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Nonlinear Field Equations and Solitons as Particles

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Abstract: Profound advances have recently interested nonlinear field theories and their exact or approximate solutions. We review the last results and point out some important unresolved questions. It is well known that quantum field theories are based upon Fourier series and the identification of plane waves with free particles. On the contrary, nonlinear field theories admit the existence of coherent solutions (dromions, solitons and so on). Moreover, one can construct lower dimensional chaotic patterns, periodic-chaotic patterns, chaotic soliton and dromion patterns. In a similar way, fractal dromion and lump patterns as well as stochastic fractal excitations can appear in the solution. We discuss in some detail a nonlinear Dirac field and a spontaneous symmetry breaking model that are reduced by means of the asymptotic perturbation method to a system of nonlinear evolution equations integrable via an appropriate change of variables. Their coherent, chaotic and fractal solutions are examined in some detail. Finally, we consider the possible identification of some types of coherent solutions with extended particles along the de Broglie-Bohm theory. However, the last findings suggest an inadequacy of the particle concept that appears only as a particular case of nonlinear field theories excitations.

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1. Introduction

Solitons and other coherent solutions of nonlinear partial differential equations (NPDEs) have been extensively studied and their importance have been recognized in quite different areas of natural sciences and especially in almost all fields of physics such as plasma physics, astrophysics, nonlinear optics, particle physics, fluid mechanics and solid state physics. Solitons have been observed with spatial scales from $10^{-9}m$ to $10^{9}m$, if we consider density waves in the spiral galaxies, the giant Red Spot in the atmosphere of

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Jupiter, the various types of plasma waves, superfluid helium, shallow water waves, structural phase transitions, liquid crystals, laser pulses, acoustics, high temperature superconductors, molecular systems, nervous pulses, population dynamics, Einstein cosmological equations, elementary particles structure and so on ([4],[11],[15], [20],[28],[25],[30],[70]).

In particular, solitons of NPDEs in 1+1 dimensions (one spatial plus one temporal dimension) possess the following properties:

(1) they are spatially localized;
(2) they maintain their localization during the time, i.e. they are waves of permanent form;
(3) when a single soliton collides with another one, both of them retain their identities and velocities after collision.

Usually mathematicians call solitons only the solutions that satisfy all the three above mentioned properties (as we will see the third property is connected with the integrability of the NPDEs) and call solitary waves solutions that satisfy only the first two properties. However, in many physics papers, the concept of soliton has been applied in a more extensive way, even if conditions ii)-iii) are not satisfied, because in the real world this concept is so useful and fruitful that one cannot afford to consider only the perfect mathematical world of soliton equations and not to use it.

For many years these solutions have been thought impossible, because a dispersive and nonlinear medium was expected to alter the wave shape over time. The first soliton observation has been given by John Scott Russell (1834) that found a solitary wave in a water channel. In 1895 J. Korteweg and G. de Vries demonstrated that for shallow water waves in a straight channel one is effectively left with a 1+1 dimensional problem and derived the appropriate nonlinear equation for the Russell soliton [27].

Subsequently, solitons have been found in many other nonlinear equations (called S-integrable equations) integrable by the inverse scattering transformation (IST) or spectral transform [1].

On the other hand, equations integrable by an appropriate transformation on the dependent/independent variables that convert them into a linear equation (integrable by the Fourier method) have been called C-integrable equations.

In the last years it has been shown that S-integrable equations are only a limited sector of nonlinear equations with solitonic solutions, because it has been demonstrated that nonintegrable equations (for example the double-sine Gordon equation and the Hasegawa-Mima system [65] and C-integrable systems have soliton solutions that satisfy conditions i-iii).

Nontrivial solutions of nonlinear equations can be found with many different methods and the inverse scattering method (S-integrable equations) has not a prominent role and it would be not useful to limit the soliton concept to a particular integration technique. Besides, there is no agreement about the concept of integrability for a nonlinear partial differential equation.

In Sect. 2 we briefly review the most important NPDEs in 1+1 dimensions and their coherent solutions, while in Sect. 3 we review the IST technique for obtaining interesting exact solutions of a nonlinear equation.

In Sect. 4 we consider S-integrable equations in 2+1 dimensions and in particular the Davey-Stewartson and Kadomtsev-Petviashvili equations. They exhibit soliton solutions that are now spatially localized in all the directions except one. However, in many nonlinear NPDEs in 2+1 dimensions, different type of coherent solutions (dromions, ring solitons as well as istantons, and breathers) are found. In particular, dromions are solutions exponentially localized in all directions, which propagate with constant velocity and are usually driven by straight line solitons, for example in the so called DS-I equation.

If we now consider the physics applications of the above treated concepts, we must begin from the fact that two different procedures can be applied in order to find physically relevant NPDEs and then determine their solutions: exact solutions of approximate model equations or approximate solutions of exact equations.

In the first case using appropriate reduction methods (and in particular the asymptotic reduction (AP) method, a very general reduction method that can be also applied to construct approximate solutions for weakly nonlinear ordinary differential equations [Mac1, Mac8, Mac12]) and introducing approximations directly into the equations describing the system under study some model nonlinear equations are obtained and their exact solutions are investigated. Usually the approximations concern with the temporal and/or spatial scale of the solutions with respect to some physical parameters. The most important nonlinear model equations are the above-mentioned S-integrable equations and this fact is not obviously a coincidence. Indeed, as it has been known for some time, very large classes of NPDEs, with a dispersive linear part, can be reduced, by a limiting procedure involving the wave modulation induced by weak nonlinear effects, to a very limited number of "universal" nonlinear evolution PDEs. These model equations appear in many applicative fields because this reduction technique is able to take into account weakly nonlinear effects. The model equations are integrable, since it is sufficient that the very large class from which they are obtainable contain just one integrable equation, because it is clear from this method that the property of integrability is inherited through the limiting technique. Using an appropriate reduction method provides a powerful tool to understand the integrability of known equations and to derive new integrable equations likely to be relevant in applicative contexts. By the AP method many new nonlinear S-integrable equations have been identified for the first time ([40],[41],[43],[45],[49],[51],[53],[54],[56]).

In Sect.5 we illustrate a powerful method for nonlinear model equations, the multilinear variable separation technique, that can be used in order to obtain solitons and other coherent solutions but also chaotic solutions, with their sensitive dependence on the initial conditions, and fractals, with their self-similar structures. Since lower dimensional arbitrary functions are present in the exact solutions of some two dimensional integrable models, we can use lower dimensional chaotic and/or fractal solutions in order to obtain solutions of higher dimensional integrable models.

However, a second procedure can be used in order to find coherent solutions for NPDEs (Sec. 6): approximate solutions for correct equations, The AP method can be applied directly to the original equations, which describe a given physical system. In this way, no model equation of the above mentioned type is obtained, because approximate solutions are directly sought. It has been demonstrated that solitons and dromions exist as approximate solutions in the particular case of ion acoustic waves in a unmagnetized or magnetized plasma, electron waves and non-resonant interacting water waves in 2+1 dimensions [Mac4, Mac6, Mac14, Mac17]. In particular, we examine a nonlinear Dirac field and demonstrate that each dromion propagates with its own group velocity and during a collision maintains its shape, because a phase shift is the only change. They are solutions of a C-integrable nonlinear partial differential system of equations describing N-interacting waves (N¿1) for modulated amplitudes Ψ_j, *j=1,. . . N.* The AP method can be applied to soliton and/or dromion propagation in nonlinear dispersive media without the complexity of the IST technique. Moreover, in 3+1 dimensions there are no known examples of S-integrable equations while the AP method is easily applicable.

In Sec. 7, we illustrate another example of the use of the AP method and consider a spontaneous symmetry breaking model and in Sec. 8 demonstrate also in this case the existence of dromions which preserve their shape during collisions, the only change being a phase shift. Moreover, other coherent solutions (line solitons, multilumps, ring solitons, instantons and breathers) are derived.

In Sec. 9 we show the existence of lower dimensional chaotic patterns such as chaotic-chaotic and periodic-chaotic patterns as well as chaotic soliton and dromion patterns. At last, we derive fractal dromion and lump solutions as well as stochastic fractal solutions.

In the conclusion we examine some important questions. From the results exposed in the previous sections we see that quantum mechanics can be considered as a first order approximation of a nonlinear theory. Moreover, dromions would correspond to extended elementary particles, in such a way to perform the de Broglie-Bohm theories, however the various type of coherent solutions suggest that elementary particles are only a particular case of nonlinear excitations.

2. Solitons and Nonlinear Equations in 1+1 Dimensions

a) John Scott Russell

The first soliton observation was performed by John Scott Russell (1808-1882) in 1834 in a canal near Edinburgh [14]. A small boat in the channel suddenly stopped and a lump of bell-shaped water formed at the front of the boat and moved forward with approximately constant speed and shape for about two miles. He called it the Great Solitary Wave and with the aid of subsequent experiments derived an empirical law for its speed

$$V = \sqrt{g(h+A)}, \tag{2.1}$$

where g is the gravity acceleration, A the wave amplitude and h the channel profundity.

b)The Korteweg-de Vries (KdV equation)

For linear equations (valid in the limit of small amplitude solutions), solitons are not possible due to the superposition principle (the sum of two solutions is yet a solution) and to the dispersion (waves with different wavelength have different velocities).

If we consider a solution formed by the sum of two waves with different wavelength that are initially superposed, then after some time they will separate. In the dispersive linear equations localized solutions cannot exist.

On the contrary, in nonlinear media (for example shallow water or plasma) the wave packet spreadness can be exactly balanced with the nonlinear terms of the equation in such a way to originate solitons and other coherent solutions. In particular, for solitons in shallow water the appropriate nonlinear equation was found by Korteweg and de Vries in 1895 [27]

$$U_t + U_{xxx} - 6UU_x = 0, \tag{2.2}$$

where $U = U(x,t)$ stand for the wave amplitude, x represents the propagation direction and t the time. The soliton solution (bell shaped and localized in space) is

$$U(x,t) = \frac{-2A^2}{\cosh^2[A(x - x_0 - 4A^2t)]}, \tag{2.3}$$

where A is a positive constant. Note that "slim" solitons are "tall" and run faster. The relation among velocity, width and amplitude is a characteristic property of solitons, while on the contrary for traveling waves of linear equations all the three quantities are usually independent of each other.

We note that if in (2.2) the nonlinear term is absent, then localized solutions would be impossible due to the dispersion. On the other hand, if the second term is absent, solution can develop a singularity in a finite time.

c) The Fermi-Pasta-Ulam experiment

In the first half of the XX century, the KdV equation was substantially forgotten but suddenly it emerged in the statistical physics and then in plasma studies and in all the phenomena with weak nonlinearities and dispersion (ion and electron waves in magnetized or unmagnetized plasma, phonon packets in nonlinear crystals).

For example, we consider an oscillator chain coupled with nonlinear forces. It is well known that the motion equations can be decoupled if we consider the normal modes that are characterized by different frequencies and evolve independently with each other. The motion is multiply periodic and the mode energy is constant over the time.

If then we introduce nonlinear forces among the oscillators, dramatic changes would appear because we expect an energy transfer among the various frequencies (stochastic behavior), as it is forecasted by the ergodic hypothesis and the energy equipartition.

With the beginning of the computer age, Fermi, Pasta and Ulam wanted to verify this prediction and numerically integrated the nonlinear equations for the oscillators. With their great surprise, they found that the prediction was wrong, because the energy concentrated on a determinate mode over the time ('recurrence'). Starting with only one oscillator excited the energy distributed itself over the modes, but returned almost completely in the first excited one. Thermodynamic equilibrium was not reached and the excitation was stable.

d) The discovery of Zabusky and Kruskal

To elucidate this behavior, we must consider the KdV equation, that is the continuous approximation of the oscillators chain. For a linear chain of atoms with a quadratic interaction, the motion equation is

$$m\ddot{y}_i = k\left(y_{i+1} - 2y_i + y_{i-1}\right) + k\alpha\left[\left(y_{i+1} - y_i\right)^2 - \left(y_i - y_{i-1}\right)^2\right], \tag{2.4}$$

where $y_i = y_i(t)$, $i = 1,..N$, N is the total number of atoms and moreover we assume that $y_{N+1} = y_1$. By means of the fourth order Taylor expansion in a, where a is the lattice constant, the motion equation becomes

$$y_{t't'} = y_{x'x'} + \varepsilon y_{x'} y_{x'x'} + \beta y_{x'x'x'x'} + O(\varepsilon a^2, a^4), \tag{2.5}$$

where $\varepsilon = 2a\alpha$, $\beta = a^2/12$, $t' = \omega t$, $\omega = \sqrt{k/m}$, $x' = x/a$ and $x = ia$.

With the variable change $T = \varepsilon t/2$, $X = x - t$, equation (2.5) yields

$$\varepsilon\left(V_{TX} + V_X V_{XX}\right) + \beta V_{XXXX} = 0, \tag{2.6}$$

where $y(x', t') = V(X, T)$. Taking $U = V_X$, one arrives to the KdV equation (2.2).

In 1965 Norman Zabusky and Martin Kruskal [86] numerically studied the KdV equation and found elastic collisions among localized solutions (that they called solitons) that preserve their identities. Note however that an analytic (and then not numeric) expression for the elastic collision in the sine-Gordon equation (see subsect. *f*) was known from 1962 [69], but it was been ignored.

After the Zabusky-Kruskal's discovery there was an explosion of papers about nonlinear waves. It was demonstrated that the KdV equation is integrable by the IST technique (see Sect. 3) and in the subsequent years many other applicative (and integrable) equations were found.

e) The nonlinear Schrödinger equation

The most important nonlinear equation is perhaps the nonlinear Schrödinger (NLS) equation, that takes into account the slow modulation of a monochromatic plane wave with weak amplitude, in a strongly dispersive and weakly nonlinear medium:

$$i\Psi_t + \Psi_{xx} + s\left|\Psi\right|^2\Psi = 0, \tag{2.7}$$

where $\Psi(x, t)$ is a complex function and $s = \pm 1$.

The soliton solution exists only for $s = 1$ and is given by

$$\Psi(x,t) = \Psi_0 \mathrm{sech}\left[\Psi_0 \frac{(x - at)}{\sqrt{2}}\right] \exp\left[i\left(\frac{a(x - bt)}{2}\right)\right], \tag{2.8}$$

where a and b are arbitrary constants, the envelope and phase velocity, respectively. The NLS equation is integrable by the IST method and has been applied in many fields (deep water, self-focusing of laser in dieletrics, optical fibers, vortices in fluid flow, etc.).

f) The sine-Gordon equation

The sine-Gordon (sG) equation,

$$U_{xx} - U_{tt} = sinU, \tag{2.9}$$

where $U = U(x,t)$ is a real function, was studied for the first time by Bianchi, Backlund and Darboux, because it describes pseudospherical surfaces with constant negative gaussian curvatiure. It was probably known to Gauss, being the reduction of the fundamental equation of differential geometry.

There are three types of coherent solutions:

$$i) \quad kink, \qquad U = 4\text{arc}\tan\left\{\exp\left[\frac{(x - vt - x_0)}{\sqrt{1 - v^2}}\right]\right\} \tag{2.10}$$

$$ii) \quad antikink, \qquad U = 4\text{arc}\tan\left\{\exp\left[-\frac{(x - vt - x_0)}{\sqrt{1 - v^2}}\right]\right\} \tag{2.11}$$

iii) breather (it is not a traveling wave, but a bound state formed by a kink-antikink couple)

$$U = 4\text{arc}\tan\left\{(\tan a)\sin\left[(\cos a)(t - t_0)\right]\text{sech}\left[(\sin a)(x - x_0)\right]\right\}, \tag{2.12}$$

being v (¡1) and a arbitrary constants.

The collision of a kink-antikink couple is described by

$$U = 4\text{arc}\tan\left\{ v\frac{\sinh\left[\frac{x}{\sqrt{1-v^2}}\right]}{\cosh\left[\frac{vt}{\sqrt{1-v^2}}\right]}\right\}, \tag{2.13}$$

that is not also in this case a traveling wave.

We now expose a simple method for obtaining soliton solutions of the SG equation that is valid also for other nonlinear equations. We take a traveling wave with velocity v as solution of (2.9),

$$U = U(x - vt) = U(T). \tag{2.14}$$

Substituting in (2.9), we obtain

$$(1 - v^2)U_{TT} = \sin U = -\frac{\partial V}{\partial U} = -\frac{\partial}{\partial U}\left(1 + \cos U\right). \tag{2.15}$$

For v¡1, the equation (2.15) describes the motion of a particle, with mass $m = 1 - v^2$, in a periodic potential. The kink solution corrisponds to a solution that passes from a maximum of the potential to the other in an infinite time (the antikink solution moves in the opposite direction). In the corresponding phase space, the soliton is constituted by the separatrices. There are also multisolitonic solutions characterized by a passage through varios maxima. On the contrary, if we assume v¿1, then $m = v^2 - 1$, and the potential becomes *V=1-cosU* and also in this case we get soliton solutions.

The sG equation, integrable by the IST, describes crystal dislocations (Frenkel-Kontorova solitons), magnetic walls, liquid crystals, magnetic fluxes in Josephson junctions, etc. Moreover, it is the Lorentz invariant and can be used in elementary particle physics, if we want to identify solitons with extended particles (in this case per v¿1, one obtain tachyons, particles with superluminal velocity).

g) Topological and non topological coherent solutions
In relativistic local field theories it is important the distinction between topological and non-topological solutions. In the first case, the boundary conditions at infinity are topologically the same for the vacuum as for the coherent solution. On the contrary, in topological solitons the boundary conditions at infinity are topologically different for the coherent solution than for a physical vacuum state.

We consider a simple example of topological solution, the kink solution of a nonlinear Klein-Gordon equation in 1+1 dimensions,

$$U_{xx} - U_{tt} = -\frac{dV}{dU}, \quad V(U) = \frac{\lambda}{4}\left[\left(\frac{m^2}{\lambda}\right) - U^2\right]^2. \tag{2.16}$$

The potential has two vacuum states

$$U = \pm\frac{m}{\sqrt{\lambda}}. \tag{2.17}$$

Since a moving solution is easily found by boosting (Lorentz transformation) a stationary solution, we consider only the latter and obtain

$$U = \pm\frac{m}{\sqrt{\lambda}}\tanh\left[\frac{m}{\sqrt{2}}(x - x_0)\right], \tag{2.18}$$

with plus (minus) sign for the (anti) kink. These solutions are topological because they connect the two different vacuum states. A moving kink,

$$\Phi = \frac{m}{\sqrt{\lambda}}\tanh\left[\frac{m\gamma}{\sqrt{2}}(x - x_0 - vt)\right], \quad \gamma = (1 - v^2)^{-\frac{1}{2}}, \tag{2.19}$$

can be shown to collide with an anti-kink in a not shape conservative way.

3. Solving Methods for Nonlinear Equations

The IST method can be considered an extension to the linear case of the Fourier method for linear partial differential equations. Given a generic NPDE, there is no general method that can establish if soliton solutions exist and how can be constructed. However the IST is the most important in the solitons seeking. The method was set up by Gardner, Green, Kruskal e Miura [18] in 1967 in order to solve the KdV equation and it was subsequently applied to many other NPDEs. In 1971 Zakharov and Shabat [87] applied this method to the NLS equation, while in 1974 the equation sG was resolved by Ablowitz, Kaup, Newell and Segur [3].

In 1968, Lax [31] demonstrated that the S-integrability of an equation is equivalent to the identification of an operator (Lax) couple (L, A) in such a way that the equation is obtained, for example in the 1+1 dimensions case, as a compatibility condition of the system:

$$Lf = \lambda f, \tag{3.1}$$

$$f_t + Af = 0, \tag{3.2}$$

with $f = f(x,t)$. We consider for example the KdV equation,

$$U_t + U_{xxxx} - 3(U^2)_x = 0, \tag{3.3}$$

where the operators L and A are

$$L = -\partial_x^2 + U, \tag{3.4}$$

$$A = -4\partial_x^3 + 6U\partial_x + 3U_x. \tag{3.5}$$

A simple calculation shows that the compatibility of the equations (3.1) e (3.2) is equivalent to (3.3). The equation KdV is the first of a hierarchy of equations where L is always given by the Schr**ö**dinger equation, while the temporal evolution operator A changes. The principal drawback of the IST technique is that there is no method for finding the Lax couple (if any) of a given NPDE and then to discover integrable equations.

The IST technique can be considered the nonlinear generalization of the Fourier transform. We take for example the equation (3.1), i. e. the spectral problem for the Schrödinger operator,

$$\left(-\partial_x^2 + U(x)\right) f(K,x) = \lambda f(K,x) = K^2 f(K,x), \tag{3.6}$$

where $K^2 \geq 0$ corresponds to the continuous spectrum and K^2¡0 to the discrete spectrum. It is well known that we can define a reflection coefficient $R(K)$ and a transmission coefficient$T(K)$,

$$f(K,x) \to \exp(-iKx) + R(K)\exp(iKx), \quad per \quad x \to +\infty \tag{3.7}$$

$$f(K,x) \to T(K)\exp(-iKx), \quad per \quad x \to +\infty \tag{3.8}$$

We now consider the eigenfunctions corresponding to the discrete eigenvalues $K^2 = -p_n^2$ and define the normalization constant ρ_n, through the relation

$$\lim_{x\to\infty} (\exp(2p_n x)\,[f(ip_n,x)]^2 = \rho_n \quad . \tag{3.9}$$

If we know the initial condition $U(x,t_0)$, we must insert it in (3.6) and calculate the spectral transform

$$S(K,t_0) = \left[R(K,t_0), -\infty < K < \infty, p_n, \rho_n(t_0), n = 1,2...N\right], \tag{3.10}$$

where $R(K,t_0)$ is the reflection coefficient, N the number of discrete eigenvalues $K^2 = -p_n^2$, with p_n¿0, and ρ_n the normalization constant (3.9).

At this point the function (3.10) is considered in the spectral space and it is demonstrated that the temporal evolution is

$$p_n(t) = p_n(t_0), \quad \rho_n(t) = \rho_n(t_0)\exp(8p_n^3(t-t_0)), \quad R(K,t) = R(K,t_0)\exp(8iK^3(t-t_0)). \tag{3.11}$$

We now antitransform in order to obtain $U(x,t)$, by a procedure that can be synthesized as follows. We define the function

$$M(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dK \exp(iKz)R(K) + \sum_{n=1}^{N} \rho_n \exp(-p_n z), \tag{3.12}$$

which satisfies the Gelfand-Levitan-Marchenko equation,

$$K(x,x') + M(x+x') + \int_x^{\infty} dx'' K(x,x'')M(x''+x') = 0, \quad x' \geq x, \tag{3.13}$$

where

$$U(x) = -2\frac{dK(x,x)}{dx}. \tag{3.14}$$

The spectral transform is formed by three steps: i) construction of the spectral transform (3.10); ii) evolution in the spectral space, (3.11); iii) antitransformation with (3.13-3.14). The IST method has been able to find the correct language for the description of many nonlinear equations. For example, in the KdV equation the discrete spectrum p_n corresponds to the localized solutions (solitons) and the continuous spectrum to solutions subject to dispersion (the so-called *background*).

In 1971 Zakharov e Faddeev demonstrated that the equation KdV is a hamiltonian system with infinite freedom degrees and found the relative angle-action variables. For this reason the KdV equation is called completely integrable.

A generalization of the Lax couple can be obtained if we take a NPDE as compatibility condition for a overdetermined system of PDE for a vectorial wave function:

$$\Psi_x = A(u,\lambda)\Psi, \quad \Psi_t = B(u,\lambda)\Psi, \tag{3.15}$$

$$A_t - B_x + [A,B] = 0. \tag{3.16}$$

Zakharov and Shabat [87] have demonstrated that the spectral problem can be reduced to the solution of a Hilbert-Riemann matricial problem.

The IST can be extended with some difficulties in the 2+1 dimensional case [1], while until now there is no known nonlinear equation in 3+1 dimensions, integrable through a spectral problem in 3 dimensions. Other important solving techniques are the Darboux, Backlund and Hirota methods ([13],[24],[66]).

4. Solitons and Coherent Solutions for Nonlinear Model Equations in 2+1 Dimensions

a) The Kadomtsev-Petviashvili equation

In the field of nonlinear equations in 2+1 dimensions, Kadomtsev and Petviashvili [26] derived a new S-integrable nonlinear equation considering the stability of KdV solitons with respect to transversal perturbations

$$\left(U_t + U_{xxx} + 3(U^2)_x\right)_x + sU_{yy} = 0, \tag{4.1}$$

where $s = \pm 1$. If $s = +1$, (equation KP-1), then we obtain the soliton

$$U(x, y, t) = 2a^2 \sec h \left\{ a \left[x + b\sqrt{3}y - \left(3b^2 + 4a^2\right) t + x_0 \right] \right\}, \tag{4.2}$$

that moves with arbitrary velocity in the plane (x, y). The soliton interaction is characterized by overtaking collisions as for the KdV equation [25]. If $s = -1$, we get the so-called KP-2 equation with a localized (but not exponentially) solution,

$$U(x, y, t) = 4\frac{\left(3a^2 y^2 - (x + a^{-1} - 3a^2 t)^2 + a^{-2}\right)}{\left(3a^2 y^2 + (x + a^{-1} - 3a^2 t)^2 + a^{-2}\right)^2}, \tag{4.3}$$

but instable. The KP equation has been applied to superficial water waves and to ion-acoustic plasma waves.

b) The Davey-Stewartson (DS) equation The S-integrable Davey-Stewartson (DS-I) equation [AnFr, DaSt]:

$$i\psi_t = (b - a)\psi_{xx} + (b + a)\psi_{yy} - \frac{s}{2}(b - a)\psi\varphi_1 - \frac{s}{2}(b + a)\psi\varphi_2, \tag{4.4a}$$

$$\varphi_{1,y} = (|\psi|^2)_x \quad \varphi_{2,x} = (|\psi|^2)_y, \tag{4.4b}$$

has been discovered in hydrodynamics and its canonical form corresponds to $a = 0, b = 1$. An alternative form is

$$i\psi_t = (b - a)\psi_{xx} + (b + a)\psi_{yy} + w\psi, \tag{4.5a}$$

$$w_{xy} = -\frac{s}{2}(b - a)(|\psi|^2)_{xx} - \frac{s}{2}(b + a)(|\psi|^2)_{yy}, \tag{4.5b}$$

that is obtained with the *ansatz*

$$w = -\frac{s}{2}(b - a)\varphi_1 - \frac{s}{2}(b + a)\varphi_2. \tag{4.6}$$

Another form (it is necessary a 45^0 rotation of the spatial axes) is

$$i\psi_t + \frac{1}{2}(\psi_{xx} + \psi_{yy}) + \alpha |\psi|^2 \psi - v\psi = 0, \tag{4.7a}$$

$$v_{xx} - v_{yy} - 2\alpha(|\psi|^2)_{xx} = 0, \tag{4.7b}$$

where α is a real parameter. The equation (4.7) is the limit in shallow water of the Benney-Roskes equation [6], where $\psi = \psi(x, y, t)$is the amplitude of a surface wave packet and $v = v(x, y, t)$ characterizes the medium motion generated by the surface wave.

The equation DS-II is

$$i\psi_t = (b - a)\psi_{zz} + (b + a)\psi_{vv} - \frac{s}{2}(b - a)\psi\varphi_1 - \frac{s}{2}(b + a)\psi\varphi_2, \tag{4.8a}$$

$$\varphi_{1,v} = (|\psi|^2)_z \quad \varphi_{2,z} = (|\psi|^2)_v, \tag{4.8b}$$

where *z=x+iy* e *v=x-iy* and its canonical form corresponds to $a = 0, b = 1$.

Finally, the equation DS-III ([76],[88]), is given by

$$i\psi_t = (a-b)\psi_{xx} - (b+a)\psi_{yy} - \frac{s}{2}(a-b)\psi\varphi_1 + \frac{s}{2}(b+a)\psi\varphi_2, \tag{4.9a}$$

$$\varphi_{1,y} = (|\psi|^2)_x \quad \varphi_{2,x} = (|\psi|^2)_y, \tag{4.9b}$$

and its canonical form is obtained with $a = 1, b = 0$.

The S-integrable ([7],[16],[17]) DS equation is important in plasma physics [67] and in quantum field theory ([1],[75],[29],[68]). Other valuable properties of this equation are the Darboux transformations [66] , a special bilinear form [21] and soliton and dromion solutions ([7],[16],[17],[66]).

5. Variable Separation Method for Nonlinear Equations in 2+1 Dimensions

In the last years it has been developed a very interesting technique for obtaining exact (and in particular coherent) solutions of nonlinear model systems, the multilinear variable separation approach. This method was first established for the DS system [34] and then developed for many other nonlinear equations, for example the Nizhnik-Novikov-Veselov (NNV) equation [36], asymmetric NNV equation [35], DS equation [37], dispersive long wave equation ([81],[82]), Broer-Kaup-Kupershmidt system [85], nonintegrable or integrable KdV equations in 2+1 dimensions ([80],[32]) and a general (N+M)-component Ablowitz-Kaup-Newell-Segur system [33]. In particular, it can be demonstrated that the solution for many nonlinear equations can be written in the form

$$U = \frac{-2\Delta q_y p_x}{(a_0 + a_1 p + a_2 q + a_3 pq)^2}, \quad \Delta = a_0 a_3 - a_1 a_2 \quad , \tag{5.1}$$

where a_0, a_1, a_2, a_3 are arbitrary constants, p=p(x,t) is an arbitrary function and q=q(y,t) is an arbitrary function for some equations (for example the DS equation) or an arbitrary solution of the Riccati equation in other cases. Different selections of the functions p and q correspond to different selections of boundary conditions and then in some sense coherent solutions can be remote controlled by some other quantities which have nonzero boundary conditions. Subsequently the method has been used for deriving chaotic and fractal solutions. Indeed, the solution (5.1) for an integrable NPDE with two or more dimensions is characterized by some arbitrary functions of lower dimensionality. As consequence a generic chaotic and/or fractal solution with lower dimension can be used to construct solutions of the given NPDE ([38],[89],[83]). The variety of solutions of (2+1)-dimensional nonlinear equations results from the fact that arbitrary exotic behaviours can transmit along the special characteristic functions p and q. For the moment in the method there are only two characteristic functions and it is an open question how to introduce more characteristic functions.

6. A Nonlinear Dirac Equation

The asymptotic perturbation method can be used for constructing approximate solutions of NPDEs and has been applied to particle-like solutions for a nonlinear relativistic scalar complex field model in 3+1 dimensions [47] and non-resonant interacting waves for the nonlinear Klein-Gordon equation [48] . The method has been later extended in order to demonstrate the existence of solitons trapping and dromion bound states for the nonlinear Klein-Gordon equation with appropriate potentials ([57],[58]). Non trivial solutions can be also obtained for relativistic vectorial fields[59] and nonlinear Dirac equation [62].

In order to illustrate this powerful technique, we seek coherent or chaotic or fractal approximate solutions of a nonlinear Dirac equation. It is well known that, in relativistic quantum mechanics, a free electron is represented by a wave function $\Psi(\underline{x}, t)$, with

$$i\hbar\Psi_t = -i\hbar c\underline{\alpha}.\nabla\Psi + \beta mc^2\Psi, \tag{6.1}$$

where c denotes the speed of light, m the mass of the electron and $\hbar$ is the Planck's constant [12]. The standard form of the $4X4$ matrices $\underline{\alpha}, \beta$(in $2X2$ blocks) is

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \underline{\alpha} = \begin{pmatrix} 0 & \underline{\sigma} \\ \underline{\sigma} & 0 \end{pmatrix}, \tag{6.2}$$

where $\underline{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices.

We seek approximate localized solutions for a particular version of the nonlinear Dirac equation

$$i\gamma^\mu\partial_\mu\Psi - m\Psi + \lambda\gamma^0\left(\overline{\Psi}\Psi\right)\Psi = 0, \tag{6.3}$$

where λ <<1 is a weak nonlinear parameter.

We use the asymptotic reduction (AP) method based on the spatio-temporal rescaling

$$\underline{\xi} = \varepsilon\underline{X}, \quad \tau = \varepsilon t, \tag{6.4}$$

and focus on a solution that, due to the weak nonlinearity (λ is a small parameter, $\lambda \to \varepsilon\lambda$), is close to a superposition of N several dispersive waves ($\varepsilon$is a bookkeeping device which will be set to unity in the final analysis).

In the linear limit the solution is

$$\sum_{j=1}^{N} A_j \exp(iz_j), \quad z_j = \underline{K}_j.\underline{X} - \omega_j t, \quad j = 1, ...N, \quad N > 1, \tag{6.5}$$

where A_j are the complex amplitudes, $\underline{K}_j \equiv (K_{1,j}, K_{2,j}, K_{3,j})$ the wave vectors and the (circular) frequency ω_j is furnished by the dispersion relation $\omega_j = \omega_j(\underline{K}_j)$. The amplitudes of these N non-resonant dispersive waves (constant in the linear limit) are slowly modulated by the non linear term of the nonlinear Dirac equation (6.3).

We will demonstrate the existence of dromions which preserve their shape during collisions, the only change being a phase shift. In addition, some special coherent solutions (line solitons, dromions, multilump solutions, ring solitons, instanton solutions and breathers) are derived. Moreover, we will show the existence of lower dimensional chaotic patterns such as chaotic-chaotic patterns, periodic-chaotic patterns, chaotic line soliton patterns, chaotic dromion patterns, fractal dromion and lump patterns as well as stochastic fractal solutions.

The linearized version of (6.3) is the well-known Dirac equation for spin 1/2 particles,

$$i\Psi_t = -i\underline{\alpha}.\nabla\Psi + \beta m\Psi, \tag{6.6}$$

satisfied by Fourier modes with constant amplitudes,

$$A_j \exp i(\underline{K}_j.\underline{X} - \omega_j t), \tag{6.7}$$

if the following dispersion relation is verified

$$\omega_j = \pm\sqrt{m^2 + \underline{K}^2}. \tag{6.8}$$

The group velocity $\underline{U}_j$ (the speed with which a wave packet peaked at that Fourier mode would move) is

$$\underline{U}_j = \frac{d\omega_j}{d\underline{K}_j} = \frac{\underline{K}_j}{\omega_j}. \tag{6.9}$$

In the following we consider a superposition of N dispersive waves, characterized by different group velocities not close to each other. Weak nonlinearity induces a slow variation of the amplitudes of these dispersive waves and the AP method derives the nonlinear system of equations for the Fourier modes amplitudes modulation, obviously in appropriate "slow" and "coarse-grained" variables defined by equations (6.4). Since the amplitudes of Fourier modes are not constant, higher order harmonics appear and in order to construct an approximate solution of the nonlinear equation (6.3) we introduce the asymptotic Fourier expansion for the positive-energy solutions (i.e. we consider the plus sign in (6.8))

$$\Psi(\underline{X}, t) = \sum_{\underline{n}(odd)} \varepsilon^{\gamma_{\underline{n}}} \begin{pmatrix} \varphi_{\underline{n}} \\ \chi_{\underline{n}} \end{pmatrix} \exp\left[i \sum_{j=1}^{N} n_j \left(\underline{K}_j.\underline{X} - \omega_j t\right)\right], \tag{6.10}$$

where the index $\underline{n}$ stands for the set $\{n_j; j = 1, 2, ..., N\}$ withn_j =*0, 1, 3,..* and $\underline{n} \neq (0, ..0)$. The functions, $\varphi_{\underline{n}}\left(\underline{\xi}, \tau, \varepsilon\right)$, $\chi_{\underline{n}}\left(\underline{\xi}, \tau, \varepsilon\right)$ depend parametrically on ε and we assume that their limit for $\varepsilon \to 0$ exists and is finite. We moreover assume that there hold the conditions

$$\gamma_{\underline{n}} = \sum_{j=1}^{N} \frac{f(n_j) - 1}{2}, \tag{6.11a}$$

$$f(n_j) = n_j, \quad for \quad n_j > 0. \tag{6.11b}$$

This implies that we obtain the main amplitudes if one of the indices n_j has unit modulus and all the others vanish. In the following we use the notation

$$\varphi_j = \varphi_{\underline{n}}(\varepsilon \to 0) \quad if \quad n_j = 1 \quad and \quad n_m = 0 \quad for \quad j \neq m, \tag{6.12}$$

and similar notations for $\chi_{\underline{n}}$.

Taking into account (6.11-6.12), the Fourier expansion (6.10) can be written more explicitly in the following form

$$\Psi(\underline{X}, t) = \sum_{j=1}^{N} \left[\begin{pmatrix} \varphi_j \\ \chi_j \end{pmatrix} \exp(i(\underline{K}_j.\underline{X} - \omega_j t)) \right] + O(\varepsilon). \tag{6.13}$$

Substituting (6.13) in equation (6.3) and considering the different equations obtained for every harmonic, we obtain for $n_j = 1$, $n_m = 0$, if $j \neq m$ at the lowest order of approximation

$$\chi_j = c_j \varphi_j = \frac{\sigma K_j}{m + \omega_j} \varphi_j, \tag{6.14}$$

and at the order of approximation of ε:

$$i\varphi_{j,\tau} = -i\underline{\sigma} \nabla \chi_j + \underline{\sigma K} \tilde{\chi}_j - \lambda \sum_{m=1}^{N} (\varphi_m^+ \varphi_m - \chi_m^+ \chi_m) \varphi_j, \tag{6.15a}$$

$$i\chi_{j,\tau} + \omega_j \tilde{\chi}_j = -i\underline{\sigma} \nabla \varphi_j - m\tilde{\chi}_j - \lambda \sum_{m=1}^{N} (\varphi_m^+ \varphi_m - \chi_m^+ \chi_m) \chi_j, \tag{6.15b}$$

where $\tilde{\chi}_j$ is the correction of order ε to χ_j. After some calculations, we arrive at a system of equations for the N modulated amplitudes φ_j,

$$\varphi_{j,\tau} + \underline{U}_j \nabla \varphi_j - i\lambda \sum_{m=1}^{N} b_m \left|\varphi_m\right|^2 \varphi_j = 0, \tag{6.16}$$

where $\underline{U}_j$ is the group velocity and b_m is a constant coefficient given by

$$b_m = \frac{2m}{m + \omega_m} \quad . \tag{6.17}$$

The system of equations (6.16) is C-integrable by means of an appropriate transformation of the dependent variables. We set

$$\varphi_j(\underline{\xi}, \tau) = \rho_j(\underline{\xi}, \tau) \exp\left[i\vartheta_j(\underline{\xi}, \tau)\right], \quad j = 1, ..N, \tag{6.18}$$

with $\rho_j = \rho_j(\underline{\xi}, \tau) > 0$ and $\vartheta_j = \vartheta_j(\underline{\xi}, \tau)$ real functions. Then equation (6.16) yields

$$\rho_{j,\tau} + \underline{U}_j \nabla \rho_j = 0, \tag{6.19a}$$

$$\vartheta_{j,\tau} + \underline{U}_j . \nabla \vartheta_j - \lambda \sum_{m=1}^{N} b_m \rho_m^2 = 0. \tag{6.19b}$$

The general solution for the Cauchy problem of (6.19a) reads

$$\rho_j(\underline{\xi},\tau) = \rho_j\left(\underline{\xi} - \underline{U}_j\tau\right), \tag{6.20}$$

where the N real functions $\rho_j\left(\underline{\xi}\right)$, which represent the initial shape, can be chosen arbitrarily. A simple particular case is the solution

$$\rho_j = \rho_j(\underline{a}_j\underline{\xi} + b_j\tau), \tag{6.21a}$$

where $b_j, \underline{a}_j$ are real constants which satisfy the relation

$$\underline{a}_j\underline{U}_j + b_j = 0. \tag{6.21b}$$

The general solution of (6.19b) is

$$\vartheta_j(\underline{\xi},\tau) = \delta_j(\underline{\xi} - \underline{U}_j\tau) + \lambda\sum_{m=1}^{N} b_m \int_0^{\tau} (\rho_m(\underline{\xi} - \underline{U}_j(\tau - \tilde{\tau}),\tilde{\tau}))^2 d\tilde{\tau}, \tag{6.22}$$

where the N arbitrary functions $\delta_j(\underline{\xi})$ are fixed by the initial data. The particular solution corresponding to (6.21b) is

$$\vartheta_j(\underline{\xi},\tau) = \delta_j(\underline{a}_j\underline{\xi} + \tilde{b}_j\tau) + \lambda\sum_{m=1}^{N} b_m \int_0^{\tau} (\rho_m(\underline{\xi} - \underline{U}_j(\tau - \tilde{\tau}),\tilde{\tau}))^2 d\tilde{\tau}, \tag{6.23a}$$

where

$$\underline{\tilde{a}}_j\underline{U}_j + \tilde{b}_j = 0. \tag{6.23b}$$

The approximate solution for the system of equations (6.3) is

$$\Psi(\underline{X},t) = \sum_{j=1}^{N} \begin{pmatrix} 1 \\ c_j \end{pmatrix} \rho_j \exp\left[i(\vartheta_j + \underline{K}_j\underline{X} - \omega_j t)\right] + O(\varepsilon). \tag{6.24}$$

where c_j is given by (6.14). The corrections of order to the approximate solution depend on higher harmonics and can be easily calculated by the AP method.

a) Solitons. The C-integrable nature of the system (6.19) implies the existence of more interesting solutions, because of the existence of arbitrary functions in the seed solutions. It is possible the existence of N solitons, with fixed speeds but arbitrary shapes, which interact each other preserving their shapes and propagate with the relative group velocity. The collision of two solitons does not produce a change in the amplitude ρ_j of each of them, but only a change in the phase given by equation (6.22).

For instance, we take

$$\rho_j(\underline{\xi},\tau) = \frac{2A_j}{ch\left(2A_j(\underline{a}_j\underline{\xi} + b_j\tau)\right)}, \tag{6.25}$$

$$\delta_j = 0 \quad for \quad \text{j=1 .. N}, \tag{6.26}$$

where A_j, for $j = 1...N$, are real constants, and the phase ϑ_j is given by (6.22). Substituting (6.27) in (6.24) we obtain the approximate solution. Each soliton advances with a constant velocity (the group velocity) before and after collisions. Only the phase is changed during collisions owing to the presence of the other solitons.

b) Dromions. The existence of localized solutions is possible also for C-integrable systems, because dromion solutions are not exclusive characteristics of equations integrable by the inverse scattering method.

A particular solution of the model system (6.19) is given by

$$\rho_j(\underline{\xi}, \tau) = A_j \exp(-B_j \left|\underline{\xi} - U_j\tau\right|), \tag{6.27}$$

$$\delta_j = 0 \quad for \quad \text{j=1, 2....N}, \tag{6.28}$$

where A_j, B_j are real constants (note that the functions ρ_j (3.4a) are localized) and ϑ_j is given by equations (6.22).

c) Lumps. It is well known that in high dimension, in addition to the dromion solutions, other interesting localized solutions, formed by rational functions, are the multiple lumps. Obviously, there are many possible choices in order to obtain multilump solutions. For instance, we take

$$\rho_j = \frac{A_j}{B_j + C_j \left|\underline{\xi} - U_j\tau\right|^2}, \tag{6.29a}$$

$$\delta_j = 0 \quad for \quad \text{j=1, ...N}, \tag{6.29b}$$

where A_j, B_j and C_j are arbitrary constants.

d) Ring solitons. The multiple ring solitons are solutions that are not equal to zero at some closed curves and decay exponentially away from the closed curves. A possible selection is

$$\rho_j = A_j \exp(-B_j f_j(R_j)), \tag{6.30a}$$

$$\delta_j = 0 \quad for \quad \text{j=1 .. N}. \tag{6.30b}$$

where

$$R_j = \left|\underline{\xi} - U_j\tau\right|, \tag{6.30c}$$

$$f_j(R) = (R - R_{0,j})^2, \tag{6.30d}$$

and A_j, B_j and $R_{0,j}$ are arbitrary constants. In Fig. 1 we show a collision between two ring solitons: the initial condition is showed in Fig. 1a, then the two ring solitons collide (Fig. 1b) and then separate (Fig. 1c). We can see that these solutions preserve their shapes but with a phase shift.

e) Instantons. If we choose a decaying function of time, we obtain also multiple instanton solutions, for example,

$$\rho_j = A_j \exp(\underline{a}_j \underline{\xi} - \lambda_j \tau), \tag{6.31a}$$

$$\delta_j = 0 \quad for \quad \text{j=1 .. N}, \tag{6.31b}$$

where A_j, $\alpha_{1,j}$ are arbitrary constants and

$$\lambda_j = \alpha_{1,j} U_{1,j} + \alpha_{2,j} U_{2,j}. \tag{6.31c}$$

f) Moving breather-like structures. Finally, if we choose some types of periodic functions of time in the above mentioned solutions, then we obtain breathers. For example, we take

$$\rho_j = A_j \cos(\underline{a}_j \underline{\xi} - \Omega_j \tau) \exp\left[-B_j \left|\underline{\xi} - U_j \tau\right|\right], \tag{6.32a}$$

$$\delta_j = 0 \quad for \quad \text{j=1 .. N}, \tag{6.32b}$$

where A_j, B_j, $\alpha_{1,j}$are arbitrary constants and

$$\Omega_j = \underline{a}_j \underline{U}_j. \tag{6.32c}$$

g) Chaotic-chaotic and chaotic-periodic patterns. If we select at least one of the arbitrary functions in order to contain some chaotic solutions of nonintegrable equations, then we obtain some type of space-time chaotic patterns, the so-called chaotic-chaotic (in all spatial directions) patterns. For example, we choose the arbitrary function as solution of the chaotic Lorenz system

$$X_T = -c(X - Y), \quad Y_T = X(a - Z) - Y, \quad Z_T = XY - bZ, \tag{6.33a}$$

with$a = 60$, $b = 8/3$, $c = 10$,or of the R**ö**ssler system

$$X_T = -Y - Z, \quad Y_T = X + aY, \quad Z_T = b + Z(X - c), \tag{6.33b}$$

with $a = 0.15$, $b = 0.2$, $c = 10$and $T = \xi - U_1\tau$ (or $T = \xi - U_1\tau$or $T = \zeta - U_3\tau$). A phase and amplitude chaotic-chaotic pattern is given by

$$\rho_j(\underline{\xi}, \tau) = X(\xi - U_{1,j}\tau) Y(\eta - U_{2,j}\tau) Z(\zeta - U_{3,j}\tau) \tag{6.34a}$$

$$\delta_j = 0 \quad for \quad j = 1, \ldots .N, \tag{6.34b}$$

while ϑ_j is given by equations (6,22). An example is given in Fig. 2.

On the contrary, we obtain a phase chaotic-chaotic pattern, if we choose the function (6.34b) as solution of the Lorenz system. Finally, if we select a chaotic-periodic solution which is chaotic in one (or two) direction and periodic in the other(s) direction(s). then we obtain the so-called chaotic-periodic patterns.

h) Chaotic line soliton solutions

If we consider the soliton line solution (6.25-6.26) we can easily deduce a chaotic solution when we select A_j as solution of the Lorenz system,

$$\rho_j(\underline{\xi}, \tau) = \frac{2A_j(\underline{\xi}, \tau)}{ch\left(2(\underline{a}_j\underline{\xi} + b_j\tau)A_j(\underline{\xi}, \tau)\right)}, \tag{6.35a}$$

$$\delta_j = 0 \quad for \quad \text{j=1 .. N}, \tag{6.35b}$$

where the phase ϑ_j is given as usual by (6.22), for $j = 1...N$, and the functions $A_j = A_j(T_j) = A_j(\underline{a}_j\underline{\xi} + b_j\tau)$ satisfy the third order ordinary differential equation equivalent to the Lorenz system,

$$\begin{aligned} A_{j,TTT} + (b+c+1)A_{j,TT} + (bc+b+A_j^2)A_{j,T} + c(b-ab+A_j^2)A_j \\ -\frac{A_{j,TT}A_{j,T} + (c+1)A_{j,T}^2}{A_j} = 0. \end{aligned} \tag{6.35c}$$

i) Chaotic dromion and lump patterns

If we consider the dromion solution (6.27-6.28), we can transform it into a chaotic pattern with an appropriate choice for A_j and/or B_j,

$$\rho_j(\underline{\xi}, \tau) = A_j \exp(-B_j \left|\underline{\xi} - U_j\tau\right|), \tag{6.36a}$$

$$\delta_j = 0 \quad for \quad \text{j=1, 2....N}, \tag{6.36b}$$

where $A_j = A_j(T_j) = A_j(\underline{a}_j\underline{\xi} + b_j\tau)$ and/or $B_j = B_j(T_j) = B_j(\underline{a}_j\underline{\xi} + b_j\tau)$ are solutions of the Lorenz equation and ϑ_j is given by equations (6.22). We obtain an amplitude (A_j chaotic) or a shape (B_j chaotic) or an amplitude and shape (A_j and B_j chaotic) dromion chaotic pattern. Similar considerations can be applied to the lump solutions (6.29).

j) Nonlocal fractal solutions. If we choose

$$\rho_j(\underline{\xi}, \tau) = \prod_{m=1}^{3} T_{m,j}\left|T_{m,j}\right| \left\{\sin\left[\ln\left(T_{m,j}^2\right)\right] - \cos\left[\ln\left(T_{m,j}^2\right)\right]\right\} \tag{6.37}$$

with $\underline{T} = (T_1, T_2, T_3)$, $\underline{T}_j = \underline{\xi} - \underline{U}_j\tau$, we get a nonlocal fractal structure for small $\underline{T}_j$. It is well known that if we plot the structure of the solution at smaller regions we can obtain the same structures.

k) Fractal dromion and lump solutions. A fractal dromion (lump) solution is exponentially (algebraically) localized in large scale and possesses self-similar structure near the center of the dromion. We consider for example an amplitude fractal dromion

$$\rho_j(\underline{\xi}, \tau) = A_j \exp(-B_j \left|\underline{\xi} - U_j\tau\right|), \tag{6.38a}$$

$$\delta_j = 0 \quad for \quad \text{j=1, 2....N}, \tag{6.38b}$$

where ϑ_j is given by equations (6.22) and $A_j = A_j(T_j) = A_j(\underline{a}_j\underline{\xi} + b_j\tau)$ is given by

$$A_j = 2 + \sin\left\{\ln\left[T_j^2\right]\right\}. \tag{6.38c}$$

By a similar choice for B_j or $\delta_j$we obtain shape or phase fractal dromion.

l) Stochastic fractal dromion and lump excitations. It is well known the stochastic fractal property of the continuous but nowhere differentiable Weierstrass function

$$W(x) = \sum_{k=1}^{N} (c_1)^k \sin\left[(c_2)^k x\right], \quad N \to \infty, \tag{6.39a}$$

with c_2 odd and

$$c_1 c_2 > 1 + \frac{3\pi}{2}. \tag{6.39b}$$

A stochastic fractal solution is (see Fig. 3 for an example)

$$\rho_j(\underline{\xi}, \tau) = \prod_{m=1}^{3} A_{m,j}(\xi_m - U_{m,j}\tau), \tag{6.40a}$$

$$\delta_j = 0 \quad for \quad \text{j=1, 2....N}, \tag{6.40b}$$

where ϑ_j is as usual given by equations (6.22), $\underline{A}_j = (A_{1,j}, A_{2,j}, A_{3,j})$, $\xi_1 = \xi$,$\xi_2 = \eta$,$\xi_3 = \zeta$, and $\underline{A}_j = \underline{A}_j(\underline{\xi} - \underline{U}_j\tau)$ is given by

$$\underline{A}_j = W(\underline{\xi} - \underline{U}_j\tau) + (\underline{\xi} - \underline{U}_j\tau)^2. \tag{6.40c}$$

m) Stochastic fractal dromion and lump excitations. In order to obtain a stochastic amplitude fractal dromion we choose

$$\rho_j(\underline{\xi}, \tau) = A_j \exp(-B_j \left|\underline{\xi} - U_j\tau\right|), \tag{6.41a}$$

$$\delta_j = 0 \quad for \quad \text{j=1, 2....N}, \tag{6.41b}$$

where ϑ_j is given by equations (6.22) $A_j = A_j(T_j) = A_j(\underline{a}_j\underline{\xi} + b_j\tau)$ is given by

$$A_j = W(T_j) + T_j^2 \tag{6.41c}$$

By similar methods we obtain shape or phase stochastic fractal dromion as well as stochastic fractal lump solutions.

7. Spontaneous Symmetry Breaking Model

We now illustrate in some detail the use of the AP method and consider a scalar complex field $\Phi = \Phi(x)$, $x = (x^0 = t, \underline{x})$, coupled with a massless vectorial gauge field $A_\mu = A_\mu(x)$, $A^\mu = (A^0, \underline{A})$, and seek approximate localized solutions for a spontaneous symmetry breaking (or hidden symmetry or Higgs) model ([19],[22],[23]) with Lagrangian [63]

$$L = [(\partial^\mu + iqA^\mu)\,\Phi]^* \,[(\partial^\mu + iqA^\mu)\,\Phi] - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{a^2}{2b^2}(\Phi^*\Phi)^2 + \frac{a^2}{2}(\Phi^*\Phi), \tag{7.1a}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{7.1b}$$

The Lagrangian (7.1) is invariant under the local transformation

$$\Phi(x) \to \Phi'(x) = \exp\left(-i\alpha(x)\Phi(x)\right), \tag{7.2a}$$

with accompanied by the gauge transformation on the potentials

$$A^\mu(x) \to A'^\mu(x) = A^\mu(x) + \frac{1}{q}\partial^\mu\alpha(x). \tag{7.2b}$$

We note that this model contains four field degrees of freedom, two in the complex scalar Higgs field and two in the massless gauge field. The field equations are

$$\partial_\mu\partial^\mu A^\nu - \partial^\nu(\partial_\mu A^\mu) = J^\nu, \tag{7.3a}$$

$$J^\nu = iq\left[\Phi^*\left(\partial^\nu\Phi\right) - \left(\partial^\nu\Phi^*\right)\Phi\right] - 2q^2A^\nu\left|\Phi\right|^2, \tag{7.3b}$$

$$\partial_\mu\partial^\mu\Phi - \frac{a^2}{2}\Phi = -\frac{a^2}{b^2}\left|\Phi\right|^2\Phi, \tag{7.3c}$$

where $\partial_\mu\partial^\mu = \partial_t^2 - \nabla^2$, a, b and q are parameters. The potential for the scalar field is

$$V(\Phi) = -\frac{a^2}{2}\left|\Phi\right|^2 + \frac{a^2}{2b^2}\left|\Phi\right|^4, \tag{7.4}$$

and the physical vacuum is identified by the condition

$$\left|\Phi\right|^2 = b^2. \tag{7.5}$$

We consider the interaction and eventually the collisions among coherent solutions with different velocities that are not close to each other and use the asymptotic reduction (AP) method based on the spatio-temporal rescaling

$$\xi = \varepsilon^2 x, \tag{7.6}$$

where

$$\xi = (\xi^0, \underline{\xi}), \quad x = (x^0, \underline{x}). \tag{7.7}$$

and the small positive nondimensional parameter ε is artificially introduced to serve as bookkeeping device and will be set equal to unity in the final analysis. The linear evolution is most appropriately described in terms of Fourier modes, which have a constant amplitude and a well defined group velocity (the speed with which a wave packet peaked at that Fourier mode would move in ordinary space). We study the modulation, in terms of the variables defined above, of the amplitude of the Fourier mode. The modulation (that would remain constant in the absence of nonlinear effects) is best described in terms of the rescaled variables, ξ, that account for the need to look on larger space and time scales, to obtain a not negligible contribution from the nonlinear term. The reduction method focuses on a solution that is small and is close to a superposition of N several dispersive waves, with different group velocities.

In the linear limit the solution is a linear combination of dispersive waves. For example for the scalar field the linear solution is

$$\sum_{j=1}^{N} C_j \exp\left(-is_j\right), \quad \tilde{s}_j = \tilde{k}_{j,\mu}x^\mu = \tilde{\omega}_j t - \underline{\tilde{K}}_j.\underline{x}, \tag{7.8}$$

where C_j are the complex amplitudes, $\tilde{k}_j^\mu = (\tilde{\omega}_j, \underline{\tilde{K}}_j)$, $\underline{\tilde{K}}_{\mathrm{j}} \equiv (\tilde{K}_{1,j}, \tilde{K}_{2,j}, \tilde{K}_{3,j})$ the wave vectors and the (circular) frequency $\tilde{\omega}_j$ is furnished by the dispersion relation $\tilde{\omega}_j = \tilde{\omega}_j(\underline{\tilde{K}}_j)$. The amplitudes of these N non-resonant dispersive waves (constant in the linear limit) are slowly modulated by the nonlinear terms. We derive a model system of equations for the slow modulation of the Fourier modes amplitudes and, subsequently, show that it is C-integrable. The Cauchy problem is resolved, just by quadratures, and explicit nontrivial solutions are constructed.

We introduce two real Higgs fields $U = U(\underline{x}, t)$ e $W = W(\underline{x}, t)$ and set

$$\Phi = \frac{1}{\sqrt{2}}\left(b + U + iW\right). \tag{7.9}$$

In the following we use the covariant ‘t Hooft gauge [tHo], which for this Abelian model is (M=qb)

$$\partial_\mu A^\mu = \lambda M W, \tag{7.10}$$

where λ is an arbitrary parameter. For any finite λ, we obtain R gauges, that are manifestly renormalisable, but involve unphysical Higgs fields such as W. We recall that in the limit $\lambda \to \infty$ we obtain the U (unitary) gauge, where only physical particles appear.

Using (7.1-7.8), the equations (7.9-7.11) yield

$$\left(\partial_\mu\partial^\mu + M^2\right) A^\nu - \left(1 - \frac{1}{\lambda}\right) \partial^\nu(\partial_\mu A^\mu) = J^\nu, \tag{7.11}$$

$$J^\nu = q\left[\frac{\left(\partial^\nu U - U\partial^\nu\right)\left(\partial_\mu A^\mu\right)}{\lambda M}\right] - q^2 A^\nu \left[U^2 + 2bU + \frac{\left(\partial_\mu A^\mu\right)^2}{\lambda^2 M^2}\right], \tag{7.12}$$

$$\left(\partial_\mu\partial^\mu + M^2\right) U = -\frac{a^2U^2}{2b} - \frac{a^2}{2b}\left(1 + \frac{U}{b}\right)\left(U^2 + \frac{\left(\partial_\mu A^\mu\right)^2}{\lambda^2 M^2}\right). \tag{7.13}$$

We consider now a superposition of N dispersive waves, characterized by different values of the wave vector $\underline{K}_j$ and by group velocities not close to each other. Weak nonlinearity induces a slow variation of the amplitudes of these dispersive waves and the AP method derives the nonlinear system of equations for the Fourier modes amplitudes modulation, obviously in appropriate "slow" and "coarse-grained" variables defined by equations (7.7). Since the amplitude of Fourier modes are not constant, higher order harmonics appear and in order to construct an approximate solution that is small of order ε and that is close in the limit of small ε to the linear solution (7.8), we introduce the asymptotic Fourier expansion

$$U(x^\mu) = \sum_{\underline{n}=-\infty}^{\infty} \exp\left(i\sum_{j=1}^{N} n_j \tilde{s}_j\right) \varepsilon^{\gamma_{\underline{n}}} \varphi_{\underline{n}}(\xi^\mu;\varepsilon), \quad \tilde{s}_j = \tilde{k}_{j,\mu}x^\mu = \tilde{\omega}_j t - \underline{\tilde{K}}_j.\underline{x}, \tag{7.14}$$

$$A^\nu(x^\mu) = \sum_{\underline{n}=-\infty}^{\infty} \exp\left(i\sum_{j=1}^{N} n_j s_j\right) \varepsilon^{\gamma_{\underline{n}}} \psi_{\underline{n}}(\xi^\mu;\varepsilon), \quad s_j = k_{j,\mu}x^\mu = \omega_j t - \underline{K}_j.\underline{x}, \tag{7.15}$$

where the index $\underline{n}$ stands for the set $\{n_j; j = 1, 2, ..., N\}$. In the expansion (7.14) $n_j = 0, \pm1, \pm2, ..,$ while in the expansion (7.15) n_j may assume only odd values, $n_j = \pm1, \pm3...$ The functions, $\psi^\nu_{\underline{n}}(\xi;\varepsilon)$, $\varphi_{\underline{n}}(\xi;\varepsilon)$ depend parametrically on ε and we assume that their limit for $\varepsilon \to 0$ exists and is finite. We moreover assume that there hold the conditions

$$\gamma_{\underline{n}} = \gamma_{-\underline{n}}, \quad \gamma_{\underline{n}} = \sum_{j=1}^{N} |n_j|. \tag{7.16}$$

This implies that we obtain the main amplitudes if one of the indices n_j has unit modulus and all the others vanish. We use the following notations, for $j = 1, 2, ...N$,

$$\varphi_{\underline{n}}(\xi, \varepsilon \to 0) = \varphi_j(\xi), \quad if \quad n_j = 1 \quad and \quad n_m = 0 \quad for \quad j \neq m, \tag{7.17a}$$

$$\varphi_{\underline{n}}(\xi; \varepsilon \to 0) = \varphi_0(\xi), \quad if \quad n_j = 0, \tag{7.17b}$$

$$\varphi_{\underline{n}}(\xi; \varepsilon \to 0) = \varphi_{2,j}(\xi), \quad if \quad n_j = 2 \quad and \quad n_m = 0 \quad for \quad j \neq m, \tag{7.17c}$$

$$\varphi_{\underline{n}}(\xi; \varepsilon \to 0) = \varphi_{11,jm}(\xi), \quad if \quad n_j = n_m = 1 \quad and \quad n_l = 0 \quad for \quad l \neq j, m, \quad j \neq m, \tag{7.17d}$$

$$\varphi_{\underline{n}}(\xi; \varepsilon \to 0) = \varphi_{1-1,jm}(\xi), \quad if \quad n_j = 1, n_m = -1, \quad and \quad n_l = 0 \quad for \quad l \neq j, m \quad , \quad j \neq m, \tag{7.17e}$$

while for the vectorial field we set

$$\Psi^\nu_{\underline{n}}(\xi; \varepsilon \to 0) = \Psi^\nu_j(\xi). \tag{7.18}$$

Taking into account (7.17-7.18), the Fourier expansion (7.14-7.15) can be written more explicitly in the following form

$$A^{\nu}(x) = \varepsilon \sum_{j=1}^{N} \left[\exp(is_j)\Psi_j^{\nu}(\xi) + c.c.\right] + O(\varepsilon^3), \tag{7.19}$$

$$\begin{aligned} U(x) &= \varepsilon \sum_{j=1}^{N} [\exp(is_j)\varphi_j(\xi) + \varepsilon \exp(2is_j)\varphi_{2,j}(\xi) + c.c.] + \varepsilon^2 \varphi_0(\xi) \\ &+\varepsilon^2 \sum_{j,m=1,j\neq m}^{N} [\exp(-is_j - is_m)\varphi_{11,jm}(\xi) + \exp(-is_j + is_m)\varphi_{1-1,jm}(\xi) + c.c.] + O(\varepsilon^3) \end{aligned} \tag{7.20}$$

where *c.c.* stands for complex conjugate.

The standard procedure is to consider the different equations obtained from the coefficients of the Fourier modes. Substituting (7.19-7.20) in equations (7.11-7.15) and considering the different equations obtained for every harmonic and for a fixed order of approximation in ε, we obtain for $n_j = 1$, $n_m = 0$, if $j \neq m$, to the order of ε, the following system of equations for the main modulated amplitudes,

$$(-\tilde{k}_{j,\mu}\tilde{k}_j^{\mu} + a^2)\varphi_j = 0, \tag{7.21a}$$

$$\left[(-k_{j,\sigma}k_j^{\sigma} + M^2)g^{\nu\mu} + \left(1 - \frac{1}{\xi}\right)(k_j^{\mu}k_j^{\nu})\right]\Psi_{j,\mu} = 0. \tag{7.21b}$$

From (7.21a-b) we obtain the dispersion relations

$$\tilde{\omega}_j^2 = \underline{\tilde{K}}_j^2 + a^2, \quad \omega_j^2 = \underline{K}_j^2 + M^2, \tag{7.22a}$$

with the associated group velocities

$$\tilde{V}_j = \frac{\underline{\tilde{K}}_j}{\tilde{\omega}_j}, \quad V_j = \frac{\underline{K}_j}{\omega_j}. \tag{7.22b}$$

Moreover, from (7.21b), as a consequence of the gauge invariance of the vectorial field (only three components of the fields are independent), we obtain

$$k_j^{\mu}\Psi_{j,\mu} = \omega_j \Psi_{j,0} - \underline{K\Psi} = 0. \tag{7.23}$$

We obtain for $n_j = 1$, $n_m = 0$, if $j \neq m$, to the order of ε^2,

$$\begin{aligned} &(-2i\tilde{k}_{j,\mu}\partial^{\mu})\varphi_j + \frac{3a^2}{2b}\left(2\varphi_0\varphi_j + 2\varphi_2\varphi_j^* + |\varphi_j|^2\varphi_j\right) \\ &+\frac{3a^2}{b^2}\sum_{m=1,m\neq j}^{N}\left(|\varphi_m|^2\varphi_j + \varphi_{11,jm}\varphi_j^* + \varphi_{1-1,jm}\varphi_j\right) = 0 \end{aligned} \tag{7.24a}$$

,

$$\left[(-2k_{j,\sigma}\partial^{\sigma})g^{\nu\mu}+\left(1-\frac{1}{\lambda}\right)(\partial^{\mu}k_{j}^{\nu})\right]\Psi_{j,\mu}+i\left(-2q^{2}\Psi_{j}^{\nu}\right)\left(\sum_{m=1}^{N}|\varphi_{m}|^{2}+b\varphi_{0}\right)=0,\quad(7.24b)$$

and for $n_j = 0$, to the order of ε^2,

$$\varphi_0 = A\sum_{m=1}^{N}|\varphi_m|^2,\quad A=-\frac{3}{b},\qquad(7.25)$$

and for $n_j = 2$, $n_m = 0$, if $j \neq m$, to the order of ε^2,

$$\varphi_{2,j} = B_2\varphi_j^2,\quad B_2=-\frac{1}{2b}\qquad(7.26)$$

and for $n_j = 1$, $n_m = 1$, $n_l = 0$ if $j, m \neq l$, to the order of ε^2,

$$\varphi_{11,jm} = \sum_{m=1}^{N}B_{11,jm}\varphi_j\varphi_m,\quad B_{11,jm}=\frac{3a^2}{b\left(a^2+2\tilde{k}_m^{\mu}\tilde{k}_{j,\mu}\right)},\qquad(7.27)$$

and for $n_j = 1$, $n_m = -1$, $n_l = 0$, if $j, m \neq l$, to the order of ε^2,

$$\varphi_{1-1,jm} = \sum_{m=1}^{N}B_{1-1,jm}\varphi_j\varphi_m^{*},\quad B_{1-1,jm}=\frac{3a^2}{b\left(a^2-2\tilde{k}_m^{\mu}\tilde{k}_{j,\mu}\right)}.\qquad(7.28)$$

Using (7.24a-b) then equations (7.21-7.22) yield

$$(\tilde{k}_{j,\mu}\partial^{\mu})\varphi_j + i\sum_{m=1}^{N}\left[\alpha_{jm}|\varphi_m|^2\right]\varphi_j = 0,\qquad(7.29a)$$

$$\left[(-2k_{j,\sigma}\partial^{\sigma})g^{\nu\mu}+\left(1-\frac{1}{\lambda}\right)(\partial^{\mu}k_{j}^{\nu})\right]\Psi_{j,\mu}+i\beta\sum_{m=1}^{N}\left[|\varphi_m|^2\right]\Psi_j^{\nu}=0,\qquad(7.29b)$$

where the coefficients α, β, are depending on the wave vectors of the scalar and vectorial fields,

$$\alpha_{jm} = -\frac{3a^2}{b^2}+\frac{9a^6}{b^2\left(a^4-4\left(k_m^{\mu}k_{j,\mu}\right)^2\right)},\quad j\neq m,\qquad(7.30a)$$

$$\alpha_{jj} = -\frac{9a^2}{2b^2},\quad \beta = 4q^2.\qquad(7.30b)$$

The system of equations (7.28-7.29) is C-integrable by means of an appropriate transformation of the dependent variables. We set

$$\varphi_j(\xi) = \rho_j(\xi)\exp\left[i\vartheta_j(\xi)\right],\quad j=1,..N,\qquad(7.31a)$$

$$\Psi_j^{\nu}(\xi) = \chi_j^{\nu}(\xi)\exp\left[i\delta_j^{\nu}(\xi)\right],\quad j=1,..N,\qquad(7.31b)$$

with $\rho_j = \rho_j(\xi), \chi_j^\nu = \chi_j^\nu(\xi)$ >0 and $\vartheta_j = \vartheta_j(\xi)$, $\delta_j^\nu = \delta_j^\nu(\xi)$ real functions. Then equation (7.28-7.29) yield

$$(\tilde{k}_{j,\mu}\partial^\mu)\rho_j = 0, \tag{7.32a}$$

$$(\tilde{k}_{j,\mu}\partial^\mu)\vartheta_j + \sum_{m=1}^{N}\left[\alpha_{jm}\rho_m^2\right] = 0, \tag{7.32b}$$

and

$$\left[(-2k_{j,\sigma}\partial^\sigma)g^{\nu\mu} + (1-\frac{1}{\lambda})(k_j^\nu\partial^\mu)\right]\chi_{j,\mu} = T_j^{\mu\nu}\chi_{j,\mu} = 0, \tag{7.33a}$$

$$\left[(-2k_{j,\sigma}\partial^\sigma)\chi_j^\nu g^{\nu\mu} + \left(1-\frac{1}{\lambda}\right)\chi_j^\nu(\partial^\mu k_j^\nu)\right]\delta_{j,\mu} + \beta\sum_{m=1}^{N}\left[|\varphi_m|^2\chi_j^\nu\right] = 0 \tag{7.33b}$$

We now consider a particular mode j with group velocity $\underline{V}_j = 0$, i. e. $\underline{K}_j = 0$ (see (7.22b)). This condition is equivalent to choose a frame where the solution of (7.33a) is not depending on the time (the proper frame). Equations (7.33a-b) yield

$$\chi_j^i = g_j^i(\underline{\xi}), \quad \delta_j^i = \tilde{\delta}_j^i(\underline{\xi}) + \beta\tau\sum_{m=1}^{N} g_j^i(\underline{\xi}), \quad for \quad i = 1, 2, 3 \tag{7.34}$$

where $g_j^i(\underline{\xi})$, $\tilde{\delta}_j^i(\underline{\xi})$ are arbitrary functions of the space variables. Note that $g_j^0(\underline{\xi})$ and $\tilde{\delta}_j^0(\underline{\xi})$ are fixed by the gauge condition (7.31). By a Lorentz boost we can construct the solution in a generic frame and in the following we use a frame moving con velocity $\underline{V}_j = (V_j, 0, 0)$with respect to the proper frame

$$\xi = \gamma_j(\xi' - V_j\tau'), \quad \eta = \eta', \quad \varsigma = \varsigma', \quad \tau = \gamma_j(\tau' - \frac{V_j\xi'}{c^2}), \quad \gamma_j = \left(1-\left(\frac{V_j}{c}\right)^2\right)^{-\frac{1}{2}}. \tag{7.35}$$

On a similar way we obtain the solution for the Higgs field,

$$\rho_j = f_j(\underline{\xi}), \quad \vartheta_j = \tilde{\vartheta}_j(\underline{\xi}) - \tau\sum_{m=1}^{N}\alpha_{jm}f_m^2(\underline{\xi}) \tag{7.36}$$

where $f_j(\underline{\xi})$, $\tilde{\vartheta}_j(\underline{\xi})$ are arbitrary functions of the space variables.

At last, an interesting particular solution for the Cauchy problem of (7.32a-7.33a) reads

$$\rho_j(\underline{\xi}, \tau) = \rho_j\left(\tilde{V}_j^\sigma\xi_\sigma\right), \tag{7.37a}$$

$$\chi_j^\nu(\underline{\xi}, \tau) = \chi_j^\nu\left(V_j^\sigma\xi_\sigma\right), \tag{7.37b}$$

where the $4N$ real functions $\rho_j\left(\underline{\xi}\right)$, $\chi_j^i\left(\underline{\xi}\right)$, *i=1, 2, 3*, which represent the initial shape, can be chosen arbitrarily and

$$k_{j,\mu}V_j^\mu = 0, \quad \tilde{k}_{j,\mu}\tilde{V}_j^\mu = 0. \tag{7.38}$$

Inserting (7.37b) in (7.33a) yields

$$\left[(-2k_{j,\sigma}V_j^{\sigma})g^{\nu\mu} + (1-\frac{1}{\lambda})(k_j^{\nu}V_j^{\mu})\right]\chi_{j,\mu} = T_j^{\mu\nu}\chi_{j,\mu} = 0, \tag{7.39}$$

and since

$$\det T = 8\left(2-\left(1-\frac{1}{\lambda}\right)\right)(k_{j,\mu}V_j^{\mu})^4 = 0, \tag{7.40}$$

we obtain

$$\chi_{j,\mu}V_j^{\mu} = 0. \tag{7.41}$$

The field $\chi_j^0(\xi)$ is fixed by the gauge condition (7.41).

In conclusion, the approximate solution for the system of equations (7.11-7.13) is

$$A^{\nu}(x) = 2\varepsilon\sum_{j=1}^{N}\chi_j^{\nu}\exp\left[i(\delta_j^{\nu} - k_{j,\mu}x^{\mu})\right] + O(\varepsilon^3), \tag{7.42}$$

$$U(x) = \varepsilon U_1(x) + \varepsilon^2 U_2(x) + O(\varepsilon^3), \tag{7.43a}$$

where

$$U_1(x) = 2\varepsilon\sum_{j=1}^{N}\left[\rho_j\cos(\tilde{k}_{j,\mu}x^{\mu} - \vartheta_j)\right], \tag{7.43b}$$

$$\begin{aligned} U_2(x) = 2\varepsilon^2\sum_{j=1}^{N}\left[B_2\rho_j^2\cos\left[2\left(\tilde{k}_{j,\mu}x^{\mu}-\vartheta_j\right)\right]\right] + \varepsilon^2\left(A\rho_j^2\right) \\ +\varepsilon^2\sum_{j,m=1,j\neq m}^{N}\left[B_{11,jm}\rho_j\rho_m\cos\left[\left(\tilde{k}_{j,\mu}x^{\mu}-\vartheta_j\right)+\left(\tilde{k}_{m,\mu}x^{\mu}-\vartheta_m\right)\right]\right] \\ +\varepsilon^2\sum_{j,m=1,j\neq m}^{N}\left[B_{1-1,jm}\rho_j\rho_m\cos\left[\left(\tilde{k}_{j,\mu}x^{\mu}-\vartheta_j\right)-\left(\tilde{k}_{m,\mu}x^{\mu}-\vartheta_m\right)\right]\right]. \end{aligned} \tag{7.43c}$$

The corrections of order to the approximate solution depend on higher harmonics and can be easily calculated by the AP method.

The validity of the approximate solution should be expected to be restricted on bounded intervals of the τ-variable and on time-scale $t = O(\frac{1}{\varepsilon^2})$. If one wishes to study solutions on intervals such that $\tau = O(\frac{1}{\varepsilon})$ then the higher terms will in general affect the solution and must be included.

8. Coherent Solutions

In the following we have written the solutions in the moving frame of reference, but for simplicity we have dropped the apices in the space and time variables.

i) Nonlinear wave. The most simple solution of the system (7.29) is the plane wave

$$\rho_j = A_j = \text{constant}, \qquad \vartheta_j = \underline{\tilde{K}}_j^{'}\underline{\xi} - \tilde{\omega}_j^{'}\tau \tag{8.1a}$$

$$\chi_j^\mu = B_j^\mu = \text{constant}, \qquad \delta_j^\mu = \underline{K}_j^{'}\underline{\xi} - \omega{'}_j\tau \tag{8.1b}$$

where the amplitudes and wave vectors are connected according to the nonlinear dispersion relation

$$\omega_j^{'} = \underline{V}_j.\underline{K}_j^{'} + \frac{\alpha}{\omega_j^{'}}\sum_{m=1}^{N} A_m^2, \quad \tilde{\omega}_j^{'} = \underline{\tilde{V}}_j.\underline{\tilde{K}}_j^{'} + \frac{1}{\tilde{\omega}_j}\sum_{m=1}^{N}\alpha_{jm}A_m^2, \tag{8.1c}$$

χ_j^0 and δ_j^0 are fixed by the gauge condition (7.23).

ii) Solitons. In the following we seek coherent solutions and use the gauge condition (7.23) which implies, being $\chi_j^\nu = (\chi_j^0, \underline{\chi}_j)$,

$$\chi_j^0 = \frac{\underline{K}_j \cdot \underline{\chi}_j}{\omega_j}. \tag{8.2}$$

The C-integrable nature of the system (7.29) implies the existence of more interesting solutions, because of the existence of arbitrary functions in the seed solutions. It is possible the existence of N solitons, with fixed speeds but arbitrary shapes, which interact each other preserving their shapes and propagate with the relative group velocity. The collision of two solitons does not produce a change in the amplitude ρ_j of each of them, but only a change in the phase given by equation (7.36).

For instance, we take

$$\rho_j(\underline{\xi}, \tau) = \frac{2\tilde{A}_j}{ch\left(2\tilde{A}_j\tilde{\gamma}_j(\xi - \tilde{V}_j\tau)\right)}, \tag{8.3a}$$

$$\chi_j^i(\underline{\xi}, \tau) = \frac{2A_j}{ch\left(2A_j\gamma_j(\xi - V_j\tau)\right)}, \quad for \quad \text{i=1, 2, 3}, \tag{8.3b}$$

$$\tilde{\vartheta}_j = \delta_j^\nu = 0 \quad for \quad \text{j=1 .. N}, \tag{8.3c}$$

where A_j, for $j = 1...N$, are real constants,

$$\tilde{\gamma}_j = \left(1 - \left(\frac{\tilde{V}_j}{c}\right)^2\right)^{-\frac{1}{2}}, \tag{8.3d}$$

γ_j is given by (7.35b) and the phase ϑ_j and δ_j^ν are given by (7.34-7.36). Each soliton advances with a constant velocity (the group velocity) before and after collisions. Only the phase is changed during collisions owing to the presence of the other solitons. Substituting (8.3) in (7.42-7.43) we obtain the approximate solution good to the order of ε. Each soliton advances with a constant velocity (the group velocity) before and after collisions. Only the phase is changed during collisions owing to the presence of the other solitons.

iii) Dromions. The existence of localized solutions is possible also for C-integrable systems, because dromion solutions are not exclusive characteristics of equations integrable by the inverse scattering method.

A particular solution of the model system is given by

$$\rho_j(\underline{\xi},\tau) = \tilde{A}_j \exp\left(-\tilde{B}_j\sqrt{\tilde{\gamma}_j^2(\xi-\tilde{V}_j\tau)^2+\eta^2+\varsigma^2}\right), \tag{8.4a}$$

$$\chi_j^i(\underline{\xi},t) = A_j^i \exp\left(-B_j^i\sqrt{\gamma_j^2(\xi-V_j\tau)^2+y^2+z^2)}\right), \quad for \quad \text{i=1, 2, 3} \tag{8.4b}$$

$$\tilde{\vartheta}_j = \delta_j^\nu = 0 \quad for \quad \text{j=1, 2....N}, \tag{8.4c}$$

where A_j, $\tilde{A}_j$,B_j, $\tilde{B}_j$ are real constants and ϑ_j and δ_j^ν are given by equations (7.34-7.36). Substituting the solution (8.4) in equation (7.42-7.43) and taking $N=2$ we obtain for two dromions with different shapes and amplitudes the approximate solution

$$U(\underline{x},t) = 2\sum_{j=1}^{2}\tilde{A}_j \exp\left(-\tilde{B}_j\sqrt{\tilde{\gamma}_j^2(x-\tilde{V}_jt)^2+y^2+z^2)}\right)\cos(\tilde{k}_{\mu,j}x^\mu+\vartheta_j), \tag{8.5}$$

$$A^\nu(\underline{x},t) = 2\sum_{j=1}^{2}A_j^\nu \exp\left(-B_j^\nu\sqrt{\gamma_j^2(x-V_jt)^2+y^2+z^2)}\right)\cos(k_{\mu,j}x^\mu+\delta_j). \tag{8.6}$$

In Fig. 4 we show a collision between two dromions for the Higgs field (see (7.43b) and (8.5) with identical mass $M = 100$ $GeV/c2$, $B1 = B_2 = M$, $P_1 = 1600$ TeV/c, P_2=*2000 TeV/c*): the initial condition is showed in Fig. 4a, then the two dromions collide (Fig. 4b) and then separate (Fig. 4c). We can see that dromions preserve their shapes but with a phase shift.

iv) Lumps. It is well known that in high dimension, in addition to the dromion solutions, other interesting localized solutions, formed by rational functions, are the multiple lumps. Obviously, there are many possible choices in order to obtain multilump solutions. For instance, we take

$$\rho_j(\underline{\xi},\tau) = \frac{\tilde{A}_j}{\tilde{B}_j+\tilde{C}_j\sqrt{\tilde{\gamma}_j^2\left(\xi-\tilde{V}_j\tau\right)^2+\eta^2+\varsigma^2}}, \tag{8.7a}$$

$$\chi_j^i(\underline{\xi},\tau) = \frac{A_j^i}{B_j^i+C_j^i\sqrt{\gamma_j^2\left(\xi-V_j\tau\right)^2+\eta^2+\varsigma^2}}, \quad for\text{i=1, 2, 3}, \tag{8.7b}$$

$$\tilde{\vartheta}_j = \delta_j^\nu = 0 \quad for \quad \text{j=1, ...N}, \tag{8.7c}$$

where $\tilde{A}_j$, $\tilde{B}_j$, $\tilde{C}_j$, A_j^i, B_j^i and C_j^i are arbitrary constants and ϑ_j^ν and δ_j^ν are given by equations (7.34-7.36).

v) Ring solitons. The multiple ring solitons are solutions that are not equal to zero at some closed curves and decay exponentially away from the closed curves. A possible selection is

$$\rho_j(\underline{\xi},\tau) = \tilde{A}_j \exp(-\tilde{B}_j f_j(R_j(\underline{\xi},\tau))), \tag{8.8a}$$

$$\chi_j^i(\underline{\xi},\tau) = A_j^i \exp(-B_j^i f_j(\tilde{R}_j(\underline{\xi},\tau))), \quad for \quad \text{i=1, 2, 3}, \tag{8.8b}$$

$$\tilde{\vartheta}_j = \delta_j^{\nu} = 0 \quad for \quad \text{j=1 .. N}. \tag{8.8c}$$

where

$$\tilde{R}_j = \sqrt{\tilde{\gamma}_j^2 \left(\xi - \tilde{V}_j\tau\right)^2 + \eta^2 + \varsigma^2}, \quad R_j = \sqrt{\gamma_j^2 \left(\xi - V_j\tau\right)^2 + \eta^2 + \varsigma^2}, \tag{8.9a}$$

$$f_j(R_j) = (R_j - R_{0,j})^2, \quad f_j(\tilde{R}_j) = (R_j - \tilde{R}_{0,j})^2, \tag{8.9b}$$

and A_j, A_j^i, B_j, B_j^i, $R_{0,j}$and $\tilde{R}_{0,j}$ are arbitrary constants. In Fig. 5 we show a collision between two ring solitons for the Higgs field (see (7.43b) and (8.8a) with identical mass $M = 100\ GeV/c^2$, $B_1 = B_2 = M^2$, $P_1 = P_2$=*2000 TeV/c*, $R_1 = 6/M, R_2 = 12/M$): the initial condition is showed in Fig. 5a, then the two ring solitons collide (Fig. 5b) and then separate (Fig. 5c). We can see that these solutions preserve their shapes but with a phase shift.

vi) Instantons. If we choose a decaying function of time, we obtain also multiple instanton solutions, for example,

$$\rho_j(\underline{\xi},\tau) = \tilde{A}_j \exp\left[-\tilde{B}_j\tilde{\gamma}_j\left(\xi - \tilde{V}_j\tau\right)\right], \tag{8.10a}$$

$$\chi_j^i(\underline{\xi},\tau) = A_j^i \exp\left[-B_j^i\gamma_j\left(\xi - V_j\tau\right)\right], \quad for \quad \text{i=1, 2, 3}, \tag{8.10b}$$

$$\tilde{\vartheta}_j = \delta_j^{\nu} = 0 \quad for \quad \text{j=1 .. N}, \tag{8.10c}$$

where $\tilde{A}_j$, A_j^i, $\tilde{B}_j$, B_j^i are arbitrary constants.

vii) Moving breather-like structures. Finally, if we choose some types of periodic functions of time in the above mentioned solutions, then we obtain breathers. For example, we take

$$\rho_j(\underline{\xi},\tau) = \tilde{A}_j \cos(\tilde{\gamma}_j\left(\xi - \tilde{V}_j\tau\right)) \exp\left[-\tilde{B}_j\sqrt{\tilde{\gamma}_j^2\left(\xi - \tilde{V}_j\tau\right)^2 + \eta^2 + \varsigma^2}\right], \tag{8.11a}$$

$$\chi_j^i(\underline{\xi},\tau) = A_j^i \cos(\gamma_j\left(\xi - V_j\tau\right)) \exp\left[-B_j^i\sqrt{\gamma_j^2\left(\xi - V_j\tau\right)^2 + \eta^2 + \varsigma^2}\right], \quad for \quad \text{i=1, 2, 3}, \tag{8.11b}$$

$$\tilde{\vartheta}_j = \delta_j^{\nu} = 0 \quad for \quad \text{j=1 .. N}, \tag{8.11c}$$

where A_j, A_j^i, $\tilde{B}_j$, $B_j^i$are arbitrary constants.

9. Chaotic and Fractal Solutions

i) Chaotic-chaotic and chaotic-periodic patterns. If we select at least one of the arbitrary functions of Section 7 in order to contain some chaotic solutions of nonintegrable equations, then we obtain some type of space-time chaotic patterns, the so-called chaotic-chaotic (in all spatial directions) patterns. For example, we choose the arbitrary function as solution of the chaotic Lorenz system

$$X_T = -c(X-Y), \quad Y_T = X(a-Z)-Y, \quad Z_T = XY - bZ, \tag{9.1a}$$

with$a = 60$, $b = 8/3$, $c = 10$,or of the R**ö**ssler system

$$X_T = -Y-Z, \quad Y_T = X + aY, \quad Z_T = b + Z(X-c), \tag{9.1b}$$

with $a = 0.15$, $b = 0.2$, $c = 10$and $T = \gamma(\xi - V\tau)$ (or $T = \eta$ or $T = \varsigma$). A phase and amplitude chaotic-chaotic pattern is given by

$$\rho_j(\underline{\xi},\tau) = X(\tilde{\gamma}_j(\xi - \tilde{V}_j\tau))Y(\eta)Z(\zeta), \tag{9.2a}$$

$$\chi_j^i(\underline{\xi},\tau) = X_j^i(\gamma_j(\xi - V_j\tau))Y_j^i(\eta)Z_j^i(\zeta), \quad for \quad \text{i=1, 2, 3,} \quad \chi_j^0 = \frac{\underline{K}_j\cdot\underline{\chi}_j}{\omega_j} \tag{9.2b}$$

$$\vartheta_j = \tilde{\delta}_j^{\nu} = 0 \quad for \quad \text{j=1, 2....N}, \tag{9.2c}$$

while ϑ_j and δ_j^{ν} is given by equations (7.34-7.36). An example is given in Fig. 6, for the Higgs field ((7.43b) and (9.2a)) with $M = 100\ GeV/c^2$ and $\gamma = 50$.

On the contrary, we obtain a phase chaotic-chaotic pattern, if we choose the function (9.2c) as solution of the Lorenz system. Finally, if we select a chaotic-periodic solution which is chaotic in one (or two) direction and periodic in the other(s) direction(s). then we obtain the so-called chaotic-periodic patterns.

ii) Chaotic line soliton solutions

If we consider the soliton line solution (8.3), we can easily deduce a chaotic solution when we select A_j as solution of the Lorenz system,

$$\rho_j(\underline{\xi},\tau) = \frac{2A_j(\underline{\xi},\tau)}{ch\left(2\tilde{\gamma}_j(\xi - \tilde{V}_j\tau)A_j(\underline{\xi},\tau)\right)}, \quad \vartheta_j = \tilde{\delta}_j^{\nu} = 0 \quad for \quad \text{j=1 .. N}, \tag{9.3a}$$

$$\chi_j^i(\underline{\xi},\tau) = \frac{2A_j^i(\underline{\xi},\tau)}{ch\left(2\gamma_j(\xi - V_j\tau)A_j^i(\underline{\xi},\tau)\right)}, \quad for \quad \text{i=1, 2, 3,} \quad \chi_j^0 = \frac{\underline{K}_j\cdot\underline{\chi}_j}{\omega_j} \tag{9.3b}$$

where the phases ϑ_j and δ_j^{ν} are given as usual by (7.34-7.36), for $j = 1...N$, and the functions $A_j^i = A_j^i(T_j) = A_j^i(\gamma_j(\underline{\xi} - V_j\tau))$ satisfy the third order ordinary differential equation equivalent to the Lorenz system (9.1),

$$\begin{aligned} &A_{j,TTT} + (b+c+1)A_{j,TT} + (bc+b+A_j^2)A_{j,T} + c(b-ab+A_j^2)A_j \\ &-\tfrac{A_{j,TT}A_{j,T}+(c+1)A_{j,T}^2}{A_j} = 0. \end{aligned} \tag{9.4}$$

iii) Chaotic dromion and lump patterns

If we consider the dromion solution (8.4), we can transform it into a chaotic pattern with an appropriate choice for A_j and/or B_j,

$$\rho_j(\underline{\xi},\tau) = A_j \exp(-B_j\sqrt{\tilde{\gamma}_j^2\left(\xi-\tilde{V}_j\tau\right)^2+\eta^2+\varsigma^2}), \tag{9.5a}$$

$$\chi_j^i(\underline{\xi},\tau) = A_j^i \exp\left(-B_j^i\sqrt{\gamma_j^2\left(\xi-V_j\tau\right)^2+\eta^2+\varsigma^2}\right), \quad for \quad \text{i=1, 2, 3,} \quad \chi_j^0 = \frac{\underline{K}_j\cdot\underline{\chi}_j}{\omega_j} \tag{9.5b}$$

$$\vartheta_j = \tilde{\delta}_j^\nu = 0 \quad for \quad \text{j=1, 2....N,} \tag{9.5c}$$

where the function $A_j = A_j(T_j) = A_j(\gamma_j(\xi - \tilde{V}_j\tau))$ and/or the other amplitude and shape functions $B_j = B_j(T_j) = B_j(\tilde{\gamma}_j(\xi - \tilde{V}_j\tau))$,for the scalar field, and $A_j^i = A_j^i(T_j) = A_j^i(\gamma_j(\xi - V_j\tau))$ and$B_j^i = B_j^i(T_j) = B_j^i(\gamma_j(\xi - V_j\tau))$ for the vectorial field are solutions of the Lorenz equation (9.4) and $\vartheta_j$and δ_j^ν are given by equations (7.34-7.36). We obtain an amplitude (A_j chaotic) or a shape (B_j chaotic) or an amplitude and shape (A_j and B_j chaotic) dromion chaotic pattern. Similar considerations can be applied to the lump solutions (8.7).

iv) Nonlocal fractal solutions. If we choose

$$\rho_j(\underline{\xi},\tau) = \prod_{m=1}^{3} \tilde{T}_{m,j}\left|\tilde{T}_{m,j}\right|\left\{\sin\left[\ln\left(\tilde{T}_{m,j}^2\right)\right] - \cos\left[\ln\left(\tilde{T}_{m,j}^2\right)\right]\right\} \tag{9.6a}$$

$$\chi_j^i(\underline{\xi},\tau) = \prod_{m=1}^{3} T_{m,j}\left|T_{m,j}\right|\left\{\sin\left[\ln\left(T_{m,j}^2\right)\right] - \cos\left[\ln\left(T_{m,j}^2\right)\right]\right\}, \quad for \quad \text{i=1, 2, 3,} \quad \chi_j^0 = \frac{\underline{K}_j\cdot\underline{\chi}_j}{\omega_j} \tag{9.6b}$$

with $\underline{\tilde{T}}_j = (\tilde{T}_{1,j}, \tilde{T}_{2,j}, \tilde{T}_{3,j})$, $\underline{\tilde{T}}_j = (\tilde{\gamma}_j(\xi - \tilde{V}_j\tau), \eta, \varsigma)$, $\underline{T}_j = (T_{1,j}, T_{2,j}, T_{3,j})$, $\underline{T}_j = (\gamma_j(\xi - V_j\tau), \eta, \varsigma)$we get a nonlocal fractal structure for small $\underline{T}_j$and $\underline{\tilde{T}}_j$. It is well known that if we plot the structure of the solution at smaller regions we can obtain the same structures.

v) Fractal dromion and lump solutions. A fractal dromion (lump) solution is exponentially (algebraically) localized in large scale and possesses self-similar structure near the center of the dromion. We consider for example an amplitude fractal dromion

$$\rho_j(\underline{\xi},\tau) = \tilde{A}_j \exp(-\tilde{B}_j\sqrt{\tilde{\gamma}_j^2\left(\xi-\tilde{V}_j\tau\right)^2+\eta^2+\varsigma^2}), \tag{9.7a}$$

$$\chi_j^i(\underline{\xi},\tau) = A_j \exp(-B_j\sqrt{\gamma_j^2\left(\xi-V_j\tau\right)^2+\eta^2+\varsigma^2}), \quad for \quad \text{i=1, 2, 3,} \quad \chi_j^0 = \frac{\underline{K}_j\cdot\underline{\chi}_j}{\omega_j} \tag{9.7b}$$

$$\vartheta_j = \tilde{\delta}_j^\nu = 0 \quad for \quad \text{j=1, 2....N,} \tag{9.7c}$$

where ϑ_j and δ_j^ν are given by equations (7.34-7.36) and $A_j^i = A_j^i(T_j) = A_j^i(\gamma_j(\xi - V_j\tau))$, $A_j = A_j(T_j) = A_j(\tilde{\gamma}_j(\xi - \tilde{V}_j\tau))$ are given by

$$A_j^i(T_j) = A_j(T_j) = 2 + \sin\left\{\ln\left[T_j^2\right]\right\}. \tag{9.8}$$

By a similar choice for B_j or $\tilde{B}_j$, $\tilde{\delta}_j$, $\tilde{\vartheta}_j$we obtain shape or phase fractal dromion.

vi) Stochastic fractal dromion and lump excitations. It is well known the stochastic fractal property of the continuous but nowhere differentiable Weierstrass function

$$W(x) = \sum_{k=1}^{N} (c_1)^k \sin\left[(c_2)^k x\right], \quad N \to \infty, \tag{9.9a}$$

with c_2 odd and

$$c_1 c_2 > 1 + \frac{3\pi}{2}. \tag{9.9b}$$

A stochastic fractal solution is

$$\rho_j(\underline{\xi}, \tau) = \tilde{A}_{1,j}(\tilde{\gamma}_j(\xi - \tilde{V}_j\tau))\tilde{A}_{2,j}(\eta)\tilde{A}_{3,j}(\varsigma), \tag{9.10a}$$

$$\chi_j^i(\underline{\xi}, \tau) = A_{1,j}^i(\gamma_j(\xi - V_j\tau))A_{2,j}^i(\eta)A_{3.j}^i(\varsigma), \quad for \quad \text{i=1, 2, 3}, \quad \chi_j^0 = \frac{\underline{K}_j \cdot \underline{\chi}_j}{\omega_j} \tag{9.10b}$$

$$\vartheta_j = \tilde{\delta}_j^\nu = 0 \quad for \quad \text{j=1, 2....N}, \tag{9.10c}$$

where ϑ_j and δ_j^ν are as usual given by equations (7.34-7.36), $\underline{A}_j = (A_{1,j}, A_{2,j}, A_{3,j})$, $\underline{\tilde{A}}_j = (\tilde{A}_{1,j}, \tilde{A}_{2,j}, \tilde{A}_{3,j})$, and $\underline{A}_j = \underline{A}_j(\gamma_j(\xi - V_j\tau), \eta, \varsigma)$, $\underline{\tilde{A}}_j = \underline{\tilde{A}}_j(\tilde{\gamma}_j(\xi - \tilde{V}_j\tau), \eta, \varsigma)$ are given by

$$A_{1,j}(\gamma_j(\xi - V_j\tau)) = A_{1,j}(\gamma_j(\xi - V_j\tau)) = W(\gamma_j(\xi - V_j\tau)) + \gamma_j^2(\xi - V_j\tau)^2, \tag{9.11a}$$

$$A_{2,j}(\eta) = \tilde{A}_{2,j}(\eta) = W(\eta) + (\eta)^2 \quad , \quad A_{3,j}(\varsigma) = \tilde{A}_{3,j}(\varsigma) = W(\varsigma) + (\varsigma)^2. \tag{9.11b}$$

An example is given in Fig. 4 for the Higgs field (7.43b) and (9.10a) with $M = 100$ GeV/c^2 and $\gamma = 50$.

vii) Stochastic fractal dromion and lump excitations. In order to obtain a stochastic amplitude fractal dromion we choose

$$\rho_j(\underline{\xi}, \tau) = \tilde{A}_j \exp(-\tilde{B}_j\sqrt{\tilde{\gamma}_j^2\left(\xi - \tilde{V}_j\tau\right)^2 + \eta^2 + \varsigma^2}), \tag{9.12a}$$

$$\chi_j^i(\underline{\xi}, \tau) = A_j^i \exp(-B_j^i\sqrt{\gamma_j^2(\xi - V_j\tau)^2 + \eta^2 + \varsigma^2}), \quad for \quad \text{i=1, 2, 3}, \quad \chi_j^0 = \frac{\underline{K}_j \cdot \underline{\chi}_j}{\omega_j} \tag{9.12b}$$

$$\vartheta_j = \tilde{\delta}_j^\nu = 0 \quad for \quad \text{j=1, 2....N}, \tag{9.12c}$$

where ϑ_j and δ_j^ν are given by equations (7.34-7.36) and $A_j^i = A_j^i(T_j) = A_j^i(\gamma_j(\xi - V_j\tau))$, $\tilde{A}_j = \tilde{A}_j(T_j) = \tilde{A}_j(\tilde{\gamma}_j(\xi - \tilde{V}_j\tau))$ are given by

$$A_j^i(T_j) = \tilde{A}_j(T_j) = W(T_j) + T_j^2, \tag{9.13}$$

By similar methods we obtain shape or phase stochastic fractal dromion as well as stochastic fractal lump solutions.

10. Conclusion

Many extensions of the work exposed in the precedent sections are possible, for example the investigation of nonlinear equations with solitons transporting superluminal signals [60], a simple technique for obtaining nonlinear equations with dromions of a given shape and velocity [61] and a modification of the Einstein general relativity equations that can produce various types of coherent solutions [64].

However, a major problem is the possibility of identification between dromions and elementary particles and indeed de Broglie [9], Bohm [8] and others ([71],[72],[73],[74]) hoped for the explanation of quantum mechanics through nonlinear classic effects.

Notably among others the Skyrme model ([77],[78],[79],[2]) describes nucleons and nucleon-nucleon interactions, while topological solitons give rise to quantization of charges. A localized and stable wave might be a good model for elementary, but we have seen that in nonlinear field equations there is a great variety of coherent solutions and chaotic and fractal patterns. If particles are excitations of nonlinear fields, it is clear that they are not the only possible excitations.

On the contrary, the quantization of the nonlinear solutions is complicated because there is no superposition principle. For example the shape of the dromion cannot be considered the shape of the wave function for the reason that a quantum soliton cannot be localized in space all the time and the uncertainty principle will cause a spreading. In the last years many methods have been proposed in order to realize the quantization that however seems to be possible in a satisfactory way only for weak nonlinear couplings.

In the next future, an exciting field of research will be the investigation of the physical interpretation of coherent, chaotic and fractal solutions in elementary particles physics. It is necessary to study further the behavior of the solutions, beyond the leading order in the expansion parameter, as well as the derivation of the model equations for the interactions among phase resonant waves.

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Figure Captions

Fig. 1: A ring soliton. The initial condition is represented in Fig. 1a, then the two coherent solutions undergo a collision (Fig. 1b) and separate (Fig. 1c). Note that the z-variable has been suppressed in order to construct a more clear solution representation.

Fig. 2: An amplitude chaotic-chaotic pattern. Note that the z-variable has been suppressed in order to construct a more clear solution picture. Surface plot is shown in the region X = [-100, 100], Y= [-100, 100] .

Fig. 3: A stochastic fractal solution with the Weierstrass function. Note that the z-variable has been suppressed in order to construct a more clear solution picture. Surface plot is shown in the region X = [-0.21, 0.21], Y = [-0.21, 0.21].

Fig. 4: Evolution plots of two dromions with identical shapes and amplitudes. Note that the z-variable has been suppressed in order to construct a more clear solution picture. The initial condition is represented in Fig. 4a, then the two dromions undergo a collision (Fig. 4b) and separate (Fig. 4c).

Fig. 5: A ring soliton. The initial condition is represented in Fig. 5a, then the two coherent solutions undergo a collision (Fig. 5b) and separate (Fig. 5c). Note that the z-variable has been suppressed in order to construct a more clear solution representation.

Fig. 6: An amplitude chaotic-chaotic pattern. Note that the z-variable has been suppressed in order to construct a more clear solution picture. Surface plot is shown in the XY-region defined by X = [-100, 100], Y= [-100, 100] .

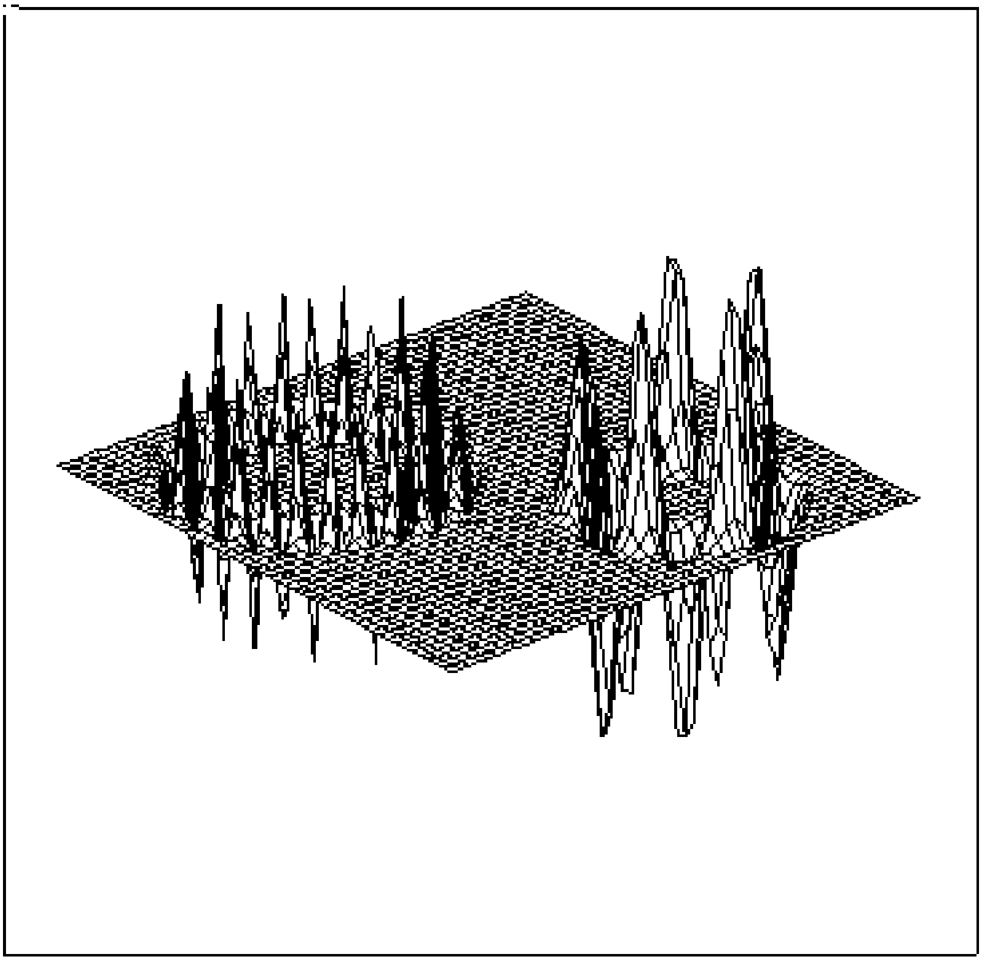

Figure 1a

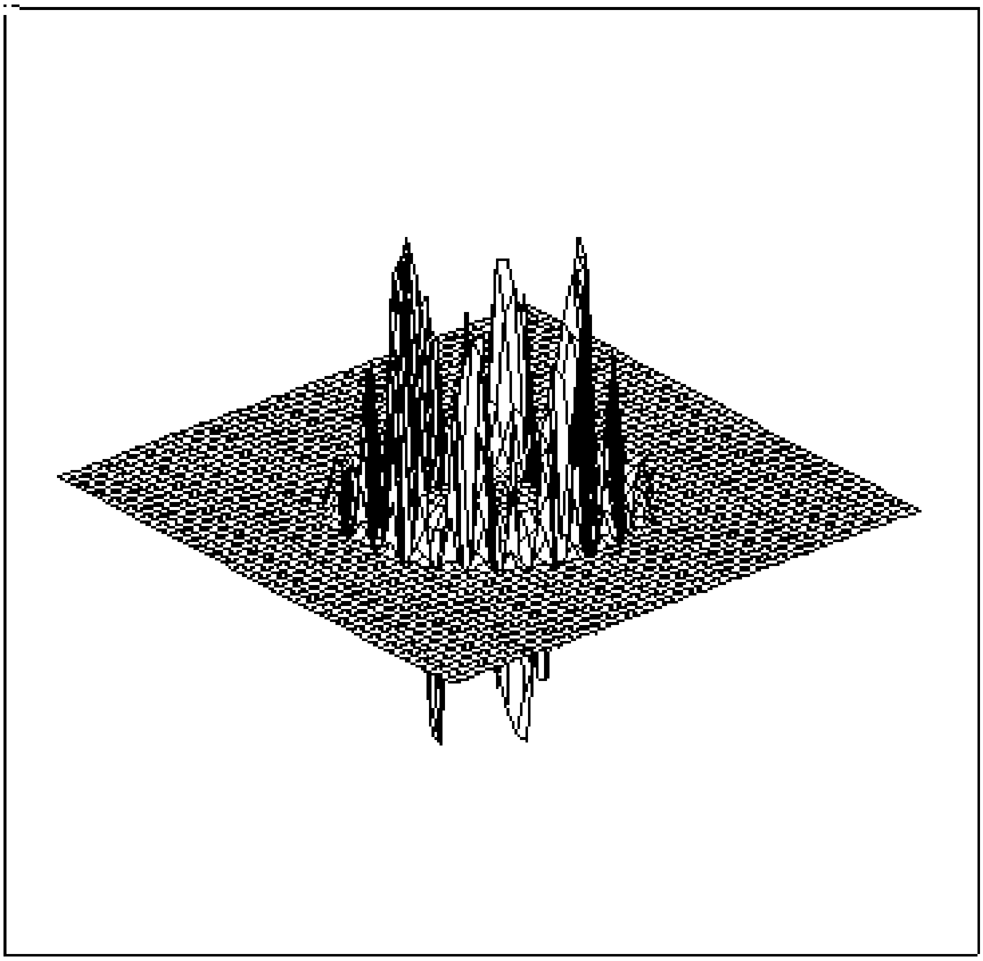

Figure 1b

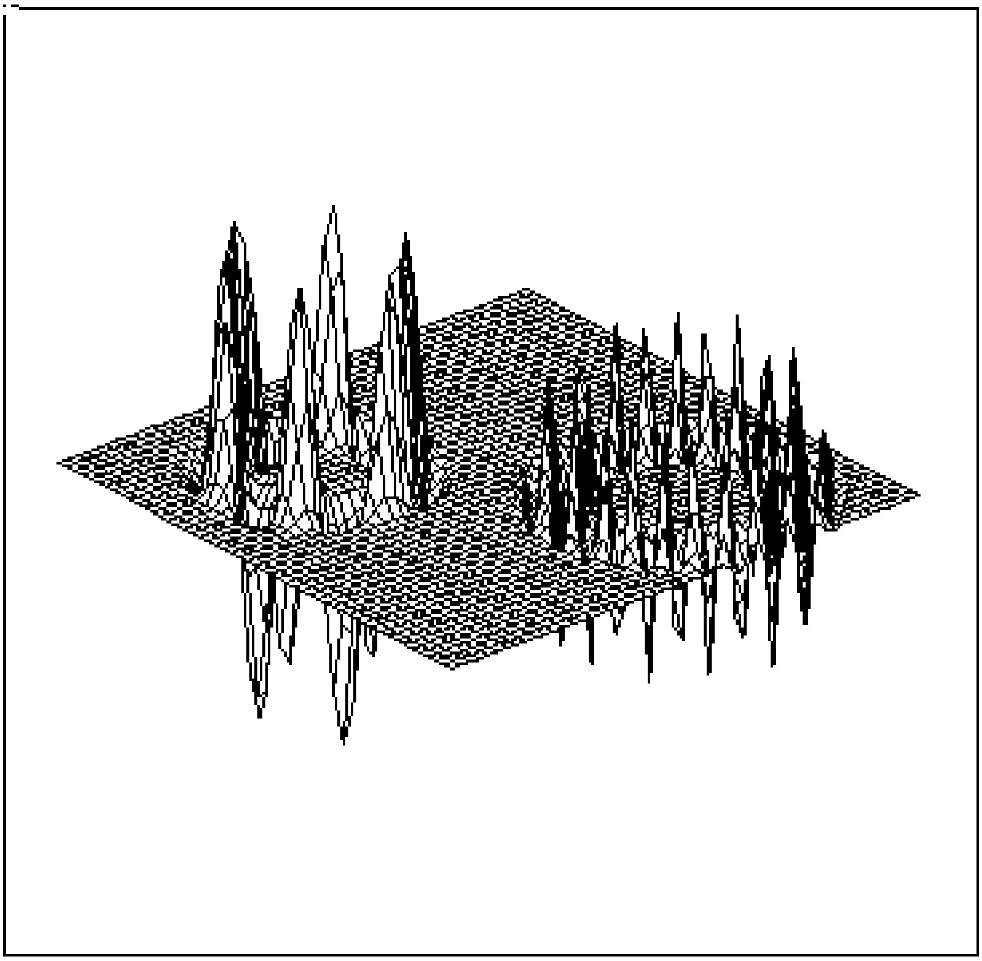

Figure 1c

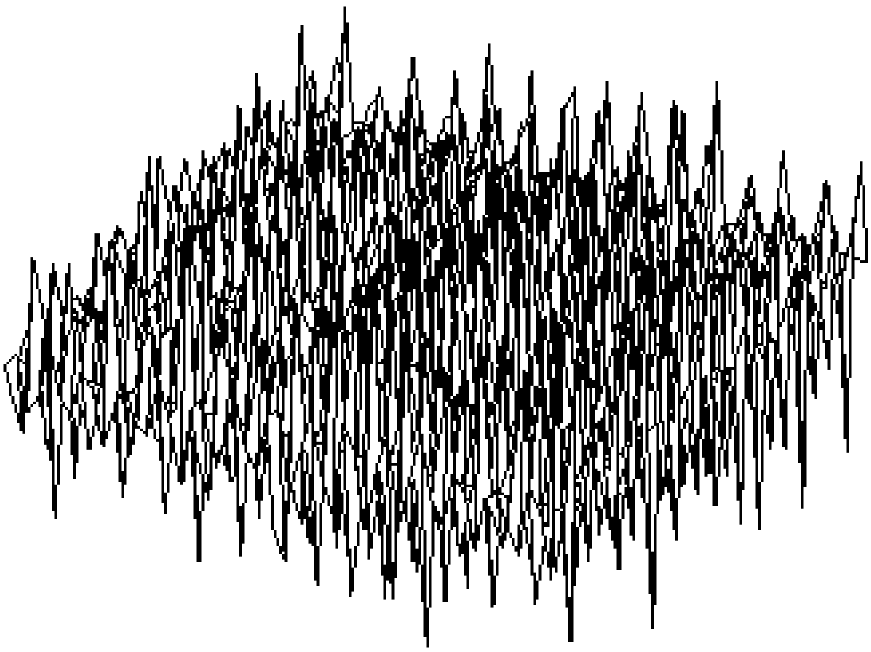

Figure 2

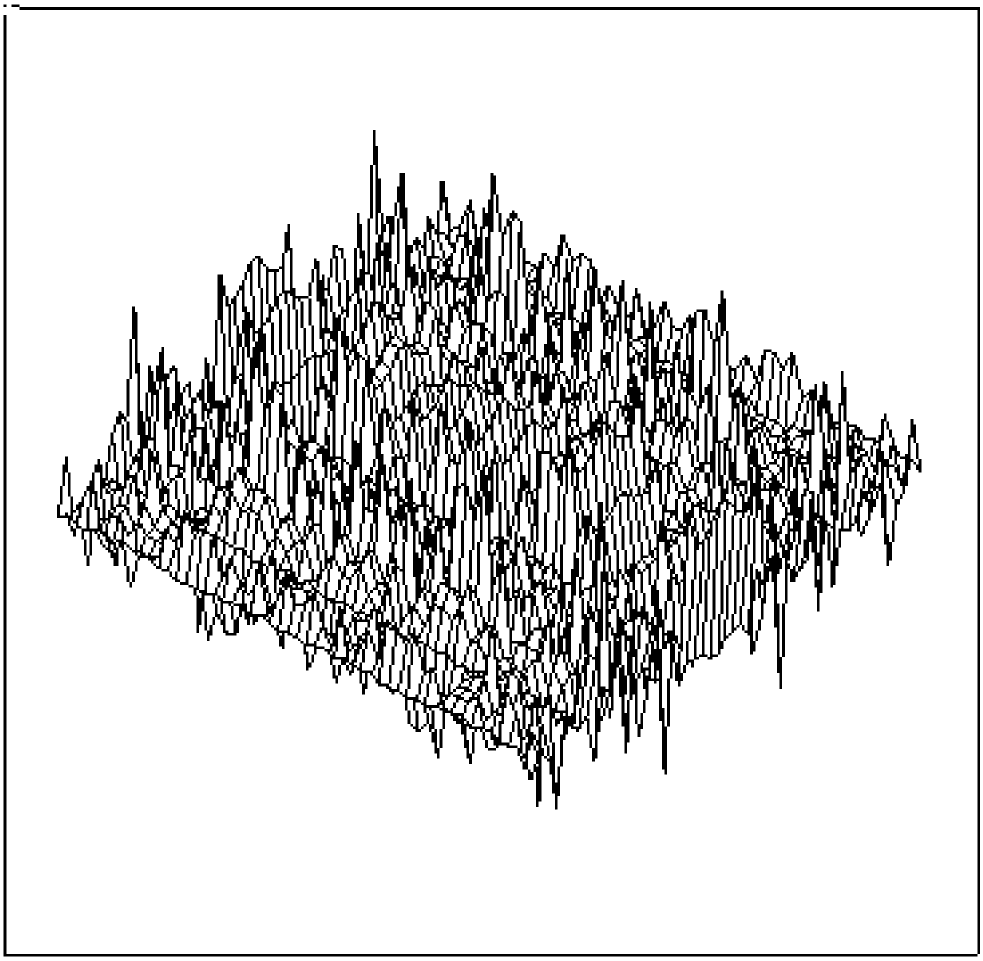

Figure 3

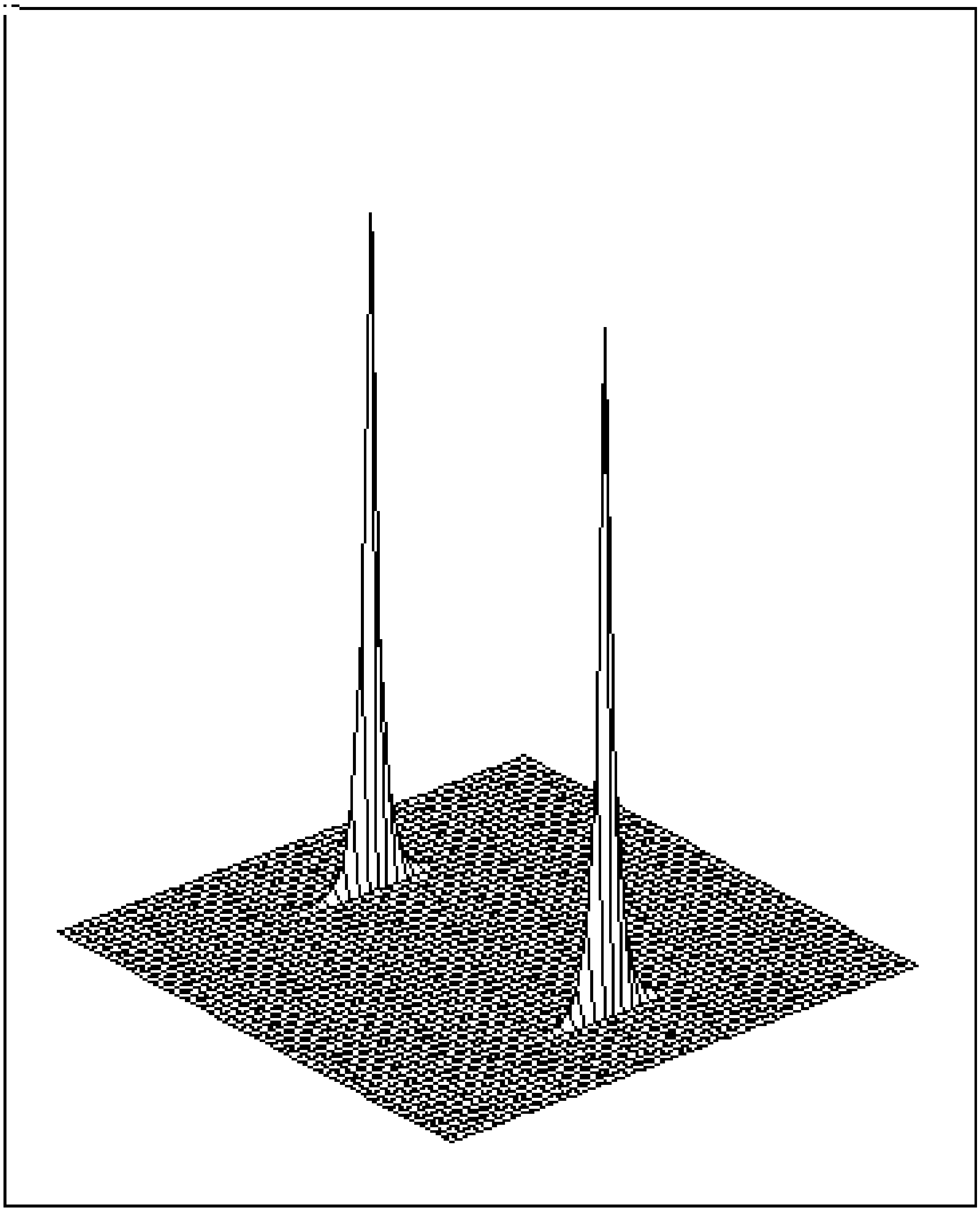

Figure 4a

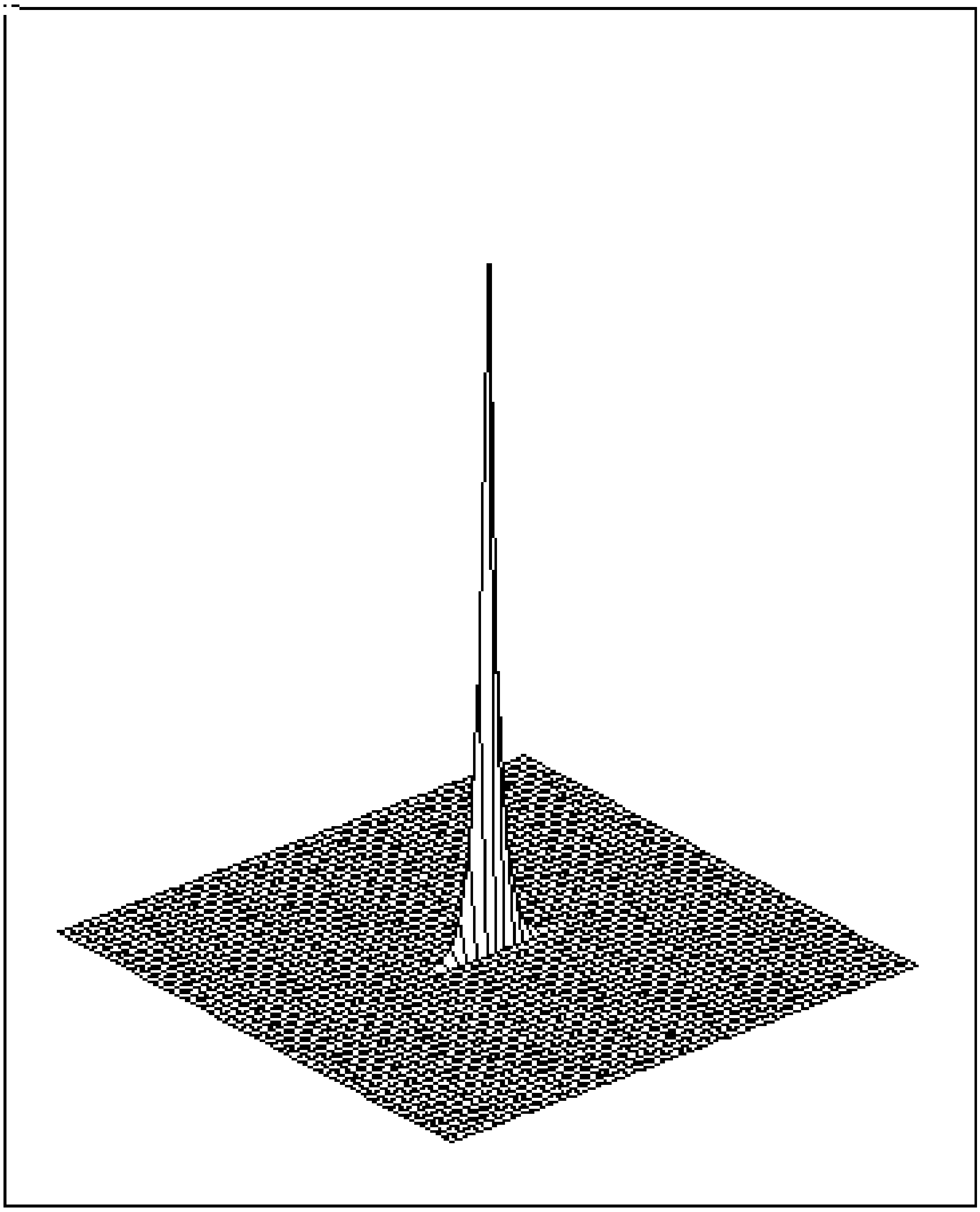

Figure 4b

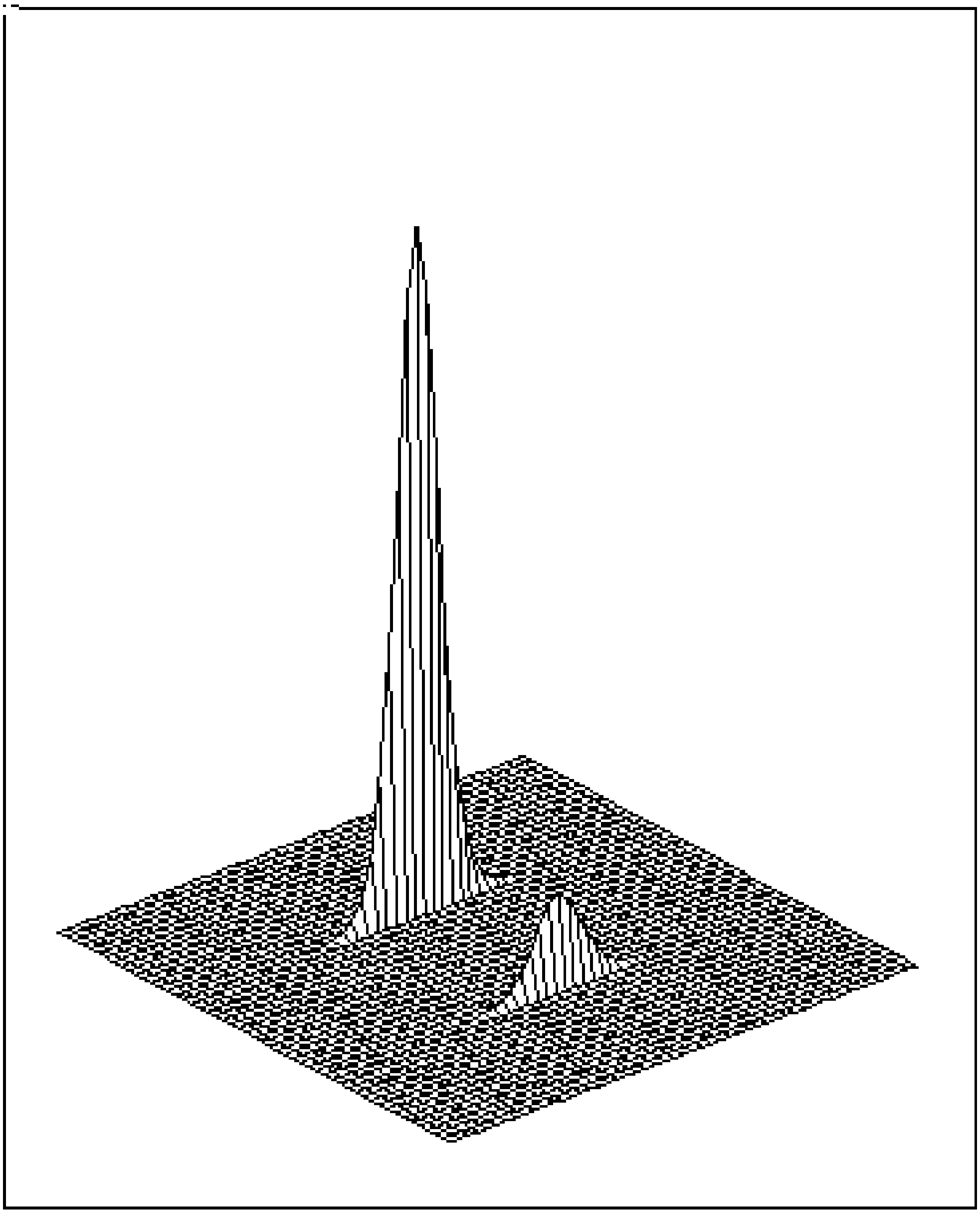

Figure 4c

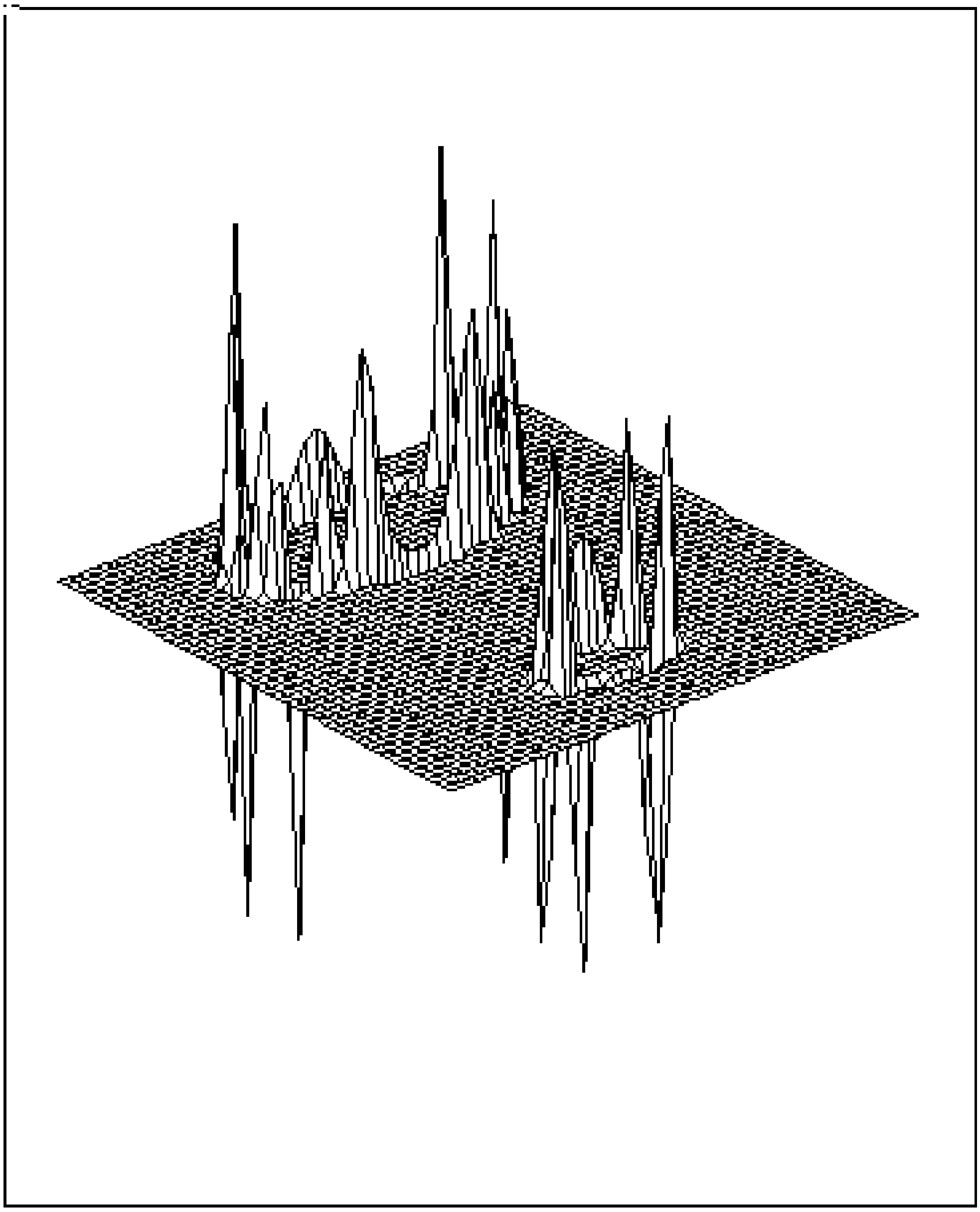

Figure 5a

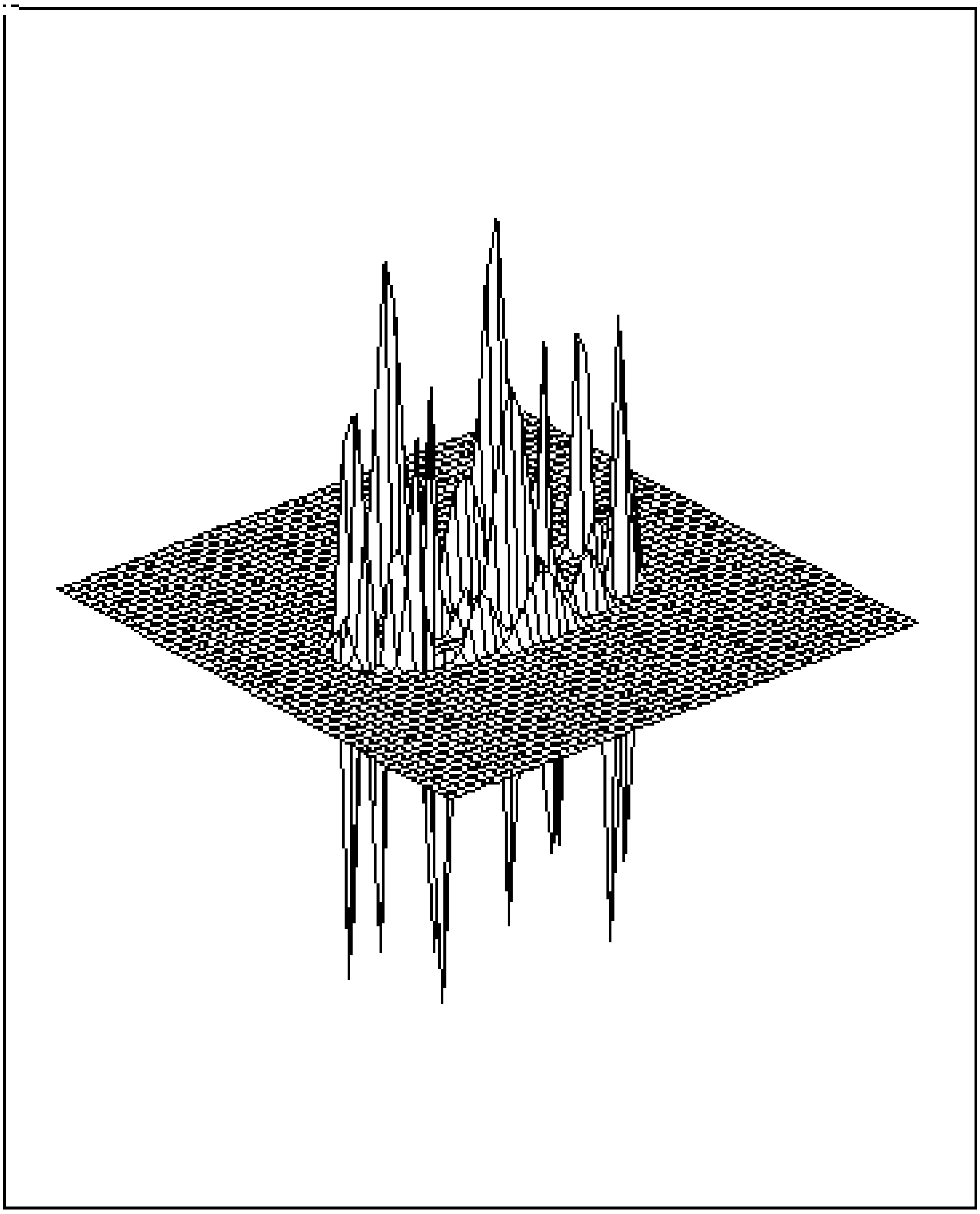

Figure 5b

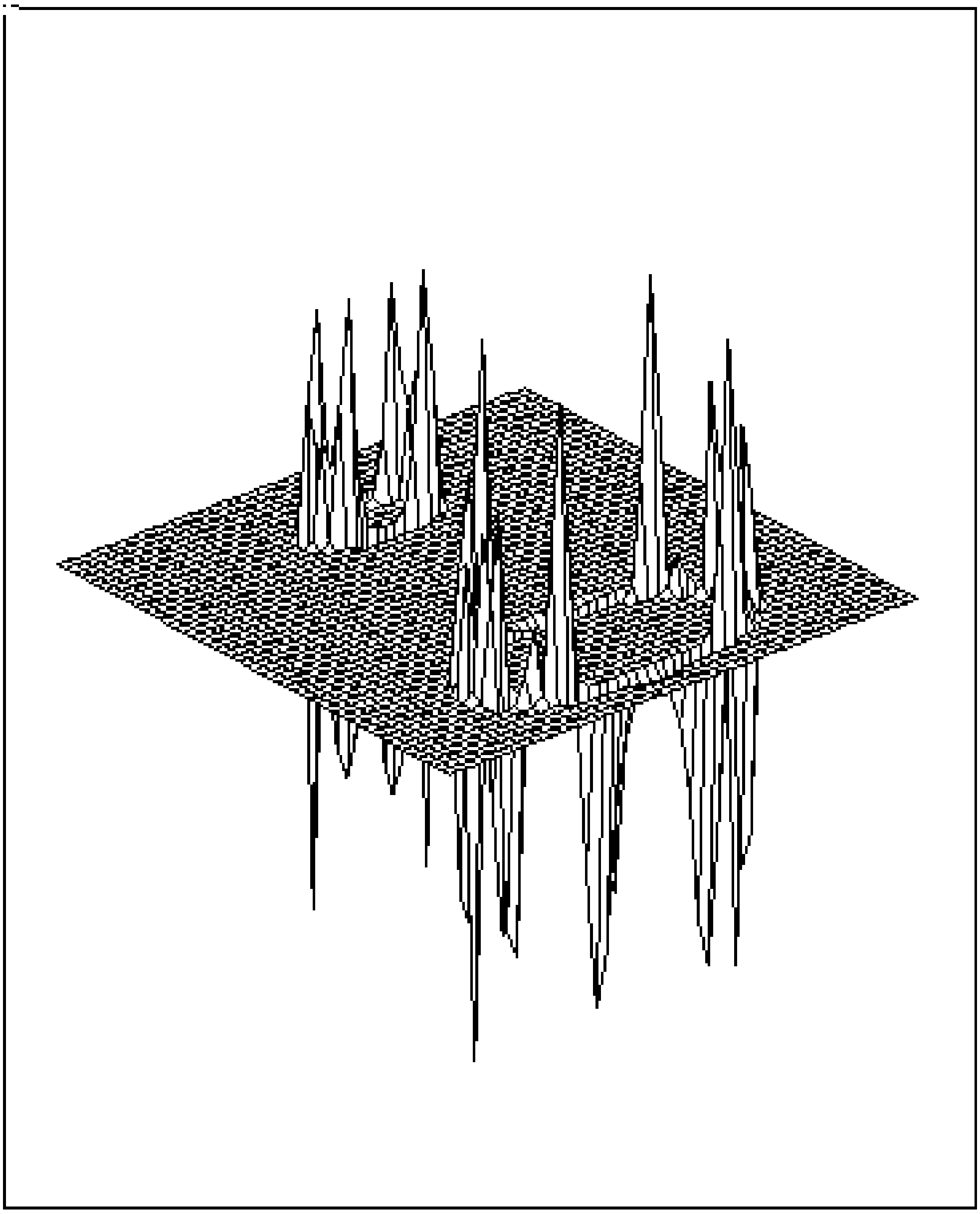

Figure 5c

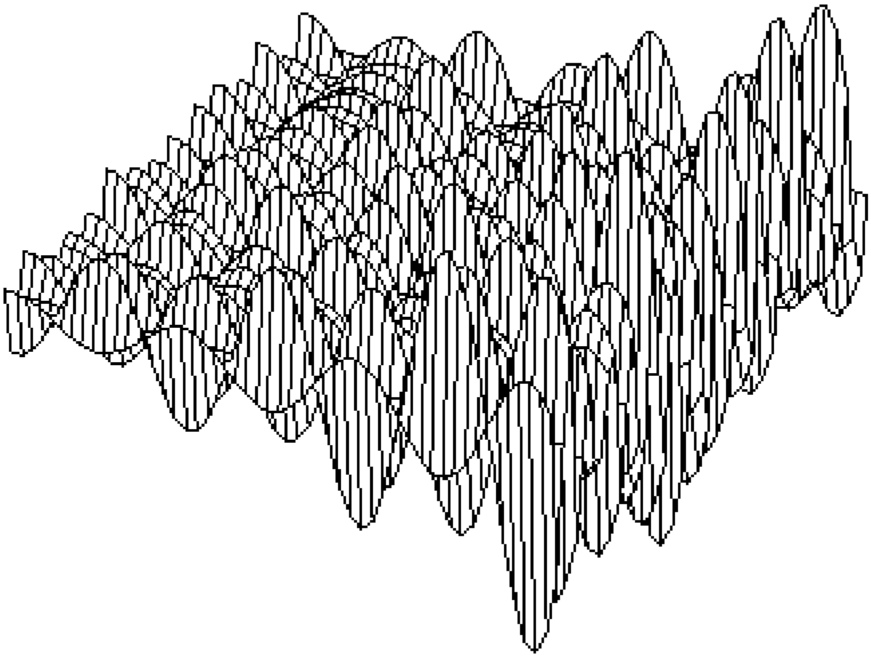

Figure 6