6$ which reduce (1) to the structure (2), and in case of success the corresponding explicit embedding will be obtained.

Another option consists in solving the differential equations of Gauss–Codazzi-Ricci (GCR) [3,6] for (1), next trying to construct the second fundamental forms ${}_{(j)}b_{ac}$, $j = 1, 2$ and the Ricci vector A_c, which determine the extrinsic geometry of R_4 with respect to E_6. If somebody were able to show that this GCR system has no solution, then the Gödel metric will not be of class two.

Yakupov [6,11-13] has shown that in every R_4 embedded into E_6, the following algebraic necessary condition has to be fulfilled:

$$Y \equiv {}^{*}R^{*tjkc}\, {}^{*}R_{arkc}\, R^{ar}_{tj} = 0, \tag{5}$$

where ${}^{*}R_{ijkc}$ and ${}^{*}R^{*}_{ijkc}$ are the duals [2,7,9,10,14-16] of the Riemann tensor. This means that if (1) had $Y \neq 0$ then would be impossible its embedding into E_6; but after a long calculation it can be checked the truth of (5) for the Gödel geometry, which do not help to decide if (1) is of class two because (5) is a necessary but not sufficient requisite for the embedding. Hence, it will be convenient to find new necessary algebraic/differential conditions alternatives to (5), rendering more information about the possibility of embedding (1) into E_6.

Lanczos spintensor

Now we shall show that in this geometry there is a symmetric tensor which generates differentially to conformal tensor and algebraically to curvature tensor. In fact, for the Gödel spacetime (1) we have [17, 18] the Lanczos potential [15]:

$$K_{aij} = \frac{1}{9}\left(R_{ji\,;\,a} - R_{ja\,;\,i}\right) \quad , \tag{6}$$

where $R_{ij} = R^c{}_{ijc}$is the Ricci tensor, being R_{acij} the curvature tensor, besides ; r means covariant derivative. On the other hand, the metric tensor satisfies $g_{ac\,;\,r} = 0$, and (1) has the constant vector $(B_r) = (0, 0, 1, 0)$,that is, $B_{r\,;\,c} = 0$. Then it is evident that (6) is equivalent to:

$$K_{aij} = \frac{\sqrt{2}}{18}\left(b_{ji\,;\,a} - b_{ja\,;\,i}\right) \ , \tag{7}$$

with

$$b_{ij} = \sqrt{2}\ \left[R_{ij} + \frac{1}{2}\left(B_i B_j - g_{ij}\right)\right] \ . \tag{8}$$

Thus we have that (8) is a differential generator, via the Lanczos potential, for the Weyl tensor:

$$C_{aijr} = K_{aij\,;r} - K_{air;j} + K_{jra\,;\,i} - K_{jri\,;\,a} + g_{ar}K_{ji} - g_{aj}K_{ri} + g_{ij}K_{ra} - g_{ir}K_{ja} \quad , \tag{9}$$

where $Kij = K^r_{ij;r}$ But b_{ij} also is an algebraical generator for the Riemann tensor because it verifies the Gauss equation [3]:

$$R_{acij} = b_{ai}b_{cj} - b_{aj}b_{ci} \ . \tag{10}$$

We must note that (8) violates the Coddazi relation [3] $b_{ac\,;\,r} - b_{ar\,;\,c} = 0$ because $b_{12\,;\,4} \neq b_{14\,;\,2}$, that is, it is impossible the embedding of (1) into E_5 [5,8].

References

[1] K. Gödel, Rev. Mod. Phys. 21 (1949) 447

[2] J. L. Synge, Relativity: the general theory, North-Holland, Amsterdam (1976)

[3] D. Kramer, H. Stephani, M. MacCallum and E. Herlt, Exact solutions of Einstein's field equations, Cambridge University Press (1980)

[4] J. Rosen, Rev. Mod. Phys. 37 (1965) 204

[5] C. D. Collinson, J. Math. Phys. 9 (1968) 403

[6] H. F. Goenner, General relativity and gravitation, A. Held, Editor, Plenum N.Y. (1980) Vol. 1, Chap. 4

[7] J. López-Bonilla, J. Morales and M. Rosales, Braz. J. Phys. 24 (1994) 522

[8] P. Szekeres, Nuovo Cim. A43 (1966) 1062

[9] R. Fuentes, J. López-Bonilla, T. Matos and G. Ovando, Gen. Rel. Grav. 21 (1989) 777

[10] G. González, J. López-Bonilla, and M. Rosales, Pramana J. Phys. 42 (1994) 85

[11] M. Sh. Yakupov, Sov. Phys. Doklady 13 (1968) 585

[12] H. F. Goenner, Gen. Rel. Grav. 6 (1975) 75

[13] V. Gaftoi, J. López-Bonilla and G. Ovando, Bull. Allahabad Math. Soc. 18 (2003) 35

[14] C. Lanczos, Ann. of Math. 39 (1938) 842

[15] C. Lanczos, Rev. Mod. Phys. 34 (1962) 379

[16] J. López-Bonilla and H. N. Núñez-Yépez, Pramana J. Phys. 46 (1996) 219

[17] J. López-Bonilla and G. Ovando, Gen. Rel. Grav. 31 (1999) 1071

[18] V. Gaftoi, J. López-Bonilla and G. Ovando, Czech. J. Phys. 52 (2002) 811

EJTP **3, No. 12** (2006) 59–65 *Electronic Journal of Theoretical Physics*

Thermopower of The Quantum Point Contacts Under the Effects of Boundary Roughness

Attia A. Awad Alla*

Physics Department, Faculty of Science,
Beni-Suef University, Egypt

Received 17 April 2006, Accepted 16 August 2006, Published 20 September 2006

Abstract: In this paper we, study the influence of scattering by boundary roughness on electron transport through quantum point contact. It is found that the thermo power of rough quantum point contact shows random and rapid fluctuations and strong with variable the Fermi energy and electrochemical potential. The thermoelectric efficiency as function of electrochemical potential and the oscillations are periodic and even in the electrochemical potential. These results agree with existing experiments and can be used as a guideline for the evaluation of the fabrication process of quantum point contact.

Keywords: Quantum Point Contacts, Boundary Roughness, Fluctuations
PACS (2006): 73.23.-b, 73.40.-c 73.63.-b,71.20.-b

1. Introduction

Recent advances in nanostructure technology have made it possible to define quasi-one-dimensional (Q1D) constructions, which lateral dimensional electron gas (2DEG) formed at semiconductor hetrostructure. Mesoscopic system has been shown to exhibit unusual conductance properties, due to quantum effects [1]. This behavior also affects the thermoelectric effect of such systems [2]. Quantized thermo power in quantum point contact [3-5] and universal thermo power fluctuations [6] have been measured. The definition of thermodynamics transport coefficients for mesoscopic systems presents theoretical challenge.

In this paper we study the thermo power effect in mesoscopic systems with the variable Fermi energy and electrochemical potential at different values of channels length and temperature respectively and thermoelectric efficiency of the quantum point contacts

* attiamd2005@yahoo.com

as function of electrochemical potential. The thermo power is sensitive to the energy dependence of the conductance.

2. Theory

We model the roughness by dividing a one-dimensional (1D), channel into a large amount of segments in the direction of transport, the segments being thin enough so that one can approximate the potential to be of transverse dependence only. A sample channel of length, L, and average width, W, with random roughness of amplitude δWon each of the two boundaries. We assume a hard wall potential in the transverse direction. We also connect the 1D channel to two semi-infinite reservoirs by smooth entries, in order to avoid the formation of longitudinal resonant states between the entrance and exit [7]. It is interesting to study the influence of scattering by boundary roughness on electron transport through quantum point contact under the variable of the Fermi energy and electrochemical potential. A current between the reservoirs is related to differences in chemical potential or temperature of the reservoirs. Within the linear-response theory, this is generally formulated as

$$J = \hat{L}\, F \tag{1}$$

Where **J** is general current traversing the system and **F** is a generalized affinity. $\hat{L}$ is the transport coefficient matrix. The corresponding net currents are heat and charge fluxes, denoted by Q and G, respectively, and Eq. (1) is written as

$$\begin{pmatrix} I \\ Q \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} \mu_L/e - \mu_R/e \\ T_L - T_R \end{pmatrix} \tag{2}$$

Where $T_L - T_R = \Delta T$, in order to calculate L_{12}, we use the approximation

$$f_L(\varepsilon, T + \Delta T) - f_R(\varepsilon, T) \approx \left(\frac{-df(\varepsilon - (\mu_F + \alpha\, eV_{sd}))}{d\varepsilon}\right)(\varepsilon - (\mu_F + \alpha\, eV_{sd}))\, \Gamma(\varepsilon)\, \Delta T/T$$

We get

$$L_{12} \equiv \left(\frac{I}{\Delta T}\right) = -\left(\frac{e}{T\,\hbar}\right)\int_{-\infty}^{\infty} d\varepsilon\, (\varepsilon - (\mu_F + \alpha\, eV_{sd}))\, \Gamma(\varepsilon)\left(\frac{df(\varepsilon - (\mu_F + \alpha\, eV_{sd}))}{d\varepsilon}\right) \tag{3}$$

By the same method we calculate L_{21},L_{11}andL_{22}; it is useful to define the following additional quantity, which provides a measure for the efficiency of the thermoelectric effect [4]

$$\Delta \equiv \left(\frac{L_{12}\, L_{21}}{L_{11}\, L_{22}}\right) \tag{4}$$

Where, Δg is a measure of the rate of entropy production in the transport process. The problem of the electron transport through the system is solved by using the scattering–matrix method developed by Xu [8]. The conductance at finite temperature, T, is then given by:-

$$G(\mu_F,T) = \left(\frac{I_{total}}{V}\right) = \left(\frac{2e^2}{h}\right)\int_0^\infty \sum_i \Gamma_i(\varepsilon)\left(-\frac{\partial f(\varepsilon-(\mu_F+\alpha\, eV_{sd})\,,T)}{\partial\varepsilon}\right)d\varepsilon \quad , \tag{5}$$

Where

$f(\varepsilon-(\mu_F+\alpha\, eV_{sd})\ ,T)$ is the Fermi –Dirac distributions function, the transmission probability is given by

$$\Gamma_i(\varepsilon) = \left[1+\frac{V_0^2}{4\varepsilon\ (V_0-\varepsilon)}Sin(K\,L)\right]^{-1} \tag{6}$$

K, is the wave vector

$$K = \left[\frac{2m^*\,(\varepsilon-\alpha\, e\, V_{sd}-E_F)}{\hbar^2}\right]^{\frac{1}{2}} \quad , \tag{7}$$

and

$$\frac{\partial f(\varepsilon-(\mu_F+\alpha\, eV_{sd})\ ,T)}{\partial\varepsilon} = (4k_BT)^{-1}\cosh^{-2}\left[\frac{(\varepsilon-(\mu_F+\alpha\, eV_{sd})-E_F)}{2k_BT}\right] \tag{8}$$

Where T, is the temperature, k_B is the Boltzmann constant, e is the electron charge, $V_{sd}$is source_drain voltage, V_0 is the potential well depth of the semiconductor heterojunction quantum point contact and $\mu_F$is electrochemical potential.

The thermo power, S (μ_F, T), for transport from one equilibrium electron reservoir to another, along a multichannel lead are given by [3, 9]

$$S(\mu_F,T) = \left[\frac{k_B}{e}\right]\ x \left\{\frac{\sum_i\int_0^\infty d\varepsilon\left[-\frac{df(\varepsilon-(\mu_F+\alpha\, eV_{sd}),T)}{d\varepsilon}\right]\Gamma_i(\varepsilon)\left[\frac{(\varepsilon-(\mu_F+\alpha\, eV_{sd})-E_F)}{2k_BT}\right]}{\sum_i\int_0^\infty d\varepsilon\left[-\frac{df(\varepsilon-(\mu_F+\alpha\, eV_{sd}),T)}{d\varepsilon}\right]\Gamma_i(\varepsilon)}\right\} \tag{9}$$

Where, ε is the transverse energy associated with the i^{th} channel in the lead, Γ_i is transmission probability from all channels into channels i, and f is the Fermi–Dirac distributions function.

3. Results and Discussion

In the following, we will present our numerical calculations for the thermo power, S (μ_F, T), (Eq.9) for the case GaAlAs/GaAs heterostructure. Our results are characterized as the follows:-

In Fig. (1) Shows that the variation of the thermo power, S(μ_F,T), with the Fermi energy, E_F, with different values of channels length, L. The thermo power is oscillations with the variable Fermi energy, and the fluctuations is random and strong, the values of thermo power at channel length, L = 500 nm is smaller than the values of thermo power at the channels length, L = 100 nm. In Fig. (2) Shows that the variation of the thermo power, S (μ_F, T), with the electrochemical potential μ_F, with different values of the temperatures, T. The thermo power is oscillations with the variable electrochemical potential, and the fluctuations is random and strong, the values of the thermo power at temperature, T= 0.5 K is higher than the values of the thermo power at the temperature, T =0.1 K. In Fig (3) Show that the thermoelectric efficiency of the quantum point contacts as function of electrochemical potential and the oscillations are periodic and even in the electrochemical potential. This result has been confirmed by us [4, 10-12] previously and by many authors [4, 13-14]. The cases in the presence roughness show the strong the thermo power suppression by the roughness.

4. Conclusion

We have studied the generic effects of boundary roughness on the thermopower, S(μ_F,T), of the quantum point contacts. We have shown that in the presence of the boundary roughness, the thermopower of long quantum point contact shows random and rapid fluctuations and strong. These effects are more pronounced on the increasing the Fermi energy, and electrochemical potential, the oscillations is caused by the rough boundaries and oscillations are due to the Coulomb blockade effect and the quantum interference of quasiparticles due to Andreev reflections processes interface. These results agree with existing experiments and can be used as a guideline for the evaluation of the fabrication process of quantum point contact.

Acknowledgment

I would like to express my deep thanks and gratitude to Prof. Dr. Adel H. Phillips for his encouragements and support facilities during preparation of this work.

References

[1] C. W. J. Beenakker and H. van Houten, in Solid State Physics: Advanced in Research and Application, edited by H. Ehrenreich and D. Turnbull (Academic, New York, 1991).

[2] Mesoscopic Phenomena in Solids, edited by B. Altshuler, P. Lee, and R. Webb (North-Holland, Amsterdam, 1991).

[3] C. R. Proetto, Phys. Rev. B 44, (1991) 9096-9099.

[4] B. E. Kane, G. R. Facer, A. S. Dzurak, N. E. Lumpkin, R. G. Clark, L. N. Pfeiffer, and K. W. West, Appl. Phys. Lett. 72, (1998) 3506-3508.

[5] Glen D. Guttman, Eshel Ben-Jacob, and David J. Bergen, Phys. Rev. B 51, (1995) 17758-17766.

[6] K. Nikolic and A. Mackinnon, Phys. Rev B 50, (1992) 11008–11017.

[7] H. Q. Xu, Phys. Rev. B 50, (1994)8469-8478.

[8] D. Csontos and H. Q. Xu, Appl. Phys. Lett. 77, (2000)2364-2366.

[9] T. Ando and H. Tamura, Phys. Rev. B 46, (1992)2332-2338.

[10] H. Q. Xu, Phys. Rev. B 47, (1993)15630-15637.

[11] Attia A. Awad Alla, Applied Science, 8, (2006)23-30.

[12] Arafa H. Aly and Adel H. Phillips, Applied Science, 7, (2005)10-15.

[13] Attia A. Awad Alla and Adel H. Phillips, Egypt. J. Sol. 36, (2004)129-138.

[14] P. Ramvall, N. Carlsson, I. Maximov, P. Omling, L. Samuelson, W. Seifert, Q. Wang and S. Lourdudoss, Appl. Phys. Lett. 71, (1997)918-920.

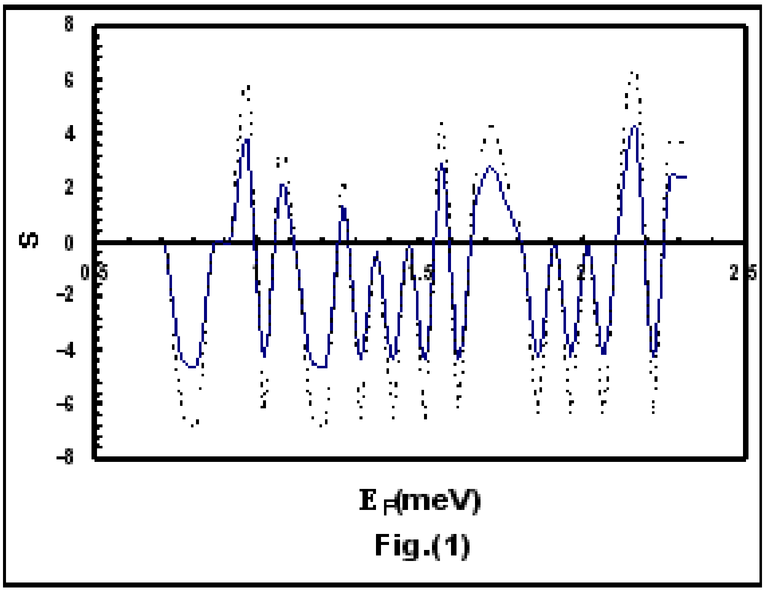

Fig. 1 The thermopower –Fermi energy dependence for different values of channels length L (Solid line L=500 nm, dashed line L=100 nm, $\alpha = 0.5, m^* = 0.047m_e$, W=100 nm).

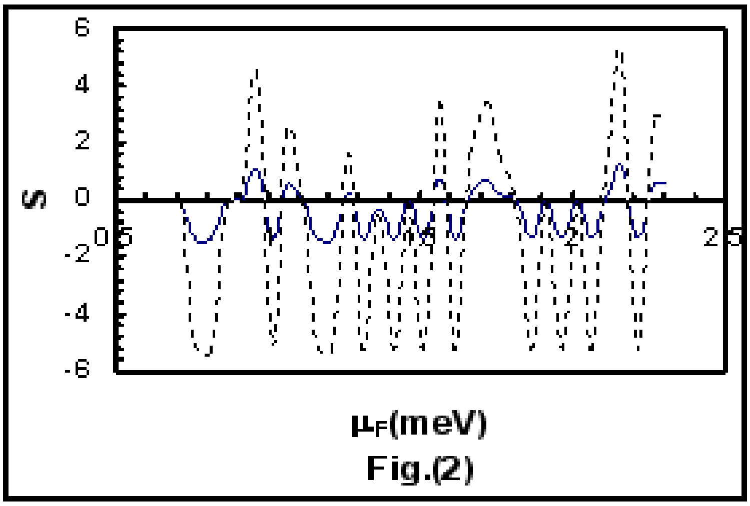

Fig. 2 The thermopower – electrochemical potential dependence for different values of temperatures (Solid line T= 0.1 K, dashed line T= 0.5 K, $\alpha = 0.5$, $m^* = 0.047m_e$, W=100 nm).

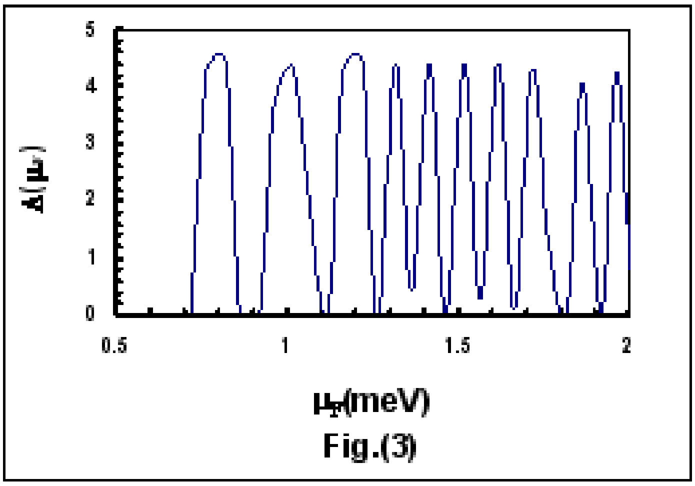

Fig. 3 The thermoelectric efficiency of the quantum point contact as function of electrochemical potential ($\alpha = 0.5$, $m^* = 0.047m_e$, W = 100 nm).

EJTP **3, No. 12** (2006) 67–70 *Electronic Journal of Theoretical Physics*

Matrix Theory and the Modified Space-Time Uncertainty

Abbas Farmany*

Department of Physics Islamic Azad University of Ilam Iran
Young Researcher Club, Azad University of Ilam Iran

Received 18 Jane 2006, Accepted 2 April 2006, Published 20 September 2006

Abstract: We consider the modified space-time uncertainty in the matrix theory point of view. First, we find a suitable theorem for the modified space-time uncertainty. Furthermore, this theorem is proved in the matrix theory compactifications

Keywords: Space-time Uncertainty, Matrix Theory
PACS (2006): 03.65.-w, 03.65.Ca, 11.25.Hf, 11.25.Mj

There has been a great deal of interest in the idea that the matrix theory may be an effective tool in physical theories. This tool plays an important role in the high-energy physics and quantum field theory. In this paper, we consider the modified space-time uncertainty in the matrix theory compactifications. In the usual space-time uncertainty, a small distance should detect by going to high and higher momenta according to $\Delta x \geq \hbar/\Delta p$. In string theory regime, Witten [1] has shown that for detection of small distances by going to very high momenta, the behavior of "*Heisenberg microscope*" changes to,

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha' \frac{\Delta p}{\hbar}, \tag{1}$$

Relation (1) is a standard result of string theory and scale inversion symmetry,

$$\frac{\Delta p\sqrt{\alpha'}}{\hbar} \leftrightarrow \frac{\hbar}{\Delta p\sqrt{\alpha'}} \tag{2}$$

which relates large and small distances,

$$\frac{\alpha'}{R} \leftrightarrow R \tag{3}$$

* a.farmany@usa.com

Relation (3) named T-duality. This symmetry relates large and small distance. Numerous calculation of the space-time uncertainty have been done from different point of view and all hint at the modified space-time uncertainty as the relation (1) [1,3,4,5,6]. An alternative solution of the modified space-time uncertainty is in the matrix theory.

Let begin with Schwarz inequality, for any $|\delta >$, and $|\gamma >$ we can write,

$$|\gamma >= (\hat{A}- < A_\alpha >)|\alpha > \tag{4}$$

$$|\delta >= (\hat{B}- < B_\alpha >)|\alpha > \tag{5}$$

In this framework the Schwarz inequality written as,

$$< \gamma|\gamma >< \delta|\delta >\geq | < \gamma|\delta > |^2 \tag{6}$$

In accordance with relation (4), (5) and (6) we have,

$$< \gamma|\gamma >= | < \alpha|(\hat{A}- < A_\alpha >)(\hat{A}- < A_\alpha >)|\alpha >= | < \alpha|(\hat{A}- < A_\alpha >)^2|\alpha > | = (\Delta A_\alpha)^2 \tag{7}$$

and,

$$< \delta|\delta >= | < \alpha|(\hat{B}- < B_\alpha >)(\hat{B}- < B_\alpha >)|\alpha >= | < \alpha|(\hat{B}- < B_\alpha >)^2|\alpha > | = (\Delta B_\alpha)^2 \tag{8}$$

From Schwarz inequality and relation (7), (8) and (6) we conclude

$$< \gamma|\gamma >< \delta|\delta >= (\Delta A_\alpha)^2(\Delta B_\alpha)^2 \geq | < \alpha|\hat{T}|\alpha > |^2 + | < \alpha|\hat{T}'|\alpha > |^2 \tag{9}$$

Where $\hat{T}, \hat{T}'$ are non-Hermitian operators [1] and,

$$\begin{aligned} \hat{T} &= (\hat{A}- < A >_\alpha)(\hat{B}- < B >_\alpha), \\ \hat{T}' &= (\hat{B}- < B >_\alpha)(\hat{A}- < A >_\alpha) \end{aligned} \tag{10}$$

Here $\hat{T}, \hat{T}'$ must be the Hermitian operators. Hence, we can analyze eqs (10) as,

$$\hat{T} = \hat{D} + \frac{1}{2}i\hat{C}, \hat{T}' = \hat{F} + \frac{1}{2}i\hat{K}. \tag{11}$$

Where $\hat{C}, \hat{D}, \hat{F}, \hat{K}$ are the Hermitian operators.

Theorem: let A be coordinate and B, be the canonical momenta. The modified uncertainty relation in matrix theory defined by,

$$(\Delta A_\alpha)(\Delta B_\alpha) \geq \frac{1}{2}(< C >^2_\alpha + < K >^2_\alpha)^{1/2}. \tag{12}$$

There is a non-commutativity relation between the coordinate and momenta, hence we can write,

$$(\hat{A}- < A >_\alpha)(\hat{B}- < B >_\alpha) \neq (\hat{B}- < B >_\alpha)(\hat{A}- < A >_\alpha) \tag{13}$$

From eqs (13) we have,

$$\hat{C} = \frac{(\hat{A}- < A >_\alpha)(\hat{B}- < B >_\alpha)}{i} - \frac{(\hat{B}- < B >_\alpha)(\hat{A}- < A >_\alpha)}{i} \tag{14}$$

$$\hat{D} = \frac{(\hat{A}- < A >_\alpha)(\hat{B}- < B >_\alpha)}{2} + \frac{(\hat{B}- < B >_\alpha)(\hat{A}- < A >_\alpha)}{2} \tag{15}$$

$$\hat{K} = \frac{(\hat{B}- < B >_\alpha)(\hat{A}- < A >_\alpha)}{i} - \frac{(\hat{A}- < A >_\alpha)(\hat{B}- < B >_\alpha)}{i} \tag{16}$$

$$\hat{F} = \frac{(\hat{B}- < B >_\alpha)(\hat{A}- < A >_\alpha)}{2} + \frac{(\hat{A}- < A >_\alpha)(\hat{B}- < B >_\alpha)}{2} \tag{17}$$

we can write,

$$(\Delta A_\alpha)^2(\Delta B_\alpha)^2 \geq \mid < T >_\alpha \mid^2 + \mid < T' >_\alpha \mid^2 = \frac{1}{4} < C >_\alpha^2 + \frac{1}{4} < K >_\alpha^2 \tag{18}$$

Consequently, from eqs. (18) we conclude that,

$$(\Delta A_\alpha)(\Delta B_\alpha) \geq \frac{1}{2}(< C >_\alpha^2 + < K >_\alpha^2)^{1/2} \quad . \tag{19}$$

Eqs (19) is the genuine modified space-time uncertainty principle in the matrix theory compactifications.

Conclusion

We incorporated the modified space-time uncertainly in the matrix theory by a theorem, and furthermore, provident of this theorem considered.

References

[1] E. Witten Phys Today April (1996) P 24-26

[2] M. B. Green, J. H. Schwarz, and E. Witten, “Superstring theory”, Cambridge University Press (1987)

[3] J. Schwarz, speaking in Stanford linear Accelerator (1996)

[4] D. Amati, M. Ciafaloni, G. Veneziano Phys Lett B 197, 81 (1987)

[5] M. Maggiore, Phys. Rev. D 49, 5182 (1994)

[6] T. Yoneya, Class. Quant. Gravity 17, 1307 (2000)

[7] N. Sasakura , hep-th/0001161

[8] A. Farmany, Proceeding 2004 Hawaiian International Conference on Science USA (2004)

[9] A. Farmany, accepted by 2005 Hawaiian International Conference on Science (USA)

EJTP **3, No. 12** (2006) 71–80 *Electronic Journal of Theoretical Physics*

Analytical One-Photon Double Differential Spectrum From In-Flight Decay of Scalar Neutral Mesons

Giuseppe Russo and Antonio Giusa

Dipartimento di Fisica e Astronomia, Università di Catania
Istituto Nazionale di Fisica Nucleare, Sezione di Catania
and Centro Siciliano di Fisica Nucleare e di Struttura della Materia
Viale A. Doria 6, I-95123 Catania, Italy

Received 23 December 2005, Accepted 18 April 2006, Published 20 September 2006

Abstract: We introduce a direct simple method to evaluate the one-photon double differential spectrum from the decay of pseudo-scalar neutral mesons. The analytical distributions of the opening angle and of the ratio of energies of the two gammas are then straightforwardly deduced. The physical interest is also outlined.

Keywords: Scalar Meson Decays, Electromagnetic spectra, Electromagnetic Decays of Elementary Particles, and eta Mesons
PACS (2006): 13.40.Hq, 14.40.Aq, 13.25.-k

1. Introduction

The study of meson decay processes provides powerful tools in the understanding of various phenomena in low- and high-energy nuclear and particle physics [1, 2].

In heavy-ion reactions at intermediate energies, the subthreshold pion production may carry important information about the initial stage of the nucleus-nucleus collision, i.e. when the projectile and target surface just begin to overlap. Theoretically, the subthreshold meson production can be described as a classical radiative process having a collective character, *i.e.* in terms of cooperative effects involving the contribution from several nucleons of the projectile and target [3]. Alternatively, in the framework of transport theory, the meson production at energies as low as 25 $MeV/nucleon$ can be interpreted as due to nucleon-nucleon collisions whose time evolution is described by the Boltzmann-Nordheim-Vlasov equation, usually solved by means of test-particle techniques. The comparison with experimental data has indicated that, at intermediate energies, the subthreshold π^0-production seems to be originated from an incoherent superposition of individual nucleon-nucleon contributions,and therefore due to Fermi momentum coupling to the relative motion of the colliding nuclei [4].

In particle physics, the pion-nucleon scattering was the first (1970s) tool adopted for the study of meson-nucleon interaction at intermediate and high energies. The performed experiments gave an estimation for the scattering probability and, moreover, the evidence for the existence of instable intermediate states, the baryonic resonances [5]. These latter can be identified by a partial-wave expansion of the interacting meson's wavefunction. The existence of baryonic resonances has been also confirmed in electromagnetic scattering on the nucleon, e.g. in the following scalar meson photoproduction reactions:

$$\gamma + N \longrightarrow N + \pi^0 \ , \tag{1}$$

$$\gamma + N \longrightarrow N + \eta \ , \tag{2}$$

where N is a nucleon (either n or p). The extraction of photo-excitation amplitudes beyond the first resonance region ($E^{cm}_{tot} \simeq 1.5\ GeV$) constitutes a powerful tool in the search for the so-called 'missing' resonances; these resonant baryonic states are predicted by QCD-inspired models, such as, for example, $SU(6) \otimes O(3)$-symmetric quark models, but not observed in conventional $\pi N \longrightarrow \pi N$ experiments. Moreover, photoproduction experiments can provide an effective criterium for the choice between the different theoretical models [6]. In the following we will focus our attention on the two-gamma decay of π^0 and η mesons, *i.e.* their dominant decay process, with branching ratios of 98% and 38% in the two cases respectively. Because of their very small mean free paths, the neutral mesons are detected indirectly, by reconstructing their kinematics from the decay products, usually via the invariant mass method. For this purpose, large efficiency detector systems with high granularity are currently operating in various laboratories [7, 8]. The differential spectra for the aforementioned processes are generally numerically deduced within Monte Carlo simulation approaches, which, as it's widely known, don't require any knowledge about the analytical expressions. Although this could be the only resource in cases of very complex calculations, we'll show here that their analytical expressions can be in general achieved by simple and straightforward mathematical considerations.

2. Double Differential Photon Spectrum

Let us consider a reaction producing a scalar neutral meson X (either π^0 or η), which in turns decays in two photons:

$$X \longrightarrow \gamma + \gamma$$

By choosing the polar axis along the direction of the X meson momentum in the laboratory-frame, the spin-0 nature of the X meson assures the process to be obviously isotropic and symmetrical in the X-center-of-mass frame, each produced photon carrying one-half of the rest energy of the X meson:

$$\theta'_1 + \theta'_2 = \pi \tag{3}$$

$$E'_{\gamma i} = \frac{m_X c^2}{2} \equiv E_0 \quad , \quad i = 1, 2 \ , \tag{4}$$

the primed quantities denoting the kinematical variables in the X-center-of-mass frame. The double-differential spectrum for the single decay photon is then trivially obtained as:

$$\frac{\partial^2 N_\gamma}{\partial E'_\gamma \partial \Omega'_\gamma} = \frac{1}{2\pi}\delta(E'_\gamma - E_0) \quad , \tag{5}$$

which is normalized to 2 in order to account for the fact that each decay process produces two photons.

By making use of (5), and of the usual Lorentz transformations for the four-momentum, the expression for the single-photon spectrum in the laboratory frame reads:

$$\frac{\partial^2 N_\gamma}{\partial E_\gamma \partial \Omega_\gamma} = \frac{E_\gamma}{E'_\gamma}\frac{\partial^2 N_\gamma}{\partial E'_\gamma \partial \Omega'_\gamma} = \frac{1}{2\pi}\frac{E_\gamma}{E'_\gamma}\delta(E'_\gamma - E_0) \tag{6}$$

Now, let us note that:

$$E'_\gamma = \gamma E_\gamma (1 - \beta \cos\theta) \ , \tag{7}$$

where γ and β have the usual meanings as in relativistic kinematics. This allows us to rewrite the (6) in terms of laboratory-frame kinematical variables only:

$$\frac{\partial^2 N_\gamma}{\partial E_\gamma \partial \Omega_\gamma} = \frac{1}{2\pi\gamma(1 - \beta\cos\theta)}\delta[\gamma E_\gamma(1 - \beta\cos\theta) - E_0] \tag{8}$$

2.1 One-photon energy spectrum

By observing that:

$$\delta[\gamma E_\gamma(1 - \beta\cos\theta) - E_0] = (\gamma\beta E_\gamma)^{-1}\delta\left[\cos\theta - \frac{1}{\beta}\left(1 - \frac{E_0}{\gamma E_\gamma}\right)\right] \quad , \tag{9}$$

the integration of (8) over the whole solid angle is straightforwardly accomplished as:

$$\frac{dN_\gamma}{dE_\gamma} = (\gamma^2\beta E_\gamma)^{-1}\int_{-1}^{1}\frac{\delta\left[\cos\theta - \frac{1}{\beta}\left(1 - \frac{E_0}{\gamma E_\gamma}\right)\right]}{(1 - \beta\cos\theta)}d\cos\theta = \frac{1}{\gamma\beta E_0}\Theta\left[1 - \frac{1}{\beta}\left|1 - \frac{E_0}{\gamma E_\gamma}\right|\right] \tag{10}$$

where Θ is the step-function (namely it is zero for negative argument and one otherwise). From this latter, and by making use of (4), we can infer the range of allowed values for the energy of the single decay photon to be:

$$E_- \leq E_\gamma \leq E_+ \ ,$$

where we defined:

$$E_\pm \equiv E_0\gamma(1 \pm \beta) = \frac{E_X}{2}(1 \pm \beta) \ ,$$

E_X being the relativistic total energy of the X meson.

The obtained flat energy spectrum (10) can easily be rewritten in a compact form as:

$$\frac{dN_\gamma}{dE_\gamma} = \frac{2}{E_X\beta}\Theta[(E_+ - E_\gamma)(E_\gamma - E_-)] \tag{11}$$

2.2 One-photon angular distribution

Let us note that:

$$\delta[\gamma E_\gamma(1-\beta\cos\theta)-E_0]=\frac{1}{\gamma(1-\beta\cos\theta)}\delta\left[E_\gamma-\frac{E_0}{\gamma(1-\beta\cos\theta)}\right] \tag{12}$$

By inserting this latter into (8), and integrating over the allowed interval of energy, we obtain:

$$\frac{dN_\gamma}{d\Omega_\gamma}=\int_{E_-}^{E_+}\frac{\partial^2 N_\gamma}{\partial E_\gamma \partial\Omega_\gamma}dE_\gamma=\frac{1}{2\pi}\frac{1-\beta^2}{(1-\beta\cos\theta)^2}\quad , \tag{13}$$

the following relation holding:

$$E_-\leq\frac{E_0}{\gamma(1-\beta\cos\theta)}\leq E_+$$

In *fig.*1 the dependence of (13) upon θ and β is shown.

3. Distribution of angular separation and of disparity of the photon pair

3.1 Angular separation distribution

Let us call θ the opening angle, in the laboratory frame, between the directions of the emitted photons, θ_1 and θ_2 being their deflection polar angles respectively. As the azimuthal angle is not affected by relativistic aberration, and the obvious relation $\phi_2-\phi_1=\pi$ holding, one easily infers:

$$\cos\theta=\frac{2\beta^2-1-\beta^2\cos^2\theta'}{1-\beta^2\cos^2\theta'}\quad , \tag{14}$$

where we made use of the usual Lorentz transformation between θ_i' and θ_i and of (3); moreover, we briefly defined $\theta'\equiv\theta_i$ for a chosen value of i . From (14) we get:

$$\cos^2\theta'=\frac{1}{\beta^2}\left(1-\frac{1-\beta^2}{\sin^2\frac{\theta}{2}}\right)\quad , \tag{15}$$

for any value of θ such that:

$$0\leq\cos\frac{\theta}{2}\leq\beta$$

From this latter (in which we took into account that $0\leq\theta\leq\pi$), the minimum value of the opening angle is easily deduced:

$$\theta_{min}=2\arccos\beta \tag{16}$$

and, from the (15):

$$d\cos\theta'=\frac{1}{4\beta\gamma^2\sin^3\frac{\theta}{2}\sqrt{\beta^2-\cos^2\frac{\theta}{2}}}d\cos\theta \tag{17}$$

It is perhaps not superflous, now, to observe, from (15), that each value of $\cos\theta$ corresponds to two (opposite) values of $\cos\theta'$, and so the expression for the angular distribution of photon pairs in the laboratory frame is retrieved as:

$$\frac{dN_{\gamma\gamma}}{d\Omega} = 2\frac{dN_{\gamma\gamma}}{d\Omega'}\frac{d\cos\theta'}{d\cos\theta} = \frac{1-\beta^2}{8\pi\beta\sin^3\frac{\theta}{2}\sqrt{\beta^2-\cos^2\frac{\theta}{2}}}\ , \tag{18}$$

where $[\theta_{min}, \pi]$ is the allowed range of values for θ.

In *fig.* 2 we plot the (18) for two different values of β. We can remark a huge rise and compression of the distribution when θ approaches θ_{min} as $\beta \to 0$; this behaviour is shown to be:

$$\lim_{\beta\to 0}\frac{dN_{\gamma\gamma}}{d\Omega} = \frac{1}{2\pi\sin\theta}\delta(\theta-\pi)$$

3.2 Disparity distribution

In order to obtain the distribution of disparity (i.e. the ratio of energies) of the two photons from the X-meson decay, let us define $u \equiv E_{\gamma\,1}/E_{\gamma\,2}$.

Then, the number of photon pairs characterized by an energy ratio not higher than ε is given [9] by:

$$\begin{aligned} N_{\gamma\gamma}(\varepsilon) &= \frac{1}{2}\int\int_{u\le\varepsilon}\frac{\partial^2 N_\gamma}{\partial E_{\gamma\,1}\partial E_{\gamma\,2}}dE_{\gamma\,1}dE_{\gamma\,2} = \\ &= \frac{1}{2}\int_{E_-}^{E_+}dE_{\gamma\,2}\int_{E_-}^{\varepsilon E_{\gamma\,2}}\frac{\partial^2 N_\gamma}{\partial E_{\gamma\,1}\partial E_{\gamma\,2}}dE_{\gamma\,1} = \\ &= \frac{1}{2}\int_{E_-/E_{\gamma\,2}}^{\varepsilon}du\int_{E_-}^{E_+}E_{\gamma\,2}\left(\frac{\partial^2 N_\gamma}{\partial E_{\gamma\,1}\partial E_{\gamma\,2}}\right)_{E_{\gamma\,1}=uE_{\gamma\,2}}dE_{\gamma\,2} \end{aligned} \tag{19}$$

Therefore,

$$\frac{dN_{\gamma\gamma}}{d\varepsilon} = \frac{1}{2}\int_{E_-}^{E_+}E_{\gamma\,2}\left(\frac{\partial^2 N_\gamma}{\partial E_{\gamma\,1}\partial E_{\gamma\,2}}\right)_{E_{\gamma\,1}=\varepsilon E_{\gamma\,2}}dE_{\gamma\,2} \tag{20}$$

In our case, the relation $E_{\gamma\,1}+E_{\gamma\,2}=E_X$ holding, one has:

$$\frac{\partial^2 N_\gamma}{\partial E_{\gamma\,1}\partial E_{\gamma\,2}} = \frac{dN_\gamma}{dE_{\gamma\,1}}\delta[E_X - E_{\gamma\,1} - E_{\gamma\,2}] \tag{21}$$

By inserting this latter in (20), and by observing that:

$$\begin{aligned} \delta(E_X - E_{\gamma\,1} - E_{\gamma\,2}) &= \delta[E_X - (1+\varepsilon)E_{\gamma\,2}] = \\ &= (1+\varepsilon)^{-1}\delta\left(E_{\gamma\,2} - \frac{E_X}{1+\varepsilon}\right)\ , \end{aligned}$$

we deduce:

$$\frac{dN_{\gamma\gamma}}{d\varepsilon} = \frac{1}{2}\frac{E_X}{(1+\varepsilon)^2}\left(\frac{dN_\gamma}{dE_{\gamma\,1}}\right)_{E_{\gamma\,1}=E_X\varepsilon/(1+\varepsilon)}\ , \tag{22}$$

from which, by using (11), we finally get:

$$\frac{dN_{\gamma\gamma}}{d\varepsilon} = \frac{1}{\beta(1+\varepsilon)^2}\Theta[(\varepsilon_+ - \varepsilon)(\varepsilon - \varepsilon_-)] \ , \tag{23}$$

where $\varepsilon_\pm \equiv (1 \pm \beta)/(1 \mp \beta)$.

In *fig.* 3 the (23) is displayed for $\beta = 0.5$.

4. Conclusions

In this paper we have introduced a direct way to retrieve analitical expressions for the energy spectrum and the angular distribution of photons from the in-flight decay of neutral scalar mesons. These have been used in order to get the angular separation and energy ratio distributions of the two decay photons. The importance of the meson production mechanisms in heavy-ion collisions at intermediate energies and in particle physics has been also outlined. In this latter context, our presented analytical expressions can be fruitfully used both in the simulation and experimental data analysis for neutral-meson experiments relying upon large solid angle acceptance and high granularity detectors.

References

[1] B. R. Martin and G.Shaw, *Particle Physics*,John Wiley & Sons (1992)

[2] D. H. Perkins, *Introduction to high energy reactions*,Addison-Wesley Publishing Co., Inc. (2000)

[3] D. Vasak, B. Müller and W. Greiner,*Physica Scripta* **22**,25 (1980)

[4] A. Bonasera, F. Gulminelli and J. Molitoris,*Phys. Rep.* **243**,1 (1994) and references therein.

[5] M. Batinić *Physica Scripta* **58**, 15 (1998), R. A. Arndt *et al.*, *Phys. Rev.* **C62**, 48 (2000), T. P. Vrana *et al.*, *Physics Reports* **328**, 181 (2000), D. M. Manley and G. M. Saleski, *Phys. Rev.* **D45**, 4002 (1992)

[6] J.-P. Bocquet, V. Kuznetsov, D. Rebreyend (Editors) *NSTAR 2004: Proceedings of the Workshop on the Physics of Excited Nucleons, Grenoble, France, 24-27 March 2004*, World Scientific Publishing, and references therein

[7] F. Ghio *et al*, *Nucl. Instr. Methods* **A404**, 71 (1998)

[8] E. Migneco *et al.*, *Nucl. Instr. Methods* **A314**,31 (1992)

[9] A. M. Mood, F. A. Graybell and D. C. Boes, *Introduction to the theory of statistics*, McGraw-Hill Inc. (1974)

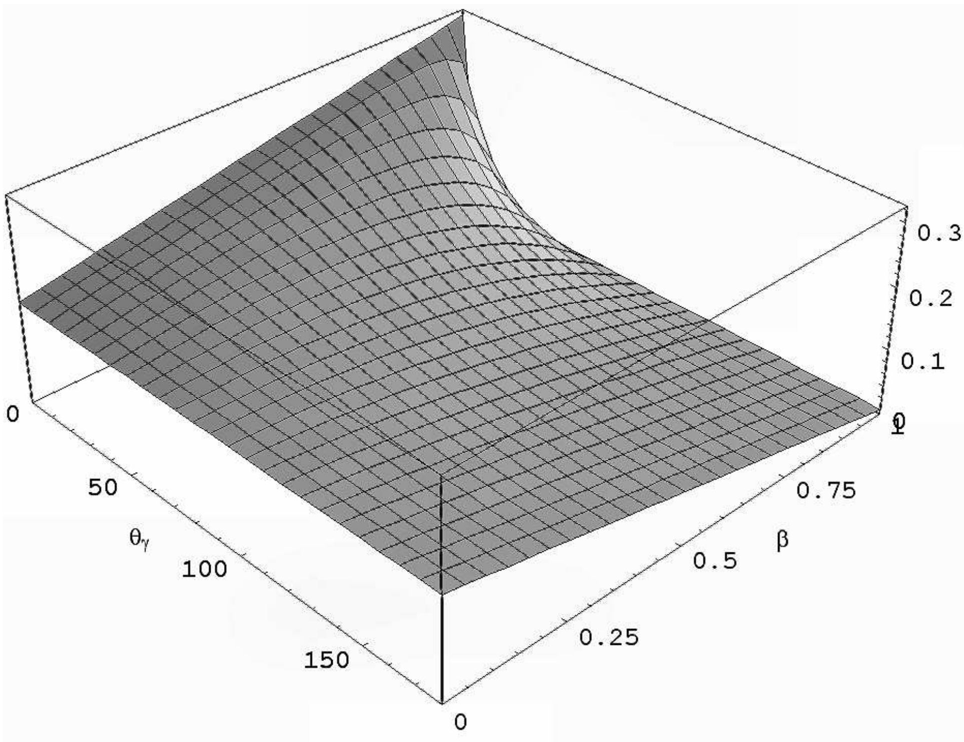

Fig. 1 One-photon angular distribution from the scalar meson decay as a function of β.

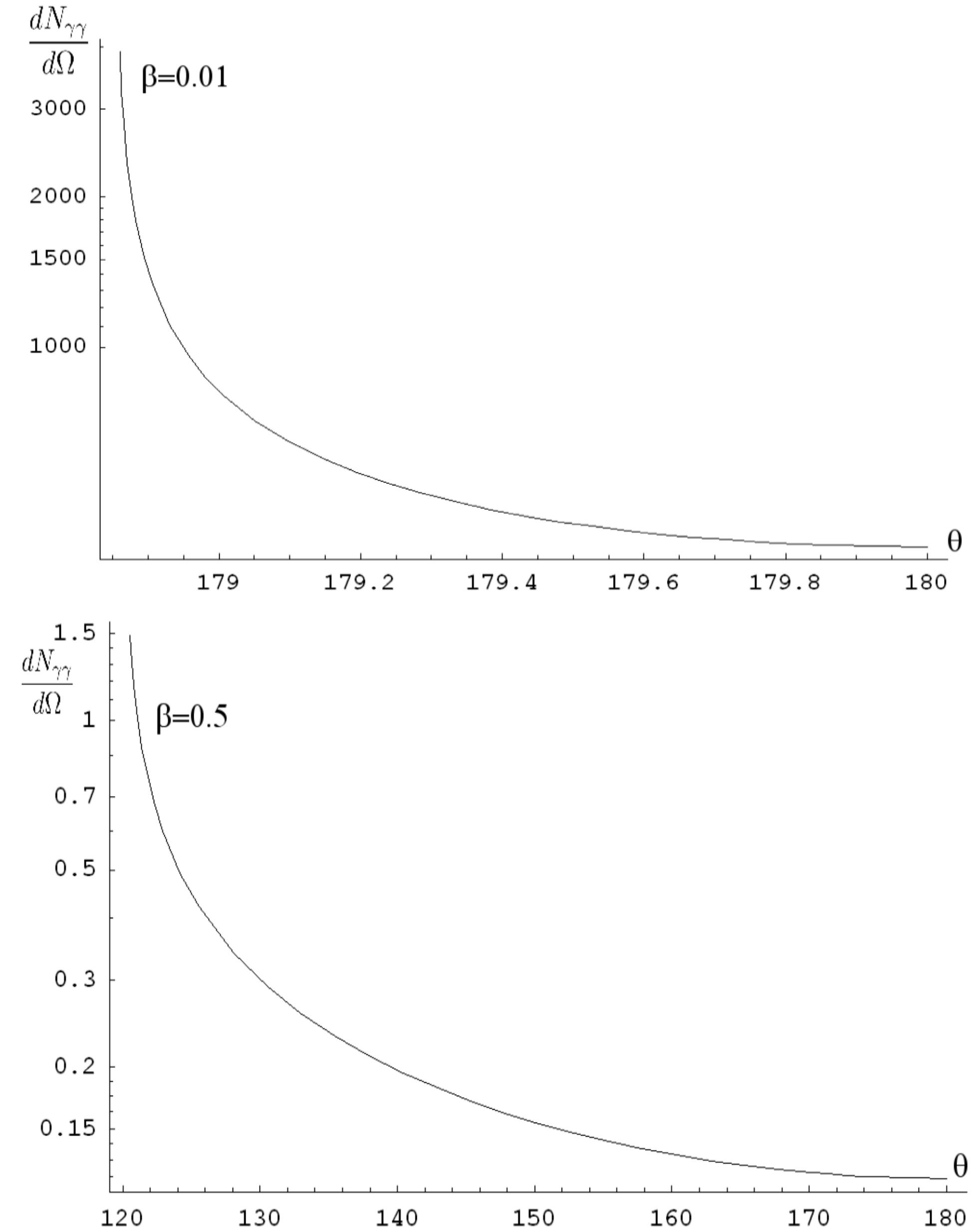

Fig. 2 Angular separation distribution of photon pairs in the laboratory frame for two different values of β .

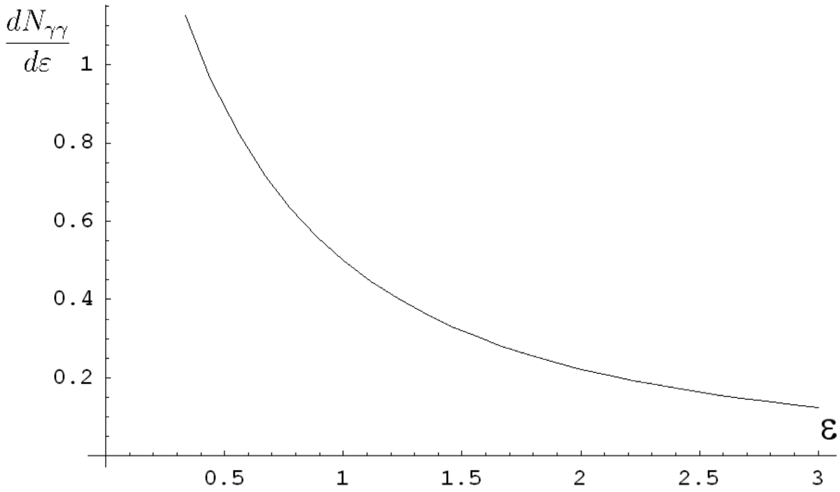

Fig. 3 Distribution of disparity for the photon pair, with $\beta = 0.5$.

On the Finite Caputo and Finite Riesz Derivatives

A. M. A. El-Sayed[1] *and M. Gaber[2]†

[1]*Faculty of Science University of Alexandria, Egypt*
[2]*Faculty of Education Al-Arish, Suez Canal University, Egypt*

Received 17 June 2006, Accepted 16 August 2006, Published 20 September 2006

Abstract: In this paper, we give some properties of the left and right finite Caputo derivatives. Such derivatives lead to finite Riesz type fractional derivative, which could be considered as the fractional power of the Laplacian operator modelling the dynamics of many anomalous phenomena in super-diffusive processes. Finally, the exact solutions of certain fractional diffusion partial differential equations are obtained by using the Adomain decomposition method and some new diffusion-wave equations are presented.

Keywords: Fractional Calculus, Fractional Differential Equations, Caputo and Riesz Derivatives, Adomian Method
PACS (2006): 45.10.Hj, 02.30.Hq, 02.50.Fz,02.30.Jr, 02.30.Gp

1. Introduction

It is known that the classical calculus provides a power tool to explain and to model many important dynamically processes in most parts of applied areas of the sciences. But the experiments and reality teach us that there are many complex systems in nature with anomalous dynamics, including biology, chemistry, physics, geology, astrophysics and social sciences, and more in particular in transport of chemical contaminant through water around rocks, dynamics of viscoelastic materials as polymers, diffusion of pollution in the atmosphere, diffusion processes involving cells, signals theory, control theory, electromagnetic theory, and many more.

In most of the above-mentioned cases, this kind of anomalous processes have a macroscopic complex behavior, and their dynamics cannot be characterized by classical derivative models. It is also important to remark that the anomalous behavior of many complex

* amasayed@hotmail.com
† mghf408@hotmail.com

processes include the multi-scaling in the time and space variables and so also the fractal characteristics of the media.

Stochastic tools have been used extensively during last 40 years to describe the dynamics of many anomalous processes, as such sub and super-diffusive processes. But the connections of these statistical models with some fractional differential equations, involving the fractional integral and derivative operators (Riemann-Liouville, Caputo, Liouville or Weyl and Riesz) have been formally established during the last 15 or 20 years and more intensively during the last 10 years by so many researches of a large list of different fields. It is possible to find so many reference of such fractional models and the applications of the fractional differential equations in the following monographic works by I. Podlubny (1999 [22]), B.J. West (1999 [26]), R. Metzler and J. Klafter (2000 [21]), E. Hilfer (Ed.) (2000 [13]), G.M. Zaslavsky (2005 [27]), and A.A. Kilbas, H.M. Sivastava and J.J. Trujillo (2006 [15]).

The Riemann-Liouville fractional integrals of order $\alpha > 0$, for suitable functions $\varphi(x)$ $(x \in \mathbb{R})$, in the interval $(a, b) \subseteq \mathbb{R}$, are well known (see for example [22, 24])

$$I_{a+}^{\alpha}\varphi(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-\zeta)^{\alpha-1} \varphi(\zeta)\ d\zeta, \tag{1.1}$$

$$I_{b-}^{\alpha}\varphi(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (\zeta-x)^{\alpha-1} \varphi(\zeta)\ d\zeta, \tag{1.2}$$

where $\Gamma(.)$ is the Gamma function. We refer to such fractional integrals I_{a+}^{α} and I_{b-}^{α} as left and right handed fractional integrals, respectively. Complementarily we define $I_{a+}^{0} = I_{b-}^{0} := I$ (Identity operator), as it is usual. Let us point out the namely the semigroup property for such fractional operators (see, for example, [24]), which can be written as follows: $\forall \alpha, \beta \geq 0$,

$$I_{a+}^{\alpha} I_{a+}^{\beta} = I_{a+}^{\beta} I_{a+}^{\alpha} = I_{a+}^{\alpha+\beta},$$
$$I_{b-}^{\alpha} I_{b-}^{\beta} = I_{b-}^{\beta} I_{b-}^{\alpha} = I_{b-}^{\alpha+\beta}.$$

In section 2. of this paper, the corresponding right Caputo fractional derivative is introduced, as the corresponding complementary operator to the well known left Caputo fractional derivative, according to the left and right handed fractional integrals given in (1.1) and (1.2). Some properties of such derivative are presented. In section 2.4, we obtain the numerical solutions of some simple examples of fractional differential equations, involving both mentioned Caputo derivatives, by using the Adomian decomposition method (ADM). As an application, in section **??**, we introduce a new fractional finite Riesz derivative. Such fractional derivative could be considerd as the fractional power of the known Laplacian operator. Finally, we establish the exact solutions of certain boundary problems for the fractional diffusion-wave partial differential equations by using the ADM. We remark that the finite Riesz derivative could play an important role in modelling the dynamics of some anomalous phenomena, for example to model the dynamics of super-diffusive processes, as a alternative to the recently models presented by several authors by using the Riesz derivative (see for example [19, 20]).

2. Left and Right Caputo Fractional Derivatives

In this section we present the fractional differentiation on finite interval in the framework of the Caputo fractional calculus. Particular attention is devoted to the ADM to solve Cauchy type problems of fractional differential equations involving the Caputo derivatives in a way accessible to applied scientists. By applying such analytic-numerical technique we shall derive the explicit solutions of some simple linear differential equations of fractional order.

Definition 1 *[10, 24] (left and right handed Riemman-Liouville fractional derivatives)*
For functions $f(x)$ given in the interval $[a,b]$, The left and the right handed Riemman-Liouville fractional derivatives are given by

$$\mathfrak{D}^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^m \int_a^x \frac{f(t)\, dt}{(x-t)^{\alpha-m+1}}, \tag{2.1}$$

$$\mathfrak{D}^{\alpha}_{b-} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^m \int_x^b \frac{f(t)\, dt}{(t-x)^{\alpha-m+1}}, \tag{2.2}$$

where $m = [\alpha]+1$, respectively. Fractional derivatives (2.1) *and* (2.2) *are usually named Riemman-Liouville fractional derivatives.*
Definition 2 *[10] (finite Weyl fractional derivative)*
The finite Weyl fractional derivative of order α, $m-1 < \alpha \leq m$ of the function $f(t) \in \mathbb{C}^m$, $t \in (0,a)$ is defined by

$$W^{\alpha}_{a} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^a (\zeta - t)^{m-\alpha-1} f^m(\zeta)\ d\zeta,$$

where $\mathbb{C}^m$ is the class of all functions $f : (0,a) \to \mathbb{X}$, with $f^{(k)}(a) = 0$, $k = 0, 1, ..., m-1$.

Let $m = 1, 2, \cdots$ and $AC^m([a,b])$ the space of functions φ which have continuous derivatives up to order $m-1$ on $[a,b]$ with $\varphi^{(m-1)} \in AC([a,b])$. Assuming that for $m-1 < \alpha < m$ and a suitable function $\varphi(x)$ (for example, $\varphi \in AC^m([a,b])$). The corresponding left and right Caputo fractional derivatives, of order $\alpha > 0$, to the Riemann-Liouville fractional integrals given in (1.1) and (1.2) are given by

$$\mathbf{D}^{\alpha}_{a+} \varphi(x) = I^{m-\alpha}_{a+} D^m \varphi(x), \tag{2.3}$$

$$\mathbf{D}^{\alpha}_{b-} \varphi(x) = (-1)^m I^{m-\alpha}_{b-} D^m \varphi(x), \tag{2.4}$$

2.1 Some properties of the left and right Caputo fractional operators

First we introduce the following Proposition:
Proposition 1 *[12]*

For a well-behaved function $\varphi(x)$ (with $x \in (a,b)$) we have

$$\left\{ \begin{array}{c} D^n I^n_{b-} \varphi(x) = (-1)^n \varphi(x), \\ I^n_{b-} D^n \varphi(x) = (-1)^n \left\{ \varphi(x) - \sum_{k=0}^{n-1} \frac{\varphi^{(k)}(b)}{k!} (b-x)^k \right\}, \end{array} \right. \quad a < x < b,$$

and

$$D^m I^m_{a+} = I \text{ and } D^m I^m_{b-} = (-1)^m I.$$

Assume that $\varphi(x)$ is a suitable function, for example $\varphi(x) \in AC^m([a,b])$. For $\alpha \in (m-1,m)$ and $D^m\varphi(x) \neq 0$, we have the following Theorem:

Theorem 1

(1) $\lim_{\alpha\to m} \mathbf{D}^\alpha_{b-}\varphi(x) = (-1)^m \varphi^{(m)}(x)$.

(2) If $\mu \geq 0$, *and* $\alpha \in (m-1,m]$, *we have*

$$\mathbf{D}^\alpha_{b-}x^\mu = \left[\prod_{k=0}^{m-1}(\mu-k)\right]\frac{(-1)^m(b-x)^{m-\alpha}b^{\mu-m}}{\Gamma(m-\alpha+1)} \times F_{1,2}(m-\mu,1;m-\alpha+1;\frac{b-x}{b}),\ b\neq 0.$$

(3)

$$I^\alpha_{b-}\mathbf{D}^\alpha_{b-}\varphi(x) = (-1)^m\varphi(x) + \sum_{k=0}^{m-1}\frac{(-1)^{m-k-1}(b-x)^k}{k!}D^k\varphi(b).$$

(4) For $\beta > m \geq \alpha > m-1$,

$$I^\beta_{b-}\mathbf{D}^\alpha_{b-}\varphi(x) = (-1)^m I^{\beta-\alpha}_{b-}\varphi(x) + \sum_{k=0}^{m-1}\frac{(-1)^{m-k-1}(b-x)^{k+\beta-\alpha}}{\Gamma(k+\beta-\alpha+1)}D^k\varphi(b).$$

(5) For $m \geq \beta > \alpha > m-1$,

$$\begin{aligned} I^\alpha_{b-}\mathbf{D}^\beta_{b-}\varphi(x) &= (-1)^m D^{\beta-\alpha}_{b-}\varphi(x) \\ &\quad + \sum_{k=0}^{m-1}\frac{(-1)^{m-k-1}(b-x)^{k+\beta-\alpha}}{\Gamma(k+\beta-\alpha+1)}D^{k+1}\varphi(b). \end{aligned}$$

(6) Assume that $\beta > m \geq \alpha > m-1$ *and* $\beta = m+\nu$. *If* $D^m I^\beta_{b\pm}\varphi(x) \neq 0$, *then*

$$\mathbf{D}^\alpha_{b\pm}I^\beta_{b\pm}\varphi(x) = (\pm 1)^m I^{\beta-\alpha}_{b\pm}\varphi(x).$$

(7) Assume that $m \geq \alpha > m-1 \geq n$ *and* $k+n=m$. *If* $D^m I^n_{b\pm}\varphi(x) \neq 0$, *then*

$$\mathbf{D}^\alpha_{b\pm}I^n_{b\pm}\varphi(x) = (\pm 1)^n\ \mathbf{D}^{\alpha-n}_{b\pm}\varphi(x).$$

Proof.

(1)

$$\begin{aligned} \lim_{\alpha\to m}\mathbf{D}^\alpha_{b-}\varphi(x) &= (-1)^m\lim_{\alpha\to m}\left(\frac{\varphi^{(m)}(b)(b-x)^{m-\alpha}}{\Gamma(m-\alpha+1)}\right. \\ &\quad \left. -\frac{1}{\Gamma(m-\alpha+1)}\int_x^b (s-x)^{m-\alpha}\varphi^{(m+1)}(s)\,ds\right) \\ &= (-1)^m\left(\varphi^{(m)}(b) - \int_x^b \varphi^{(m+1)}(s)\,ds\right) = (-1)^m\varphi^{(m)}(x). \end{aligned}$$

So, in (2.3) *and* (2.4) *we can take* $\alpha \in (m-1, m]$.

(2) From the fact that

$$I_{b-}^{\beta} x^{\ell} = \frac{(b-x)^{-\beta} b^{\ell}}{\Gamma(1-\beta)} F_{1,2}(-\ell, 1; 1-\alpha; \frac{b-x}{b}), \ \beta > 0,$$

where $F_{1,2}(a, b; c; z)$ *is the hypergeometric distribution introduced as [3]*

$$F_{1,2}(a, b; c; z) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j} \frac{z^j}{j!},$$

$c \neq 0, -1, -2, \cdots, |z| \leq 1,$ *and* $(a)_j = \frac{\Gamma(a+j)}{\Gamma(a)}$. *So, we have*

$$\mathbf{D}_{b-}^{\alpha} x^{\mu} = \left[\prod_{k=0}^{m-1} (\mu - k) \right] \frac{(-1)^m (b-x)^{m-\alpha} b^{\mu-m}}{\Gamma(m-\alpha+1)}$$

$$\times F_{1,2}(m-\mu, 1; m-\alpha+1; \frac{b-x}{b}), \ b \neq 0,$$

for $\mu \geq 0$, *and* $\alpha \in (m-1, m]$.

(3)

$$I_{b-}^{\alpha} \mathbf{D}_{b-}^{\alpha} \varphi(x) = I_{b-}^{\alpha} I_{b-}^{m-\alpha} D^m \varphi(x) = I_{b-}^{m} D^m \varphi(x)$$

$$= (-1)^m \varphi(x) + \sum_{k=0}^{m-1} \frac{(-1)^{m-k-1} (b-x)^k}{k!} D^k \varphi(b).$$

(4) For $\beta > m \geq \alpha > m-1$,

$$I_{b-}^{\beta} \mathbf{D}_{b-}^{\alpha} \varphi(x) = I_{b-}^{\beta} I_{b-}^{m-\alpha} D^m \varphi(x)$$

$$= I_{b-}^{m+\beta-\alpha} D^m \varphi(x)$$

$$= (-1)^m I_{b-}^{\beta-\alpha} \varphi(x) + \sum_{k=0}^{m-1} \frac{(-1)^{m-k-1} (b-x)^{k+\beta-\alpha}}{\Gamma(k+\beta-\alpha+1)} D^k \varphi(b).$$

(5) For $m \geq \beta > \alpha > m-1$,

$$I_{b-}^{\alpha} \mathbf{D}_{b-}^{\beta} \varphi(x) = I_{b-}^{\alpha} I_{b-}^{m-\beta} D^m \varphi(x)$$

$$= I_{b-}^{m+\alpha-\beta} D^{m-1} (D\varphi(x)) = (-1)^m D_{b-}^{\beta-\alpha} \varphi(x)$$

$$+ \sum_{k=0}^{m-1} \frac{(-1)^{m-k-1} (b-x)^{k+\beta-\alpha}}{\Gamma(k+\beta-\alpha+1)} D^{k+1} \varphi(b).$$

(6) Assume that $\beta > m \geq \alpha > m-1$ *and* $\beta = m + \nu$. *If* $D^m I_{b\pm}^{\beta} \varphi(x) \neq 0$, *then*

$$\mathbf{D}_{b\pm}^{\alpha} I_{b\pm}^{\beta} \varphi(x) = I_{b\pm}^{m-\alpha} D^m I_{b\pm}^{m} I_{b\pm}^{\nu} \varphi(x)$$

$$= (\pm 1)^m I_{b\pm}^{m-\alpha+\nu} \varphi(x) = (\pm 1)^m I_{b\pm}^{\beta-\alpha} \varphi(x).$$

(7) Assume that $m \geq \alpha > m-1 \geq n$ *and* $k + n = m$. *If* $D^m I_{b\pm}^{n} \varphi(x) \neq 0$, *then*

$$\mathbf{D}_{b\pm}^{\alpha} I_{b\pm}^{n} \varphi(x) = I_{b\pm}^{m-\alpha} D^m I_{b\pm}^{n} \varphi(x) = (\pm 1)^n \ \mathbf{D}_{b\pm}^{\alpha-n} \varphi(x).$$

■

It is easy to check that the following Lemma is true.

Lemma 1

If $\varphi^{(p)}(b) = 0$, *for all* $p = 0, 1, 2, \cdots, [\alpha]$. *Then [10]*

(a) $\lim_{\alpha\to p} \mathbf{D}^{\alpha}_{b-}\varphi(x) = \mathbf{D}^{p}_{b-}\varphi(x) = \left(\mp\frac{d}{dx}\right)^{p}\varphi(x)$ *and*

(b) *The family* $\left\{\mathbf{D}^{\alpha}_{b-};\ \alpha\in\mathbb{R}\right\}$ *is a multiplicative group.*

(c) *If* $\mu > 0$, $\alpha \in (m-1, m]$ $(m \geq 1)$*and* $b > x$, *then*

$$\mathbf{D}^{\alpha}_{x,b-}(b-x)^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(b-x)^{\mu-\alpha}.$$

2.2 The relationship between the operators $\mathbf{D}^{\alpha}_{b-}$ and $\mathfrak{D}^{\alpha}_{b-}$

The left and the right handed Riemman-Liouville fractional derivatives are given as [24],

$$\mathfrak{D}^{\alpha}_{a+}\varphi(x) = \frac{1}{\Gamma(m-\alpha)}\left(\frac{d}{dx}\right)^{m}\int_{a}^{x}\frac{\varphi(t)\,dt}{(x-t)^{\alpha-m+1}}, \tag{2.5}$$

$$\mathfrak{D}^{\alpha}_{b-}\varphi(x) = \frac{(-1)^{m}}{\Gamma(m-\alpha)}\left(\frac{d}{dx}\right)^{m}\int_{x}^{b}\frac{\varphi(t)\,dt}{(t-x)^{\alpha-m+1}}, \tag{2.6}$$

where $m = [\alpha]+1$, respectively.

When $\alpha \in (0,1)$ we have the following relationship between the operators $\mathbf{D}^{\alpha}_{b-}$ and $\mathfrak{D}^{\alpha}_{b-}$ (see [24])

$$\mathfrak{D}^{\alpha}_{b-}\varphi(x) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{\varphi(b)}{(b-x)^{\alpha}} - \mathbf{D}^{\alpha}_{b-}\varphi(x)\right].$$

Such relations can be extended easily to the case that $\alpha\in(m-1,m]$ as follows:

$$\mathfrak{D}^{\alpha}_{b-}\varphi(x) = \mathbf{D}^{\alpha}_{b-}\varphi(x) + \sum_{k=0}^{m-1}\frac{(-1)^{m-k}(b-x)^{k-\alpha}}{\Gamma(k-\alpha+1)}\left[D^{k}\varphi(x)\right]_{x=b}. \tag{2.7}$$

So, if $\alpha\in(m-1,m]$ and $\varphi^{(k)}(x) = 0$ for all $k = 0,1,2,\cdots,m-1$, then

$$\mathbf{D}^{\alpha}_{b-}\varphi(x) \equiv \mathfrak{D}^{\alpha}_{b-}\varphi(x).$$

Remark 1

(A) *From the relationship* (2.7) *between the operators* $\mathbf{D}^{\alpha}_{b-}$ *and* $\mathfrak{D}^{\alpha}_{b-}$, *we find that the finite Weyl derivative (Definition 2) coincide with the right handed Riemman-Liouville fractional derivative (Definition 1).*

(B) *By passing to the limit for* $\alpha\to m^{-}$ *and using* (2.7), *we have*

$$\begin{aligned}\lim_{\alpha\to m^{-}}\mathfrak{D}^{\alpha}_{b-}\varphi(x) &= (-1)^{m}\varphi^{(m)}(x) + \sum_{k=0}^{m-1}(-1)^{m-k}\delta^{(m-k-1)}(b-x)\varphi^{(k)}(b)\\ &\neq (-1)^{m}\varphi^{(m)}(x).\end{aligned}$$

So, from Theorem 1 (1), *we see that the right Caputo fractional derivative* $\mathbf{D}^{\alpha}_{b-}$ *have the continuation property, in other hand this property is not verified in the case of the right handed Riemman-Liouville fractional derivative.*

The following results are well known (see, for example [15]). Here we will present a proof of it by using the ADM as an example. We must mention that the ADM to solve the ordinary fractional differential equations has been used by several authors with good results (see, for example, [23, 25]).

Lemma 2 *Let $x \in (a,b) \subseteq \mathbb{R}$, $\varphi_a \in \mathbb{R}$, and $\alpha \in (0,1]$. Then the following Cauchy type problem*

$$ {}^{c}D^{\alpha}_{a+}\varphi(x) = \varphi(x) \tag{2.8}$$

$$ \varphi(a) = \varphi_a, \tag{2.9}$$

have the unique solution

$$\varphi(x) = \varphi_a E_\alpha\left((x-a)^\alpha\right). \tag{2.10}$$

Proof. First of all we can be sure that the problem (2.8)-(2.9) have a unique continuous solution by the application of the known existence theorem for Cauchy type problems involving the left Caputo derivative (see for example, [9] and [15]). Here, we apply the ADM to get the exact solution of the problem (2.8)-(2.9). In fact by the mentioned theorem we know that such problem is equivalent to the following Volterra integral equation

$$\varphi(x) = \varphi_a + I^{\alpha}_{a+}\varphi(x).$$

Now solving the last equality by using the ordinary ADM, with $\varphi_0(x) = \varphi_a$, we have

$$\begin{aligned}
\varphi_1(x) &= I^{\alpha}_{a+}\varphi_0 = \varphi_a \frac{(x-a)^\alpha}{\Gamma(1+\alpha)}, \\
&\vdots \\
\varphi_n(x) &= I^{\alpha}_{a+}\varphi_{n-1} = \varphi_a \frac{(x-a)^{n\alpha}}{\Gamma(1+n\alpha)}, \ n \geq 1. \\
&\vdots
\end{aligned}$$

Then, we can conclude that the unique solution of (2.8)-(2.9) is given by (2.10). ■

Lemma 3 *The unique continuous solution of the following Cauchy type problem*

$$ {}^{c}D^{\alpha}_{b-}\varphi(x) = \varphi(x), \ x \in (a,b) \subseteq \mathbb{R}, \ \alpha \in (0,1], \tag{2.11}$$

$$ \varphi(b) = \varphi_b \tag{2.12}$$

is given by

$$\varphi(x) = \varphi_b E_\alpha\left((b-x)^\alpha\right). \tag{2.13}$$

Proof. As in Lemma 2, we apply the corresponding existence theorem to Cauchy type problems involving the right Caputo derivative ${}^{c}D^{\alpha}_{b-}$, from which we know that the problem (2.11)-(2.12) has a unique continuous solution which is equivalent to the integral equation

$$\varphi(x) = \varphi_b - I^{\alpha}_{b-}\varphi(x).$$

Now we can apply the ADM, with $\varphi_0(x) = \varphi_b$, so we obtain that

$$\varphi_n(x) = I^{\alpha}_{b-}\varphi_{n-1} = \varphi_b\left(\frac{(b-x)^{n\alpha}}{\Gamma(1+n\alpha)}\right),$$

and we can conclude that (2.13) is the solution of (2.11)-(2.12). ■

2.3 A physical model

Consider a semi-infinite rod with a thermal source producing a temperature wave $f(t)$ localized at the front of the rod. The temperature wave intensity of the rod is registered by a detector at the times $t = 0$ and $t = b$.

An efficient way of obtaining some information about a layered medium structure and makeup is to measure its temperature wave expose reaction. The suggested technique allows us to restore the medium basic parameters with arbitrary accuracy.

Notice that in this case we study the primary phase of temperature oscillation in order to solve inverse problems. In the case of media exhibiting memory. So, we have the following boundary value problem in $\Omega = \{(x,t) : x \geq 0,\ t \in [0,b]\}$:

$$\begin{aligned}
\mathbf{D}^{\alpha}_{t,b-}\varphi(x,t) &= \lambda^2 \frac{\partial^2 \varphi(x,t)}{\partial x^2},\ \alpha \in (0,2],\ x \geq 0,\ t \in [0,b],\\
\varphi(x,0) &= p(x),\ x \geq 0,\\
\varphi(x,b) &= q(x),\ x \geq 0,\ \alpha \in (1,2],\\
\varphi(0,t) &= f(t),\ t \in [0,b]\\
\varphi_x(0,t) &= g(t),\ t \in [0,b].
\end{aligned} \tag{2.14}$$

where λ is the thermal diffusivity and α is the anomalous diffusion index, (see for example [4]).

2.4 Applications of the Adomian method to solve fractional models

The ADM has been used to solve the problem (2.14) in the case $\alpha = 1, 2$ by Lesnic [16, 17, 18]. In this section we extend the ADM given in [16, 17, 18] to obtain the explicit solution of this boundary problem in the case that $\alpha \in (0,1)$ and $\alpha \in (1,2)$ for the following examples.

Example 1 *Consider the fractional boundary value problem*

$$\begin{aligned}
\frac{\partial^2 u(x,t)}{\partial x^2} &= \mathbf{D}^{2\alpha}_{t,b-}u(x,t),\ \alpha \in (\frac{1}{2},1],\ x > 0,\ t \in (0,b),\\
u(0,t) &= \frac{2(b-t)^{2\alpha}}{\Gamma(2\alpha+1)},\ t \in (0,b),\\
u_x(0,t) &= 0,\ t \in (0,b),\\
u(x,0) &= x^2 + \frac{2b^{2\alpha}}{\Gamma(2\alpha+1)},\ u(x,b) = x^2.
\end{aligned}$$

The solution of this problem is given by

$$u(x,t) = \frac{2(b-t)^{2\alpha}}{\Gamma(2\alpha+1)} + x^2.$$

Let us define the starting term

$$u_0(x,t) = \frac{1}{2}\left[p(t) + xq(t) + f_0(x) + \left(\frac{b-t}{b}\right)^{\alpha}(f_b(x) - f_0(x))\right],$$

and the recurrence relation as [16, 17, 18],

$$u_{n+1}(x,t) = \frac{1}{2}\left[L_t^{-1}L_{xx} + L_{xx}^{-1}L_t\right]u_n,\ n \geq 0,$$

where $L_t = {}^c D_{t,b-}^{2\alpha}$, $L_{xx} = \frac{\partial^2}{\partial x^2}$,

$$\begin{aligned} L_t^{-1}w(t) &= \int_t^b \frac{(s-t)^{2\alpha-1}}{\Gamma(2\alpha)}w(s)ds + \left(\frac{b-t}{b}\right)^\alpha \int_0^b \frac{(b-s)^{2\alpha-1}}{\Gamma(2\alpha)}w(s)ds, \\ L_{xx}^{-1} &= \int_0^x dx' \int_0^{x'} dx''. \end{aligned}$$

The ADM gives the solution in the following decomposed form

$$u(x,t) = \lim_{N\to\infty}\Phi_N(x,t), \quad \Phi_N(x,t) = \sum_{n=0}^{N} u_n(x,t),\ N \geq 0.$$

In this example, we use this starting term and recurrence relation in the decomposition method to avoid the error which may be appear in our solution (see [1, 2, 16, 17, 18]). Now, using Lemma 1 (c) and Theorem 1 (3), *with* $m = 2$, *we can conclude that*

$$\begin{aligned} u(x,t) &= -\frac{2tb^\alpha}{\Gamma(2\alpha+1)} + \left(x^2 + \frac{2(b-t)^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2tb^\alpha}{\Gamma(2\alpha+1)}\right)\sum_{n=0}^{\infty}\frac{1}{2^{n+1}} \\ &= x^2 + \frac{2(b-t)^{2\alpha}}{\Gamma(2\alpha+1)}. \end{aligned}$$

We check easily that when $\alpha \to 1$, *we have* $\varphi(x,t) = x^2 + (b-t)^2$, *which is the solution of the wave equation,*

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2},\ 0 < t < b,\ x > 0,$$

with the conditions

$$\begin{aligned} u(0,t) &= (b-t)^2,\ u_x(0,t) = 0,\ 0 \leq t \leq b, \\ u(x,0) &= x^2 + b^2,\ u(x,b) = x^2,\ x > 0. \end{aligned}$$

Example 2 *Consider the backward fractional heat equation given as [11]*

$$\frac{\partial u(x,t)}{\partial t} = k\mathbf{D}_{x,b-}^{2\alpha}u(x,t),\ \alpha \in (\frac{1}{2},1],\ x \in (0,b),\ t > 0,$$

subject to the following initial and boundary conditions

$$\begin{aligned} u(x,0) &= f_0(x),\ x \in [0,b] \\ u(0,t) &= p(t),\ u(b,t) = r(t),\ t > 0, \end{aligned}$$

where $\mathbf{D}_{x,b-}^{2\alpha}u(x,t) = I_{b-}^{2-\alpha}\frac{\partial^2 u(x,t)}{\partial x^2}$. *We use the starting term*

$$u_0(x,t) = \frac{1}{2}\left[f_0(x) + p(t) + \left(\frac{b-x}{b}\right)^\alpha (r(t) - p(t))\right],$$

and the recurrence relation

$$u_{n+1}(x,t) = \frac{1}{2}\left[kL_t^{-1}L_{xx} + \frac{1}{k}L_{xx}^{-1}L_t\right]u_n, \ n \geq 0,$$

where $L_t = \frac{\partial}{\partial t}$, $L_{xx} = \mathbf{D}^{2\alpha}_{x,b-}$,

$$L_t^{-1}w(t) = \int_0^t w(s)ds,$$

$$L_{xx}^{-1}v(x) = \int_x^b \frac{(s-x)^{2\alpha-1}}{\Gamma(2\alpha)}v(s)ds + \left(\frac{b-x}{b}\right)^{\alpha}\int_0^b \frac{(b-s)^{2\alpha-1}}{\Gamma(2\alpha)}v(s)ds.$$

To illustrate the decomposition method for solving this problem, consider $k=\pm 1$, $f_0(x) = \frac{2(b-x)^{2\alpha}}{\Gamma(1+2\alpha)}$, $p(t) = \frac{2b^{2\alpha}}{\Gamma(1+2\alpha)} + 2kt$ *and* $r(t) = 2kt$, *then the given recurrence relation and starting term gives*

$$u_0(x,t) = \frac{1}{2}\left[\frac{2(b-x)^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{2(b-x)^{\alpha}b^{\alpha}}{\Gamma(1+2\alpha)} + 2kt\right],$$

$$u_n(x,t) = \frac{1}{2^{n+1}}\left[\frac{2(b-x)^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{2(b-x)^{\alpha}b^{\alpha}}{\Gamma(1+2\alpha)} + 2kt\right], \ n \geq 1,$$

and

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t) \\ &= \frac{2(b-x)^{\alpha}b^{\alpha}}{\Gamma(1+2\alpha)} + \left(\frac{2(b-x)^{2\alpha}}{\Gamma(1+2\alpha)} + 2kt - \frac{2(b-x)^{\alpha}b^{\alpha}}{\Gamma(1+2\alpha)}\right)\sum_{i=0}^{\infty}\frac{1}{2^{i+1}} \\ &= \frac{2(b-x)^{2\alpha}}{\Gamma(1+2\alpha)} + 2kt. \end{aligned}$$

As $\alpha \to 1$, *we have*

$$u(x,t) = (b-x)^2 + 2kt$$

which the solution of the forward and backward heat problem (see [17])

$$\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < b, \ t > 0, \ k = \pm 1,$$

with the initial and boundary conditions $u(x,0) = (b-x)^2$, $u(0,t) = 2kt$, $u(b,t) = b^2 + 2kt$.

3. The finite fractional powers of the second order derivative

The finite Riesz potential, for $x \in (a,b) \subseteq \mathbb{R}$, was introduced in [24], as follows:

$$\begin{aligned} I^{\alpha}_{(a,b)}\varphi(x) &= \frac{I^{\alpha}_{a+}\varphi(x) + I^{\alpha}_{b-}\varphi(x)}{2\cos\left(\frac{\alpha\pi}{2}\right)} \\ &= \frac{1}{2\Gamma(\alpha)\cos\left(\frac{\alpha\pi}{2}\right)}\int_a^b |x-\zeta|^{\alpha-1}\varphi(\zeta)\,d\zeta, \end{aligned}$$

for any $\alpha > 0$ with the exclusion of odd integer numbers for which $\cos\left(\frac{\alpha\pi}{2}\right)$ vanishes.

The finite Riesz potential given above has the semigroup only in restricted rang, e.g.

$$I^{\alpha}_{(a,b)} I^{\beta}_{(a,b)} = I^{\alpha+\beta}_{(a,b)} \text{ for } 0 < \alpha < 1,\ 0 < \beta < 1,\ \alpha + \beta < 1.$$

Now, we can introduce the finite Riesz derivative, in the framework of the Caputo derivative, as follows:

$$\begin{aligned} R^{\alpha}_{(a,b)}\varphi(x) &= I^{m-\alpha}_{(a,b)} D^m \varphi(x) = I^{m-\alpha}_{(a,b)} \left(D^m \varphi(x)\right) \\ &= -\frac{\mathbf{D}^{\alpha}_{a+}\varphi(x) + \mathbf{D}^{\alpha}_{b-}\varphi(x)}{2\cos\left(\frac{(m-\alpha)\pi}{2}\right)} \\ &= \frac{-1}{2\Gamma(\alpha)\cos\left(\frac{(m-\alpha)\pi}{2}\right)} \int_a^b |x-\zeta|^{m-\alpha-1} D^m \varphi(\zeta)\, d\zeta, \end{aligned}$$

for any positive $\alpha \in (m-1, m)$ and $x \in (a,b) \subseteq \mathbb{R}$.

In general, the Riesz fractional derivative $R^{\alpha}_{(a,b)}$ turns out to be related to the $\frac{\alpha}{2}$-power of the positive definite operator $-D^2 = -\frac{d^2}{dx^2}$ as follows:

$$R^{\alpha}_{(a,b)}\varphi(x) = \left(-\frac{d^2}{dx^2}\right)^{\frac{\alpha}{2}} \varphi(x) = \begin{cases} -\frac{\mathbf{D}^{\alpha}_{a+}\varphi(x) + \mathbf{D}^{\alpha}_{b-}\varphi(x)}{2\cos\left(\frac{(1-\alpha)\pi}{2}\right)}, & \alpha \in (0,1), \\ -\frac{\mathbf{D}^{\alpha}_{a+}\varphi(x) + \mathbf{D}^{\alpha}_{b-}\varphi(x)}{2\cos\left(\frac{(2-\alpha)\pi}{2}\right)}, & \alpha \in (1,2), \\ -\frac{1}{\pi}\int_a^b \frac{D\varphi(\zeta)}{|t-\zeta|} d\zeta, & \alpha = 1, \end{cases} \tag{3.1}$$

where $x \in (a,b) \subseteq \mathbb{R}$.

From our definition of the finite Riesz derivative it is easy to prove the following Lemma.

Lemma 4

Let $\alpha \in (1,2)$, and $\varphi(x)$ be suitable function, for example $\varphi(x) \in AC^2$, then

(a) $\lim_{\alpha\to 2} R^{\alpha}_{(a,b)}\varphi(x) = -\frac{d^2}{dx^2}\varphi(x)$.

(b) $R^{\alpha}_{(a,b)}(kx + \ell) = 0$ *for all constants* k, ℓ.

(c) *For* $\alpha \in (m-1, m]$, *we have*

$$R^{\alpha}_{(a,b)} I^m_{a+}\varphi(x) = I^{m-\alpha}_{(a,b)}\varphi(x),$$

and

(d) $R^{\alpha}_{(a,b)} I^m_{b-}\varphi(x) = (-1)^m I^{m-\alpha}_{(a,b)}\varphi(x)$.

Proof.

(a)

$$\lim_{\alpha\to 2} R^{\alpha}_{(a,b)}\varphi(x) = \lim_{\alpha\to 2} I^{2-\alpha}_{(a,b)} D^2\varphi(x)$$
$$= \lim_{\alpha\to 2}\left[-\frac{\mathbf{D}^{\alpha}_{a+}\varphi(x)+\mathbf{D}^{\alpha}_{b-}}{2\cos\left(\frac{(2-\alpha)\pi}{2}\right)}\right]\varphi(x)$$
$$= -\frac{d^2}{dx^2}\varphi(x).$$

So, in (3.1) *we can take* $\alpha = 2$.

(b) $R^{\alpha}_{(a,b)}(kx+\ell) = I^{2-\alpha}_{(a,b)} D^2 (kx+\ell) = 0$ *for all constants* k, ℓ.

(c) For $\alpha \in (m-1, m]$, *then by using Proposition 1, we have*

$$R^{\alpha}_{(a,b)} I^{m}_{a+}\varphi(x) = I^{m-\alpha}_{(a,b)} D^m \left(I^{m}_{a+}\varphi(x)\right) = I^{m-\alpha}_{(a,b)}\varphi(x),$$

(d) $R^{\alpha}_{(a,b)} I^{m}_{b-}\varphi(x) = I^{m-\alpha}_{(a,b)} D^m \left(I^{m}_{b-}\varphi(x)\right) = (-1)^m I^{m-\alpha}_{(a,b)}\varphi(x).$

■

Remark 2

The finite Riesz derivative introduced above could be so convenient to be applied in model connected with physics, engineering, and applied science (see for example, [20-23]). Also we must point out that The finite Riesz derivative is a complementary operator to the well known Riesz derivative given explicitly by Samko et al. [24], in the d-dimensional case, as follows:

$$(-\Delta)^{\alpha/2}_{d} f(x) = \frac{-\Gamma\left[(d-2+\alpha)/2\right]}{\pi^{(2-\alpha)/2} 2^{2-\alpha}\Gamma\left[(2-\alpha)/2\right]}\int_{\Omega}\frac{\Delta f(\xi)}{\|x-\xi\|^{d-2+\alpha}} d\Omega(\xi)$$
$$= -I^{2-\alpha}_{d}\left[\Delta f(x)\right].$$

where $0 < \alpha < 2$, $x \in \Omega \subset \mathbb{R}^d$, *and*

$$I^{\beta}_{d} w(x) = \frac{\Gamma\left[(d+\beta)/2\right]}{\pi^{\beta/2} 2^{\beta}\Gamma\left[\beta/2\right]}\int_{\Omega}\frac{w(\xi)}{\|x-\xi\|^{d+\beta}} d\Omega(\xi).$$

Example 3

We can consider the space-time fractional diffusion equation in a finite space domain, which is obtained from standard diffusion equation in a finite space domain, by replacing in the standard diffusion equation

$$\frac{\partial}{\partial t} u(x,t) = C\frac{\partial^2}{\partial x^2} u(x,t), \quad x \in \mathbb{R},\ t \geq 0$$

the second-order space derivative by the finite Riesz derivative of order $\beta \in (1,2]$ *and the first-order time derivative by the Caputo derivative of order* $\alpha \in (0,1]$, *which can interpreted as a space and time derivative of fractional order, we obtain a sort of generalized diffusion equation.*

$$\mathbf{D}^{\alpha}_{t} u(x,t) = C R^{\beta}_{x} u(x,t),\ x \in (a,b),\ t \geq 0.$$

The infinite space domain was investigated with respect to its scaling and similarity properties in [14].

$$\mathbf{D}_t^{\alpha} u\left(x,t\right) = C R_x^{\beta} u\left(x,t\right), \;\; x \in \mathbb{R}, \;\; t \geq 0 \tag{3.2}$$

with to the initial condition

$$u\left(x,0\right) = \delta\left(x\right), \;\; x \in \mathbb{R}, \tag{3.3}$$

where $\mathbf{D}_t^{\alpha} \equiv \mathbf{D}_{t,0+}^{\alpha}$ *is the Caputo time fractional derivative of order* α*, with respect to* t*,* $R_x^{\beta} \equiv R_{(a,b),x}^{\beta}$ *is the space finite Riesz derivative of order* β *in* (a,b)*,* C *is the positive coefficient of diffusion,* $\alpha \in (0,1]$*,* $\beta \in (1,2]$*,* $u(x,t)$ *is a real function in the time-space variables, and* $\delta\left(x\right)$ *is the Dirac delta function. So the solution of* (3.2) *with the initial condition* (3.3) *is given as [14]*

$$u_{\alpha,\beta}\left(x,t\right) = \int_{-\infty}^{\infty} G_{\alpha,\beta}\left(x-y,t\right)\delta\left(y\right)\;dy,$$

where

$$G_{\alpha,\beta}\left(x,t\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} E_{\alpha}\left[C(-ik)^{\beta} t^{\alpha}\right]\;dk.$$

and $E_{\alpha}\left(z\right)$ *is the Mittag-Leffler function defined as [22]*

$$E_{\alpha}\left(z\right) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\left(1+n\alpha\right)}.$$

Acknowledgement

The authors would like to acknowledge a useful discussion with Professor Juan J. Trujillo.

References

[1] G. Adomian, A new approach to the heat equation-an application of the decomposition method, *J. Math. Anal. Appl.* (1986), **113**, 202-209.

[2] G. Adomian, Modification of the decomposition approach to the heat equation, *J. Math. Anal. Appl.* (1987), **124**, 290-291.

[3] G. Bateman and A. Erdélyi, *Higher Transcendental Functions*, vol. 1, McGraw-Hill, New York 1953.

[4] A. N. Bondarenko, D. S. Ivaschenko, Levenberg–Marquardt method for restoration of layered medium key parameters, *Korus* (2005), 43-46.

[5] W. Chen and S. Holm, Fractional Laplacian time-space models and Lévy stable distribution for linear and nonlinear frequency-dependent lossy media, *Research Report Simula Research Lab.*, (2002), No. 2002-9.

[6] W. Chen and S. Holm, Lévy stable distribution and [0,2] power dependence of the absorption coefficient on the frequency, *Chin. Phys. Lett.*, (2005), **22**(10), 2601-2603.

[7] W. Chen and S. Holm, Fractional Laplacian time-space models for linear and nonlinear lossy media exhibiting arbitrary frequency dependency, *J Acoust Soc. Am.* (2004), **115**, 1424-1430.

[8] W. Chen, Time-space fabric underlying anomalous diffusion, *Chaos, Solitons and Fractals*, (2006), **28**, 923–929.

[9] K. Diethelm, *Fractional Differential Equations, Theory and Numerical Treatment*, manuscript, http://wwwpublic.tu-bs.de/~diethelm/lehre/f-dgl/fde-skript.ps.

[10] A.M.A. El-Sayed, Finite Weyl fractional calculus and abstract fractional differential equations, *Fract. Calc.* (1996), **9**, 59-68.

[11] A.M.A. El-Sayed, Fractional-order diffusion-wave equation, *Int. J. Theoretical Physics*, (1996), **35** (2), 311-322.

[12] R. Gorenflo and F. Mainardi, Essentials of Fractional Calculus, www.maphysto.dk/oldpages/events/ LevyCAC2000/MainardiNotes/fm2k0a.ps.

[13] R. Hilfer (Ed.), *Applications of Fractional Calculus in Physics*, World Scientific Publ. Co., Singapore, 2000.

[14] F. Hunag and F. Liu, The space-time fractional diffusion equation with Caputo derivatives, J. Appl. Math. & Computing (2005), **19**, (1-2),179-190.

[15] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies, Vol. 204, Elsevier, Amsterdam, 2006.

[16] D. Lesnic, A computational algebraic investigation of the decomposition method for time-dependent problems, *Appl. Math. Comput.* (2001), **119**, 197-206.

[17] D. Lesnic, The Cauchy problem for the wave equation using the decomposition method, *Appl. Math. Lett.* (2002), **15**, 697-701.

[18] D. Lesnic, The decomposition method for forward and backward time-dependent problems, *J. Comput. Appl. Math.* (2002), **147**, 27-39.

[19] F. Mainardi, Y. F. Luchko and G. Pagnini, The fundamental solution of the space-time fractional diffusion equation, *Fract. Calc. Appl. Anal.* (2001), **4**, 153-192.

[20] F. Mainardi and G. Pagnini, Space-time fractional diffusion: exact solutions and probability interpretation. In: *"WASCOM 2001"-11th Conference on Waves and Stability in Continuous Media*, World Sci. Publishing, River Edge, New Jersey, 2002, 296-301.

[21] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: A fractional dynamics approach, *Phys. Rep.*, (2000), **339**, 1-77.

[22] I. Podlubny, Fractional Differential Equations, Academic press, San Diego, 1999.

[23] S. Saha-Ray and R.K. Bera, Analytical solution of a fractional diffusion equation by Adomian decomposition method, *Appl. Math. and Comput.* (2006), **174**, 329-336.

[24] S. Samko, A. Kilbas, and O. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, London, 1993.

[25] N. T. Shawagfeh, Analytical approximate solutions for nonlinear fractional differential equations, *Appl. Math. and Comput.* (2002), **131**, 517-529.

[26] B. J. West, *Physiology, Promiscuity and Prophecy at the Millennium. A Tale of Tails*, World Scientific Publishing Company, Singapore, 1999.

[27] G. M. Zaslavsky, *Chaos and Fractional Dynamics*, Vol. 511 of Lect. Notes in Phys., Oxford University Press, Oxford, 2005.

EJTP **3, No. 12** (2006) 97–116 *Electronic Journal of Theoretical Physics*

Numerical Classical and Quantum Mechanical Simulations of Charge Density Wave Models

A. W. Beckwith*

Department of Physics and Texas Center for Superconductivity and Advanced Materials at the University of Houston Houston, Texas 77204-5005, USA

Received 13 February 2006, Accepted 20 April 2006, Published 20 September 2006

Abstract: First, using a driven harmonic oscillator model by a numerical scheme formulated by Littlewood, we present a computer simulation of charge density waves (CDW); next, we use this simulation to show how the dielectric model presented via this procedure leads to a blow up at the initialization of a threshold field E_T. Finding this approach highly unphysical, we initiated inquiry into alternative models. We investigate how to present the transport problem of CDW quantum mechanically, through a numerical simulation of the massive Schwinger model. We find that this single-chain quantum mechanical simulation used to formulate solutions to CDW transport is insufficient for transport of soliton-antisolitons (S-S') through a pinning gap model of CDW. We show that a model Hamiltonian with Peierls condensation energy used to couple adjacent chains (or transverse wave vectors) permits formation of S-S' that can be used to transport CDW through a potential barrier. This addition of the Peierls condensation energy term is essential for any quantum model of CDW to give a numerical simulation to tunneling behavior.

Keywords: Charge-Density-Wave Systems (CDW), Schwinger Equation, CDW Simulations, Solitons, Tunneling
PACS (2006): 03.75.Lm, 71.45.Lr, 71.55.-i , 78.20.Ci, 85.25.Cp

1. Introduction

The classical charge density wave (CDW) transport model, as presented by Gruner, answers a host of CDW questions associated with electrodynamic phenomena. However, as we show, we obtained a very non linear blow up of the calculated dielectric response of $NbSe_3$, which indicates that the Gruner model requires revision. This lead to inves-

* projectbeckwith2@yahoo.com

tigations of first a single-, then a multi-chain model of CDW based upon the massive Schwinger equation model, with results we discuss herein.

Previously, we used the integral Bogomil'nyi inequality to show how a soliton-anti soliton (S-S') pair could form [1], [2]. Here, we argue that this is equivalent to putting in a so called multi-chain interaction term with a constant term in it proportional to the Peierls gap times a cosine term representing interaction of different CDW chains in our massive Schwinger model, [3] which is highly unusual since at first glance adding in an additional potential energy term makes the problem look like a Josephon junction problem with no connection to the fate of the false vacuum hypothesis. We found that a single-chain simulation of the problem suffers from two defects. First, it does not answer what are the necessary and sufficient conditions for formation of a S-S'. More importantly, we also find through numerical simulations of the single-chain transport model that one needs additional physical conditions to permit barrier penetration. Our numerical simulation of the single chain problem for CDW involving S-S' gave a resonance condition in transport behavior over time, with no barrier tunneling. We argue that the false-vacuum hypothesis [1,2,4] is a necessary condition for the formation of S-S' pairs — and that the multi-chain term we add to a massive Schwinger equation for CDW transport is a sufficiency condition for the explicit formation of a S-S' in our CDW transport problem. We initiate the second quantum mechanical section of this monograph by a numerical simulation of the single chain model of CDW, then show how addition of the Peierls condensensation energy permits a S-S' to form. Then, we present how to numerically simulate a multi-chain CDW simulation.

2. Presenting the Classical Washboard Potential Using Littlewood'S Random Pinning Model

In 1986, Littlewood [5] presented an innovative scheme which incorporates a classical phase pinning model of Fukuyama, Lee, and Rice [6] [7] for the interaction of impurities in a one dimensional setting. We note that, in numerical form, Littlewood's scheme bears striking semblance to the Sine-Gordon equation [8] for evolution of phase values along a one dimensional crystal. The impurity sites are randomly distributed in one dimension; we have that the phase term $\phi(x)$ represents the local 'position' of a charge density wave which interacts via an interaction potential of

$$V_j(x - R_j) \equiv V \cdot \delta(x - R_j)$$

which is a short range interaction between the phase $\phi(x)$ and an impurity site R_j. V turns out to be a strength of interaction term which we set equal to unity in our simulation, and the randomly chosen position of impurities,R_j, happens to be chosen via a random number generator in our simulation, and this selection done in such a way as to avoid bunching about certain fixed numerical quantities in a one dimensional line. This necessitated an ordering insuring $R_{j+1} > R_j$. Given this, Littlewood used an overdamped equation of motion, as well as dimensionless units, in order to give an evolution equation with a first order derivative of phase with respect to time, assuming that phase $\phi(x)$ responds 'instantly' to the effects of an extremely localized interaction of phase with each impurity site given by R_j. This last assumption permits us to integrate between impurity sites so as to come up with a first order in time evolution equation for the phase $\phi(x_j) \equiv \phi_j$, where each $x_j \equiv c \cdot R_j$. The constant, c, is the impurity concentration and assumes that we have a correlation length L so that we observe $c \cdot L^d >> 1$; that is, we have weak impurity pinning (here, d is the dimensionality of the spatial integration). In our model, we set d=1.0. The remainder of this section on classical models is to look at the consequences of taking a discrete Fourier transform (DFT) of current $J \approx \left\langle \dot{\phi} \right\rangle$ to obtain computed conductivity and dielectric values due to the evolution of CDW along a one-dimensional crystal.

Several caveats are in order. First, we had to keep impurity sites from clustering too closely about the origin. When this occurs, we obtain wildly divergent computed numerical values for several computed physical quantities, especially, the derivative of phase with respect to time; this leads to spurious results for conductivity even when the applied E field is < E_{th}. In fact, the scheme became so unstable that, when we had numerous impurity sites near the origin, the derivative of phase with respect to time would blow up after only several dozen time steps from an initial time. In contrast with this instability, quite stable values of the derivative of phase with respect to time exist so long as the applied field to a quasi one-dimensional metal sample (e.g., $NbSe_3$)wa s less than a strength V and applied electric field E having their dimensions *rescaled* by variable changes to non-dimensional constants. However, it is important to note that Eq. 2.1 uses a non-uniform distribution of impurity sites, when there is an interaction between phase and ions in a one-dimensional setting. However, the $\Delta\phi_i$ term in Eq. 2.1 represents the interaction between adjacent impurity sites and shows compression (or deformation) of the CDW phase, while assuming the impurity sites as given by $X_i = cR_i$ have a random distribution of R_i values, while having $R_i > R_{i-1}$.

$$\dot{\phi}_i = \Delta^2\phi_i + \frac{1}{2}E(X_{i+1} - X_i) + V\sin(\theta_i + \phi_i) \tag{2.1}$$

Eq. 2.1 is due, in part to setting the acceleration term $\ddot{\phi}$ in the *sliding condition* (uniform spacing for impurities) for CDW equal to zero (called *deep damping* due to importance of the $\frac{1}{\tau}\dot{\phi}$ term) while then, next, randomizing the position of impurity sites that is initially set equally spaced in Eq. 2.1. The θ_i expression in Eq. 2.1 is a randomized force term that varies according to a random generation of numerical values between zero and 2π. Furthermore, although the sliding criteria for CDW mentioned in Eq. 2.2 assumes no spatial compression (meaning the presence of CDW only, but of no soliton), we can specifically show a distinct spatial behavior for the ϕ phases as generated by Eq. 2.1 above. We now refer to the uniform spacing between impurity sites equation for the evolution of phase values, by

$$\ddot{\phi} + \frac{1}{\tau}\dot{\phi} + \omega_0^2\sin\phi = e \cdot \frac{Q}{M_F} \cdot E(t) \tag{2.2}$$

Eq. 2.1 explicitly uses $\phi_i = \phi(X_i)$ where $X_{i=}$ c R $_i$ and c represents impurity concentration for each impurity site on a one-dimensional line. R $_i$ represents each place on a one-dimensional line for each impurity site and is a randomly set, monotonically increasing function for each i_{th} index that grows larger. We also used a discretized second derivative.[9], [10], [11]

$$\Delta^2\phi_i = \frac{\phi_{i+1} - \phi_i}{X_{i+1} - X_i} - \frac{\phi_i - \phi_{i-1}}{X_i - X_{i-1}} \tag{2.3}$$

If we look at the first end point of the impurity sites, this procedure leads to a re-write of Eq. 2.3, which looks like [12]

$$\Delta^2\phi_1 = \frac{\phi_2 - \phi_1}{X_2 - X_1} - \frac{\phi_1 - \phi_N}{X_1 - (X_N - L)} \tag{2.4}$$

where L is the grid length used in this simulation of CDW dynamics.

For the sake of including in both DC and AC contributions to an electric field, we can write

$$E = E_{dc} \tag{2.5}$$

and/or

$$E = E_{dc} + E_{ac}sin(\omega\tau) \tag{2.6}$$

When these electric field values are put into both Eq. 2.1, we may then examine dielectric plots which are plotted against increasing frequency according to:

$$Re\varepsilon(\omega) = 4\pi\left(\frac{Im\sigma(\omega)}{\omega}\right) \tag{2.7}$$

and

$$Im\varepsilon(\omega) = 4\pi\left(\frac{Re\sigma(\omega)}{\omega}\right) \tag{2.8}$$

$$Re\sigma(\omega) \propto g1 \cdot \sum_n \left\langle \dot{\phi} \right\rangle_n \cdot \cos(\omega t_n) \cdot \Delta t \tag{2.9}$$

as well as

$$Im\sigma(\omega) \propto g1 \cdot \sum_n \left\langle \dot{\phi} \right\rangle_n \cdot \sin(\omega t_n) \cdot \Delta t \tag{2.10}$$

As written, the derivative of phase used here is from a second-order Runge-Kutta simulation, which was chosen for robustness of simulation. Having a higher-order accurate simulation for the derivative of phase, as symbolically indicated above placed in what appears to be a first-order calculation of conductivity would effectively negate the entire purpose of improved accuracy of taking the derivative of the phase calculation, as symbolically referred to in Eq. 2.1. We must perform the DFT inside the Runge-Kutta subroutine initially chosen to analyze the left side of Eq. 2.1 accurately. Otherwise, round-off error from the first-order conductivity calculation dominates, negating the second-order calculations used for the current calculation. We find that if we re-scale dielectric measurements, we re-scale dielectric measurements versus an applied electric field by resetting $\varepsilon/\varepsilon_{initial}$ in place of just ε versus E field (applied to an experimental sample), and that as the frequency ω gets much smaller than ω_c, we observe increasingly non-linear dielectric behavior as the E field approaches E_{th}. This is seen in Figures 2a, 2b,

3. Review of the Q.M. Numerical Behavior of a Single Chain for Cdw Dynamics

Partly due to the failure of the classical model to avoid a blow up of the dielectric constant, addressed in the second section, we review alternate computational models that could provide some of the numerical behavior that has more overlap with known experimental features seen in previous device-development lab (TcSAM) experiments that were performed in the 1990s up to 2000. First, we examine. a quantum mechanical CDW model introduced by Dr. Miller, which answers certain physical issues but that we found required building in additional features.

We are modifying a one chain model of CDW transport initially pioneered by Dr. John Miller that furthered Dr. John Bardeen's work on a pinning gap presentation of CDW transport and that involves a Hamiltonian modeling how CDW would move via modeling with S-S' pairs. Qualitatively, the single-chain model is a useful way to introduce how a threshold electric field would initiate transport. We did, however, assume that the CDW would be easily modeled with a S-S' Gaussian packet, which is what we found needs further justification. With these considerations, we undertook this investigation to determine, among other things, the necessary and sufficient condition to physically justify use of a S-S' for our wave packet.

We start by using an extended Schwinger model [3], [13] with the Hamiltonian set as

$$H = \int_x \left[\frac{1}{2 \cdot D} \cdot \Pi_x^2 + \frac{1}{2} \cdot (\partial_x \phi_x)^2 + \frac{1}{2} \cdot \mu_E^2 \cdot (\phi_x - \varphi)^2 + \frac{1}{2} \cdot D \cdot \omega_P^2 \cdot (1 - \cos\phi) \right] \tag{3.1}$$

as well as working with a quantum mechanically based energy

$$E = i\hbar \frac{\partial}{\partial t} \tag{3.2a}$$

and momentum

$$\Pi = (\hbar/i) \cdot \frac{\partial}{\partial \phi(x)} \tag{3.2b}$$

The first case we are considering is a one-chain mode situation. Here, in order to introduce a time component, $\Theta \equiv \omega_D t$ was used explicitly as a driving force, while using the following difference equation due to using the Crank Nickelson [14] scheme. We should note that ω_D is a driving frequency to this physical system which we were free to experiment with in our simulations. The first index, j, is with regards to *space*, and the second, n, is with regards to *time* step. Eq. 3.3 is a numerical rendition of the massive Schwinger model plus an interaction term, where one is calling $E = i\hbar\frac{\partial}{\partial t}$ and one is using the following replacement

$$\begin{aligned} &\phi(j, n+1) = \phi(j, n-1) \\ &+ i \cdot \Delta t \cdot \begin{pmatrix} \frac{\hbar}{D} \left[\frac{\phi(j+1,n) - \phi(j-1,n) - 2\cdot\phi(j,n) + \phi(j+1,n+1) + \phi(j-1,n+1) - 2\phi(j,n+1)}{(\Delta x)^2} \right] \\ - \frac{2\cdot V(j,n)}{\hbar} \phi(j,n) \end{pmatrix} \end{aligned} \tag{3.3}$$

We use these variants of Runge-Kutta in order to obtain a sufficiently large time step interval so as to be able to finish calculations in a reasonable period of time, while avoiding an observed spectacular blow up of simulated average phase values; one so bad that one gets nearly infinite wave function values after, say 100 time steps at $\Delta t \approx 10^{-13}$. Stable Runge-Kutta simulations require $\Delta t \approx 10^{-19}$. Otherwise, one would need up to half a year on a PC in order to get the graph presented in Figure 4 below.

A second numerical scheme, the Dunford-Frankel, which is implicit [14], allows us to expand the time step even further. Then, the 'massive Schwinger model' equation has:

$$\phi\left(j,n+1\right) = \frac{2\cdot\tilde{R}}{1+2\cdot\tilde{R}}\cdot\left(\phi\left(j-1,n\right)-\phi\left(j+1,n\right)\right)+\frac{1-2\cdot\tilde{R}}{1+2\cdot\tilde{R}}\cdot\phi\left(j,n-1\right) \qquad (3.4)$$

$$-i\cdot\Delta t\frac{V(j,n)}{\hbar}\phi\left(j,n\right)$$

where one has $\tilde{R} = -i\cdot\Delta t\frac{\hbar}{2\cdot D\cdot(\Delta x)^2}$ .The advantage of this model is that it is second-order accurate, explicit, and unconditionally stable, so as to avoid numerical blow up behavior. One then gets resonance phenomena as represented by Figure 4. This is, to put it mildly, quite unphysical and necessitates making the changes that we present in this manuscript.

4. Addition of an Additional Term in the Massive Schwinger Equation to Permit Formation of A S-S' in Our Model

Initially, we show how addition of an interaction term between adjacent CDW chains will allow a S-S' to form due to analytical considerations that we outline here. Next, we show in a numerical simulation how these terms could lead to quantum tunneling. Finally we shall endeavor to show how our argument with the interaction term ties in with the fate of the false vacuum construction of S-S' terms performed when we used the Bogomil'nyi inequality [2], [15] as a necessary condition to the formation of S-S' term. Let us now first refer to how we can obtain a soliton via assuming that adjacent CDW terms can interact with each other.

There is an interesting interplay between the results of using the Bogomil'nyi inequality [2], [15] to obtain a S-S' pair which we approximate via a domain thin wall approximation [2], [16] and the nearest neighbor approximation of how neighboring chains interrelate with one another to obtain a representation of phase-evolution as an arctan function w.r.t. space and time variables. To whit, we can say that the Bogomil'nyi inequality provides for the necessity of a S-S' pair nucleating via a Gaussian approximation, while the interaction of neighboring chains of CDW material permits the existence of S-S' in CDW transport.

The Bogomil'nyi inequality [2], [15] permits the nucleation of a S-S' pair, whereas in the argument we advance here is also pertinent whether or not we have the existence of an individual S-S'. This assumes we are using Δ' as a Peierls gap [17] energy term as an upper-bound for energy coupling between adjacent CDW chains. Note that in the argument about the formation of a S-S', we use the following equation for a multi chain simulation Hamiltonian with Peierls condensation energy [3], [17] used to couple adjacent chains (or transverse wave vectors):

$$H = \sum_n \left[\frac{\Pi_n^2}{2 \cdot D_1} + E_1 \left[1 - \cos \phi_n \right] + E_2 \left(\phi_n - \Theta \right)^2 + \Delta' \cdot \left[1 - \cos \left(\phi_n - \phi_{n-1} \right) \right] \right] \quad (4.1a)$$

with ‘momentum ‘ we define as

$$\Pi_n = (h/i) \cdot \frac{\partial}{\partial \phi_n} \quad (4.1b)$$

We can reverse engineer this Hamiltonian to come up with an equation of motion which leads to a soliton, via use of taking the potential in Eq. 3.1a and then use a nearest neighbor approximation to use a Lagrangian based calculation of a chain of pendulums coupled by harmonic forces to obtain a differential equation which has a soliton solution. To do this, if we say that the nearest neighbors of the adjacent chains make the primary contribution, we may write the interaction term in the potential of this problem to be [3]

$$\Delta' \left(1 - \cos \left[\phi_n - \phi_{n-1} \right] \right) \rightarrow \frac{\Delta'}{2} \cdot \left[\phi_n - \phi_{n-1} \right]^2 + \text{ very small H.O.T.s} \quad (4.2)$$

and then considered a nearest neighbor interaction behavior via

$$V_{n.n.} (\phi) \approx E_1 \left[1 - \cos \phi_n \right] + E_2 \left(\phi_n - \Theta \right)^2 + \frac{\Delta'}{2} \cdot \left(\phi_n - \phi_{n-1} \right)^2 \quad (4.3)$$

Here, we have that $\Delta' >> E_1 >> E_2$, so then we had a round off of

$$V_{n.n.} (\phi) \Big|_{first\ order\ roundoff} \approx E_1 \left[1 - \cos \phi_n \right] + \frac{\Delta'}{2} \cdot \left(\phi_{n+1} - \phi_n \right)^2 \quad (4.4)$$

which then permits us to write

$$U \approx E_1 \cdot \sum_{l=0}^{n+1} \left[1 - \cos \phi_l \right] + \frac{\Delta'}{2} \cdot \sum_{l=0}^{n} \left(\phi_{l+1} - \phi_l \right)^2 \quad (4.5)$$

which allowed us, eventually, to obtain using $L = T - U$ a differential equation of

$$\ddot{\phi}_i - \omega_0^2 \left[\left(\phi_{i+1} - \phi_i \right) - \left(\phi_i - \phi_{i-1} \right) \right] + \omega_1^2 \sin \phi_i = 0 \quad (4.6)$$

with

$$\omega_0^2 = \frac{\Delta'}{m_{e^-} l^2} \quad (4.7)$$

and

$$\omega_1^2 = \frac{E_1}{m_{e^-} l^2} \quad (4.8)$$

where we assume the chain of pendulums, each of which is of length l actually will lead to a kinetic energy

$$T = \frac{1}{2} \cdot m_{e^-} l^2 \cdot \sum_{j=0}^{n+1} \dot{\phi}_j^2 \quad (4.9)$$

where we neglect the E_2 value. However, as we state in our derivation of the formation of a S-S' pair, having $E_2 \to \varepsilon^+ \approx 0^+$ would tend to lengthen the distance between a S-S' pair nucleating, with a tiny value of $E_2 \to \varepsilon^+ \approx 0^+$, indicating that the distance L between constituents of an S-S' pair would get very large. We did, however, find that it was necessary to have a large Δ' for helping us obtain a Sine-Gordon equation. This is so that when we set the horizontal distance of the pendulums to be d, the chain is of length $L' = (n+1)d$. Then, if mass density is $\rho = m_{e^-}/d$ and we model this problem as a chain of pendulums coupled by harmonic forces, we set an imaginary bar with a quantity η as being the modulus of torsion of the imaginary bar, and $\Delta' = \eta/d$. We have an invariant quantity, which we will designate as: $\omega_0^2 d^2 = \frac{\eta}{\rho \cdot l^2}$ =v^2, which, as n approaches infinity, allows us to write a Sine-Gordon

$$\frac{\partial^2 \phi(x,t)}{\partial t^2} - v^2 \frac{\partial \phi(x,t)}{\partial x^2} + \omega_1^2 \sin\phi(x,t) = 0 \tag{4.10}$$

with a way to obtain soliton solutions. In order to obtain soliton solitons, we introduce dimensionless variables of the form $z = \frac{\omega_1}{v} \cdot x$ ' $\tau = \omega_1 \cdot t$, leading us to finally obtain a dimensionless Sine–Gordon equation, which we write as:

$$\frac{\partial^2 \phi(z,\tau)}{\partial \tau^2} - \frac{\partial^2 \phi(z,\tau)}{\partial z^2} + \sin\phi(z,\tau) = 0 \tag{4.11}$$

so that

$$\phi_\pm(z,\tau) = 4 \cdot \arctan\left(\exp\left\{\pm \frac{z + \beta \cdot \tau}{\sqrt{1-\beta^2}}\right\}\right) \tag{4.12}$$

where we can vary the value of $\phi_\pm(z,\tau)$ between 0 to $2 \cdot \pi$. Below is an example of how one can do just that: When one is looking at $\phi_+(z,\tau)$ and set $\beta = -.5$, where one has $\tau = 0$ one can have $\phi_+(z << 0, \tau = 0) \approx \varepsilon \approx 0$ and, also, have $\phi_+(z = 0, \tau = 0) = \pi$; whereas for sufficiently large z one can have $\phi_+(z, \tau = 0) \to 2 \cdot \pi$. In a diagram with z as the abscissa and $\phi_+(z,\tau)$ as the ordinate, this soliton field from 0 to $2 \cdot \pi$ propagates with increasing time in the positive z direction and with a dimensionless velocity of β. In terms of the original variables, the soliton so modeled moves with velocity $v \cdot \beta$ in either the positive or negative x direction. One gets a linkage with the original pendulum model linked together by harmonic forces by allowing the pendulum chain as an infinitely long rubber belt whose width is l and which is suspended vertically. What we have described is a flip over of a vertical strip of the belt from $\phi = 0$ to $\phi = 2 \cdot \pi$ which moves with a constant velocity along the rubber belt. This motion is typical of the soliton we have managed to model mathematically from our potential terms above. It is very important to keep in mind the approximations used above. First, we are using the nearest neighbor approximation to simplify equation 4.4. Then, we are assuming that the contribution to the potential due to the driving force $E_2(\phi_n - \Theta)^2$ is a second order effect. All of this in its own way makes for an unusual physical picture, namely that the 'capacitance' effect given by $E_2(\phi_n - \Theta)^2$ will not be a decisive influence in deforming the solution, and is a second order effect which is enough to influence the energy band structure the soliton will be tunneling through but is not enough to break up the soliton itself.

5. Computer Simulation Work for Multi Chain Representations of Cdw Transport

Now, our Peierls gap energy [3], [17] was added to the massive Schwinger equation model [13] precisely due to the prior resonance behavior with a one-chain computer simulation. We can now look at the situation with more than one chain. To do so, take a look at a Hamiltonian with Peierls condensation energy used to couple adjacent chains (or transverse wave vectors)[3], [18]:

$$H = \sum_n \left[\frac{\Pi_n^2}{2 \cdot D_1} + E_1 \left[1 - \cos \phi_n \right] + E_2 \left(\phi_n - \Theta \right)^2 + \Delta' \cdot \left[1 - \cos \left(\phi_n - \phi_{n-1} \right) \right] \right] \quad (5.1)$$

and $\Pi_n = \left(\hbar/_i \right) \cdot \partial/_{\partial \phi_n}$ and when we will use wave functions which are

$$\Psi = N \cdot \prod_j \left(a_1 \exp\left(-\alpha \cdot \phi_j^2 \right) + a_2 \exp\left(-\alpha \left(\phi_j - 2 \cdot \pi \right)^2 \right) \right) \quad (5.2a)$$

with a two chain analogue of

$$\Psi_{two\,chains} = N \cdot \prod_{n=1}^{2} \left(a_1 \exp\left(-\alpha \cdot \phi_j^2 \right) + a_2 \exp\left(-\alpha \left(\phi_j - 2 \cdot \pi \right)^2 \right) \right) \quad (5.2b)$$

If so, we put in the requirement of quantum degrees of freedom so that one has for each chain for a two dimensional case

$$\left| a_1 \right|^2 + \left| a_2 \right|^2 = 1 \quad (5.3)$$

that provides coupling between nearest neighbor chains. In doing so, we are changing the background potential of this to a different situation where one has multiple soliton pairs that are due to the Δ' term in which has huge cusps given which permit the existence of tunneling due to the band structure we will present as given in Figure 5, which we will describe in the next paragraph. We will first describe a two band structure and then generalize to a five band structure we will graph in Figure 5 later on. First for a two cusp band situation with dynamical structure we have two chain interactions which we will describe here first. We should note that in tandem with $NbSe_3$ being quasi one dimensional that $\alpha \approx \frac{1}{\sqrt{soliton\,width}}$. For phase co-ordinate ϕ_j, $\exp\left(-\alpha \cdot \phi_j^2 \right)$ is an unrenormalized Gaussian representing a S-S' centered at $\phi_j = 0$, and a probability of being centered there given by $|a_1|^2$. Similarly, $\exp\left(-\alpha \cdot (\phi_j - 2 \cdot \pi)^2 \right)$. is an unrenormalized Gaussian representing a 'soliton'(anti-soliton) centered at $\phi_j = 2 \cdot \pi$ with a probability of occurrence at this position given by $|a_2|^2$. We can use Eq. 5.3 to represent the total probability that one has some sort of tunneling through a potential given by Eq. 5.1 dominated by the term Δ' which dominates the dynamics we can expect due to Eq. 5.1. We then are working with

$$E\left(\Theta \right) = \left\langle \Psi_{two\,chains} \right| H_{two\,chains} \left| \Psi_{two\,chains} \right\rangle \quad (5.4)$$

We observe a band structure of sorts given by this minimum 'energy surface' given in the graph of Eq. 5.4 And we find that the term $\Delta^{'}$ given in Eq. 4.4 is needed in order to obtain a band structure in the first place. The situation in which we have a band structure with $\Delta^{'}(1-\cos[\phi_2-\phi_1])$ included [3],[18] becomes complicated when we use Fortran 90, since this would ordinarily imply coupled-differential equations, which are extremely unreliable to solve numerically. For a number of reasons, one encounters horrendous round off errors with coupled-differential equations solved numerically in Fortran. Thus, when the problem was completed, instead, using Mathematica software which appears to avoid the truncation errors Fortran 90 presents us if we use a PC. with standard techniques. Here is how the problem was presented before being coded for Mathematica: where one has $E_1 = E_p =$ pinning energy, $E_2 = E_c =$ charging energy, and $\Delta^{'} \cdot [1-\cos(\phi_2-\phi_1)]$ represents coupling between "degrees of freedom" of the two chains. For higher number of interacting chains, we generalize to $\Delta^{'} \cdot [1-\cos(\phi_n-\phi_{n-1})]$ When we had five interacting chains, the wave function was set to a different value than given in either Eq. 5.2 or Eq. 5.2a

$$\Psi_m(\phi_i) = \sum_{m=-2}^{2} b_m \exp(-\alpha(\phi_i - 2\cdot\pi\cdot m)) \tag{5.5}$$

with:

$$\sum_{m=-2}^{2} b_m^2 = 1 \tag{5.6}$$

we obtained a minimum energy 'band structure' with five adjacent parabolic arcs [3], [18]. We obtain a minimum energy out of this we can write as:

$$E = E_{\min} = \langle\Psi|\,\hat{H}\,|\Psi\rangle \tag{5.7}$$

where $D_1 = 174.091, E_p = .00001, E_c = .000001$ and $\Delta^{'} = .005$ for Hamiltonian

$$\hat{H}_{two\,chains} = \sum_{n=1}^{2}\left[\frac{\Pi_n^2}{2\cdot D_1} + E_1[1-\cos\phi_n] + E_2(\phi_n-\Theta)^2 + \Delta'\cdot[1-\cos(\phi_n-\phi_{n-1})]\right] \tag{5.8}$$

where minimum energy curves are set by the coefficients of the two wave functions, which are set as $b_{-2}, b_{-1}, b_0, b_1, b_2; c_{-2}, c_{-1}, c_0, c_1, c_2; \alpha$(which happens to be the wave parameter for Eq.5.6.

This leads to an energy curve given in Figure 5 where there are five local minimum potential energy values. It is a reasonable guess that for additional chains (i.e. if m bracketed by numbers > 2) that the number of local minimum values will go up, provided that one uses a modified version of numerical simulation wave function as given in Eq. 5.5. We performed the following to plot an average <phi> value, which we will represent in equation 5.10. The easiest way to put in a time dependence in the Hamiltonian (Eq. 5.8) is to provisionally set $\Theta = \omega_D t$ for the graphics presented, $\omega_D = 0.67$ M Hz. If we set $\Psi \equiv \Psi(\phi_1, \phi_2, \Theta)$ which has an input from the Hamiltonian $\hat{H}_{two\,chains}$

then we can set up an average phase, which we will call:

$$\Phi = \frac{1}{2}(\phi_1 + \phi_2) \tag{5.9}$$

where we calculate a mean value of phase given by [3], [18]

$$\langle \Phi(\Theta) \rangle = \int_{-\eta\pi}^{\eta\pi} \int_{-\eta\pi}^{\eta\pi} d\phi_1 d\phi_2 \frac{1}{2} \cdot (\phi_1 + \phi_2) \left| \Psi(\phi_1, \phi_2, \Theta) \right|^2 \tag{5.10}$$

The integral $\langle \Phi(\Theta) \rangle$ was evaluated by 'Nintegrate' of Mathematica, and was graphed against Θ in Figure 6, with $\eta = 20$

These total sets of graphs put together are strongly suggestive of tunneling when one has $\Delta \neq 0$ in $\hat{H}_{two\,chains}$.

The simulation results of Figure 6 are akin to a thin wall approximation leading to a specific shape of the S-S'pair in *phase*-space, which is also akin to when we have abrupt but finite transitions after long periods of stability [1], [2]. We can link this sort of abrupt transitions to what happens when we have a 'thin wall approximation ' as spoken of by Sidney Coleman in his "fate of the false vacuum" hypothesis [4] for *instanton* transitions. We do, however, need to verify if or not that the soliton solution to this problem is optimal for tunneling. Trying to show this will be the main reason for the next section treatment of how a multi-chain interaction will be a necessary condition for formation of S-S' in CDW transport problems. This is when we will be working with wave functionals of the form given by initial and final wave functionals looking like

$$\begin{aligned} \Psi_i[\phi(x)]|_{\phi\equiv\phi_{ci}} &= c_i \cdot \exp\left\{-\alpha \cdot \int dx \left[\phi_{ci}(x) - \phi_0(x)\right]^2\right\}, \\ \Psi_f[\phi(x)]|_{\phi\equiv\phi_{cf}} &= c_f \cdot \exp\left\{-\alpha \int dx \left[\phi_{cf}(x) - \phi_0(x)\right]^2\right\} \end{aligned} \tag{5.11a,b}$$

6. Conclusion: Setting up the Framework for a Field Theoretical Treatment of Tunneling

We have, in the above identified pertinent issues needed to be addressed in an analytical treatment of CDW transport. First, we should try to have a formulation of the problem of tunneling that has congruence with respect to the Sidney Coleman false-vacuum hypothesis. We make this statement based upon the abrupt transitions made in a multi-chain model of CDW tunneling that are identical in form to what we would expect in a thin-wall approximation of a boundary between true and false vacuums. Secondly, we can say that it is useful to keep a S-S' representation of solutions for CDW transport. Figure 5 and Figure 6 address minimum conditions for the formation of a S-S'; what we have outlined here, however, concerns when we want to have a band structure pertinent to tunneling analysis; then we should keep the Δ' term necessitated in coupling chains together in CDW transport analysis.

We explicitly argue that a tunneling Hamiltonian based upon functional integral methods is essential for satisfying the necessary and sufficient conditions for the formation of a S-S' pair. The Bogomil'nyi inequality stresses the importance of the relative unimportance of the driving force $E_2 \cdot (\phi_n - \Theta)^2$, which we drop out in our formation of a S-S' in our multi chain calculation. In addition, we argue those normalization procedures, plus assuming a net average value of the $\Delta' \left(1 - \cos\left[\phi_n - \phi_{n-1}\right]\right) \to \frac{\Delta'}{2} \cdot \left[\phi_n - \phi_{n-1}\right]^2 +$ small terms as seen in our analysis of the contribution to the Peierls gap contribution to S-S' pair formation in our Gaussian[1,2] wave functional Eqs. 5.11a,b, where the normalizing term $c_{i,f}$ would allow us to scale out an averaged out value of the $\Delta' \left(1 - \cos\left[\phi_n - \phi_{n-1}\right]\right) \to \frac{\Delta'}{2} \cdot \left[\phi_n - \phi_{n-1}\right]^2$, and representation of how S-S' pairs interact in a multi-chain model evolve in a pinning gap transport problem for CDW dynamics. This would allow us, if done, to have S-S' pairs being used in equation 5.13 due to their formation in our problem and due the Peierls gap term in the Hamiltonian [3], [18].

Then the fate of the false vacuum hypothesis used [1], [2], [3]so that the S-S' pairs can have nucleation behavior as seen in Figure 7.

is consistent with a Gaussian wave functional representation of transport behavior [1], [2] leading to [1], [2] matching with experimentally observed current behavior [2], [3] as seen in Figure 8. This would permit us to form necessary and sufficient conditions for permitting a Gaussian wave functional to use S-S' pairs to form the current experimentally observed in Figure 8, where after a long derivation [1], [2], [3]we have Figure 8 below.

where we write

$$I \propto \tilde{C}_1 \cdot \left[\cosh\left[\sqrt{\frac{2 \cdot E}{E_T \cdot c_V}} - \sqrt{\frac{E_T \cdot c_V}{E}}\right]\right] \cdot \exp\left(\frac{-E_T \cdot c_V}{E}\right) \tag{6.1}$$

with $\tilde{C}_1 \equiv \frac{C_1 \cdot C_2}{2 \cdot m^*}$

which is a significant improvement over a prior expression derived to qualitatively fit experimental data [19]

$$I \propto G_P \cdot (E - E_T) \cdot \exp\left(\frac{-E_T}{E}\right) \text{ if } E > E_T \tag{6.2}$$

We have, in the last several pages , indicated how the classical model breaks down. Specifically, a driven harmonic oscillator model gives a resonance phenomena which leads to singular dielectric and conductivity behavior which has no physical counter part in laboratory experiments. The computer simulations of a quantum model, essentially assuming a multi chain model , i.e. a modified Josephon junction , give correct behavior w.r.t. tunneling, as noted by our computer simulations with step behavior of the phase in potential. This is equivilent to the current model so noted at the end which improves upon a Zenier curve fit by giving non negative current values for an applied electric field below the threshold electric field.

That last result, called in TcSAM the "diamond" by those in the know at TcSAM of this problem, has lead to speculations of topological domain wall contributions to density wave physics. Needless to say all those theoretical developments would have been still born and not even brought up had these computer simulations not shown the necessity of a make over of the classical Washboard potential. What is presented here is the pre cursor of these developments, and essential physics which we have used to extrapolate what is a new re make of density wave evolution in quasi one dimensional metallic solids.

References

[1] A. W.Beckwith, J.H.Miller,Jr, arxiv : math-ph/0406048 (Tunneling Hamiltonian Representation of False Vacuum Decay I .Comparison with Bogomol'nyi Inequality ; A.W. Beckwith, J.H.Miller)

[2] A.W. Beckwith , J.H. Miller,jr ; arxiv :math-ph/0406053 (Tunneling Hamiltonian Representation of False Vacuum Decay II. Application to Soliton–Anti Soliton Pair Creation ; A.W. Beckwith, J.H.Miller)

[3] A.W. Beckwith 'Classical and Quantum models of Density wave transport: A comparative study' PhD dissertation , U. of Houston physics department , December 2001

[4] S. Coleman , Phys. Rev. D 15,2929(1977)

[5] B Littlewood, Phys. Rev B 33, 6694 (1986)

[6] H. Fukuyama and P.A. Lee , Phys. Rev B 17, 535 (1977)

[7] P. A. Lee and T.M. Rice, Phys. Rev B 19,3970 (1979)

[8] S. Kagoshima, H. Nagasawa, T. Sambongi ' One dimensional Conductors', Springer – Verlag, 1987, pp 32-34, in particular equation 2.74

[9] P.B Littlewood, Phys. Rev B 33, 6694 (1986), see equation 2.5

[10] P.B Littlewood, Phys. Rev B 33, 6694 (1986), see equation 2.6

[11] L. Pietronero and S. Strassler, Phys. Rev B 28, 5863 (1983);H. Matsukawa and H. Takayama, Solid State Commun. 50, 283(1984)

[12] Private communications with Dr. Bill Mayes II

[13] J. H. Miller,Jr. , C. Ordonez, and E. Prodan, Phys. Rev. Lett 84, 1555(2000)

[14] G. Smith,' Numerical Solutions of Partial Differential equations;Finite difference Methods' , third edition, Oxford, 1986 ; K. Morton Morton and D.Mayers; 'Numerical Solution of Partial Differential equations: An introduction'

[15] A. Zee, ' Quantum field theory in a nutshell' ; Princeton university press , 2003 pp. 279-80

[16] J. H. Miller, Jr., G. Cardenas, A. Garcia-Perez, W. More, and A. W. Beckwith, J. Phys. A: Math. Gen. 36, 9209 (2003).

[17] S. Kagoshima,H. Nagaswawa, T. Sambongi, ' One dimensional conductors', Springer Verlag 1987

[18] J. H. Miller, Jr., G. Cardenas, A. Garcia-Perez, W. More, and A. W. Beckwith, J. Phys. A: Math. Gen. 36, 9209 (2003).

[19] J.H. Miller,J. Richards,R.E. Thorne,W.G.Lyons and J.R. Tucker ; Phys.Rev. Lett. 55,1006 (1985)

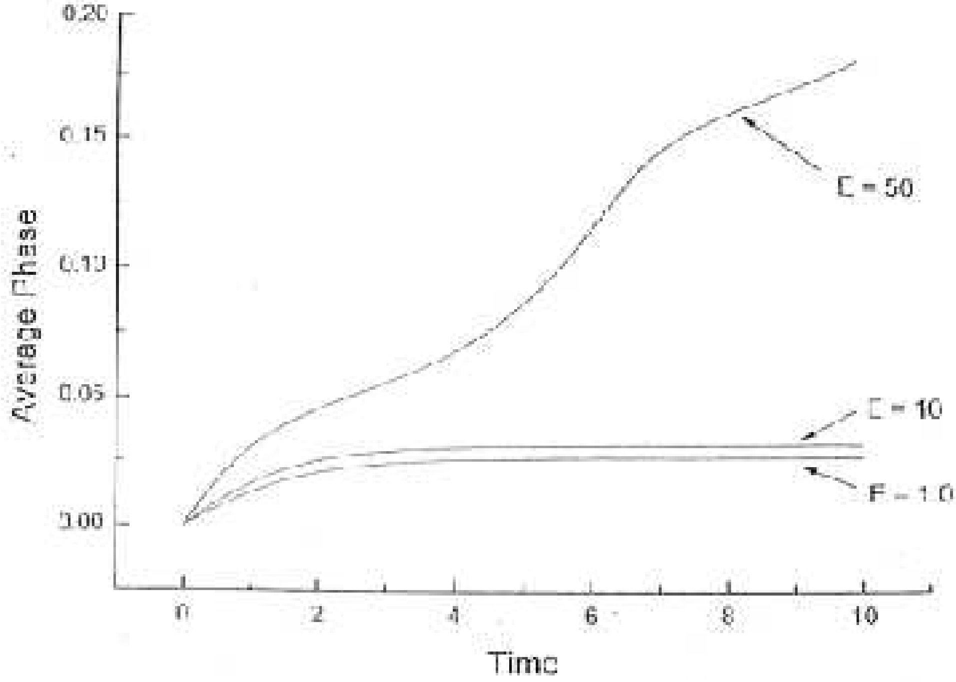

Fig. 1 Average phase $\langle\phi\rangle$ plotted against time (for E_{dc}) with $\langle\phi\rangle$ stabilizing if $E_{dc} < E_{th}$ and $\langle\phi\rangle$ monotonically increasing if $E_{dc} > E_{th.}$

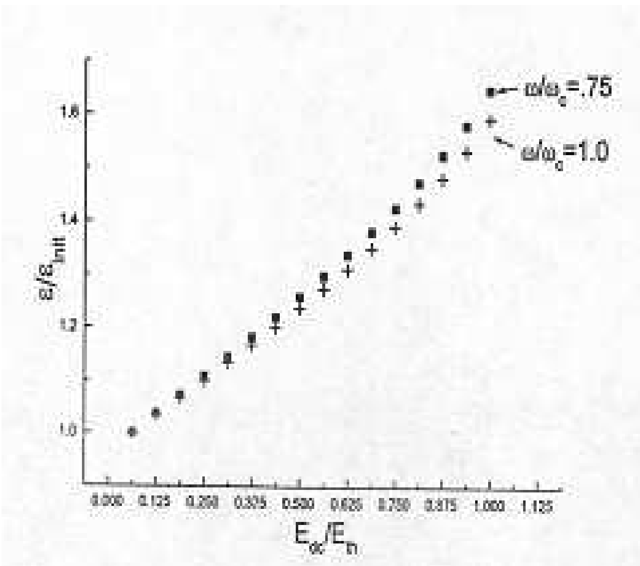

Fig. 2 a Comparison of scaled dielectric values when one has signal frequency $\omega \leq \omega_c$ i.e. near a critical value ω_c .**Figure 2a** applies to low frequency plots. One obtains the situation that there is a blow up of the dielectric response if one has the electric field exceeding a threshold value, which could not be duplicated numerically. The dielectric is infinite valued when $E=E_{th}$

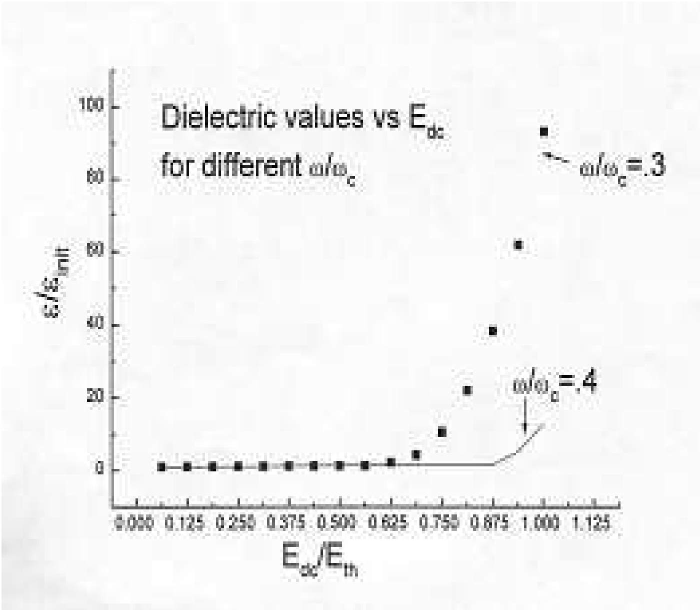

Fig. 2 b Comparison of scaled dielectric values when one has signal frequency $\omega \leq \omega_c$ i.e. near a critical value ω_c . **Figure 2b** to high frequency plots. One obtains the situation that there is a blow up of the dielectric response if one has the electric field exceeding a threshold value, which could not be duplicated numerically. The dielectric is infinite valued when E=E_{th}

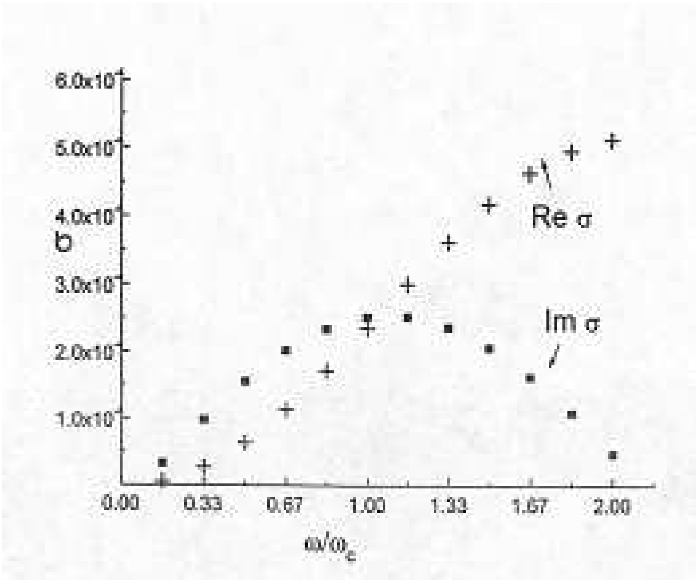

Fig. 3 This conductivity plot shows the origins of how we pick critical value ω_c

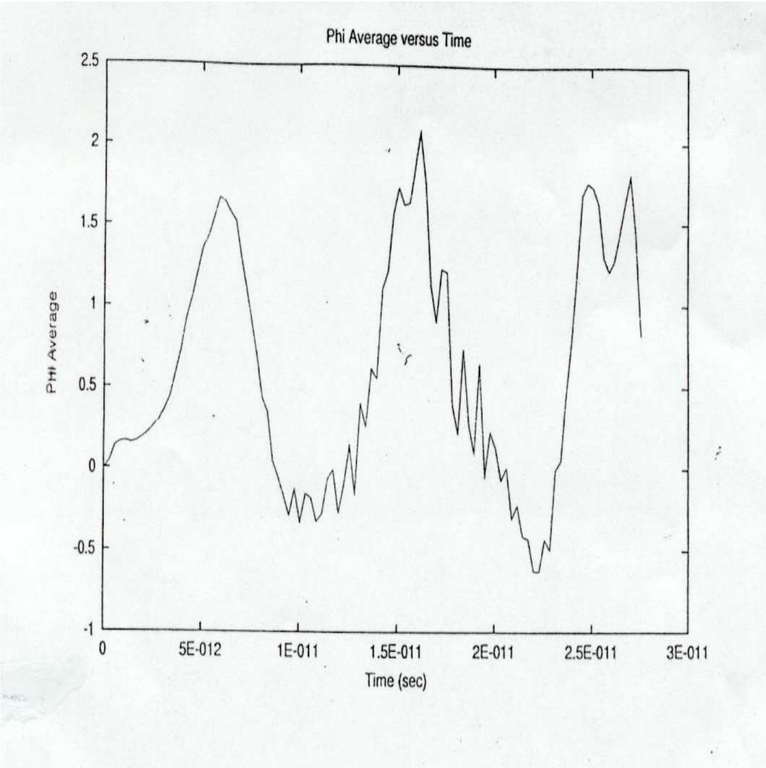

Fig. 4 Beginning of resonance phenomena in single chain quantum dynamics due to using the traditional Crank–Nickelson numerical iteration scheme of the one-chain model.

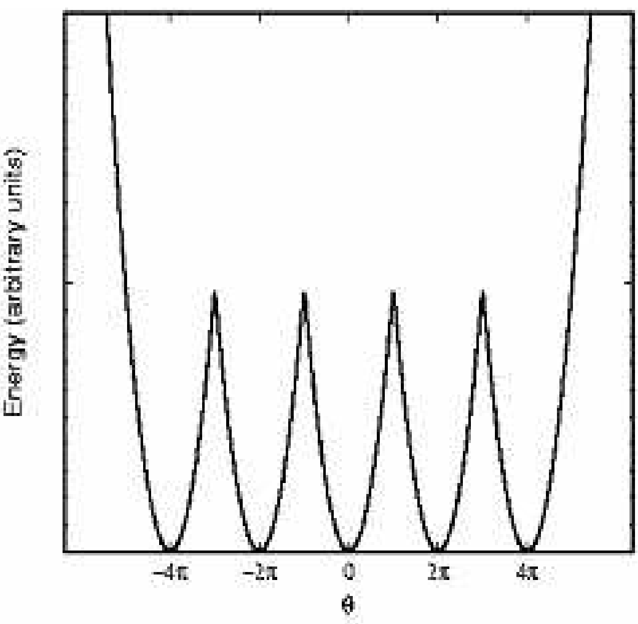

Fig. 5 Determining band structure via a *Mathematica 8* program, with wave functions given by Eq. 4.6.

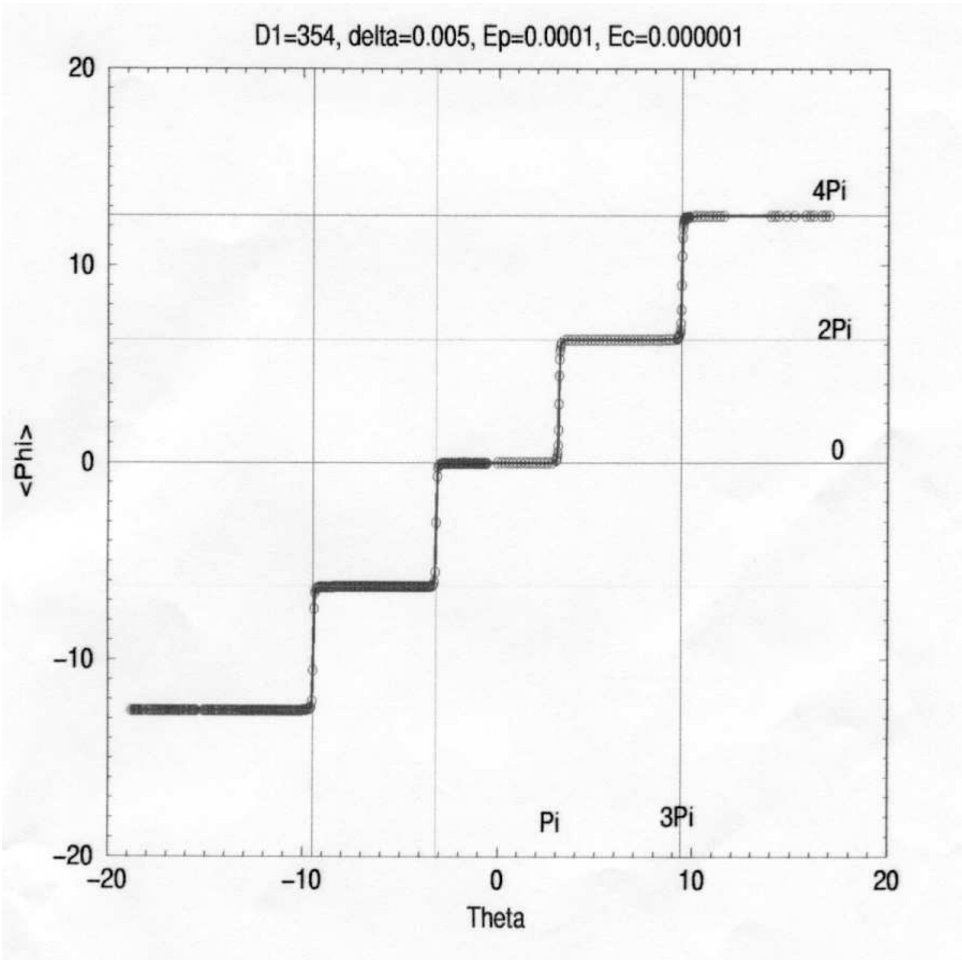

Fig. 6 Phase vs. Θ, according to the predictions of the 'multi-chain'-tunneling model.

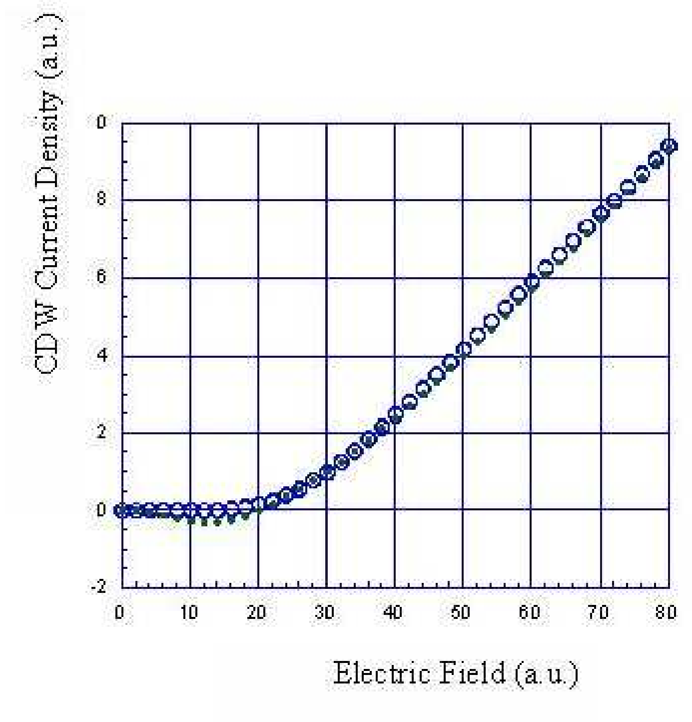

Fig. 7 Experimental and theoretical predictions of current values. The dots represent a Zenier curve fitting polynomial, whereas the blue circles are for the S-S' transport expression derived with a field theoretic version of a tunneling Hamiltonian.

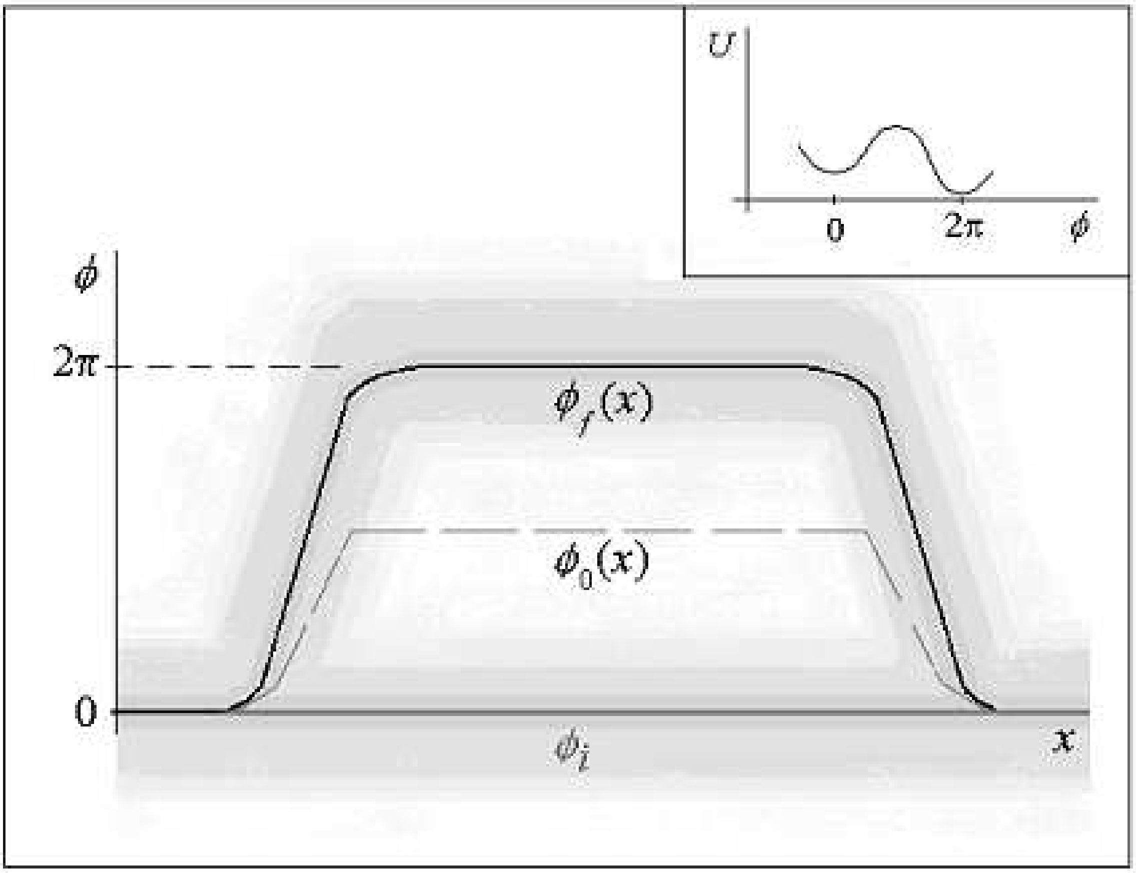

Fig. 8 Evolution from an initial state ϕ_i to a final state $\phi_f$for a double-well potential (inset) in a 1-D model, showing a kink-antikink pair bounding the nucleated bubble of true vacuum. The shading illustrates quantum fluctuations about the classically optimum configurations of the field $\phi_i\,(x)$= 0 and $\phi_f\,(x)$, while ϕ_0(x) represents an intermediate field configuration inside the tunnel barrier

EJTP **3, No. 12** (2006) 117–144 *Electronic Journal of Theoretical Physics*

A New Wave Quantum Relativistic Equation from Quaternionic Representation of Maxwell-Dirac Isomorphism as an Alternative to Barut-Dirac Equation

V. Christianto*

V. Christianto Via Florentin Smarandache,
Department of Mathematics, University of New Mexico Gallup, NM 87301, USA

Received 5 November 2005, Accepted 5 January 2006, Published 20 September 2006

Abstract: It is known that Barut's equation could predict lepton and hadron mass with remarkable precision. Recently some authors have extended this equation, resulting in Barut-Dirac equation. In the present article we argue that it is possible to derive a new wave equation as alternative to Barut-Dirac's equation from the known exact correspondence (isomorphism) between Dirac equation and Maxwell electromagnetic equations via biquaternionic representation. Furthermore, in the present note we submit the viewpoint that it would be more conceivable if we interpret the *vierbein* of this equation in terms of superfluid velocity, which in turn brings us to the notion of topological electronic liquid. Some implications of this proposition include quantization of celestial systems. We also argue that it is possible to find some signatures of Bose-Einstein cosmology, which thus far is not explored sufficiently in the literature. Further experimental observation to verify or refute this proposition is recommended.

Keywords: Relativistic Quantum Mechanics, Barut's equation, Barut-Dirac's equation
PACS (2006): 03.65.Pm, 03.65.Ca, 11.10.-z

1. Introduction

It is known that Barut's equation could predict lepton and hadron mass with remarkable precision [1]. A plausible extension of Barut's equation is by using Barut-Dirac 's model via inclusion of electron self-field. Furthermore, a number of authors has extended

* vxianto@yahoo.com

this equation using non-linear field theory [2a][5][5a]. Barut's equation is as follows [5a]:

$$\left[i\gamma_\nu\partial_\nu - a\partial_\mu^2/m + \kappa\right]\Psi = 0 \tag{1}$$

where $\partial_\nu = \partial/\partial x_\nu$ and repeated indices imply a summation [5a]. The remaining parameters come from substitution of variables: $m = \kappa/\alpha_1$ and $a/m = -\alpha_2/\alpha_1$ [5a]. In the meantime Barut-Dirac-Vigier's equation could be written as:

$$\left[c\alpha.p - E + \beta(mc^2 + \in e^2/r)\right]\Psi = -[(\in \alpha\hbar e^2)/(4\pi mc^2 r^2)]i\beta\alpha\Psi \tag{2}$$

Despite this apparently remarkable result of Barut's equation, nonetheless there is question concerning the physical meaning of his equation, in particular from the viewpoint of non-linear field theory [2a]. This question seems very interesting, in particular considering the unsolved question concerning the physical meaning of *wavefunction* in Quantum Mechanics [4a]. It is known that some proponents of 'realism' interpretation of Quantum Mechanics predict that there should be a complete 'realism' description of physical model of electron, where non-local hidden variables could be included [4][1a]. We consider that this question remains open for discussion, in particular in the context of plausible analog between classical electrodynamics and non-local quantum interference effect, via Aharonov-Casher effect [8].

In the present article we argue that it is possible to derive a new wave quantum relativistic equation as an alternative to Barut-Dirac-Vigier's equation. Our description is based on the known exact correspondence (isomorphism) between Dirac equation and Maxwell electromagnetic equations via biquaternionic representation. In fact, we will discuss five approaches as alternative to Barut-Dirac equation. And we would argue that the question of which of these approaches is the most consistent with experimental data remains open. Our proposition of alternative to Barut(-Dirac) equation was based on characteristics of Barut equation:

- it is a second-order differential equation (1);
- it shall include the physical meaning of vierbein in quantum mechanical equation;
- it has neat linkage with other known equations in Quantum Mechanics including Dirac equation [5a], while its solution could be different from Dirac approach [11];
- our observation asserts that it shall also include a proper introduction of Lorentz force, and acceleration from relativistic fluid dynamics.

Furthermore, in the present note we submit the viewpoint that it would be more conceivable if we interpret the *vierbein* of this equation in terms of superfluid velocity [12][13], which in turn brings us to the notion of topological electronic liquid [27]. Its implications to quantization of celestial systems lead us to argue in favor of signatures of Bose-Einstein cosmology, which thus far has not been explored sufficiently yet in the literature [49a][49b].

What we would argue in the present note is that one could expect to extend further this quaternion representation into the form of unified wave equation, in particular using Ulrych's representation [7]. While such an attempt to interpret *vierbein* of Dirac equation has been made by de Broglie (in terms of 'Dirac fluid' [41]), it seems that an exact representation in terms of superfluid velocity has never been made before. From this viewpoint one could argue that the superfluid vierbein interpretation will make the picture resembles superfluid bivacuum model of Kaivarainen [20][21]. Furthermore, this proposition seems to support previous hypothetical argument by Prof. J-P. Vigier on the further development of theoretical Quantum Mechanics [6]:

"..a revival, in modern *covariant form*, of the ether concept of the founding fathers of the theory of light (Maxwell, Lorentz, Einstein, *etc.*). This is a crucial question, and it now appears that the vacuum is a real physical medium, which presents surprising properties (superfluid, *i.e.* negligible resistance to inertial motions) ... "

Provided this proposition of unified wave equation in terms of superfluid velocity *vierbein* corresponds to the observed facts, and then it could be used to predict some new observations, in particular in the context of condensed-matter analog of astrophysics [16][17][18]. Therefore in the last section we will extend this proposition to argue in favor of signatures of Bose-Einstein cosmology, including some recent relevant observation supporting this argument.

While quaternionic Quantum Mechanics has been studied before by Adler etc. [14c][28], and also biquaternionic Quantum Mechanics [2][3], it seems that interpreting the right-hand-side of the unified wave equation as superfluid 4-velocity has not been considered before, at least not yet in the context of cylindrical relativistic fluid of Carter and Sklarz-Horwitz.

In deriving these equations we will not rely on exactitude of the solutions, because as we shall see the known properties, like fine structure constant of hydrogen, can be derived from different approaches [11][15][19][22a]. Instead, we will use 'correspondence between physical theories' as a guiding principle, i.e. we argue that it is possible to derive some alternatives to Barut equation via generalization of various wave equations known in Quantum Mechanics. More linkage between these equations implies consistency.

Further experimental observation to verify or refute this proposition is recommended.

2. Biquaternion, Imaginary algebra, Unified relativistic Wave Equation

Before we discuss biquaternionic Maxwell equations from unified wave equation, first we should review Ulrych's method [7] by defining imaginary number representation as follows [7]:

$$x = x_0 + j.x_1, \quad j^2 = -1 \tag{3}$$

This leads to the multiplication and addition (or substraction) rules for any number, which is composed of real part and imaginary number:

$$(x \pm y) = (x_0 \pm y_0) + j.(x_1 \pm y_1), \tag{4}$$

$$(xy) = (x_0 y_0 + x_1 y_1) + j.(x_0 y_1 + x_1 y_0). \tag{5}$$

From these basic imaginary numbers, Ulrych [7] argues that it is possible to find a new *relativistic algebra*, which could be regarded as modified form of standard quaternion representation.

Once we define this imaginary number, it is possible to define further some relations as follows [14]. Given $w = x_0 + j.x_1$, then its D-conjugate of w could be written as:

$$\bar{w} = x_0 - j.x_1 \tag{6}$$

Also for any given two imaginary numbers $w_1, w_2 \in D$, we get the following relations [14]:

$$\overline{w_1 + w_2} = \bar{w}_1 + \bar{w}_2 \tag{7}$$

$$\overline{w_1 \bullet w_2} = \bar{w}_1 \bullet \bar{w}_2 \tag{8}$$

$$|w|^2 = \bar{w} \bullet w = x_0^2 - x_1^2 \tag{9}$$

$$|w_1 \bullet w_2|^2 = |w_1|^2 \bullet |w_2|^2 \tag{10}$$

All of these provide us nothing new. For extension of these imaginary numbers in Quantum Mechanics, see [33]. Now we will review a few elementary definitions of quaternions and biquaternions, which are proved to be useful.

It is known that biquaternions could describe Maxwell equations in its original form, and some of the use of biquaternions was discussed in [2][34].

Quaternion number, Q is defined by [33][60]:

$$Q = a + b.i + c.j + d.k \quad a, b, c, d \in R, \tag{11a}$$

where

$$i^2 = j^2 = k^2 = ijk = -1 \tag{11b}$$

Alternatively, one could extend this quaternion number to Clifford algebra [3a][3][6][25][41], because higher-dimensions Clifford algebra and analysis give the possibility to generalize the factorisations into higher spatial dimensions and even to space-time domains [70a]. In this regard quaternions H $\sim$ $C\ell_{0,2}$, while standard imaginary numbers C$\sim$ $C\ell_{0,1}$ [70a].

Biquaternion is an extension of this quaternion number, and it is described here using Hodge-bracket operator, in lieu of known Hodge operator ($** = -1$) [5a]:

$$\{Q\}* = (a + iA) + (b + iB).i + (c + iC).j + (d + iD).k, \tag{12a}$$

where the second part (A,B,C,D) is normally set to zero in standard quaternions [33].

For quaternion differential operator, we define quaternion Nabla operator:

$$\nabla^q \equiv c^{-1}.\partial/\partial t + (\partial/\partial x)i + (\partial/\partial y)j + (\partial/\partial z)k = c^{-1}.\partial/\partial t + \vec{i}.\vec{\nabla} \tag{12b}$$

And for biquaternion differential operator, we define a quaternion Nabla-Hodge-bracket operator:

$$\{\nabla^q\}* \equiv (c^{-1}.\partial/\partial t + c^{-1}.i\partial/\partial t) + \{\vec{\nabla}\}* \tag{12c}$$

where Nabla-Hodge-bracket operator is defined as:

$$\{\vec{\nabla}\}* \equiv (\partial/\partial x + i\partial/\partial X).i + (\partial/\partial y + i\partial/\partial Y).j + (\partial/\partial z + i\partial/\partial Z).k. \tag{13a}$$

It is worthnoting here that equations (4)-(10) are also applicable for biquaternion number. While equations (3)-(12a) are known in the existing literature [33][59], and sometimes called 'biparavector' (Baylis), we prefer to call it 'imaginary algebra' with emphasis on the use of Hodge-bracket operator. It is known that determinant and differentiation of quaternionic equations are different from standard differential equations [59], therefore solution for this problem has only been developed in recent years.

The Hodge-bracket operator proposed herein could become more useful if we introduce quaternion number (11a) in the paravector form [70]:

$$\vec{q} = \sum_{k=0}^{3} q_k.e_k \text{ when } \{q_k\} \subset C, \{e_k\,|k = 1, 2, 3\} \tag{13b}$$

and e_0 is the unit. Therefore, biquaternion number could be written in the same form [70]:

$$\{\vec{q}\}* = \vec{q} + i\vec{q} = \sum_{k=0}^{3} q_k.e_k + i\{\sum_{k=0}^{3} q_k.e_k\} \tag{13c}$$

Now we are ready to discuss Ulrych's method to describe unified wave equation [7], which argues that it is possible to define a unified wave equation in the form [7]:

$$D\phi(x) = m_\phi^2.\phi(x), \tag{14}$$

where unified (wave) differential operator D is defined as:

$$D = \left[(P - qA)_\mu \left(\bar{P} - qA\right)^\mu\right]. \tag{15}$$

To derive Maxwell equations from this unified wave equation, he uses free photon fields expression [7]:

$$DA(x) = 0, \tag{16}$$

where potential A(x) is given by:

$$A(x) = A^0(x) + jA^1(x), \tag{17}$$

and with electromagnetic fields:

$$E^i(x) = -\partial^0 A^i(x) - \partial^i A^0(x), \tag{18}$$

$$B^i(x) = \in^{ijk} \partial_j A_k(x). \tag{19}$$

Inserting these equations (17)-(19) into (16), one finds Maxwell electromagnetic equation [7]:

$$\begin{aligned} &-\nabla \bullet E(x) - \partial^0 C(x) \\ &+ij\nabla \bullet B(x) \\ &-j(\nabla x B(x) - \partial^0 E(x) - \nabla C(x)) \\ &-i(\nabla x E(x) + \partial^0 B(x)) = 0 \end{aligned} \tag{20}$$

The gauge transformation of the vector potential A(x) is given by [7]:

$$A'(x) = A(x) + \nabla\Lambda(x)/e, \tag{21}$$

where $\Lambda(x)$ is a scalar field. As equations (17)-(18) only use simple definitions of imaginary numbers (3)-(5), then an extension from (20) and (21) to biquaternionic form of Maxwell equations is possible [2][34].

In order to define biquaternionic representation of Maxwell equations, we could extend Ulrych's definition of *unified differential operator* [7] to its biquaternion counterpart, by using equation (12a), to become:

$$\{D\}* \equiv \left[(\{P\} * -q\{A\}*)_\mu \left(\{\bar{P}\} * -q\{A\}*\right)^\mu\right], \tag{22a}$$

or by definition $P = -i\hbar\nabla$and (13a), equation (22a) could be written as:

$$\{D\}* \equiv \left[(-\hbar\{\nabla\} * -q\{A\}*)_\mu \,(-\hbar\{\nabla\} * -q\{A\}*)^\mu\right], \tag{22b}$$

where each component is now defined in its biquaternionic representation. Therefore the biquaternionic form of unified wave equation takes the form:

$$\{D\} * \phi(x) = m_\phi^2.\phi(x), \tag{23}$$

if we assume the wavefunction is not biquaternionic, and

$$\{D\} * \{\phi(x)\}* = m_\phi^2.\{\phi(x)\} * . \tag{24}$$

if we suppose that the wavefunction also takes the same biquaternionic form.

Now, biquaternionic representation of free photon fields could be written in the same way with (16), as follows:

$$\{D\} * A(x) = 0 \tag{25}$$

We will not explore here complete solution of this biquaternion equation, as it has been discussed in various literatures aforementioned above, including [2][33][34][59]. However, immediate implications of this biquaternion form of Ulrych's unified equation can be described as follows.

Ulrych's fermion wave equation in the presence of electromagnetic field reads [7]:

$$\left[(P - qA)_\mu \left(\bar{P} - qA\right)^\mu \psi\right] = -m^2.\psi, \tag{26}$$

which asserts c=1 (conventionally used to write wave equations). In accordance with Ulrych [7] this equation implies that the differential operator of the quantum wave equation (LHS) is composed of the momentum operator P multiplied by its dual operator, and taking into consideration electromagnetic field effect qA. And by using definition of momentum operator:

$$P = -i\hbar\nabla. \tag{27}$$

So we get three-dimensional relativistic wave equation [7]:

$$[(-i\hbar\nabla_\mu - qA_\mu)(-i\hbar\nabla^\mu - qA^\mu)\psi] = -m^2.c^2.\psi. \tag{28}$$

which is Klein-Gordon equation. Its 1-dimensional version has also been derived by Nottale [67, p,29]. A plausible extension of equation (28) using biquaternion differential operator defined above (22a) yields:

$$[(-\hbar\{\nabla_\mu\} * -q\{A_\mu\}*)(-\hbar\{\nabla^\mu\} * -q\{A^\mu\}*)\psi] = -m^2.c^2.\psi, \tag{29}$$

which could be called as 'biquaternionic' Klein-Gordon equation.

Therefore we conclude that there is neat correspondence between Ulrych's fermion wave equation and Klein Gordon equation, in particular via biquaternionic representation. It is also worthnoting that it could be shown that Schrodinger equation could be derived from Klein-Gordon equation [11], and Klein-Gordon equation also neatly corresponds to Duffin-Kemmer-Petiau equation. Furthermore it could be proved that modified (quaternion) Klein-Gordon equation could be related to Dirac equation [7]. All of these linkages seem to support argument by Gursey and Hestenes who find plenty of interesting features using quaternionic Dirac equation. In this regard, Meessen has proposed a method to describe elementary particle from Klein-Gordon equation [30].

By assigning imaginary numbers to each component [7, p.26], equation (26) could be rewritten as follows (by writing c=1):

$$\left[(P - qA)_\mu (P - qA)^\mu - eE^i ij\sigma_i - eB^i\sigma_i + m^2\right]\psi = 0, \tag{30}$$

where Pauli matrices σ_i are written explicitly. Now it is possible to rewrite equation (30) in complete tensor formalism [7], if Pauli matrices and electromagnetic fields are expressed with antisymmetric tensor, so we get:

$$\left[(P - qA)_\mu \left(\bar{P} - qA\right)^\mu - e\sigma_{\mu\nu}F^{\mu\nu} + m^2\right]\psi = 0, \tag{31}$$

where

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu). \tag{32}$$

Note that equation (31) is formal identical to *quadratic form* of Dirac equation [7], which supports argument suggesting that modified (quaternion) Klein-Gordon equation could be related to Dirac equation. Interestingly, equation (31) is also known in the literature as Feynman-Gell-Mann's equation, and its implications will be discussed in subsequent section [5]. Interestingly, if we neglect contribution of the electromagnetic field (q and e) component, and using only 1-dimensional of the partial differentiation, one gets a wave equation from Feynman rules [56, p.6]:

$$\left(\partial_\mu\partial^\mu + m^2\right)\Psi = 0, \tag{33}$$

which has been used to describe quantum-electrodynamics without renormalization [56].

Further extension of equation (28) could be made by expressing it in terms of 4-velocity:

$$[(-i\hbar\nabla_\mu - qA_\mu)(-i\hbar\nabla^\mu - qA^\mu)\psi] = -p_\mu p^\mu.\psi. \tag{34}$$

In the context of relativistic fluid [10][11], one could argue that this 4-velocity corresponds to superfluid vierbein [13][16][17]. Therefore we could use Carter-Langlois' equation [12]:

$$\mu_\rho.\mu^\rho = -c^2.\mu^2, \tag{35}$$

by replacing m with the effective mass variable μ. This equation has the meaning of cylindrically symmetric superfluid with known metric [12]:

$$g_{\rho\sigma}.dx^\rho.dx^\sigma = -c^2.dt^2 + dz^2 + r^2.d\phi^2 + dr^2. \tag{36}$$

Further extension of equation (35) is possible, as discussed by Fischer [13], where the effective mass variable term also appears in the LHS of velocity equation, by defining momentum of the continuum as:

$$p_\alpha = \mu.u_\alpha. \tag{37}$$

Therefore equation (35) now becomes:

$$\mu^2.u_\alpha.u^\alpha = -c^2.\mu^2, \tag{38}$$

where the effective mass variable now acquires the meaning of chemical potential [13]:

$$\mu = \partial \in /\partial\rho, \tag{39}$$

and

$$\rho.p_\alpha/\mu = \left(K/\hbar^2\right)p_\alpha = j_\alpha, \tag{40}$$

$$K = \hbar^2(\rho/\mu). \tag{41a}$$

The quantity K is defined as the stiffness coefficient against variations of the order parameter phase. Alternatively, from macroscopic dynamics of Bose-Einstein condensate containing vortex lattice, one could write the chemical potential in the form [57]:

$$\mu = \mu_0.\left[1 - (\Omega_0/\omega_\perp)^2\right]^{2/5} \tag{41b}$$

where the quantity Ω corresponds to the angular frequency of the sample and is assumed to be uniform, ω is the oscillator frequency, and chemical potential in the absence of rotation is given by [57]:

$$\mu_0 = (\hbar\omega_{ho}/2)(Na/0.0667a_{ho})^{2/5} \tag{41c}$$

and N represents the number of atoms and a is the corresponding oscillator length [57]:

$$a_{ho} = \sqrt{\hbar/M\omega_{ho}} \tag{41d}$$

Now the sound speed c_s could be related to the equations above, for a barotropic fluid [13], as:

$$c_s = d(\ln \mu) / d(\ln \rho) = \left(K/\hbar^2\right) d^2 \in / d\rho^2. \tag{42}$$

Using this definition, then equation (42) could be rewritten as follows:

$$p_\alpha = \left(K^{-1}\hbar^2\right) j_\alpha = (j_\alpha / c_s).d^2 \in / d\rho^2, \tag{43}$$

Introducing this result (43) into equation (34), we get:

$$\left[(-i\hbar\nabla_\mu - qA_\mu)(-i\hbar\nabla^\mu - qA^\mu)\psi\right] = -\left((j_\alpha/c_s).d^2 \in / d\rho^2\right)^2 .\psi \tag{44}$$

which is an alternative expression of relativistic wavefunction in terms of superfluid sound speed, c_s. Note that this equation could appear only if we interpret 4-velocity in terms of *superfluid vierbein* [11][12]. Therefore this equation is Klein-Gordon equation, where vierbein is defined in terms of superfluid velocity. Alternatively, in condition without electromagnetic charge, then we can rewrite equation (44) in the known form of standard Klein-Gordon equation [36]:

$$\left[D_\mu D^\mu \psi\right] = -\left((j_\alpha/c_s).d^2 \in / d\rho^2\right)^2 .\psi. \tag{45}$$

Therefore, this alternative representation of Klein-Gordon equation (45) has the physical meaning of relativistic wave equation for superfluid phonon [37][38].

A plausible extension of (44) is also possible using our definition of biquaternionic differential operator (22a):

$$\boxed{\{D\} * \psi = -\left((j_\alpha/c_s).d^2 \in / d\rho^2\right)^2 \psi} \tag{46}$$

which is an alternative expression from Ulrych's [7] unified relativistic wave equation, where the vierbein is defined in terms of superfluid sound speed, c$_s$. This is the main result of this section. As alternative, equation (46) could be written in compact form:

$$[\{D\} * + \Gamma]\Psi = 0, \tag{47}$$

where the operator Γ is defined according to the quadratic of equation (43):

$$\Gamma = \left((j_\alpha/c_s).d^2 \in / d\rho^2\right)^2 . \tag{48a}$$

For the solution of equation (44)-(47), one could refer for instance to alternative description of quarks and leptons via SU(4) symmetry [28][58]. As we note above, equation (31) is also known in the literature as Feynman-Gell-Mann's equation, and it has been argued that it has neat linkage with Barut equation [5]. This assertion could made more conceivable by noting that equation (31) is quadratic form of Dirac equation. In this regard, recently Kruglov has considered a plausible generalization of Barut equation via third-order differential extension of Dirac equation [60]:

$$(\gamma_\mu \partial_\mu + m_1)(\gamma_\nu \partial_\nu + m_2)(\gamma_\alpha \partial_\alpha + m_3)\psi = 0. \tag{48b}$$

It is also interesting to note that in his previous work, Kruglov [60a] has argued in favor of Dirac-Kahler equation:

$$(d - \delta + m)\,\psi = 0, \tag{48c}$$

where the operator $(d - \delta)$ is the analog of Dirac operator $\gamma_\mu \partial^\mu$. It seems plausible, therefore, in the context of Kruglov's recent attempt to generalize Barut equation [60] to argue that further generalization to biquaternionic form is possible by rewriting equation (47) in the third-order equation, by using our definition (12c):

$$[\{\vec{\nabla}^q_\mu\} * + p_\mu][\{\vec{\nabla}^q_\nu\} * + p_\nu][\{\vec{\nabla}^q_\alpha\} * + p_\alpha]\Psi = 0. \tag{48d}$$

Therefore, we could consider this equation as the first alternative to (generalized) Barut equation. Note that we use here equation (12c) instead of (22a), in accordance with Kruglov [60] definition:

$$\partial_\nu = \partial/\partial x_\nu = (\partial/\partial x_m, \partial/\partial(it)) \tag{48e}$$

In subsequent sections, we will consider a number of other plausible alternatives to Barut-Dirac's equation, in particular from the viewpoint of superfluid vierbein.

3. Alternative #2: Barut-Dirac-Feynman-Gell-Mann Equation

It is argued [5, p. 4] that Barut equation is the sum of Dirac equation and Feynman-Gell-Mann's equation (31). But from the aforementioned argument, it should be clear that the Feynman-Gell-Mann's equation is nothing more than Ulrych's fermion wave equation, which is indeed a quadratic of Dirac equation. Therefore, it seems that there should be other route to derive Barut-Dirac type equation. In this regard, we submit the viewpoint that the introduction of electron self-field would lead to an alternative of Barut equation.

First, let us rewrite equation (31) with assigning the real c in lieu of c=1:

$$\left[(P - qA)_\mu \left(\bar{P} - qA\right)^\mu - e\sigma_{\mu\nu}F^{\mu\nu} + m^2c^2\right]\psi = 0, \tag{49}$$

By using equation (34), then Feynman-Gell-Mann's equation becomes:

$$[(-i\hbar\nabla_\mu - qA_\mu)(-i\hbar\nabla^\mu - qA^\mu) - e\sigma_{\mu\nu}F^{\mu\nu} + p_\mu p^\mu]\,\Psi = 0, \tag{50}$$

or

$$[(-i\hbar\nabla_\mu - qA_\mu)(-i\hbar\nabla^\mu - qA^\mu) + p_\mu p^\mu]\,\Psi = (e\sigma_{\mu\nu}F^{\mu\nu})\Psi, \tag{51}$$

which can be called Feynman-Gell-Mann's equation with superfluid vierbein interpretation, in particular if we then introduce equation (43) into the LHS.

In this regard, we can introduce Ibison's description of electron self-energy from ZPE [38]:

$$e\sigma_{\mu\nu}F^{\mu\nu} = m_0 a^\mu - m_0\tau_0\left[da^\lambda/d\tau + a^\lambda a_\lambda u^\mu/c^2\right] \tag{52}$$

where

$$\tau_0 = e^2/6\pi\varepsilon_0 m_0 c^3 \tag{53}$$

The first term in the right hand side of equation (52) could be written in the Lorentz form [42] [24a, p.12]:

$$m_0 a^{\mu} = m[dv/dt] = e[E + vxB] \tag{54}$$

where:

$$E = -\nabla\phi, \tag{55}$$

$$B = \nabla xA. \tag{56}$$

Therefore, by defining a new parameter [24a, p.12]:

$$\forall = e[E + vxB]^{\mu} - m_0(e^2/6\pi\varepsilon_0 m_0 c^3)\left[da^{\lambda}/d\tau + a^{\lambda}a_{\lambda}u^{\mu}/c^2\right], \tag{57}$$

one could rewrite equation (51) in term of equation (43):

$$\left[(-i\hbar\nabla_{\mu} - qA_{\mu})(-i\hbar\nabla^{\mu} - qA^{\mu}) + \left((j_{\alpha}/c_s).d^2 \in /d\rho^2\right)^2\right]\Psi = \forall\Psi, \tag{58}$$

which could be regarded as a second alternative expression of Barut equation. Therefore we propose to call it Barut-Dirac-Feynman-Gell-Mann equation. Implications of this equation should be verified via experiments, in particular with condensed-matter physics.

4. Alternative #3: Second Order Differential Form of Schrödinger-Type Equation

It is known that Barut equation is a typical second-order differential equation, which is therefore non-linear. Therefore a good alternative to Barut equation could be derived from similar approach with Schrödinger's original equation, but this time it should be differentiated twice.

In this regard, it seems worthnoting here that it is more proper to use Noether's expression of total energy in lieu of standard derivation of Schrödinger's equation ($E = \vec{p}^2/2m$). According to Noether's theorem [39], the total energy of the system corresponding to the time translation invariance is given by:

$$E = mc^2 + (cw/2).\int_0^{\infty}\left(\gamma^2.4\pi r^2.dr\right) = k\mu c^2 \tag{59}$$

where k is *dimensionless* function. It could be shown, that for low-energy state the total energy could be far less than $E = mc^2$. Interestingly Bakhoum [22] has also argued in favor of using $E = mv^2$ for expression of total energy, which expression could be traced back to Leibniz. Therefore it seems possible to argue that expression $E = mv^2$ is more generalized than the standard expression of special relativity, in particular because the total energy now depends on actual velocity [39].

From this new expression, it is plausible to rederive quantum relativistic wave equation in second-order differential expression, and it turns out the new equation should also include a Lorentz-force term in the same way of equation (57). This feature is seemingly interesting, because these equations are derived from different approach from (57).

We start with Bakhoum's assertion that it is more appropriate to use $E = mv^2$, instead of more convenient form $E = mc^2$. This assertion would imply [22]:

$$H^2 = p^2.c^2 - m_o^2.c^2.v^2. \tag{60}$$

Therefore, for phonon speed (c_s) in the limit $p \to 0$, we write [37]:

$$E(p) \equiv c_s. \left|p\right|. \tag{61}$$

A bit remark concerning Bakhoum's expression, it does not mean to imply or to interpret $E = mv^2$ as an assertion that it implies zero energy for a rest mass. Actually the problem comes from 'mixed' interpretation of what we mean with 'velocity'. In original Einstein's paper (1905) it is defined as 'kinetic velocity', which can be measured when standard 'steel rod' has velocity approximates the speed of light. But in quantum mechanics, we are accustomed to make use it deliberately to express 'photon speed'=c. According to Bakhoum, to get a consistent interpretation between special relativity and quantum mechanics, we should treat this definition of velocity according to its context, in particular to its linkage with electromagnetic field. Therefore, in special relativity 1905 paper, it should be better to interpret it as 'speed of free electron', which approximates c. For muon, Spohn [42] has obtained v=0.9997c which is very near to c, but not exactly =c. For hydrogen atom with 1 electron, the electron occupies the first excitation (quantum number n=1), which implies that their speed also approximate c, which then it is quite safe to assume $E \sim mc^2$. But for atoms with large amount of electrons occupying large quantum numbers, as Bakhoum showed that electron speed could be far less than c, therefore it will be more exact to use $E = mv^2$, where here v should be defined as 'average electron speed'. Furthermore, in the context of relativistic fluid, we could use $E_\alpha = \mu.u_\alpha u_\alpha$ from equation (37).

In the first approximation of relativistic wave equation, we could derive Klein-Gordon-type relativistic equation from equation (60), as follows. By introducing a new parameter:

$$\zeta = i(v/c), \tag{62}$$

then we can rewrite equation (60) in the known procedure of Klein-Gordon equation:

$$E^2 = p^2.c^2 + \zeta^2 m_o^2.c^4, \tag{63}$$

where $E = mv^2$. [22] By using known substitution:

$$E = i\hbar.\partial/\partial t, \quad p = \hbar\nabla/i, \tag{64}$$

and dividing by $(\hbar c)^2$, we get Klein-Gordon-type relativistic equation:

$$-c^{-2}\partial\Psi/\partial t + \nabla^2\Psi = {k_o'}^2\Psi, \tag{65}$$

where

$$k_o' = \zeta m_o c/\hbar. \tag{66}$$

One could derive Dirac-type equation using similar method. But the use of new parameter (62) seems to be indirect, albeit it simplifies the solution because here we can use the same solution from Klein-Gordon equation [30].

Alternatively, one could derive a new quantum relativistic equation, by noting that expression of total energy $E = mv^2$ is already relativistic equation. We will derive here two approaches to get relativistic wave equation from this expression of total energy.

The first approach, is using Ulrych's [7] method as follows:

$$E = mv^2 = p.v \tag{67}$$

Taking square of this expression, we get:

$$E^2 = p^2.v^2 \tag{68}$$

or

$$p^2 = E^2/v^2 \tag{69}$$

Now we use Ulrych's substitution [7]:

$$\left[(P - qA)_\mu \left(\bar{P} - qA\right)^\mu\right] = p^2, \tag{70}$$

and introducing standard substitution in Quantum Mechanics (64), one gets:

$$\left[(P - qA)_\mu \left(\bar{P} - qA\right)^\mu\right] \Psi = v^{-2}.(i\hbar.\partial/\partial t)^2\Psi, \tag{71}$$

or

$$\left[(-i\hbar\nabla_\mu - qA_\mu)\left(-i\hbar\nabla^\mu - qA^\mu\right) - (i\hbar/v.\partial/\partial t)^2\right] \Psi = 0. \tag{72a}$$

which can be called as Noether-Ulrych-Feynman-Gell-Mann's (NUFG) equation. This is the third alternative to Barut-Dirac equation.

Alternatively, by using standard definition p=m.v, we can rewrite equation (71) in form of equation (43):

$$\left[(P - qA)_\mu \left(\bar{P} - qA\right)^\mu\right] \Psi = m^2 \left((j_\alpha/c_s).d^2 \in /d\rho^2\right)^{-2}.(i\hbar.\partial/\partial t)^2\Psi. \tag{72b}$$

In order to verify that we can use the same method with Schrödinger equation to derive nonlinear wave equation, let us consider Oleinik's nonlinear wave equation. It is argued that the proper equation of motion is not the Dirac or Schrödinger equation, but an equation with a new self-energy term [24]. This would mean that there is a pair wavefunction to include electron interaction with its surrounding medium. Therefore, the standard Schrödinger equation becomes nonlinear equations of motion [24]:

$$\left[i\partial/\partial t + \bar{\nabla}^2/2m - U(x)\right] \begin{pmatrix} \Psi(x) \\ \bar{\Psi}(x) \end{pmatrix} = 0 \tag{73}$$

where we use $\hbar = 1$ for convenience.

From this equation, one can get the relativistic version corresponding to Dirac equation [24]. Interestingly, Froelich [66] has considered equation of motion for the few-body systems associated with the hydrogen-antihydrogen pairs using radial Schrödinger-type equation. Therefore, it seems interesting to consider equation (73) also in the context of hydrogen-antihydrogen molecule.

And because equation (73) is derived from the standard definition of total energy $E = \vec{p}^2/2m$, then our method to use equation (60) seems to be a logical extension of Oleinik's method. To get nonlinear version similar to equation (73), then we could rewrite equation (72a) as:

$$\left[(-i\hbar\nabla_\mu - qA_\mu)(-i\hbar\nabla^\mu - qA^\mu) - (i\hbar/v.\partial/\partial t)^2\right]\begin{pmatrix}\Psi(x)\\ \bar{\Psi}(x)\end{pmatrix} = 0. \tag{74}$$

What's more interesting here, is that Oleinik [24a, p.12] has shown that equation (73) could lead to an expression of Newtonian-Lorentz force similar to equation (54):

$$m_0 a^\mu = m[d^2r/dt^2] = e[E + v \times B] \tag{75}$$

This verifies our aforementioned proposition that a good alternative to Barut's equation should include a Lorentz-force term in wave equation. In other words, from equation (73) we find neat linkage between Schrödinger equation, nonlinear wave, and Lorentz-force. We will use this linkage in the following section. It turns out that we can find a proper generalization of Barut's equation via introduction of Newtonian-acceleration from velocity of the relativistic fluid in similar form of Lorentz force.

5. Alternative #4: Lorentz-force & Newtonian Acceleration Method

For the fourth method, we will introduce Leibniz rule [40] into equation (67) via differentiation with respect to time, which yields:

$$dE/dt = d[p.v]/dt = v.[dp/dt] + p.[dv/dt] \tag{76}$$

The next step is taking derivation of the known substitution in QM:

$$dE/dt = i\hbar.\partial^2/\partial t^2, \tag{77}$$

$$dp/dt = d(-i\hbar\nabla)/dt = -i\hbar\dot{\nabla}$$

Now, substituting back equation (77) and (64) into equation (76), we get:

$$(i\hbar.\partial^2/\partial t^2)\Psi = (v.[-i\hbar\dot{\nabla}] - [dv/dt].i\hbar\nabla)\Psi. \tag{78}$$

At this point, we note that the second term in the right hand side of equation (78) could be written in the Lorentz force form [42], and following equation (54):

$$[dv/dt] = e/m.(E + vxB) \tag{79}$$

where:

$$E = -\nabla\phi, \tag{80}$$

$$B = \nabla x A. \tag{81}$$

Therefore, we can rewrite equation (78) in the form:

$$(i\hbar.\partial^2/\partial t^2)\Psi = (v.[-i\hbar\dot{\nabla}] - e/m.[E + vxB].i\hbar\nabla)\Psi, \tag{82}$$

which is a new wave relativistic quantum equation as alternative to Barut equation. To our present knowledge, this alternative wave equation (82) has never been derived elsewhere.

As an alternative to equation (79), we can rewrite Lorentz form in term of Newtonian acceleration. In this regard, it is worthnoting that the definition of acceleration of relativistic fluid is not widely accepted yet [10]. Therefore we will use here result from relativistic field equations from Poisson process [46], from which we get an expression of acceleration [46]:

$$dv/dt = \hbar/2m.(\partial^2 u/\partial x^2) - v.\partial u/\partial x + u.\partial v/\partial x - m^{-1}.\partial V/\partial x = \exists \tag{83}$$

Therefore, by substituting this equation into (78), we get:

$$(i\hbar.\partial^2/\partial t^2)\Psi = (v.[-i\hbar\dot{\nabla}] - \exists.i\hbar\nabla)\Psi, \tag{84}$$

which can be considered as a better alternative to equation (82).

6. Alternative #5: Schrödinger-Ginzburg-Landau Equation and Quantization of Celestial Systems

In the preceding section (#4), we have found the neat linkage between Schrödinger equation, nonlinear wave, and Lorentz-force, which indicates a possibility to be considered as alternative to Barut equation. Now, as the fifth alternative method, it will be shown that we can expect to generalize Schrödinger equation to describe quantization of celestial sytems. While this notion of macro-quantization is not widely accepted yet, as we will see the logarithmic nature of Schrödinger equation is sufficient to ensure its applicability to larger systems. As alternative, we will also discuss an outline for deriving Schrödinger equation from simplification of Ginzburg-Landau equation. It is known that Ginzburg-Landau equation exhibits fractal character.

First, let us rewrite Schrödinger equation (73) in its common form:

$$\left[i\partial/\partial t + \bar{\nabla}^2/2m - U(x)\right]\Psi = 0 \tag{85}$$

where we use $\hbar = 1$for convenience, or

$$(i\partial/\partial t)\Psi = H.\Psi \tag{86}$$

Now, it is worthnoting here that Englman & Yahalom [4a] argue that this equation exhibits logarithmic character:

$$\ln\Psi(x,t) = \ln\left(|\Psi(x,t)|\right) + i.\arg(\Psi(x,t)) \tag{87}$$

Schrödinger already knew this expression in 1926, which then he used it to propose his equation called '*eigentliche Wellengleichung*' [4a]. Therefore equation (85) can be rewritten as follows:

$$2m(\partial \ln |\Psi| /\partial t) + 2\bar{\nabla} \ln |\Psi| .\bar{\nabla} \arg[\Psi] + \bar{\nabla}.\bar{\nabla} \arg[\Psi] = 0 \quad (88)$$

Interestingly, Nottale's scale-relativistic method [43][44] was also based on generalization of Schrödinger equation to describe quantization of celestial systems. It is known that Nottale-Schumacher's method [45] could predict new exoplanets in good agreement with observed data. Nottale's scale-relativistic method is essentially based on the use of first-order scale-differentiation method defined as follows [43][44]:

$$\partial V/\partial(\ln \delta t) = \beta(V) = a + bV + ... \quad (89)$$

Now it seems clear that the logarithmic derivation, which is essential in scale-relativity approach, also has been described properly in Schrödinger's original equation [4a]. In other word, its logarithmic form ensures applicability of Schrödinger equation to describe macroquantization of celestial systems.

To emphasize this assertion of the possibility to describe quantization of celestial systems, let us return for a while to the preceding section where we use Fischer' description [13] of relativistic momentum of 4-velocity (37)-(38). Interestingly Fischer [13] argues that the circulation leading to equation (37)-(38) is in the relativistic dense superfluid, defined as the integral of the momentum:

$$\gamma_s = \oint p_\mu dx^\mu = 2\pi.N_v\hbar, \quad (90)$$

and is quantized into multiples of Planck's quantum of action. This equation is the covariant Bohr-Sommerfeld quantization of γ_s. And then Fischer [13] concludes that the *Maxwell equations of ordinary electromagnetism can be cast into the form of conservation equations of relativistic perfect fluid hydrodynamics* [10], in good agreement with Vigier's guess as mentioned above. Furthermore, the topological character of equation (90) corresponds to the notion of topological electronic liquid, where compressible electronic liquid represents superfluidity [27].

It is worthnoting here, because here vortices are defined as elementary objects in the form of stable topological excitations [13], then equation (90) could be interpreted as signatures of Bohr-Sommerfeld quantization from topological quantized vortices. Fischer [13] also remarks that equation (90) is quite interesting for the study of superfluid rotation in the context of gravitation. Interestingly, application of Bohr-Sommerfeld quantization to celestial systems is known in literature [47][48], which here in the context of Fischer's arguments it seems plausible to suggest that quantization of celestial systems actually corresponds to superfluid-quantized vortices at large-scale [27]. In our opinion, this result supports known experiments suggesting neat correspondence between condensed matter physics and various cosmology phenomena [16]-[19].

To make the conclusion that quantization of celestial systems actually corresponds to superfluid-quantized vortices at large-scale a bit conceivable, let us consider an illustration of quantization of celestial orbit in solar system.

In order to obtain planetary orbit prediction from this hypothesis we could begin with the Bohr-Sommerfeld's conjecture of quantization of angular momentum. This conjecture may originate from the fact that according to BCS theory, superconductivity can exhibit macroquantum phenomena [16][65]. In principle, this hypothesis starts with observation that in quantum fluid systems like superfluidity, it is known that such vortexes are subject to quantization condition of integer multiples of 2π, or $\oint v_s.dl = 2\pi.n\hbar/m_4$. As we know, for the wavefunction to be well defined and unique, the momenta must satisfy Bohr-Sommerfeld's quantization condition:

$$\oint_{\Gamma} p.dx = 2\pi.n\hbar \tag{91}$$

for any closed classical orbit Γ. For the free particle of unit mass on the unit sphere the left-hand side is [49]:

$$\int_0^T v^2.d\tau = \omega^2.T = 2\pi.\omega \tag{92}$$

where $T=2\pi/\omega$ is the period of the orbit. Hence the quantization rule amounts to quantization of the rotation frequency (the angular momentum):$\omega = n\hbar$. Then we can write the force balance relation of Newton's equation of motion [49]:

$$GMm/r^2 = mv^2/r \tag{93}$$

Using Bohr-Sommerfeld's hypothesis of quantization of angular momentum, a new constant g was introduced:

$$mvr = ng/2\pi \tag{94}$$

Just like in the elementary Bohr theory (before Schrödinger), this pair of equations yields a known simple solution for the orbit radius for any quantum number of the form [49]:

$$r = n^2.g^2/(4\pi^2.GM.m^2) \tag{95}$$

which can be rewritten in the known form [43][44]:

$$r = n^2.GM/v_o^2 \tag{96}$$

where r, n, G, M, v_o represents orbit radii, quantum number (n=1,2,3,...), Newton gravitation constant, and mass of the nucleus of orbit, and specific velocity, respectively. In this equation (96), we denote:

$$v_o = (2\pi/g).GMm \tag{97}$$

The value of m is an adjustable parameter (similar to g). [43][44]

Using this equation (96), we could predict quantization of celestial orbits in the solar system, where for Jovian planets we use least-square method and define M in terms of reduced mass $\mu = (M_1.M_2)/(M_1 + M_2)$. From this viewpoint the result is shown in Table 1 below [49]:

Table 1. Comparison of prediction and observed orbit distance of planets in Solar system (in 0.1 AU unit) [49]

Object	No.	Bode	Nottale	CSV	Observed	Δ(%)
	1		0.4	0.428		
	2		1.7	1.71		
Mercury	3	4	3.9	3.85	3.87	0.52
Venus	4	7	6.8	6.84	7.32	6.50
Earth	5	10	10.7	10.70	10.00	-6.95
Mars	6	16	15.4	15.4	15.24	-1.05
Hungarias	7		21.0	20.96	20.99	0.14
Asteroid	8		27.4	27.38	27.0	1.40
Camilla	9		34.7	34.6	31.5	-10.00
Object	No.	Bode	Nottale	CSV	Observed	Δ(%)
Jupiter	2	52		45.52	52.03	12.51
Saturn	3	100		102.4	95.39	-7.38
Uranus	4	196		182.1	191.9	5.11
Neptune	5			284.5	301	5.48
Pluto	6	388		409.7	395	-3.72
2003EL61	7			557.7	520	-7.24
Sedna	8	722		728.4	760	4.16
2003UB31	9			921.8	970	4.96
Unobserved	10			1138.1		
Unobserved	11			1377.1		

For comparison purpose, we also include some recent observation by M. Brown *et al.* from Caltech [50][51][52][53]. It is known that Brown *et al.* have reported not less than four new planetoids in the outer side of Pluto orbit, including 2003EL61 (at 52AU), 2005FY9 (at 52AU), 2003VB12 (at 76AU, dubbed as *Sedna.)* And recently Brown and his team reported a new planetoid finding, called 2003UB31 (97AU). This is not to include *Quaoar* (42AU), which has orbit distance more or less near Pluto (39.5AU), therefore this object is excluded from our discussion. It is interesting to remark here that all of those new 'planetoids' are within 8% bound from our prediction of celestial quantization based on the above Bohr-Sommerfeld quantization hypothesis (Table 1). While this prediction is not so precise compared to the observed data, one could argue that the 8% bound limit also corresponds to the remaining planets, including inner planets. Therefore this 8% uncertainty could be attributed to macroquantum uncertainty and other local factors.

While our previous prediction only limits new planet finding until n=9 of Jovian planets (outer solar system), it seems that there are enough reasons to suppose that more planetoids are to be found in the near future. Therefore it is recommended to extend further the same quantization method to larger n values. For prediction purpose, we include in Table 1 new expected orbits based on the same quantization procedure we outlined before. For Jovian planets corresponding to quantum number n=10 and n=11, our method suggests that it is likely to find new orbits around 113.81 AU and 137.71 AU, respectively. It is recommended therefore, to find new planetoids around these predicted orbits.

As an interesting alternative method supporting this proposition of quantization from superfluid-quantized vortices (90), it is worthnoting here that Kiehn has argued in favor of re-interpreting the square of the wavefunction of Schrödinger equation as the vorticity distribution (including topological vorticity defects) in the fluid [61]. From this viewpoint, Kiehn suggests that there is *exact mapping* from Schrödinger equation to Navier-Stokes equation, using the notion of quantum vorticity [61]. Interestingly, de Andrade & Sivaram [62] also suggest that there exists formal analogy between Schrödinger equation and the Navier-Stokes viscous dissipation equation:

$$\partial V/\partial t = \nu.\nabla^2 V \tag{98}$$

where ν is the kinematic viscosity. Their argument was based on propagation torsion model for quantized vortices [62]. While Kiehn's argument was intended for ordinary fluid, nonetheless the neat linkage between Navier-Stokes equation and superfluid turbulence is known in literature [63][64][21].

Therefore, it seems interesting to consider a plausible generalization of Schrödinger equation in particular in the context of viscous dissipation method. First, we could write Schrödinger equation for a charged particle interacting with an external electromagnetic field [61] in the form of equation (28) and (85):

$$\left[\left(-i\hbar\nabla_\mu - qA_\mu\right)\left(-i\hbar\nabla^\mu - qA^\mu\right)\Psi\right] = \left[-i2m.\partial/\partial t + 2mU(x)\right]\Psi. \tag{99}$$

In the presence of electromagnetic potential [69], one could include another term into the LHS of equation (99):

$$\left[\left(-i\hbar\nabla_\mu - qA_\mu\right)\left(-i\hbar\nabla^\mu - qA^\mu\right) + eA_o\right]\Psi = 2m\left[-i\partial/\partial t + U(x)\right]\Psi. \tag{100}$$

This equation has the physical meaning of Schrödinger equation for a charged particle interacting with an external electromagnetic field, which takes into consideration Aharonov effect [69]. Topological phase shift becomes its immediate implication, as already considered by Kiehn [61].

Therefore, in the context of quaternionic representation of Schrödinger equation [70], one could write equation (100) in terms of equation [22a]:

$$[\{D\} * +eA_o]\,\Psi = 2m\,[-i\partial/\partial t + U(x)]\,\Psi. \tag{101}$$

In the context of topological phase shift [69], it would be interesting therefore to find the scalar part of equation (101) in experiments [8].

As described above, one could also derive equation (96) from scale- relativistic Schrödinger equation [43][44]. It should be noted here, however, that Nottale's method [43][44] differs appreciably from the viscous dissipative Navier-Stokes approach of Kiehn, because Nottale only considers his equation in the Euler-Newton limit [67][68]. Nonetheless, as we shall see, it is possible to find a generalization of Schrödinger equation from Nottale's approach in similar form with equation (101). In order to do so, first we could rewrite Nottale's generalized Schrödinger equation via diffusion method [67][71]:

$$\begin{aligned} &i2m\gamma\left[-\left(i\gamma + a(t)/2\right)\left(\partial\psi/\partial x\right)^2\psi^{-2} + \partial\ln\psi/\partial t\right] \\ &+ i\gamma a(t).\left(\partial^2\psi/\partial x^2\right)/\psi = \Phi + a(x) \end{aligned} \tag{102}$$

where ψ,$a(x)$, Φ, γ each represents classical wave function, an arbitrary constant, scalar potential, and a constant, respectively. If the function f(t) is such that

$$a(t) = -i2\gamma, \quad \alpha(x) = 0, \tag{103}$$

$$\gamma = \hbar/2m \tag{104}$$

then one recovers the original Schrödinger equation (85).

Further generalization is possible if we rewrite equation (102) in quaternion form similar to equation (101):

$$\begin{aligned} &i2m\gamma\left[-\left(i\gamma + a(t)/2\right)(\{\nabla\}*)^2\psi^{-2} + \partial\ln\psi/\partial t\right] \\ &+ i\gamma.a(t).\left(\{\nabla'\}*\right)/\psi = \Phi + a(x) \end{aligned} \tag{105}$$

Alternatively, with respect to our superfluid dynamics interpretation [13], one could also get Schrödinger equation from simplification of Ginzburg-Landau equation. This method will be discussed subsequently. It is known that Ginzburg-Landau equation can be used to explain various aspects of superfluid dynamics [16][17][18].

According to Gross, Pitaevskii, Ginzburg, wavefunction of N bosons of a reduced mass m* can be described as [55]:

$$-(\hbar^2/2m*).\nabla^2\psi + \kappa\left|\psi\right|^2\psi = i\hbar.\partial\psi/\partial t \tag{106}$$

For some conditions (where the temperature dependence of the density of Cooper pairs, n_s, is just the square of order parameter. Or $\left|\psi\right|^2 \approx n_s = A(T_c - T)$), then it is possible to replace the potential energy term in equation (106) with Hulthen potential. This substitution yields:

$$-(\hbar^2/2m*).\nabla^2\psi + V_{Hulthen}.\psi = i\hbar.\partial\psi/\partial t \tag{107}$$

where

$$V_{Hulthen}(r) = \kappa\,|\psi|^2 \approx -Ze^2.\delta.e^{-\delta r}/(1-e^{-\delta r}) \tag{108}$$

This equation (108) has a pair of exact solutions. It could be shown that for small values of δ, the Hulthen potential (108) approximates the effective Coulomb potential, in particular for large radius [14b]:

$$V^{eff}_{Coulomb} = -e^2/r + \ell(\ell+1).\hbar^2/(2mr^2) \tag{109}$$

Therefore equation (109) could be rewritten as:

$$-\hbar^2\nabla^2\psi/2m* + \left[-e^2/r + \ell(\ell+1).\hbar^2/(2mr^2)\right].\psi = i\hbar.\partial\psi/\partial t \tag{110}$$

For large radii, second term in the square bracket of LHS of equation (110) reduces to zero [54],

$$\ell(\ell+1).\hbar^2/(2mr^2) \to 0 \tag{111}$$

so we can write equation (110) as:

$$(-\hbar^2\nabla^2\psi/2m* + U).\psi = i\hbar.\partial\psi/\partial t \tag{112}$$

where Coulomb potential can be written as:

$$U = -e^2/r \tag{113}$$

This equation (112) is nothing but Schrödinger equation (85). Therefore we have re-derived Schrödinger equation from simplification of Ginzburg-Landau equation, in the limit of small screening parameter. Calculation shows that introducing this Hulthen effect (108) into equation (107) will yield different result only at the order of 10^{-39} m compared to prediction using equation (110), which is of course negligible. Therefore, we conclude that for most celestial quantization problems the result of TDGL-Hulthen (110) is *essentially* the same with the result derived from equation (85). Now, to derive equation (96) from Schrödinger equation, the reader is advised to see Nottale's scale-relativistic method [43][44].

What we would emphasize here is that this derivation of Schrödinger equation from Ginzburg-Landau equation is in good agreement with our previous conjecture that equation (90) implies macroquantization corresponding to superfluid-quantized vortices. This conclusion is the main result of this section. It is also worthnoting here that there is recent attempt to introduce Ginzburg-Landau equation in the context of microtubule dynamics [72], which implies wide applicability of this equation.

In the following section, we would extend this argument by noting that macroquantization of celestial systems implies the *topological* character of superfluid-quantized vortices, and cosmic microwave background radiation is also an indication of such topological superfluid vortices.

7. Further Note: Signatures of Bose-Einstein Cosmology

It is known that CMBR temperature (2.73K) is conventionally assumed to come from the hot early Universe, which then cools adiabatically to the present epoch. Nonetheless this description is not without problems, such as how to consider the small temperature fluctuations of CMBR as the seeds that give rise to large-scale structure such as galaxy formation [73]. Furthermore it is known that CMBR follows Planck radiation law with high precision, so one could argue whether it also indicates that large-scale structures obey quantum-mechanical principles. Therefore we will consider here some alternative hypothesis, which support the idea of low-energy quantum mechanics corresponding to superfluid vortices described in the preceding section.

In recent years, there are alternative arguments suggesting that the Universe indeed resembles the dynamics of N number of Planckian oscillators. Using similar assumption, for instance Antoniadis *et al.* [74] argue that CMBR temperature could be derived using *conformal invariance* symmetry, instead of using Harrison-Zel'dovich spectrum. Other has derived CMBR temperature from Weyl framework [74a]. Furthermore, if the CMBR temperature 2.73K could be interpreted as low-energy part of the Planck distribution law, then it seems to indicate that the Universe resembles Bose-Einstein condensate [75]. Pervushin *et al.* also argued that CMBR temperature could be derived from conformal cosmology with relative units [76]. These arguments seem to support Winterberg's hypothesis that superfluid phonon-roton aether could explain the origin of cosmic microwave background radiation [18][19].

Of course, it does not mean that CMBR data fits perfectly with Planck distribution law. It has been argued that CMBR data more corresponds to q-deformed Planck radiation distribution [77]. However, this argument requires further analysis. What interests us here is that there are reasons to believe that a quantum universe based on Planck scale is not merely a pure hypothetical notion, in particular if we consider known analogy between superfluidity and various cosmology phenomena [16][17].

Another argument comes from fractality argument. It has been discovered by Feynman that the typical quantum mechanical paths are non-differentiable and fractal [67]. In this regard, it has been argued that the Universe is embedded in Cantorian fractal spacetime having non-integer Hausdorff dimension [78], and from this viewpoint it could be inferred that the correlated fluctuations of the fractal spacetime is analogous to the Bose-Einstein condensate phenomenon. Interestingly, there is also hypothesis suggesting that Hausdorff dimension could be related to temperature of ideal Bose gas [79].

From these aforementioned arguments, it seems plausible to suppose that that CMBR temperature 2.73K could be interpreted as a signature of Bose-Einstein condensate cosmology. In particular, one could consider [22b] that "this relationship comes directly from Boltzmann's law N= B.k.T, where N is the background noise power; T is the background temperature in degrees Kelvin; and B is the bandwidth of the background radiation. It follows that the ratio (N/kB) for the cosmic background radiation is approximately equal to "e", because we usually convert the equation to decibels by taking natural logarithm.

The relationship is a solid one in fact." From this viewpoint, it seems quite conceivable to explain why CMBR temperature 2.73K is near enough to known number e= 2.71828. . . , which seems to suggest that the logarithmic form of Schrödinger equation ('eigentliche Wellengleichung') [4a] *may have* a deep linkage with this number e= 2.71828. . .

Nonetheless, we recognize that this proposition requires further analysis before we could regard it as conclusive. But we can describe here some arguments to support the new interpretation supporting this Bose-Einstein cosmology argument:

- From Fischer's argument [13] we know that Bohr-Sommerfeld quantization from superfluid vortice could exhibit at all scales, including celestial quantization. This proposition comes directly from his assertion of *the topological character* of superfluid vortices, because superfluid is topological electronic liquid [27].
- Extending further the aforementioned hypothesis of topological superfluid vortices, then it seems interesting to compare it with topological analysis of COBE-DMR data. G. Rocha *et al.* [80] argue using wavelet approach with Mexican Hat potential that it is possible to interpret the data as clue for a finite torus Universe, albeit not conclusive enough.
- Interestingly, this conjecture could be related to Bulgadaev's argument [81] suggesting that topological quantum number could be related to torus structure as stable *soliton* [81a]. In effect, this seems to imply that the basic structure of physical phenomena throughout all scales could take the form of topological torus.

In other words, the topological character of superfluid vortices implies that it is possible to generalize superfluid vortices to large scales. And the topological character of CMBR data seems to support our proposition that the universe indeed exhibits topological structures. It follows then that CMBR temperature is topological [80] in the sense that the superfluid nature of background temperature [18][19] could be explained from topological superfluid vortices.

Interestingly, similar argument has been pointed out by a number of authors by mentioning non-Gaussian part of CMBR spectrum. However, further discussion on this issue requires another note.

8. Concluding Remarks

It is known that Barut equation could predict lepton mass (and also hadron mass) with remarkable precision. Therefore, in the present article, we attempt to find plausible linkage between Dirac-Maxwell's isomorphism and Barut-Dirac-Vigier equation. From this proposition we could find a unified wave equation in terms of superfluid velocity (vierbein), which then could be used as basis to derive some alternative descriptions of Barut equation. Further experiment is required to verify which equation is the most reliable.

In the present note we submit the viewpoint that it would be more conceivable if we interpret the *vierbein* of the unified wave equation in terms of superfluid velocity, which in turn brings us to the notion of topological electronic liquid. Nonetheless, the proposed imaginary algebra discussed herein is only at its elementary form, and it requires further analysis in particular in the context of [5a][7][14][28]. It is likely that this subject will become the subject of subsequent paper.

Furthermore, the notion of topological electronic liquid could lead to topological superfluid vortices, which may explain the origin of macroquantization of celestial systems and perhaps also topological character of Cosmic Microwave Background Radiations. Nonetheless, such a proposition requires further analysis before it can be considered as conclusive.

Provided the aforementioned propositions of using superfluid velocity (*vierbein*) to describe unified wave equation correspond to the observed facts, and then in principle it seems to support arguments in favor of possibility to observe condensed-matter hadronic reaction.

Acknowledgement

The author would thank to Profs. C. Castro, M. Pitkänen, R.M. Kiehn, Ezzat G. Bakhoum, A. Kaivarainen, P. LaViolette, F. Smarandache, and E. Scholz, for insightful discussions and remarks. Special thanks go to Prof. C. Castro for suggesting finding linkage between quaternionic Klein-Gordon equation and Duffin-Kemmer-Petiau equation, to Prof. Ezzat G. Bakhoum for his remark on Boltzmann distribution and its linkage to CMBR temperature.

References

[1] Barut, A.O., *Phys. Rev. Lett.* Vol. 42 (1979). [1a] A.O. Barut, "On the status of hidden variable theories in Quantum mechanics," *Apeiron* Vol. 2 No. 4, Oct. 1995, p. 97.

[2] Gsponer, A., & J-P. Hurni, *Found. Phys. Lett.* Vol. 14 No.1 (2001) 77-85, preprint arXiv:math-ph/0201049. [2a] Gsponer, A., & J-P. Hurni, hep-ph/0201193. [2b] Gsponer, A., *Int. J. Theor. Phys.* 41 (2002) 689-694, preprint arXiv:math-ph/0201053.

[3] Varlamov, V.V., *Annales Fondation Louis de Broglie* Vol. 27 no. 2(2002), [3a] arXiv:hep-th/9709051 (1997).

[4] Post, E.J., *Annales Fondation Louis de Broglie* no. 2, 217-240 (2002). [4a] Englman, R., & H. Yahalom, arXiv:physics/0406149 (2004).

[5] Dvoeglazov, V.V., arXiv:math-ph/0503008 (2005). [5a] Kruglov, S.I., *Annales Fondation Louis de Broglie* Vol. 29 (2004) arXiv:quant-ph/0408056 (2004)

[6] Vigier, J-P., "Fundamental problems in quantum physics," *Apeiron* Vol. 2 No. 4, Oct. 1995, (1995) p.114.

[7] Ulrych, S, arXiv:hep-th/9904170 (1999); [7a] Ulrych, S., physics/0009079.

[8] Aharonov, Y., *et al.*, arXiv:quant-ph/0311155 (2003).

[9] Hofer, W.A., "Beyond Uncertainty: the internal structure of electron and photon," arXiv:physics/9611009 (1996).

[10] Sklarz, S., & L. Horwitz, arXiv:physics/0104019 (2001).

[11] Cui, H.Y., arXiv:physics/0103019 (2001); [11a] arXiv:physics/0408025 (2004); [11b] arXiv:physics/0409023 (2004).

[12] Carter, B., & D. Langlois, arXiv:hep-th/9507059 (1995). [12a] Carter, B., & D. Langlois, *Phys. Rev. D* **51** (1995), 5855-64.

[13] Fischer, U.W., arXiv:cond-mat/9907457 (1999).

[14] Antonuccio, F., arXiv:hep-th/9812036 (1998); [14a] arXiv:hep-th/9408166; [14b] Alhaidari, A., arXiv:math-ph/0405022 (2004); [14c] Adler, S.L., arXiv:hep-th9306009.

[15] Oudet, X., "The quantum state and the doublets," *Annales Fondation Louis de Broglie* Vol. **25** No.1, 24p (2000).

[16] Zurek, W., "Cosmological experiments in superfluids and superconductors," in *Proc. Euroconference Formation and Interaction of Topological Defects*, A. Davis & R. Brandenberger (eds.) Plenum (1995); preprint at arXiv:cond-mat/9502119. [16b] Barcelo, C., *et al.*, "Analogue gravity from Bose-Einstein condensate," *Class. Quantum Grav.* **18,** 1137-1156 (2001).

[17] Volovik, G., "Superfluid analogies of cosmological phenomena," arXiv:gr-qc/0005091 (2000a); [17b] Volovik, G., "Links between gravity and dynamics of quantum liquids," *Int. Conf. "Cosmology. Relativistic Astrophysics. Cosmoparticle Physics" (COSMION-99)* (2000) preprint at arXiv:gr-qc/0004049; [17c] Volovik, G., arXiv:gr-qc/0104046 (2001); [17d] Nozieres, P., & D. Pines, *The theory of quantum liquids: Superfluid Bose Liquid.* Wesley Publ. Co. Inc., 116-124 (1990).

[18] Winterberg,F., *Z. Naturforsch.,* **57a** (2002) 202-204; presented at *the 9*th *Canadian Conf. on General Relativity and Relativistic Astrophysics,* Edmonton, May 24-26, 2001.

[19] Winterberg, F., "Maxwell's Aether, the Planck Aether hypothesis, and Sommerfeld's Fine Structure Constant, "http://www.cet.sunderland.ac.uk (2002), http://www.znaturforsch.com/56a/56a0681.pdf .

[20] Kaivarainen, A., arXiv:physics/0207027 (2002)

[21] Kaivarainen, A., "Hierarchic models of turbulence, superfluidity and superconductivity," arXiv:physics/0003108 (2000). Also in http://www.karelia.ru

[22] Bakhoum, E., "Fundamental disagreement of wave mechanics with relativity," arXiv:physics/0206061 (2002) p. 5-7. [22a] Bakhoum, E., physics/0207107. [22b] Bakhoum, E., email communication, Sept. 24^{th} 2005.

[23] Goenner, H.F.M., "On the history of the unified field theories", *Living Rev. Relativity* Vol. 7, 102 (2004) http://www.livingreviews.org/lrr-2004-2

[24] Oleinik, V.P., *Nonlin. Math. Phys.,* Vol. 4 No. 1-2 (1997) 180-189. [24a] Oleinik, V.P. *et al.,* "On quantum dynamics of the self-acting electron," ( ).

[25] Castro., C., & M. Pavsic, "The extended relativity in Clifford-spaces," commissioned review to appear in *Int. J. Mod. Phys.* D (2004).

[26] Feynman, R., *et al.*, *The Feynman Lectures on Physics*, Addison-Wesley, Reading, MA, Vol. II, (1964) p. 1-3.

[27] Wiegmann, P., arXiv:cond-mat/9808004 (1998).

[28] Dahm, R., arXiv:hep-ph/9601207 (1996).

[29] Rowlands, P., arXiv:hep-ph/0301071 (2003).

[30] Meessen, A., *Found. of Phys.*, no. 29 (2000) 281-316.

[31] Henley, E., & J. Schiffer, arXiv:nucl-th/9807041 (1998) p.11.

[32] Baez, J., arXiv:math.RA/0105155 (2001).

[33] Waser, A., "Quaternions in electrodynamics," (2001) http://www.aw-verlag.ch

[34] Kassandrov, V.V., *Grav. & Cosmology* Vol.1(1995) arXiv:gr-qc/0007027 (2000); [34a] arXiv:gr-qc/00070276 (2000).

[35] Baylis, W.E., arXiv:quant-ph/0202060 (2002).

[36] van Dongen, J., arXiv:gr-qc/0009087 (2000) p.5

[37] Consoli, M., arXiv:gr-qc/0204106 (2002). [37a] Consoli, M., & E. Costanzo, arXiv:hep-ph/0311317 (2003).

[38] Ibison, M., "Electrodynamics in the Zero-Point Field: On the equilibrium spectral energy distribution, and the origin of inertial mass." *Found. Phys. Letter* Vol. 16 No. 1 (2003) 83-90.

[39] Ying-Qiu, G., "Some properties of the spinor soliton," *Advances in Applied Clifford Algebras* **8** No. 1 (1998) 17-29.

[40] Woon, S., DAMTP-R-97/33, arXiv:hep-th/9707206 (1997) p. 18-21.

[41] Fauser, B., arXiv:hep-th/9908200 (1999).

[42] Spohn, H., arXiv:quant-ph/9911065 (1999).

[43] Nottale, L., *et al., Astron. Astrophys.* **322**, (1997) 1018.

[44] Nottale, L.*, Astron. Astrophys.* **327,** (1997) 867-889.

[45] Nottale, L., G. Schumacher, & E.T. Levefre, *Astron. Astrophys.* **361**, (2000) 379-387; preprint at http://www.daec.obspm.fr/users/nottale

[46] Ohba, I., "Some stochastic aspect of quantization", *Pramana J.Phys.*, Indian Academy of Sciences, Vol. 59. No. 2 (2002) 397-404.

[47] Rubcic, A., & J. Rubcic, "The quantization of solar-like gravitational systems," *Fizika* B **7** Vol. 1, 1-13 (1998).

[48] Agnese, A.G., & R. Festa, "Discretization of the cosmic scale inspired from the Old Quantum Mechanics," *Proc. Workshop on Modern Modified Theories of Gravitation and Cosmology* 1997, preprint arXiv:astro-ph/9807186 (1998).

[49] Christianto, V., *Apeiron* Vol. 11 No. 3 (2004), preprint at http://reachme.at/coolbit

[49a] Christianto, V., AFLB Vol. 31 no. 1 (2006)

URL:http://www.ensmp.fr/aflb/AFLB-311/aflb311m370.pdf

[49b] Christianto, V., Apeiron VOl. 11 (2004)

http://www.slac.stanford.edu

[50] NASA News Release (Jul 2005), http://www.nasa.gov/vision/universe/solarsystem/newplanet-072905.html

[51] BBC News (Oct 2004), http://news.bbc.co.uk/1/hi/sci/tech/4730061.stm

[52] Brown, M., *et al.*, *ApJ. Letters* (Aug. 2004). Preprint arXiv: astro-ph/0404456 (2004); *ApJ.* forthcoming issue (2005), astro-ph/0504280 (2005).

[53] Brown, M. (Jul. 2005), http://www.gps.caltech.edu/‘mbrown/planetlila/

[54] Pitkänen, M., http://www.physics.helsinki.fi/∼matpitka/articles/nottale.pdf

[55] Infeld, E., *et al.,* arXiv:cond-mat/0104073 (2001).

[56] Lodder, J.J., submitted to *Physica A*, arXiv:hep-th/9405265 (1994).

[57] Cozzini, M., & S. Stringari, arXiv:cond-mat/0211294 (2002).

[58] Marchuk, N., "A model of the composite structure of quarks and leptons with SU(4) gauge symmetry," arXiv:hep-ph/9801382 (1998).

[59] Cohen, N., *et al., Electronic Journal of Linear Algebra* **7** (2000), arXiv:math-ph/9907015 (1999).

[60] Kruglov, S.I., arXiv:hep-ph/0507027 (2005); [60a] Kruglov, S.I., "Dirac-Kahler equation," arXiv:hep-th/0110060; [60b] Kruglov, S.I., hep-th/0110251 (2001).

[61] Kiehn, R.M., "An interpretation of wavefunction as a measure of vorticity," http://www22.pair.com/csdc/pdf/cologne.pdf (1989).

[62] de Andrade, L.G., & C. Sivaram, arXiv:hep-th/9811067 (1998). [62a] de Andrade, L.G., gr-qc/0405062.

[63] Kivotides, D., arXiv:physics/0405033 (2004).

[64] Godfrey, S.P. *et al.*, “A new interpretation of oscillating flow experiments in superfluid Helium II,” *J. Low Temp. Physics*, Nos. 112 (Oct 2001).

[65] Schrieffer, J.R., “Macroscopic quantum phenomena from pairing in superconductors,” Lecture, December 11^{th}, 1972.

[66] Froelich, P., *et al.,* “Hydrogen-antihydrogen molecule and its properties,” *Few-Body Systems* (2003), at the 3^{rd} *Workshop on the Dynamics and Systems of Critically Stable Few-Body Systems*, ECT, Trento, Italy, 2003.

[67] Nottale, L., “Scale relativity and fractal spacetime,” *Chaos, Solitons, Fractals*, Vol. **7** No. 6 (1996) p. 29.

[68] Coles, P., arXiv:astro-ph/0209576 (2002).

[69] Anandan, J.S., in *Quantum Coherence and Reality, Proc. Conf. Fundamental Aspects of Quantum Theory*, Columbia SC., edited by J.S. Anandan & J.L. Safko, (World Scientific 1994). arXiv:gr-qc/9504002 (1995).

[70] Kravchenko, V., & V.V. Kravchenko, arXiv:math-ph/0305046 (2003). [70a] Bernstein, S., Factorization of the nonlinear Schrodinger equation and applications, (199..) p. 2-3.

[71] Zakir, Z., arXiv:gr-qc/9906079 (1999).

[72] Rosu, H.C., *et al.*, arXiv:quant-ph/0204099 (2002).

[73] Baryshev, V., *et al.*, *Vistas in Astronomy*, Vol. **38** Part 4, (1994); preprint arXiv:astro-ph/9503074 (1995) p. 10.

[74] Antoniadis, I., P. Mazur, & E. Mottola, *Phys.Rev.Lett.* **79** (1997) 14-17; preprint at arXiv: astro-ph/9611208. [74a] E. Scholz, arXiv:astro-ph/0403446.

[75] Trucks, M., arXiv:gr-qc/9811043 (1998).

[76] Pervushin, V.N., *et al.,* arXiv:hep-th/0209070 (2002). [76a] Blaschke, D., *et al.*, arXiv:hep-th/0110003 (2001).

[77] Martinez, S., *et al.*, arXiv:cond-mat/0201565 (2002). [77a] Lubo, M., hep-th/0009162 (2000).

[78] Castro, C., *et al.*, arXiv:hep-th/0004152 (2000).

[79] Kim, S-H, *et al.*, arXiv:cond-mat/9908086 (1999) p.5.

[80] Rocha, G., *et al.,* arXiv:astro-ph/0205155 (2002).

[81] Bulgadaev, S., Landau Institute preprint, arXiv:hep-th/9901036 (1999). [81a] Faddeev, L., & A.J. Niemi, arXiv:hep-th/97051776.

EJTP 3, No. 12 (2006) 145–150 *Electronic Journal of Theoretical Physics*

Dynamics of Charged Spherically Symmetric Thick Shell

A. Eid*

Department of Astronomy,
Faculty of Science,
Cairo University, Egypt

Received 3 June 2006, Accepted 11 August 2006, Published 20 September 2006

Abstract: We Consider a spherically symmetric thick shell in two different space times. We have used the equation of motion for thick shell, developed by Khakshournia and Mansouri, to obtain the equation of motion of a charged spherical shell. We Expand the dynamical equation of motion of thick shell, to the first order of its thickness, to compare it with the dynamics of charged thin shell. It is shown that the effect of thickness is to speed up the collapse of the shell.

Keywords: General Relativity, Spherically Symmetric Thick Shell
PACS (2006): 04.50.+h, 04.20.q, 04.20.Fy,

1. Introduction

The thin shell formalism of general relativity (GR) has found wide applications in GR and cosmology. This formalism was first developed in [1] and applied to the gravitational collapse problem [2]. Studies on gravitational collapse, dynamics of bubbles and domain walls in inflationary models, wormhole, signature changes, structure and dynamics of voids in the large scale structure of the universe are some of the applications (cf [3] and references their in). Thin shells are considered as zero thickness objects with a δ- function singularity in their energy-momentum and Einstein's tensors.

However the dynamics of a real thick shell has been rarely discussed in the literature because of the complexity to define it within GR and to find its exact dynamical equations. The outstanding paper that modifies the Israel thin shell equations to treat the motion of spherical and Planar thick domain walls is that of Garfinkle and Gregory [4], see also [5].

* aeid06@yahoo.com

According to the results of that paper, the effect of thickness in the first approximation is to reduce effectively the energy density of the wall compared to the corresponding thin domain wall, and therefore to increase the collapse velocity of the wall in vacuum.

I use the formalism developed by Mansouri and Khakshournia(MK) [6] to treat a charged thick shell. The organization of this paper is as follows. In Section 2 I give a brief introduction to MK junction condition of thick shell. I describe the dynamics of charged spherically symmetric thick shell Section 3. A general conclusion is given in Section 4.

2. The Junction Conditions

Consider a spherically symmetric thick shell with two boundaries Σ_1 and Σ_2 dividing the space –time into three regions: M_{in} inside the inner boundary Σ_1, M_{out} outside the outer boundary Σ_2, and M for the thick shell having two boundaries Σ_1 and Σ_2. First of all , write down the appropriate junction condition on each boundary $\Sigma_j (j = 1, 2)$ treated as a three dimensional timelike hypersurface. The continuity of the second fundamental form of Σ_j, or the extrinsic curvature tensor K_{ab} of Σ_j, so that consider $\Sigma_1(\Sigma_2)$ as a boundary surface separating M region from $M_{in}(M_{out})$. This crucial requirement is formulated as

$$[K_{ab}] \overset{\Sigma_j}{=} 0 \quad (j = 1, 2), \tag{1}$$

where the square bracket indicates the jump of K_{ab} across Σ_j, ($[K_{ab}] = K^+_{ab} - K^-_{ab}$). Latin indices range over the intrinsic coordinates of Σ_j denoted by (τ_j, θ, ϕ), where τ_j is the proper time of Σ_j. In particular, the angular component of equation (1) on each boundary is written as

$$K^{\theta+}_{\theta}|_{\Sigma_1} - K^{\theta-}_{\theta}|_{\Sigma_1} = 0 \tag{2}$$

$$K^{\theta+}_{\theta}|_{\Sigma_2} - K^{\theta-}_{\theta}|_{\Sigma_2} = 0 \tag{3}$$

where the superscript +(-) refers to the side of Σ_j towards which the corresponding unit space like normal vector $n^\alpha(-n^\alpha)$ points. It means that on $\Sigma_1(\Sigma_2)$, the superscript + refers to the region $M(M_{out})$ and the superscript - refers to the region $M_{in}(M)$. Adding equations (2) and (3), to get

$$K^{\theta+}_{\theta}|_{\Sigma_2} - K^{\theta-}_{\theta}|_{\Sigma_1} + K^{\theta+}_{\theta}|_{\Sigma_1} - K^{\theta-}_{\theta}|_{\Sigma_2} = 0. \tag{4}$$

In the next section, this general equation will be applied to the case of a collapsing charged shell.

3. Dynamics of Charged Thick Shell

Consider a spherical thick shell immersed in two different spherically symmetric space-times. The space-time outside the shell is described by Reissner-Nordstrom (RN) metric:

$$ds^2_o = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2 \tag{5}$$

where

$$f = 1 - \frac{2m}{r} + \frac{e^2}{r^2} = 1 - \frac{F(r)}{r} = 1 + \Phi(r) \tag{6}$$

where $F(r)$ or $\Phi(r)$ is an real function, and e is the electric charge.

The space-time inside the shell is described by Minkowski metric:

$$ds_i^2 = -dT^2 + dr^2 + r^2 d\Omega^2 \tag{7}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

is the line element on the unit sphere. The induced intrinsic metric on Σ_j may be represented as

$$ds^2\left|_{\Sigma_j} = -d\tau_j^2 + R_j^2(\tau_j)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad (j = 1, 2),\right.$$

where $R_j(\tau_j)$ being the proper radius of Σ_j. Now, define the constant commoving thickness of the shell as follows $2\delta = r_2 - r_1$ where r_1 and r_2 are commoving radii of the boundaries Σ_1 and Σ_2 respectively. Using the metric (5),(7) the relevant extrinsic curvature tensors in the region M are

$$K_\theta^{\theta+}|_{\Sigma_1} = \frac{1}{R_1}\sqrt{1 + \dot{R}_1^2 - \frac{F(r_1)}{R_1}}$$

$$K_\theta^{\theta-}|_{\Sigma_2} = \frac{1}{R_2}\sqrt{1 + \dot{R}_2^2 - \frac{F(r_2)}{R_2}} \tag{8}$$

and the relevant extrinsic curvature tensors in the regions M_{in} and M_{out} are

$$K_\theta^{\theta+}|_{\Sigma_2} = \frac{1}{R_2}\sqrt{1 + \dot{R}_2^2 - \frac{R(r_2)}{R_2}}$$

$$K_\theta^{\theta-}|_{\Sigma_1} = \frac{1}{R_1}\sqrt{1 + \dot{R}_1^2} \tag{9}$$

where $R_j \equiv R(r_j, \tau)$ and $R(r_2)$ is the radius of the spherical shell within the commoving surface r_2.

To obtain the dynamical equation of the thick shell, expand the following quantities in a Taylor series around (r_0), the mean commoving radius of the thick shell,

$$R(r_j, \tau) = R(r_0, \tau) + \varepsilon_j \delta R'(r_0, \tau) + 0\left(\delta^2\right) \tag{10}$$

$$F(r_j) = F(R_0) + \varepsilon_j \delta F'(r_0) + 0\left(\delta^2\right) \tag{11}$$

$$R(r_2) = R(r_0) + \delta R'(r_0) + 0\left(\delta^2\right) \tag{12}$$

where $\varepsilon_1 = -1$ and $\varepsilon_2 = +1$. Using equations (10), (11), (12) in the expressions (8) and (9), and keeping only terms up to the first order of δ, to get

$$K_\theta^{\theta-}|_{\Sigma_1} = \frac{1}{R_0}\sqrt{1 + \dot{R}_0^2}(1 + \delta(\frac{R_0'}{R_0} - \frac{\dot{R}_0\dot{R}_0'}{1 + \dot{R}_0^2})),$$

$$K_{\theta}^{\theta+}|_{\Sigma_2} = \frac{1}{R_0}\sqrt{1+\dot{R}_0^2-\frac{R(r_0)}{R_0}}(1-\delta(\frac{R_0'}{R_0}-\frac{\dot{R}_0\dot{R}_0'-\frac{R'(r_0)}{2R_0}+\frac{R_0'R(r_0)}{2R_0^2}}{1+\dot{R}_0^2-\frac{R(r_0)}{R_0}})),$$

$$K_{\theta}^{\theta+}|_{\Sigma_1} = \frac{1}{R_0}\sqrt{1+\dot{R}_0^2-\frac{F(r_0)}{R_0}}(1+\delta(\frac{R_0'}{R_0}-\frac{\dot{R}_0\dot{R}_0'-\frac{F'(r_0)}{2R_0}+\frac{R_0'F(r_0)}{2R_0^2}}{1+\dot{R}_0^2-\frac{F(r_0)}{R_0}})),$$

$$K_{\theta}^{\theta-}|_{\Sigma_2} = \frac{1}{R_0}\sqrt{1+\dot{R}_0^2-\frac{F(r_0)}{R_0}}(1+\delta(\frac{R_0'}{R_0}-\frac{\dot{R}_0\dot{R}_0'-\frac{F'(r_0)}{2R_0}+\frac{R_0'F(r_0)}{2R_0^2}}{1+\dot{R}_0^2-\frac{F(r_0)}{R_0}})), \tag{13}$$

where $R_0 \equiv R(r_0,\tau)$. Substituting equation (13) into equation (4) and noting that $F(r_0) \equiv R(r_0)$, then the thick shell's equation of motion written up to the first order in δ is

$$\alpha-\beta = 2\delta\frac{F'(r_0)}{2\beta R_0}-\delta\left[\frac{R_0'}{R_0}(\alpha-\beta)+\frac{\dot{R}_0\dot{R}_0'}{\alpha\beta}(\alpha-\beta)+\frac{1}{2\beta R_0}\left(R'(r_0)+\frac{R_0'F(r_0)}{R_0}\right)\right] \tag{14}$$

where $\alpha = \sqrt{1+\dot{R}_0^2}$

and $\beta = \sqrt{1+\dot{R}_0^2-\frac{R(r_0)}{R_0}} \equiv \sqrt{1+\dot{R}_0^2-\frac{F(r_0)}{R_0}}$.

This is the generalization of thin shell dynamical equation up to the first order of the thickness.

To verify the thin shell limit of this thick shell dynamical equation, consider the following definition for the surface energy density of the infinitely thin shell [1],

$$\sigma = \int_{-\varepsilon}^{+\varepsilon}\rho(r,\tau)\,dn \tag{15}$$

Where n is the proper distance in the direction of the normal n_μ and 2ε is the physical thickness of the shell. With the metric (5) equation (15) takes the form

$$\sigma = \int_{-\delta}^{+\delta}\frac{\rho(r,\tau)}{\sqrt{1+\dot{R}^2-\frac{F(r)}{R}}}dr \tag{16}$$

Since $\frac{\partial m}{\partial R} = 4\pi R^2\rho G$, then equation (16) can be written as

$$8\pi G\sigma = \int_{-\delta}^{+\delta}\frac{2\frac{\partial m}{\partial R}dr}{R^2\sqrt{1+\dot{R}^2-\frac{F(r)}{R}}} = \int_{-\delta}^{+\delta}\frac{F'(r)}{R^2\sqrt{1+\dot{R}^2-\frac{F(r)}{R}}}dr \tag{17}$$

Substituting equations (10) and (11) into equation (17) and integrate it up to the first order in δ to get

$$8\pi G\sigma = 2\delta\frac{F'(r_0)}{R_0^2\sqrt{1+\dot{R}_0^2-\frac{F(r_0)}{R_0}}}+0\left(\delta^2\right) \tag{18}$$

Substituting equations (18) into equation (14) to get

$$\alpha-\beta=4\pi G\sigma R_0-\delta\left[\frac{R_0'}{R_0}(\alpha-\beta)+\frac{\dot{R}_0\dot{R}_0'}{\alpha\beta}(\alpha-\beta)+\frac{1}{2\beta R_0}\left(R'(r_0)+\frac{R_0'F(r_0)}{R_0}\right)\right] \quad (19)$$

When δ tends to zero the second term on the right hand side goes to zero, then this equation reduced to the equation of motion of charged thin shell.

Rewrite the dynamical equation of thick shell (19) in the form

$$\alpha-\beta=4\pi GR_0\tilde{\sigma} \quad (20)$$

where

$$\tilde{\sigma}=\sigma-\frac{\delta}{4\pi GR_0}\left[\frac{R_0'}{R_0}(\alpha-\beta)+\dot{R}_0\dot{R}_0'\left(\frac{\alpha-\beta}{\alpha\beta}\right)+\frac{1}{2\beta R_0}\left(R'(r_0)+\frac{R_0'F(r_0)}{R_0}\right)\right] \quad (21)$$

It has the same form of the equation of thin shell with the effective surface density $\tilde{\sigma}$.

From equation (21) note that the shell starting its collapse at rest when the velocity $\dot{R}$ is negative during the collapse, it becomes more negative with r so that $\dot{R}_0' \prec 0$, so $\dot{R}\dot{R}_0'$ must be positive. Also, the radius of the shell layers is increased with $r$so that $R'(r_0)\rangle 0$. Therefore all terms within the bracket on the right hand side of equation (21) are positive. This leads to the result $\tilde{\sigma}\langle\sigma$. Solving equation (20) for $\dot{R}^2$ to get

$$\dot{R}_0^2=-1+4\pi^2G^2\tilde{\sigma}^2R_0^2+\frac{F(r_0)}{2R_0}+\frac{F^2(r_0)}{64\pi^2G^2\tilde{\sigma}^2R_0^4}. \quad (22)$$

It follows that $\dot{R}^2$ becomes larger with smaller $\tilde{\sigma}$ and $R_0\rangle R(r_0)$. Substituting by (6) into equation (22) to get

$$\begin{aligned}\dot{R}_0^2=&-1+4\pi^2G^2\tilde{\sigma}^2R_0^2+\tfrac{m}{R_0}+\tfrac{m^2}{16\pi^2G^2\tilde{\sigma}^2R_0^4}\\&+\tfrac{e^2}{2R_0^2}(-1-\tfrac{m}{8\pi^2G^2\tilde{\sigma}^2R_0^3}+\tfrac{e^2}{32\pi^2G^2\tilde{\sigma}^2R_0^4})\end{aligned}$$

Therefore the first order thickness corrections to the Israel thin shell approximation speed up the collapse of the shell.

4. Conclusion

I applied the modified Israel formalism which developed by MK to the case of the collapse of a charged thick shell in RN and obtained the zero thickness limit of the charged thin shell equation, and Israel thin shell equation with e =0.

It has been shown that the effect of thickness up to the first order in the shell thickness, is to speed up the collapse of the shell.

References

[1] Israel W., Nuovo Cimento. 44 B (1966) 1.

[2] Frolov V.P., Sov. Phys. JETP 38 (1974) 393.

[3] Berezin V.A., Kuzmin V.A. and Tkachev I.I., Phys. Rev. D 36 (1987) 2919.

[4] Garfinkle D. and Gregory R., phys. Rev. D 41 (1990) 1889.

[5] Bonjour F., Charmousis C. and Gregory R., phys. Rev. D 62 (2000) 083504.

[6] Mansouri R. and Khakshournia S., Gen. Rel. Grav. 34 (2002) 1847.