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A Toy Model of Financial Markets

J. P. Singh and S. Prabakaran [*] [†]

Department of Management Studies
Indian Institute of Technology Roorkee
Roorkee 247667, India

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Abstract: Several techniques of fundamental physics like quantum mechanics, field theory and related tools of non-commutative probability, gauge theory, path integral etc. are being applied for pricing of contemporary financial products and for explaining various phenomena of financial markets like stock price patterns, critical crashes etc.. In this paper, we apply the well entrenched methods of quantum mechanics and quantum field theory to the modeling of the financial markets and the behaviour of stock prices. After defining the various constituents of the model including creation & annihilation operators and buying & selling operators for securities, we examine the time evolution of the financial markets and obtain the Hamiltonian for the trading activities of the market. We finally obtain the probability distribution of stock prices in terms of the propagators of the evolution equations.

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1. Introductionn

The specialty of "physics" is the study of interactions between the various manifestations of matter and its constituents. The development of this subject over the last several centuries has led to a gradual refining of our understanding of natural phenomena. Accompanying this has been a spectacular evolution of sophisticated mathematical tools for the modeling of complex systems. These analytical tools are versatile enough to find application not only in point processes involving particles but also aggregates thereof leading to field theoretic generalizations and condensed matter physics.

Furthermore, with the rapid advancements in the evolution and study of disordered

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[*] jatinfdm@iitr.ernet.in

[†] Jatinder_pal2000@yahoo.com

systems and the associated phenomena of nonlinearity, chaos, self organized criticality etc., the importance of generalizations of the extant mathematical apparatus to enhance its domain of applicability to such disordered systems is cardinal to the further development of science.

A considerable amount of work has already been done and success achieved in the broad areas of q-deformed harmonic oscillators [1], representations of q-deformed rotation and Lorentz groups [2-3]. q-deformed quantum stochastic processes have also been studied with realization of q-white noise on bialgebras [4]. Deformations of the Fokker Planck's equation [5], Langevin equation [6] and Levy processes [7-8] have also been analysed and results reported.

Though at a nascent stage, the winds of convergence of physics and finance are unmistakably perceptible with several concepts of fundamental physics like quantum mechanics, field theory and related tools of non-commutative probability, gauge theory, path integral etc. being applied for pricing of contemporary financial products and for explaining various phenomena of financial markets like stock price patterns, critical crashes etc. [8-19]. The origin of the association between physics and finance, though, can be traced way back to the seminal works of Pareto [20] and Batchlier [21], the former being instrumental in establishing empirically that the distribution of wealth in several nations follows a power law with an exponent of 1.5, while the latter pioneered the modeling of speculative prices by the random walk and Brownian motion. The cardinal contribution of physicists to the world of finance came from Fischer Black & Myron Scholes through the option pricing formula [22] which bears their epitaph and which won them the Nobel Prize for economics in 1997 together with Robert Merton [23]. They obtained closed form expressions for the pricing of financial derivatives by converting the problem to a heat equation and then solving it for specific boundary conditions.

The theory of stochastic processes constitutes the “golden thread” that unites the disciplines of physics and finance. Modeling of non relativistic quantum mechanics as energy conserving diffusion processes is, by now, well known [24]. Unification of the general theory of relativity and quantum mechanics to enable a consistent theory of quantum gravity has also been attempted on “stochastic spaces” [25]. Time evolution of stock prices has been, by suitable algebraic manipulations, shown to be equivalent to a diffusion process [26].

Contemporary empirical research into the behavior of stock market price /return patterns has found significant evidence that financial markets exhibit the phenomenon of anomalous diffusion, primarily superdiffusion, wherein the variance evolves with time according to a power law $t^\alpha$with $\alpha > 1.0$. The standard technique for the study of superdiffusive processes is through a stochastic process that evolves according to a Langevin equation and whose probability distribution function satisfies a nonlinear Fokker Planck equation [27].

There is an intricate yet natural relationship between the power law tails observed in stock market data and probability distributions that emanate as the solution of the Fokker Planck equation. The Fokker Planck equation is known to describe anomalous diffusion under time evolution. Empirical results [28-31] establish that temporal changes of several financial market indices have variances that that are shown to undergo anomalous super diffusion under time evolution.

One of the most exhaustive set of studies on stock market data in varying dimensions has been reported in [32-36]. In [36], a phenomenological study was conducted of stock price fluctuations of individual companies using data from two different databases covering three major US stock markets. The probability distributions of returns over varying timescales ranging from 5 min. to 4 years were examined. It was observed that for timescales from 5 minutes upto 16 days the tails of the distributions were well described by a power law decay. For larger timescales results consistent with a gradual convergence to Gaussian behaviour was observed. In another study [32] the probability distributions of the returns on the S & P 500 were computed over varying timescales. It was, again, seen that the distributions were consistent with an asymptotic power law behaviour with a slow convergence to Gaussian behaviour. Similar findings were obtained on the analysis of the NIKKEI and the Hang –Sang indices [32].

Stock market phenomena are assumed to result from complicated interactions among many degrees of freedom, and thus they were analyzed as random processes and one could go to the extent of saying that the Efficient Market Hypothesis [37-38] was formulated with one primary objective – to create a scenario which would justify the use of stochastic calculus [39] for the modeling of capital markets.

The Efficient Market Hypothesis contemplates a market where all assets are fairly priced according to the information available and neither buyers nor sellers enjoy any advantage. Market prices are believed to reflect all public information, both fundamental and price history and prices move only as sequel to new information entering the market. Further, the presence of large number of investors is believed to ensure that all prices are fair. Memory effects, if any at all, are assumed to be extremely short ranging and dissipate rapidly. Feedback effects on prices are, thus, assumed to be marginal. The investor community is assumed rational as benchmarked by the traditional concepts of risk and return.

An immediate corollary to the Efficient Market Hypothesis is the independence of single period returns, so that they can be modeled as a random walk and the defining probability distribution, in the limit of the number of observations being large, would be Gaussian.

Anomalous diffusion is a hallmark of several intensively studied physical systems. It is observed, for example, in the chaotic dynamics of fluid in rapidly rotating annulus [40], conservative motion in a periodic potential [41], transport of fluid in a porous media [42], percolation of gases in porous media [43], crystal growth spreading of thin films under gravity [44], radiative heat transfer [45], systems exhibiting surface to surface growth [46] and so on.

Several analogies between physical systems and financial processes have been explored in the last decade, some of which have already been mentioned above. Perhaps, the most striking one is that between financial crashes witnessed in stock markets and critical phenomena like phase transitions is discussed here to place the main theme of this paper in its proper perspective.

Stock market crashes are believed to exhibit log periodic oscillations which are characteristic of systems exhibiting discrete scale invariance i.e. invariance through rescaling by integral powers of some length scale like the Serpinski triangle and other similar fractal shapes. In the years preceding the infamous crash of October 19, 1987, the S & P market index was seen to fit the following expression exceedingly precisely [47-48],

$$(S\,\&\,P)_t = \Omega + \Gamma\left(t_c - t\right)^{\gamma}\left\{1 + \Xi\cos\left[\theta\ln\left(t_c - t\right) + \phi\right]\right\}$$

Physicists working in solid state and condensed matter physics would immediately recognize the analogy of the above expression with the one obtained for critical phenomenon in spin model of ferromagnetism [49]. We briefly elucidate the salient features of this model. Crystalline solids comprise of atoms arranged in a lattice. Each such atom generates a magnetic field parallel to the direction of the atom's spin. In the case of substances that do not exhibit ferromagnetic character, these spin directions are randomly oriented so that the aggregate magnetic field vanishes. However, in ferromagnetic substances these spins are polarized in a particular direction resulting in a nonzero aggregate field. Ferromagnetic substances usually exhibit two distinct phases. one in which the spins orient themselves in a particular direction resulting in an aggregate magnetic moment at temperatures below a well defined critical temperature t_c and the other where the spins are disoriented with a zero aggregate moment above the critical temperature. At temperatures below t_c, the coupling force between neighboring atoms predominates resulting in an alignment of spins whereas above t_c the additional energy manifests itself in disorienting (randomizing) the spins.

Renormalization group theory enables us to group these atoms in blocks of spins whose composite spins are equal to the algebraic sum of the spins of the atoms constituting the block. It then provides that a model involving interactions between these composite spins of a block can be constructed that replicates the macroscopic properties of the block and yet cannot depend on the size of the block. That is, the system would exhibit a scaling symmetry, which is discrete, if we allow for the finite size of the atom and continuous otherwise. The magnetic susceptibility of such a magnetic substance defined by $\chi\left(T\right) = \frac{\partial M}{\partial B}\big|_{B=0}$, where the symbols have their usual meaning, obeys a power law of the form $\chi\left(T\right) = Re\left[\left(T - T_c\right)^{\alpha+i\beta}\right]$or equivalently $\chi\left(T\right) = \left(T - T_c\right)^{\alpha}\left\{1 + \beta\cos\left[\ln\left(T - T_c\right)\right] + O\left(\beta^2\right)\right\}$which is reminiscent of op cited expression for log periodic oscillations in financial crashes.

Furthermore, the access to enhanced computing power during the last decade has enabled analysts to try refined methods like the phase space reconstruction methods for determining the Lyapunov Exponents [50] of stock market price data, besides doing Rescaled Range Analysis [51] etc. A set of several studies has indicated the existence of strong evidence that the stock market shows chaotic behavior with fractal return structures and positive Lyapunov exponents. Results of these studies have unambiguously established the existence of significant nonlinearities and chaotic behavior in these time series [52-55].

In this paper, we attempt one such model. The objective is to apply the well entrenched methods of quantum mechanics and quantum field theory to the study of the financial markets and the behaviour of stock prices. Section 2, which forms the essence of this paper, arrives at various results for financial markets by modeling them as quantum Hamiltonian systems. The probability distribution for stock prices in efficient markets is also obtained. Section 3 concludes.

1.1 Quantum Model of Financial Markets

We consider an “isolated” financial market comprising of n investors and m type of securities. The market is “isolated” in the sense that new types of securities are neither created nor are existing ones destroyed. Further, the number of investors is also constant. The investor i, $i = 1, 2, 3.......n$ is assumed to possess a cash balance of $x_i, i = 1, 2, 3.......n$ (which may be negative, representing borrowings) and $y_{ij}(z), i = 1, 2, 3.....n; j = 1, 2, 3....m$ units of security j at a unit price of z . Obviously, $y_{ij} \geq 0, \forall i, j$.

Towards constructing a basis for our Hilbert space representing the financial market, we define a pure state of the system as

$$|\Psi_i\rangle = |\{x_i, \{y_{ij}(z), j = 1, 2, ...m\}, i = 1, 2, ...n\}\rangle \tag{1}$$

Thus, a pure state represents a state of the market where the entire holdings of cash and securities of every investor are known with certainty. This represents a complete measurement of the market and hence, is in conformity with the standard definition of “pure state” of a system.

A basis for our Hilbert space may then be constituted by the set of all the pure states of the type (1) i.e.

$$\Psi = \{|\{x_i, \{y_{ij}(z), j = 1, 2, ...m\}, i = 1, 2, ...n\}\rangle\} \tag{2}$$

The elements of this basis set Ψ satisfy the orthogonality condition $\langle\Psi_i \mid \Psi_j\rangle = \delta_{ij}$ with respect to the scalar product defined in the sequel. The orthogonality condition makes sense in the financial world – it implies that if a market is in a pure state $|\Psi_i\rangle$ then it cannot be in any other pure state.

However, a complete measurement of the market is, obviously, not practicable in real life. At any point in time, we are likely to have certain information only about a fraction of the market constituents. Hence, the instantaneous state of the market $|\psi(t)\rangle$ may be represented by a linear combination of the pure states $|\Psi_l(t)\rangle$ i.e.

$$|\psi(t)\rangle = \sum_l C_l |\Psi_l(t)\rangle \tag{3}$$

We endow our Hilbert space H with the scalar product

$$\langle\psi(t) \mid \xi(t)\rangle = \sum_{l,m} C_l^* D_m \langle\Psi_l(t) \mid \Psi_m(t)\rangle = \sum_{l,m} C_l^* D_m \delta_{lm} = \sum_l C_l^* D_l \tag{4}$$

where we have assumed the orthogonality of the pure states.

The components of the state space vector $|\psi(t)\rangle$are given by$C_l = \langle \Psi_l(t) \mid \psi(t)\rangle$ and are related to the probability of finding the market in the pure state $|\Psi_l(t)\rangle$.

Since our basis comprises of all possible measurable pure states, the completeness of the basis is ensured so that

$$I = \sum_l |\Psi_l(t)\rangle \langle \Psi_l(t)| \tag{5}$$

In analogy with the no particle state or ground state in quantum mechanics, we can define a ground state of our financial market as

$$|0\rangle = |x_i = 0, y_{ij}(z) = 0 \forall i, j, z\rangle \tag{6}$$

i.e. the ground state is the market state in which no investor has any cash balances nor any securities. This state is, obviously, a pure state being fully measurable and would also not evolve in time since no trade can take place in this market.

We define the cash and security coordinate operators $\hat{x}_i$& $\hat{y}_{ij}(z)$ by their action on the basis state (1) to provide respectively the balances of cash and the j^{th} security (at price z) with the i^{th} investor as the eigenvalues i.e.

$$\hat{x}_i |\{x_i, \{y_{ij}(z), j = 1, 2, ...m\}, i = 1, 2, ...n\}\rangle = x_i |\{x_i, \{y_{ij}(z), j = 1, 2, ...m\}, i = 1, 2, ...n\}\rangle \tag{7}$$

$$\hat{y}_{ij}(z) |\{x_i, \{y_{ij}(z), j = 1, 2, ...m\}, i = 1, 2, ...n\}\rangle = y_{ij}(z) |\{x_i, \{y_{ij}(z), j = 1, 2, ...m\}, i = 1, 2, ...n\}\rangle \tag{8}$$

A cash translation operator $\hat{T}_i(z)$ is also defined by the following

$$\hat{T}_i(z') |\{x_i, \{y_{ij}(z), j = 1, 2, ...m\}, i = 1, 2, ...n\}\rangle = |\{x_i + z', \{y_{ij}(z), j = 1, 2, ...m\}, i = 1, 2, ...n\}\rangle \tag{9}$$

i.e. it transfers an amount of cash z to the i^{th} investor.

The operator $\hat{T}_i(z)$ obviously satisfies the following properties

$$\hat{T}_i(z_1)\hat{T}_i(z_2) = \hat{T}_i(z_1 + z_2) \tag{10}$$

$$\hat{T}_i(0) = \hat{I} \tag{11}$$

$$\left[\hat{T}_i(z), \hat{x}_j\right] = \hat{T}_i(z)\hat{x}_j - \hat{x}_j\hat{T}_i(z) = -z\delta_{ij}\hat{T}_i(z) \tag{12}$$

$$\hat{T}_i^{\dagger}(z) = \hat{T}_i(-z) \tag{13}$$

Towards obtaining an explicit representation of the cash translation operator, we assume $\hat{p}_i = \left.\frac{d\hat{T}_i(z)}{dz}\right|_{z=0}$ as the generator of infinitesimal cash translations dz to the investor i. Expanding $\hat{T}_i(z)$ as a Taylor's series and using eqs. (10), (11) we have

$$\frac{d\hat{T}_i(z)}{dz} = \lim_{dz \to 0} \frac{\hat{T}_i(z + dz) - \hat{T}_i(z)}{dz} = \lim_{dz \to 0} \frac{\left[\hat{T}_i(dz) - 1\right]\hat{T}_i(z)}{dz} = \lim_{dz \to 0} \frac{\left[\hat{T}_i(0) + \left.\frac{d\hat{T}_i(z)}{dz}\right|_{z=0} dz ... - 1\right]\hat{T}_i(z)}{dz} = \hat{p}_i\hat{T}_i(z) \tag{14}$$

with the solution $\hat{T}_i\left(z\right) = e^{z\hat{p}_i}$. Furthermore, we have (suppressing the y_{ij} indices for the sake of brevity)

$$\begin{aligned}&\left|\left\{x_i + dz, i = 1, 2, ...n\right\}\right\rangle = \hat{T}_i\left(dz\right)\left|\left\{x_i, i = 1, 2, ...n\right\}\right\rangle = \left[\hat{T}_i\left(0\right) + \left.\frac{d\hat{T}_i(z)}{dz}\right|_{z=0} dz...\right]\left|\left\{x_i, i = 1, 2, ...n\right\}\right\rangle \\ &= \left[I + \hat{p}_i dz...\right]\left|\left\{x_i, i = 1, 2, ...n\right\}\right\rangle\end{aligned} \tag{15}$$

Hence,

$$\begin{aligned}&\frac{\partial\left\langle\left\{x_i, i=1,2,...n\right\}\middle|\psi\right\rangle}{\partial x_i} = \lim_{dz\to 0}\frac{\left\langle\left\{x_i+dz, i=1,2,...n\right\}\middle|\psi\right\rangle - \left\langle\left\{x_i, i=1,2,...n\right\}\middle|\psi\right\rangle}{dz} \\ &= \hat{p}_i^{\uparrow}\left\langle\left\{x_i, i = 1, 2, ...n\right\} \mid \psi\right\rangle = -\hat{p}_i\left\langle\left\{x_i, i = 1, 2, ...n\right\} \mid \psi\right\rangle \Rightarrow \hat{p}_i = -\frac{\partial}{\partial x_i}\end{aligned} \tag{16}$$

so that $\hat{T}_i\left(z\right) = e^{-z\frac{\partial}{\partial x_i}}$. The following commutation rule holds between $\hat{x}_i$and $\hat{p}_i$

$$\left[\hat{x}_i, \hat{p}_j\right] = \delta_{ij} \tag{17}$$

The condition of an isolated market ensures that the basis and hence the Hilbert space does not depend on time. This implies that the temporal evolution of the system is unitary.

Creation & Annihilation Operators for Securities

We define $\hat{a}_{ij}\left(z\right)$as the annihilation operator of the security j from the portfolio of investor i for a price z i.e. when operator $\hat{a}_{ij}\left(z\right)$ acts on a state, the number of units of security j is reduced by one from the portfolio of investor i for a price z.Similarly, we define creation operators $\hat{a}^{\uparrow}_{ij}\left(z\right)$ as the adjoint of the annihilation operators that increase the number of units of security j in the portfolio of investor i for a price z.The precise action of these operators on a state vector is defined by the following

$$\hat{a}_{ij}(z)\left|\left\{x_i, \left\{y_{ij}\left(z\right), j = 1, 2, ...m\right\}, i = 1, 2, ...n\right\}\right\rangle = \sqrt{y_{ij}(z).z}\left|\left\{x_i, \left\{y_{ij}\left(z\right) - 1, j = 1, 2, ...m\right\}, i = 1, 2, ...n\right\}\right\rangle \tag{18}$$

and

$$\hat{a}^{\uparrow}_{ij}(z)\left|\left\{x_i, \left\{y_{ij}\left(z\right), j = 1, 2, ...m\right\}, i = 1, 2, ...n\right\}\right\rangle = \sqrt{(y_{ij}(z) + 1)z}\left|\left\{x_i, \left\{y_{ij}\left(z\right) + 1, j = 1, 2, ...m\right\}, i = 1, 2, ...n\right\}\right\rangle \tag{19}$$

where the factor ‘z’ has been introduced in the eigenvalues to ensure “scale invariance” of the theory.

These operators satisfy the following commutation relations:-

$$\left[\hat{a}_{ij}(z), \hat{a}^{\uparrow}_{ij}(z')\right] = z\delta_{zz'}\delta_{ik}\delta_{jl} \tag{20}$$

and

$$\left[\hat{a}_{ij}(z), \hat{a}_{kl}(z')\right] = \left[\hat{a}^{\uparrow}_{ij}(z), \hat{a}^{\uparrow}_{ij}(z')\right] = 0 \tag{21}$$

$$\left[\hat{T}_i\left(z\right), \hat{a}_{jk}\left(z'\right)\right] = \left[\hat{T}_i\left(z\right), \hat{a}^{\uparrow}_{jk}\left(z'\right)\right] = 0 \tag{22}$$

$$\left[\hat{T}^{\uparrow}_i\left(z\right), \hat{a}_{jk}\left(z'\right)\right] = \left[\hat{T}^{\uparrow}_i\left(z\right), \hat{a}^{\uparrow}_{jk}\left(z'\right)\right] = 0 \tag{23}$$

Further more

$$\hat{a}^{\uparrow}_{ij}\left(z\right)\hat{a}_{ij}\left(z\right)\left|\left\{x_i,\left\{y_{ij}\left(z\right),j=1,2,...m\right\},i=1,2,...n\right\}\right\rangle = zy_{ij}\left(z\right)\left|\left\{x_i,\left\{y_{ij}\left(z\right),j=1,2,...m\right\},i=1,2,...n\right\}\right\rangle \tag{24}$$

which implies that the number operator would be

$$\hat{y}_{ij}\left(z\right)=\frac{\hat{a}^{\uparrow}_{ij}\left(z\right)\hat{a}_{ij}\left(z\right)}{z} \tag{25}$$

Using the aforesaid operators we can construct an arbitrary basis state from the ground state as follows

$$\left|\left\{x_i,\left\{y_{ij}\left(z\right),j=1,2,...m\right\},i=1,2,...n\right\}\right\rangle=\prod_{i=1}^{n}\hat{T}_i\left(x_i\right)\prod_{j=1}^{m}\prod_{\left\{z,y_{ij}\left(z\right)\in\mathrm{N}\right\}}\left(\hat{a}^{\uparrow}_{ij}\left(z\right)\right)^{y_{ij}\left(z\right)}\left|0\right\rangle \tag{26}$$

Buying and selling operators

The buying (selling) operation of a security is, in each case, a composite operation consisting of the following:-

i. the creation (annihilation) of a security at the relevant price z; and

ii. the decrease (increase) in the cash balance by z of the investor undertaking the trade.

Hence we can define the buying (selling) operator as composite of the cash translation operator and the creation (annihilation) operators for securities as follows:-

$$\hat{b}^{\uparrow}_{ij}\left(z\right)=\hat{a}^{\uparrow}_{ij}\left(z\right)\hat{T}^{\uparrow}_{i}\left(z\right)=\hat{a}^{\uparrow}_{ij}\left(z\right)\hat{T}_i\left(-z\right) \tag{27}$$

for the “buying” operation and

$$\hat{b}_{ij}\left(z\right)=\hat{a}_{ij}\left(z\right)\hat{T}_i\left(z\right) \tag{28}$$

for the “selling” operation. These operators satisfy the following commutation rules

$$\left[\hat{b}_{ij}\left(z\right),\hat{b}^{\uparrow}_{kl}\left(z'\right)\right]=z\delta_{zz'}\delta_{ik}\delta_{jl} \tag{29}$$

$$\left[\hat{b}_{ij}\left(z\right),\hat{b}_{kl}\left(z'\right)\right]=\left[\hat{b}^{\uparrow}_{ij}\left(z\right),\hat{b}^{\uparrow}_{kl}\left(z'\right)\right]=0 \tag{30}$$

$$\left[\hat{b}_{ij}\left(z\right),\hat{T}_k\left(z'\right)\right]=\left[\hat{b}^{\uparrow}_{ij}\left(z\right),\hat{T}_k\left(z'\right)\right]=0 \tag{31}$$

$$\left[\hat{b}^{\uparrow}_{ij}\left(z\right),\hat{x}_k\right]=z\delta_{ik}\hat{b}^{\uparrow}_{ij}\left(z\right) \tag{32}$$

$$\left[\hat{b}_{ij}\left(z\right),\hat{x}_k\right]=-z\delta_{ik}\hat{b}_{ij}\left(z\right) \tag{33}$$

1.2 Temporal Evolution of Financial Markets

In analogy with quantum mechanics, we mandate that the state of the market at a given instant of time 't', is represented by a vector in the Hilbert space H whose components determine the statistical nature of the market. Hence the temporal evolution of the market in essentially determined by the evolution of this vector with the flow of time. In the Schrödinger picture, the time evolution of a system can be characterized through a unitary evolution operator $\hat{U}(t,t_0)$ in H, that acts on the initial state $|\psi(t_0)\rangle$ to transform it to $|\psi(t)\rangle$ i.e

$$|\psi(t)\rangle = \hat{U}(t,t_0)\,|\psi(t_0)\rangle \tag{34}$$

The assumption of the market being isolated and hence $\Psi = \{|\{x_i, \{y_{ij}(z), j = 1, 2, ...m\}, i = 1, 2, ...n\}\rangle\}$ being a complete basis at all times, and the conservation of probability i.e. $\sum_l |C_l(t)|^2 = 1, \forall t$ together with the group property of $\hat{U}(t,t_0)$ implies that the temporal evolution is unitary i.e.

$$U(t,t_0)\,U^{\uparrow}(t,t_0) = U^{\uparrow}(t,t_0)\,U(t,t_0) = 1 \tag{35}$$

Furthermore $\hat{U}(t_0,t_0) = 1$. Defining the Hamiltonian $\hat{H}(t) = i\frac{\partial}{\partial t}\hat{U}(t,t_0)\Big|_{t=t_0}$ as the infinitesimal generator of time translations (evolution) we obtain, through a Taylor's expansion up to first order $\hat{U}(t+\delta t,t_0) = \hat{U}(t,t_0) + \frac{\partial \hat{U}}{\partial t}(t+\delta t,t)\Big|_{\delta t=0}\hat{U}(t,t_0)\,\delta t + ...$or

$$\frac{\partial \hat{U}(t,t_0)}{\partial t} = \lim_{\delta t\to 0}\frac{\hat{U}(t+\delta t,t_0) - \hat{U}(t,t_0)}{\delta t} = -i\hat{H}(t)\,\hat{U}(t,t_0) \tag{36}$$

with the immediate solution $\hat{U}(t,t_0) = e^{-\int_{t_0}^{t}\hat{H}(t)dt}$ where time ordering of the operators constituting the Hamiltonian is assumed.

Before progressing further with the development of the model, some observations are in order about the theory developed thus far.

(1) In standard quantum mechanics, $\hat{H}(t)$ is usually a bounded operator and hence the exponential series in $\hat{U}(t,t_0) = e^{-\int_{t_0}^{t}\hat{H}(t)dt}$converges so that its approximation to first order is acceptable giving $i\frac{\partial|\psi(t)\rangle}{\partial t} = \hat{H}(t)\,|\psi(t)\rangle$ which is the Schrödinger equation of wave mechanics. This may not always be the case in financial markets.

(2) Since time evolution of financial market, essentially, occurs through trades in securities, it is appropriate to infer that the Hamiltonian represents the trading activities of the market.

(3) In order that the evolution operator $\hat{U}(t,t_0)$ is well defined, we mandate that the Hilbert space His so constructed that the kernel of $\hat{U}(t,t_0)$ is empty.

1.3 Modeling Time Value of Money

Time value of money and interest rate instruments are classically modeled through the first order differential equation $\frac{dB(t)}{dt} = r(t)\,B(t)$with the solution $B(t) = B(0)\,e^{\int r(t)dt}$.

A possible candidate for the Hamiltonian function H(in the classical picture) that would generate this temporal development as the equations of motion is

$$H\left(x,p;t\right)=\sum_{i=1}^{n}H_i\left(x_i,p_i;t\right)=\sum_{i=1}^{n}r_i\left(t\right)x_i\left(t\right)p_i\left(t\right) \tag{37}$$

This Hamiltonian leads to the following equations of motion

$$\frac{dx_i\left(t\right)}{dt}=\frac{\partial H_i\left(x_i,p_i;t\right)}{\partial p_i}=r_i\left(t\right)x_i\left(t\right), \quad \frac{dp_i\left(t\right)}{dt}=-\frac{\partial H_i\left(x_i,p_i;t\right)}{\partial x_i}=-r_i\left(t\right)p_i\left(t\right) \tag{38}$$

While the interpretation of first of these equations is straightforward being the growth of cash reserves of the i^{th}investor with the instantaneous rate $r_i\left(t\right)$, the implications of second equation are more subtle. To provide a financial logic to this equation, we note that p_i is the infinitesimal generator of cash translations in the classical picture and hence $\frac{dp_i(t)}{dt}=-\frac{\partial H_i(x_i,p_i;t)}{\partial x_i}=-r_i\left(t\right)p_i\left(t\right)$ represents the rate of change of the cash translations generator which, given a fixed rate of growth of cash, would decrease with the amount of cash translations.

Using the Weyl formalism for transformation from the classical to the quantum picture, we require that the quantum mechanical analog of $H\left(x,p;t\right)$ be Hermitian and symmetric in its component operators. Hence, we postulate the ansatz

$$\hat{H}\left(\hat{x}\left(t\right),\hat{p}\left(t\right);t\right)=\sum_{i=1}^{n}\hat{H}\left(\hat{x}_i\left(t\right),\hat{p}_i\left(t\right);t\right)=\sum_{i=1}^{n}\frac{ir_i\left(t\right)}{2}\left(\hat{x}_i\hat{p}_i+\hat{p}_i\hat{x}_i\right)=\sum_{i=1}^{n}ir_i\left(t\right)\hat{x}_i\left(\hat{p}_i+\frac{1}{2}\hat{I}\right) \tag{39}$$

for the quantum mechanical Hamiltonian representing the time value of money, so that the time development operator is

$$\hat{U}\left(t,t_0\right)=e^{\left[-i\int\limits_{t_0}^{t}\hat{H}(t)dt\right]}=e^{\sum\limits_{i=1}^{n}\int\limits_{t_0}^{t}r_i(t)x_i\left(\hat{p}_i+\frac{1}{2}\hat{I}\right)dt} \tag{40}$$

which may be evaluated using standard methods like Green's functions and Feynmann propagator theory.

1.4 Representation of Trading Activity

Let us consider a deal in which an investor 'i' buys a security 'j' at a price of 'z' units and immediately thereafter sells the same security to another investor 'k' at a price of 'z'' units and credits/debits the difference $z'-z$ to his cash account. The composite transaction will, in our operator formalism, take the form $\hat{b}_{ij}\left(z'\right)\hat{b}_{ij}^{\dagger}\left(z\right)$. In analogy with this argument, we can represent the Hamiltonian for trading activity of the market as

$$H_{Tr}\left(t\right)=\sum_{i,j,k,l}\int\limits_{0}^{\infty}\frac{dz}{z}\int\limits_{0}^{\infty}\frac{dz'}{z'}h_{ijkl}\left(\xi,t\right)\hat{b}_{ij}^{\dagger}\left(z\right)\hat{b}_{kl}\left(z'\right) \tag{41}$$

where $\xi=ln\frac{z'}{z}$ensures that the amplitudes are scale invariant.

1.5 Probability Distribution of Stock Prices

We now derive the probability distribution of stock prices in market equilibrium and show that the prices follow a lognormal distribution, thereby vindicating the efficacy of this model.

For this purpose, we assume that an investor $i = \alpha$ buys one unit of a security $j = \beta$ at time $t = t_i$ for a price z. We need to ascertain the probability $P_T\left(z' \left| z\right.\right)$i.e. the probability of the security $j = \beta$ having a price z' at time $t_f = t_i + T$.We assume that during the period $t_f - t_i$, investor α holds exactly one unit of β and that before t_i and after t_f,α holds no unit of β.

Let $\left|\psi^z_{\alpha\beta}\left(t_i\right)\right\rangle$be the state that represents investor α holding exactly one unit of β at a price z at time $t_i$in the Hilbert space H. Hence, we have $\left|\psi^z_{\alpha\beta}\left(t_i\right)\right\rangle = \hat{b}^{\dagger}_{\alpha\beta}\left(z\right)\left|\overline{\psi_{\alpha\beta}\left(t_i\right)}\right\rangle$where $\left|\overline{\psi_{\alpha\beta}\left(t_i\right)}\right\rangle$is the state that represents investor α not holding any unit of β. This also implies that $\hat{b}_{\alpha\beta}\left(z\right)\left|\overline{\psi_{\alpha\beta}\left(t_i\right)}\right\rangle = 0$.Let us assume that the final state corresponding to the initial state $\left|\psi^z_{\alpha\beta}\left(t_i\right)\right\rangle$ is represented by $\left|\psi^z_{\alpha\beta}\left(t_f\right)\right\rangle$so that

$$\left|\psi^z_{\alpha\beta}(t_f)\right\rangle = \hat{U}(t_i,t_f)\left|\psi^z_{\alpha\beta}(t_i)\right\rangle = e^{-i\int\limits_{t_i}^{t_f}\hat{H}dt}\hat{b}^{\dagger}_{\alpha\beta}(z)\left|\overline{\psi_{\alpha\beta}(t_i)}\right\rangle \tag{42}$$

The amplitudes of $\left|\psi^{z'}_{\alpha\beta}\left(t_f\right)\right\rangle$ are determined in the usual way by taking scalar product $\left\langle\psi^{z'}_{\alpha\beta}\left(t_f\right)\right|\left.\psi^z_{\alpha\beta}\left(t_f\right)\right\rangle$ and we have, for the matrix elements of the propagator

$$G\left(z',t_f;z,t_i\right) = \left\langle\overline{\psi_{\alpha\beta}(t_f)}\right|\hat{b}_{\alpha\beta}\left(z'\right)e^{-i\int\limits_{t_i}^{t_f}\hat{H}dt}\hat{b}^{\dagger}_{\alpha\beta}(z)\left|\overline{\psi_{\alpha\beta}(t_i)}\right\rangle \tag{43}$$

In this case, the trading Hamiltonian will contain creation and annihilation operators relating to the investor α and those relating to the securityβ i.e., it will be of the form

$$\hat{H}_{Tr}\left(t\right) = \sum_{k,l}\int\limits_0^{\infty}\frac{dz}{z}\int\limits_0^{\infty}\frac{dz'}{z'}h_{\alpha\beta kl}\left(\xi,t\right)\hat{b}^{\dagger}_{\alpha\beta}\left(z\right)\hat{b}_{kl}\left(z'\right) \tag{44}$$

We further make the assumption that the amplitudes can be approximated by their first two moments about $\xi = 0$, being sharply peaked about $z' = z$since, in the timescales being considered, most trades would occur around z. Hence, we have

$$h_{\alpha\beta kl} \sim \left[\Omega_{\alpha\beta kl}\left(t\right) - i\xi^{-1}\Xi_{\alpha\beta kl}\left(t\right)\right]\delta\left(\xi\right) \tag{45}$$

Noting that $\xi = ln\frac{z'}{z}$, we have $\xi^{-1} = \left(\ln\frac{z'}{z}\right)^{-1} = \left(\frac{z'}{z} - 1\right)^{-1} = \frac{z}{z'-z}$to first order and $\delta\left(\xi\right) = \delta\left(\ln\frac{z'}{z}\right) = \delta\left(z' - z\right)\left[\frac{d\left(\ln\frac{z'}{z}\right)}{dz'}\right]^{-1} = z'\delta\left(z' - z\right)$. Using these results and eqs. (44) & (45), we obtain

$$\hat{H}_{Tr}\left(t\right) \sim \sum_{k,l}\int\limits_0^{\infty}\frac{dz}{z}\int\limits_0^{\infty}\frac{dz'}{z'}z\delta\left(z' - z\right)\left[\Omega_{\alpha\beta kl}\left(t\right) - i\frac{z}{z' - z}\Xi_{\alpha\beta kl}\left(t\right)\right]\hat{b}^{\dagger}_{\alpha\beta}\left(z\right)\hat{b}_{kl}\left(z'\right)$$

$$= \sum_{k,l} \int_0^\infty \frac{dz}{z} \hat{b}^\dagger_{\alpha\beta}(z) \left[\Omega_{\alpha\beta kl}(t) + iz\Xi_{\alpha\beta kl}(t) \frac{\partial}{\partial z}\right] \hat{b}_{kl}(z') \tag{46}$$

We note that this expression for the Hamiltonian is linear in $\frac{\partial}{\partial z}$ and hence it can be diagonalized in the "momentum space" through a Fourier transformation and we have

$$\hat{H}_{Tr}(t) = \frac{1}{2\pi} \sum_{k,l} \int_0^\infty \frac{dz}{z} \int_0^\infty \frac{dz'}{z'} \int_{-\infty}^\infty dp \hat{b}^\dagger_{\alpha\beta}(z) \left[\Omega_{\alpha\beta kl}(t) + i\Xi_{\alpha\beta kl}(t) p\right] \hat{b}_{kl}(z') e^{ip\xi} \tag{47}$$

The assumption of market equilibrium implies that the Hamiltonian should be independent of time over the relevant timescales that would be much smaller than those determining aggregate market behaviour so that we may write eq. (43) as

$$G(z', t_f; z, t_i) = \left\langle \overline{\psi_{\alpha\beta}(t_i)} \right| \hat{b}_{\alpha\beta}(z') e^{-i\hat{H}(t_i)T} \hat{b}^\dagger_{\alpha\beta}(z) \left| \overline{\psi_{\alpha\beta}(t_i)} \right\rangle \tag{48}$$

Because of the Hamiltonian being diagonal in momentum space, it is more convenient to work in momentum space for evaluating the propagators and we have, for the equivalent of eq. (48) in momentum space as

$$\tilde{G}(p', p; T, t_i) = \left\langle \overline{\psi_{\alpha\beta}(t_i)} \right| \left[\int_0^\infty \frac{dz'}{z'} e^{ip' ln(z'/\kappa)} \hat{b}_{\alpha\beta}(z')\right] e^{-i\hat{H}(t_i)T} \left[\int_0^\infty \frac{dz}{z} e^{-ipln(z/\kappa)} \hat{b}^\dagger_{\alpha\beta}(z)\right] \hat{b}^\dagger_{\alpha\beta}(z) \left| \overline{\psi_{\alpha\beta}(t_i)} \right\rangle \tag{49}$$

To solve the problem further, we make use of second order perturbation theory. The first step is to split the Hamiltonian into components as follows

$$\hat{H}(t) = \sum_l \int_0^\infty \frac{dz}{z} \hat{b}^\dagger_{\alpha\beta}(z) \left[\Omega_{\alpha\beta\alpha l}(t) + \Xi_{\alpha\beta\alpha l}(t) p\right] \hat{b}_{\alpha\beta}(z) + \sum_{k,l,k\neq\alpha} \int_0^\infty \frac{dz}{z} \hat{b}^\dagger_{\alpha\beta}(z) \left[\Omega_{\alpha\beta kl}(t) + i\Xi_{\alpha\beta kl}(t) p\right] \hat{b}_{kl}(z') \tag{50}$$

Let $E_i$be the energy eigenstate of the unperturbed Hamiltonian i.e. of the state of the market before the purchase of security β by the investor α , then the energy eigenstate of the Hamiltonian $\hat{H}(t_i)$i.e. after the purchase of security β by the investor α will be of the form $E_p = E_i + \sum\limits_l \left[\Omega_{\alpha\beta\alpha l}(t_i) + i\Xi_{\alpha\beta\alpha l}(t_i) p\right] - ip^2\sigma^2$where the second term represents the impact on the energy eigenstates of the transactions involving investor α or security β and the last term is the second order perturbation term due to the overall fluctuations of the market. Substituting this value of E_p in eq. (49) and noting that the Hamiltonian and hence the propagator $\tilde{G}(p', p; T, t_i)$is also diagonal in "momentum space", we have

$$\tilde{G}(p', p; T, t_i) \sim 2\pi\delta(p' - p) e^{-iTE_p} = 2\pi\delta(p' - p) e^{-iT\left[E_i + \Omega(t_i) + i\Xi(t_i)p - ip^2\sigma^2\right]} \tag{51}$$

where $\sum\limits_l \Omega_{\alpha\beta\alpha l}(t_i) = \Omega(t_i)$, $\sum\limits_l \Xi_{\alpha\beta\alpha l}(t_i) = \Xi(t_i)$.

Inverting back to "coordinate space", we obtain

$$G(z', t_f; z, t_i) = \frac{1}{2\pi} \int_{-\infty}^\infty dp e^{-iT\left[E_i + \Omega(t_i) + i\Xi(t_i)p - ip^2\sigma^2\right] - ip \ln z'/z} \sim \frac{e^{iT(E_i + \Omega)}}{2\sigma\sqrt{\pi T}} e^{\left[\frac{(ln(z'/z) + \Xi T)^2}{4\sigma^2 T}\right]} \tag{52}$$

The probability $P_T\left(z'\left|z\right.\right)$i.e. the probability of the security $j = \beta$ having a price z' at time $t_f = t_i + T$ will then be proportional to the square of the above amplitude and hence, we finally obtain

$$P_T\left(z'\left|z\right.\right)\alpha\left|G\left(z',t_f;z,t_i\right)\right|^2 = \left(4\pi\sigma^2T\right)^{-1}e^{\left[\frac{\left(In\left(z'/z\right)+\Xi T\right)^2}{2\sigma^2T}\right]} \tag{53}$$

which agrees perfectly with the standard stochastic theory of finance wherein stock returns are modeled extensively through lognormal distributions.

2. Conclusions

The following interesting observations emanate from the above analysis:-

(1) Eq. (53), on comparison with the standard expression for probability distribution of stock price in the conventional stochastic calculus based approach to the Black Scholes formula, identifies Ξ with the expected return on stock. This return is independent of the eigenvalue $E_i$and hence, the state of the market. A financial interpretation of this could be that the stock returns are dependent on the performance of the company and independent of market dynamics.

(2) Independence of stock returns of the market dynamics would, however, mandate that the stock volatility measured by the standard deviation σ is related to the stock market dynamics which seems justified since higher trading volumes would imply greater volatility and vice versa.

(3) If we define the uncertainty of measurement of a random variate by its standard deviation, then, from eqs. (52) & (53), we have the uncertainty for the stock price process $z$and its Fourier conjugate p, after a time T, as $\sigma_z = \sigma\sqrt{T}$and $\sigma_p = \frac{1}{2\sigma\sqrt{T}}$so that $\sigma_z\sigma_p = \frac{1}{2}$ as it should be, since the distribution of z is assumed Gaussian in the aforesaid analysis.

(4) Furthermore, $\frac{\sigma_z}{\sigma_p} = 2\sigma^2T$ which enables the identification of σ^2 as the reciprocal of the mass and hence, the inertia of the stock price process. It is intriguing to note that the same analogy follows through another completely independent analysis i.e. the Black Scholes equation for the option price in its standard form is given by $\frac{\partial C}{\partial t} = -rS_t\frac{\partial C}{\partial S} - \frac{1}{2}\sigma^2S_t^2\frac{\partial^2C}{\partial S^2} + rC$, where $C\left(S_t,t\right)$ denotes the instantaneous price of a call option with exercise price E at any time t before maturity when the price per unit of the underlying is S_t. Making the substitution $S_t = e^x$we obtain $\frac{\partial C}{\partial t} = -r\frac{\partial C}{\partial x} - \frac{1}{2}\sigma^2\frac{\partial^2C}{\partial x^2} + \frac{1}{2}\sigma^2\frac{\partial C}{\partial x} + rC$ which, when compared with the standard quantum mechanical Hamiltonian in one degree of freedom identifies σ^2 as the reciprocal of the mass of the underlying system.

Contemporary quantitative finance is dominated by stochastic modeling of market behaviour. These models are essentially in the nature of tools of data analysis that aim to predict future events by applying probabilistic methods to historical data. Empirical evidence testifies that probability distributions of stock returns are negatively skewed, have fat tails and show leptokurtosis [56]. Some of the features of empirical distributions of stock prices are modeled through Levy distributions [57-60], stochastic volatility [61] or cumulant expansions around the lognormal case. Each of these models, however, attempts to empirically attune the model parameters to fit observed data and hence, is equivalent to interpolating or extrapolating observed data in one form or the other.

Hence, stochastic models fail to take cognizance of causal factors that get submerged in the superficial patterns exhibited by the avalanche of data being analysed. In actual fact, every new price determination of a security and hence, the fluctuation of prices is attributable to a new trade in the relevant security at that price. The “trading process” therefore manifests itself as a price history of a security. The fundamental limitation of stochastic tools in simulating extended memory effects is circumvented by this approach. An attempt has also been made through this “toy model” to establish that a quantum mechanical version of financial markets results in a temporal evolution of the probability distribution analogous to that of simple stochastic systems. Stochastic models also lack ability to accommodate collective effects like phase coherency in lasers that could, possibly, be built into this quantized description.

It need be emphasized here that the above is purely a phenomenological model for modeling stock behavior. It is fair to say that the current stage of research in financial processes is dominated by the postulation of phenomenological models that attempt to explain a limited set of market behavior. There is a strong reason for this. A financial market consists of a huge number of market players. Each of them is endowed with his own set of beliefs about rational behavior and it is this set of beliefs that govern his actions. The market, therefore, invariably generates a heterogeneous response to any stimulus. Furthermore, “rationality” mandates that every market player should have knowledge and understanding about the “rationality” of all other players and should take full cognizance in modeling his response to the market. This logic would extend to each and every market player so that we have a situation where every market player should have knowledge about the beliefs of every other player who should have knowledge of beliefs of every other player and so on. We, thus, end up with an infinitely complicated problem that would defy a solution even with the most sophisticated mathematical procedures. Additionally, unlike as there is in physics, financial economics does not possess a basic set of postulates like General Relativity and Quantum Mechanics that find homogeneous applicability to all systems in their domain of validity.

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