

Rayleigh process and matrix elements for the one-dimensional harmonic oscillator

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Abstract: We show that, the matrix elements $\langle m | e^{-\gamma x} | n \rangle$ for the one-dimensional harmonic oscillator have application in Markov process theory, permitting thus to resolve the Fokker-Planck equation for the two-dimensional probability density corresponding to Rayleigh case.

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1. Introduction

In [1-4] were calculated the matrix elements:

$$f(\gamma) = \langle m | e^{-\gamma x} | n \rangle = \int_{-\infty}^{\infty} \psi_m^*(x) e^{-\gamma x} \psi_n(x) dx \quad (1)$$

for the harmonic oscillator in one dimension, where $\gamma \geq 0$ is an arbitrary parameter. Then, it was deduced the following result for $m \geq n$:

$$f(\gamma) = \sqrt{\frac{n!}{m!}} \left(-\frac{\gamma}{\sqrt{2}} \right)^{m-n} e^{\gamma^2/4} L_n^{m-n} \left(-\frac{\gamma^2}{2} \right) \quad (2)$$

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in terms of the associated Laguerre polynomials L_n^q .

It is interesting to observe that the 2nd order differential equation [5-8] defining to L_q^p permits to show, via (2), that $f(\gamma)$ satisfies the equation:

$$\frac{d^2 f}{d\gamma^2} + \frac{1}{\gamma} \frac{df}{d\gamma} - \frac{1}{4\gamma^2} (\gamma^4 + 4A\gamma^2 + 4Q) f = 0 \quad (3)$$

where $A = m + n + 1$ and $Q = (m - n)^2$. That is, (2) is solution of (3), with which it is possible to obtain [9] the Morse's radial wave function [10].

In Sec. 2, $f(\gamma)$ is employed to resolve the nonstationary stochastic Fokker-Planck equation (FPE) [11] for the Rayleigh distribution.

2. Two-Dimensional Probability Density Associated to Rayleigh Process.

The equation (3) has the structure:

$$D_1(\gamma)f = \left(\frac{\gamma^2}{4} + A + \frac{Q}{\gamma^2} \right) f \quad (4)$$

where it appears the important Bessel's operator [12]:

$$D_C(\gamma) = \frac{d^2}{d\gamma^2} + \frac{C}{\gamma} \frac{d}{d\gamma} \quad (5)$$

for the case $C = 1$. The operator (5) has interesting applications in hydrodynamics, the theory of subharmonic functions, electrostatics, the Euler-Poisson-Darboux equation, elasticity, the generalized radiation problem, quantum mechanics and generalized axially symmetric potential. Here we shall show that, the Bessel's operator D_1 is useful for to determine the probability density ω associated to Rayleigh process in two dimensions, because the FPE can adopt a form similar to (4).

In fact, the nonstationary stochastic FPE for the Rayleigh distribution is given by [11] p. 73:

$$\dot{\omega} = \frac{\partial}{\partial x} \left[\left(\beta x - \frac{k}{2x} \right) \omega \right] + \frac{k}{2} \frac{\partial^2 \omega}{\partial x^2} \quad (6)$$

being k and β positive parameters, then the corresponding eigenfunctions $X(x)$ are solutions of:

$$\sigma^2 \frac{d^2 X}{dx^2} + \left(x - \frac{\sigma^2}{x} \right) \frac{dX}{dx} + \left(1 + \frac{\sigma^2}{x^2} + \frac{\lambda}{\beta} \right) X = 0 \quad (7)$$

with $\sigma^2 = \frac{k}{2\beta}$. The operator D_1 participates when in (7) we make the following change of functions:

$$F(x) = x^{-1} e^{\frac{x^2}{4\sigma^2}} X \quad (8)$$

obtaining thus the relation:

$$D_1(x)F = \left(\frac{x^2}{4\sigma^4} - \frac{1}{\sigma^2} - \frac{\lambda}{\sigma^2\beta} \right) F \quad (9)$$

On the other hand, in (4) it is possible to realize the formal change of variable:

$$\gamma = \frac{i}{\sigma}x \quad , \quad i = \sqrt{-1} \quad (10)$$

then it results the equation:

$$D_1(x)f = \left(\frac{x^2}{4\sigma^4} + \frac{Q}{x^2} - \frac{A}{\sigma^2} \right) f \quad , \quad (11)$$

with the same structure as (9); therefore $Q = 0$, that is $m = n$, and $A = 2n + 1 = 1 + \frac{\lambda}{\beta}$, then $\lambda = 2n\beta$, $n = 0, 1, 2, \dots$. Besides, F is proportional to f given by (2) with the change (10):

$$F \propto e^{-\frac{x^2}{4\sigma^2}} L_n \left(\frac{x^2}{2\sigma^2} \right) \quad (12)$$

then (8), (12) and factors of normalization lead us to the eigenfunctions:

$$X_n(x) = \frac{x}{n!\sigma^2} e^{-\frac{x^2}{2\sigma^2}} L_n \left(\frac{x^2}{2\sigma^2} \right) \quad (13)$$

which are solutions of (7) for $\lambda = 2n\beta$. The result (13) is our principal aim because it permits to write immediately the two-dimensional probability density associated to Rayleigh case, to see [11].

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