Wave Equations, Renormalization and Meaning of the Planck's Mass: Some Qualitative Considerations

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Abstract: The five-dimensional version of the quantum relativistic Klein-Gordon wave equation is assumed to be a more fundamental description for the dynamics of the single particle without spin. The meaning of the renormalization procedure in QFT and the Planck's mass one are briefly discussed from this point of view.

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1. Introduction

The formulation of relativistically covariant wave equations was one of the first decisive steps towards the (not yet reached) unification of quantum mechanics and relativity. Maiorana has given various important, and may be not completely understood to date, contributes to this subject (refs. 1, 2, 3, 4). Therefore, it seems right to speak about the wave equations in a publication dedicated to his memory. We will express some elementary qualitative considerations on the relationship between this classical subject and some more recent questions of the modern quantum field theory (QFT) and the quantum gravity.

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2. The Klein-Gordon Equation

The prototype of the quantum-relativistic wave equations consists of well known four-dimensional Klein-Gordon equation (KG4):

$$\left[\hbar^2\Box^2 + m^2c^2\right]\psi(x_0, x_1, x_2, x_3) = 0 \tag{1}$$

where m is the particle mass. We will limit our line of reasoning to this particular wave equation, since it will be immediately applicable to any other similar equation.

The KG4 is a quantum translation of the ordinary dispersion law for free relativistic particles and then it is valid for this kind of particles. It may be derived from the general pentadimensional Klein-Gordon equation (KG5) :

$$\left[\hbar^2\Box^2 - (\hbar/c)^2\partial_\tau^2\right]\varphi(x_0, x_1, x_2, x_3, \tau) = 0 \tag{2}$$

if

$$\varphi(x_0, x_1, x_2, x_3, \tau) = \psi(x_0, x_1, x_2, x_3)e^{\frac{-imc^2\tau}{\hbar}} \tag{3}$$

in which τ is a scalar of Universe. In general, the KG5 does not satisfy the usual dispersion law for the free relativistic particle and is, therefore, applicable to real interacting particles or to virtual particles.

If one defines the mass operator as $i(\hbar/c^2)\partial_\tau$, the wavefunctions represented by the eq.(3) are eigenfunctions of this operator with eigenvalue m. These eigenfunctions satisfy the eq.(1) and are associated with free real particles having a definite mass.

A generic linear superposition:

$$\varphi(x_0, x_1, x_2, x_3, \tau) = \sum_j \psi_{m_j}(x_0, x_1, x_2, x_3)e^{\frac{-im_jc^2\tau}{\hbar}} \tag{4}$$

will be associated with a particle having a non definite mass (real interacting particle or virtual particle). The Fourier analysis of the (4) gives the following relationship between the dispersion of the mass, Δm and the dispersion of the τ variable, $\Delta\tau$:

$$(\Delta mc^2)(\Delta\tau) \approx \hbar \tag{5}$$

In the case of the real interacting particles, considered only within the limits of the interaction area, the dispersion Δmc^2 is of the same order as the interaction energy E_{int}, thus one can write $E_{int}(\Delta\tau) \approx \hbar$. Therefore, if we assume the KG5 to be the fundamental equation, the KG4 is obtained as approximation in the limit $(\Delta m)/m << 1$, that is $E_{int} << mc^2$. In other terms, the KG4 is an approximation which is no more valid when the energy exchanged with other fields or particles during the interaction is sufficiently high to create one or more copies of the particle under consideration.

The parameter τ must not be considered as a fifth spacetime coordinate; it admits an immediate physical interpretation in the ordinary four-dimensional spacetime. In fact, if one assumes the existence of a set of paths $\{x_\mu = x_\mu(\tau)\ ;\ \mu = 0, 1, 2, 3\}$ in the ordinary spacetime, such as $dx_\mu/cd\tau = \gamma_\mu$ (so called Breit equation), the KG5 takes the form:

$$(iD_\tau)^2\varphi(x_0, x_1, x_2, x_3, \tau) = 0 \tag{6}$$

which is a constraint on the variation of φ along these paths. This is if the quadrivelocities (Dirac matrices) $\gamma_\mu$don't depend explicitly on τ. D is the operator of total derivation.

Therefore, these paths consist in series of infinitesimal "jumps" at light speed along the spacetime, a sort of generalization of the Zitterbewegung.

The paths orientation along the t axis is not, in general, definite. In other terms, both the past and the future lightcones having their origin in an arbitrarily chosen point on a certain path, do not always contain portions of that path. But, if the eq. (3) is valid:

$$\frac{d\tau}{dt} = [\tau, H] + \partial_t \tau = \partial_t \tau = \left(\frac{dx_0}{cd\tau} \right)^{-1} = \gamma_0 \tag{7}$$

since in this case H is the ordinary Klein-Gordon Hamiltonian, which not depend on τ. Consequently, $t = \gamma_0 \tau + t_0$, that is, the paths have a fixed time orientation given by the sign of the eigenvalue of γ_0. So we arrive to the conclusion: in respect of the interactions localizing the particle with an uncertainty greater than $\hbar$/mc (that is for interaction of energies $E_{int} << mc^2$), the particle is a time orientated process, its dynamics being described by the KG4. Otherwise it is not a t-orientated process (creations and annihilations of the particle take place, which diffuse it along the spatial radius $\hbar$/mc) and its dynamics is plausibly described by the KG5.

The equations (1), (2) may be immediately generalized to a curve spacetime, by expressing the Dalembertian operator in the appropriate coordinates. If one admits that the gravitation is described by the spacetime curvature, as it is, for example, in the general relativity, one has immediately a description of the gravitational field effect on the particle. This effect is manifested even for m = 0, as in any metric theory of gravitation. Nevertheless, in the terms of the KG5 the gravitation is merely the dependence of the quadrivelocities γ_μ on x_μ.

3. Renormalization: A heuristic Justification

From the above reasoning follows that during the high energy interactions ($E_{int} >> mc^2$) the particle dynamics would be described by using the KG5 (for example, by adding the proper terms describing the interaction with the external field). The formulation of the modern quantum field theory (QFT) starts with a translation of the KG4 into the second quantization formalism. This option enables to assume a definite time orientation for all the interacting particles; after all, the time coordinate used by the observer in order to coordinate the events is t, not τ. This results in necessity to renormalize the calculated physical observables[1]. In fact, at each instant t_0, the particle now appears to be dispersed in the cloud of its various locations at that time t_0 [the values of $x_i(\tau)$, with i = 1, 2, 3, corresponding to the various values of τ for which one has $t(\tau) = t_0$], extended upon a space region of size $\hbar$/mc. Naturally, the charge, the mass and the total quantum

[1] For a discussion about the possibility to treat the problem involving several interacting particles by using a formalism of first quantization, remaining at the ontological level of the KG5, one may see ref. 5.

numbers of the cloud and the single particle ones are the same. In fact, any time when the particle passes through the hyperplane t = constant in the opposite directions, the contributions to these observables elide each other.

An external interaction distorts the "form" of this cloud. Since the only physical reality consists of interaction events with the outside, the physically relevant values must be calculated by subtracting the "unperturbed" cloud from the "distorted" cloud one. Any renormalization procedure is based on this principle. So, the renormalization is far from being an "arbitrary" procedure or one introduced "ad hoc" only in order to arrive to physically defined results, but it is a necessity resulting from the option to have a t-oriented theory. In the terms of the QFT, it consists of subtracting of the unmodified terms, which express the particle "self interaction" by through its own fields, from those modified by the external interaction.

4. Limits of Applicability and Planck's Mass

The particular expression of the limit $E_{int} \approx mc^2$, in which the KG5 is substituted by the KG4, certainly depends on the field that mediates the interaction. For the electrostatic field, this limit is reached when the distance r between two particles of charge e goes below the value r_0 at which the potential energy is equal to the rest energy : $e^2/r_0 = mc^2$. If the interaction is mediated by a photon exchange, the limit is reached when the photon wavelength λ goes below the value λ_0 at which the photon energy is equal to the rest energy: $\hbar c/\lambdabar_0 = mc^2$, where $\lambdabar_0 = \lambda_0/2\pi$. Then, in the case of electromagnetic interaction we have two different values of the collision parameter at which one passes from the KG5 description to the KG4 one: $r_0 = e^2/mc^2$ (the so called classical radius) for the static interaction, $\lambdabar_0 = \hbar/mc$ (the so called Compton wavelength) for the radiative interaction. Their ratio is a fundamental constant of the Universe : the fine structure constant α, and then it can't be changed. Since α is much less than 1, the KG4 is not more valid, in general, at distances comparable to $\lambdabar_0$. In order to describe the dynamics at shorter distances, it is necessary to use the KG5 or, according to the consolidated practice, to pass to QFT formalism by using the second quantization.

Now let's consider the gravitational interaction. One has now the static limit for $Gm^2/r_0 = mc^2$, that is for $r_0 = Gm/c^2$. In the case of graviton exchange the limit is still expressed by $\hbar c/\lambdabar_0 = mc^2$, that is $\lambdabar_0 = \hbar/mc$, since gravitons and photons manifest the same dispersion laws.

The ratio of r_0 and $\lambdabar_0$ is now $Gm^2/\hbar c$ and it depends on m; one has $r_0 = \lambdabar_0$ for $m = M_{Planck} = (\hbar c/G)^{1/2}$. For the mass values less than Planck's mass is $r_0 < \lambdabar_0$, while for masses greater than Planck's mass is $r_0 > \lambdabar_0$. In general, $r_0/\lambdabar_0 = (m/ M_{Planck})^2$.

From this point of view, the Planck's mass is a parameter which controls the passage between two different modes of violation of the KG4. In the terms of this reasoning, the fundamental description of the dynamics is given by the KG5, which is a relativistically invariant, continuous equation. Therefore, do not seem to emerge, for $E_{int} > M_{Planck}c^2$, transitions to finite or discrete geometries with the appearance of quantized spacetime intervals or breaking of the relativistic invariance.

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