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Scattering of an α Particle by a Radioactive Nucleus

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Abstract: *In the following we reproduce, translated into English, a section of Volumetto II, a notebook written by Majorana in 1928 when he was still a Physics student at the University of Rome (see S. Esposito, E. Majorana jr, A. van der Merwe and E. Recami (eds.)* Ettore Majorana: Notes on Theoretical Physics*, Kluwer, New York, 2003). This study was performed by the author when he was preparing his Thesis work on "The Quantum Theory of Radioactive Nuclei" (unpublished), whose supervisor was E. Fermi.*

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Let us consider the emission of an α particle by a radioactive nucleus and assume that such a particle is described by a quasi-stationary wave. As Gamov has shown, after some time this wave scatters at infinity. In other words, the particle spends some time near the nucleus but eventually ends up far from it. We now begin to study the features of such a quasi-stationary wave, and then address the *inverse* of the problem studied by Gamov.[1] Namely, we want to determine the probability that an α particle, colliding with a nucleus that has just undergone an α radioactive transmutation, will be captured by the nucleus so as to reconstruct a nucleus of the element preceding the original one in the radioactive genealogy. This issue has somewhat been addressed by Gudar, although not deeply enough. It is directly related to our hypothesis according to which, under conditions rather different from the ones we are usually concerned with, a process can take place that reconstitutes the radioactive element.

Following Gamov, let us suppose that spherical symmetry is realized, so that the azimuthal quantum of the particle near the nucleus is zero. For simplicity, we neglect for the moment the overall motion of the other nuclear components. The exact formulae will have to take account of that motion, and thus the formulae that we shall now derive will have to be modified; but this does not involve any major difficulty. For the spherically

[1] The author is referring here to G.Gamov, *Z. Phys.* **41** (1928) 204. [N.d.T]

symmetric stationary states, setting, as usual, $\psi = \chi/x$, we shall have

$$\frac{d^2\chi}{dx^2} + \frac{2m}{\hbar^2}(E - U)\,\chi = 0. \tag{1}$$

Beyond a given distance R, which we can assume to be of the order of the atomic dimensions, the potential U practically vanishes. The functions χ will then be symmetric for $E > 0$. For definiteness, we require U to be exactly zero for $x > R$, but it will be clear that no substantial error is really introduced in this way in our calculations. For the time being, let us consider the functions χ to depend only on position, and —as it is allowed— to be real. Furthermore, we use the normalization condition

$$\int_0^R \chi^2\,dx = 1. \tag{2}$$

Let us now imagine that it exists a quasi-stationary state such that it is possible to construct a function u_0 which vanishes for $x > R$, satisfies the constraint

$$\int_0^R |u_0|^2\,dx = 1, \tag{3}$$

and approximately obeys[2] the differential equation (1) at the points where its value is large. This function u_0 will be suited to represent the α particle at the initial time. It is possible to expand it in terms of the functions χ that are obtained by varying E within a limited range. Let us then set

$$E = E_0 + W. \tag{4}$$

The existence of such a quasi-stationary state is revealed by the fact that for $x < R$ the functions χ, normalized according to Eq. (2), and their derivatives are small for small W.

In first approximation, we can set, for $x < R$,

$$\begin{aligned} \chi_W &= \chi_0 + W\,y(x), \\ \chi'_W &= \chi'_0 + W\,y'(x), \end{aligned} \tag{5}$$

and these are valid (as long as U has a reasonable behavior) with great accuracy and for all values of W in the range of interest. In particular, for $x = R$:

$$\begin{aligned} \chi_W(R) &= \chi_0(R) + W\,y(R), \\ \chi'_W(R) &= \chi'_0(R) + W\,y'(R). \end{aligned} \tag{6}$$

[2] For an approximately determined value of t, while being almost real.

Bearing in mind that Eq. (1) simply reduces for $x > R$ to

$$\frac{\mathrm{d}^2\chi_W}{\mathrm{d}x^2} + \frac{2m}{\hbar^2}\,(E_0 + W)\,\chi_W = 0, \tag{7}$$

for $x > R$ we get

$$\begin{aligned} \chi_W &= (a + bW)\,\cos\frac{1}{\hbar}\sqrt{2m(E_0+W)}(x-R) \\ &\quad + (a_1 + b_1 W)\,\sin\frac{1}{\hbar}\sqrt{2m(E_0+W)}(x-R), \end{aligned} \tag{8}$$

having set

$$\begin{aligned} a &= \chi_0(R), \quad b = y(R), \\ a_1 &= \frac{\hbar\,\chi_0'(R)}{\sqrt{2m(E_0+W)}}, \quad b_1 = \frac{\hbar\,y'(R)}{\sqrt{2m(E_0+W)}}. \end{aligned} \tag{9}$$

Note that a_1 and b_1 are not strictly constant but, to the order of approximation for which our problem is determined, we can consider them as constant and replace them with

$$a_1 = \frac{\hbar\,\chi_0'(R)}{\sqrt{2mE_0}}, \quad b_1 = \frac{\hbar\,y'(R)}{\sqrt{2mE_0}}. \tag{10}$$

Moreover, since E_0 is not completely determined, we shall fix it in order to simplify Eq. (8); with this aim, we can shift R by a fraction of wavelength $h/\sqrt{2mE_0}$. It will then be found that Eq. (8) can always be replaced with the simpler one

$$\begin{aligned} \chi_W &= \alpha\,\cos\sqrt{2m(E_0+W)}\,(x-R)/\,\hbar \\ &\quad + \beta W\,\sin\sqrt{2m(E_0+W)}\,(x-R)/\,\hbar. \end{aligned} \tag{11}$$

We set

$$\sqrt{2m(E_0+W)}\,/\,\hbar = \sqrt{2mE_0}\,/\,\hbar + 2\pi\,\gamma = C + 2\pi\,\gamma, \tag{12}$$

and, in first approximation, the following will hold:

$$2\pi\,\gamma \simeq \frac{W}{\hbar\sqrt{2E_0/m}} = \frac{W}{\hbar v}, \tag{13}$$

v being the (average) speed of the emitted α particles. On substituting into Eq. (11), we approximately find

$$\begin{aligned} \chi_W &= \alpha\,\cos(C+2\pi\gamma)(x-R) \\ &\quad + \beta'\,\gamma\,\sin(C+2\pi\gamma)(x-R), \end{aligned} \tag{14}$$

with

$$\beta' \;=\; \beta\, 2\pi\hbar\,\sqrt{2E_0/m}. \tag{15}$$

For the moment, the χ_W functions are normalized as follows:

$$\int_0^R \chi_W^2\,\mathrm{d}x \;=\; 1.$$

We denote by η_W the same eigenfunctions normalized with respect to $\mathrm{d}\gamma$. For $x > R$, we then get

$$\begin{aligned}\eta_W &= \frac{2}{\sqrt{\alpha^2+\beta'^2\gamma^2}}\,[\alpha\,\cos(C+2\pi\gamma)(x-R)\\ &+\,\beta'\,\gamma\,\sin(C+2\pi\gamma)(x-R)] \;=\; \frac{2}{\sqrt{\alpha^2+\beta'^2\gamma^2}}\,\chi_W.\end{aligned} \tag{16}$$

We expand u_0, which represents the α particle at the initial time, as a series in η_W, and get

$$u_0 \;=\; \int_{-\infty}^{\infty} K_\gamma\,\eta_W\,\mathrm{d}\gamma. \tag{17}$$

Now, since $u_0 = \chi_W$ for $x \leq R$ and therefore

$$K_\gamma \;=\; \int_0^{\infty} \eta_W\,u_0\,\mathrm{d}x \;=\; \frac{2}{\sqrt{\alpha^2+\beta'^2\gamma^2}}\int_0^R \chi_W^2\,\mathrm{d}x \;=\; \frac{2}{\sqrt{\alpha^2+\beta'^2\gamma^2}}, \tag{18}$$

on substituting into Eq. (17), we obtain

$$u_0 \;=\; \int_{-\infty}^{\infty} \frac{4\,\chi_W}{\alpha^2+\beta'^2\gamma^2}\,\mathrm{d}\gamma. \tag{19}$$

For small values of x, the different functions χ_W actually coincide and are also equal to u_0; it must then be true that

$$1 \;=\; \int_{-\infty}^{\infty} \frac{4}{\alpha^2+\beta'^2\gamma^2}\,\mathrm{d}\gamma \;=\; -\,\frac{4\pi}{\alpha\beta'}, \tag{20}$$

and, consequently,

$$\beta' \;=\; -\,\frac{4\pi}{\alpha} \tag{21}$$

must necessarily hold. Because of Eq. (13), if we introduce the time dependence, we approximately get

$$u \;=\; \mathrm{e}^{iEt/\hbar}\int_{-\infty}^{\infty} \frac{4\chi_W\,\exp\left\{2\pi i\sqrt{2E_0/m}\;\gamma t\right\}}{\alpha^2+16\pi^2\gamma^2/\alpha^2}\,\mathrm{d}\gamma. \tag{22}$$

For small values of x the χ_W's can be replaced with u_0, and we have

$$u \;=\; u_0\,\mathrm{e}^{iE_0t/\hbar}\,\exp\left\{-\alpha^2\sqrt{2E_0/m}\,t/2\right\}. \tag{23}$$

This can be written as

$$u \;=\; u_0\, \mathrm{e}^{iE_0 t/\hbar}\, \mathrm{e}^{-t/2T}, \tag{24}$$

quantity T denoting the time-constant ("mean-life")

$$T \;=\; \frac{1}{\alpha^2\sqrt{2E_0/m}} \;=\; \frac{1}{\alpha^2 v}. \tag{25}$$

In this way, and using also Eq. (21), both α and β' can be expressed in terms of T:

$$\alpha = \frac{\pm 1}{\sqrt{vT}} \;=\; \frac{\pm 1}{\sqrt[4]{2(E/m)T^2}}, \tag{26}$$

$$\beta' = \mp 4\pi\,\sqrt{vT} \;=\; \mp 4\pi\,\sqrt[4]{2(E/m)T^2}. \tag{27}$$

It will be clear that only one stationary state corresponds to a hyperbolic-like orbit in the classical theory. The revolution period or, more precisely, the time interval between two intersections of the orbit with the spherical surface of radius r, is given by

$$P_W \;=\; \frac{4}{(\alpha^2 + \beta'^2\gamma^2)v}, \tag{28}$$

and the maximum value is reached for $W = 0$:

$$P_W \;=\; \frac{4}{\alpha^2 v} \;=\; 4T. \tag{29}$$

As a purely classical picture suggests, the probabilities for the realization of single stationary states are proportional to the revolution periods (see Eq. (18)), and T itself can be derived from classical arguments. Indeed, if a particle is on an orbit W and inside the sphere of radius R, on average it will stay in this orbit for a time $T_W = (1/2)P_W = (2/v)/(\alpha^2 + \beta'^2\gamma^2)$, and the mean value of T_W will be

$$\overline{T_W} \;=\; \int_0^\infty T_W^2\,\mathrm{d}\gamma \Big/ \int_0^\infty T_W\,\mathrm{d}\gamma \;=\; \frac{1}{\alpha^2 v} \;=\; T. \tag{30}$$

However, we must caution that, by pushing the analogy even further to determine the expression for the survival probability, we would eventually get a wrong result.

The eigenfunction u takes the form in Eq. (23) only for small values of x. Neglecting what happens for values of x that are not too small, but still lower than R, and considering, moreover, even the case $x > R$, from Eqs. (15) and (19) we have

$$u = \mathrm{e}^{iE_0 t/\hbar}\left[\int_0^\infty \frac{4\alpha\cos(C + 2\pi\gamma)(x-R)}{\alpha^2 + \beta'^2\gamma^2}\,\mathrm{e}^{2\pi i v\gamma t}\,\mathrm{d}\gamma \right.$$
$$\left. - \int_0^\infty \frac{4\beta'\gamma\sin(C + 2\pi\gamma)(x-R)}{\alpha^2 + \beta'^2\gamma^2}\,\mathrm{e}^{2\pi i v\gamma t}\,\mathrm{d}\gamma\right], \tag{31}$$

where α and β' depend on T according to Eqs. (26), (27). Equation (31) can be written as

$$u = \mathrm{e}^{iE_0 t/\hbar}\left[\mathrm{e}^{iC(x-R)}\int_0^\infty \frac{(2\alpha - 2i\beta'\gamma)}{\alpha^2 + \beta'^2\gamma^2}\,\mathrm{e}^{2\pi i(vt + x - R)\gamma}\,\mathrm{d}\gamma \right.$$
$$\left. + \,\mathrm{e}^{-iC(x-R)}\int_0^\infty \frac{(2\alpha + 2i\beta'\gamma)}{\alpha^2 + \beta'^2\gamma^2}\,\mathrm{e}^{2\pi i[vt - (x-R)]\gamma}\,\mathrm{d}\gamma\right]. \tag{32}$$

Since α and β' have opposite signs and, for $t > 0$ and $x > R$, one has $vt + x - R > 0$, the first integral is zero, while the second one equals

$$\int_0^\infty \frac{(2\alpha + 2i\beta'\gamma)}{\alpha^2 + \beta'^2\gamma^2}\, \mathrm{e}^{2\pi i[vt-(x-R)]\gamma}\, \mathrm{d}\gamma \;=\; 2\int_0^\infty \frac{\mathrm{e}^{2\pi i[vt-(x-R)]\gamma}}{\alpha - i\beta'\gamma}\, \mathrm{d}\gamma$$

$$= \begin{cases} -\dfrac{4\pi}{\beta'}\, \mathrm{e}^{2\pi(\alpha/\beta')[vt-(x-R)]} \;=\; -\dfrac{4\pi}{\beta'}\, \mathrm{e}^{-(\alpha^2/2)[vt-(x-R)]}, \\ \\ 0, \end{cases} \tag{33}$$

for $vt - (x - R) > 0$ and $vt - (x - R) < 0$, respectively. On substituting into Eq. (34) and recalling that, from Eq. (12), $C = mv/\hbar$, we finally find

$$u \;=\; \begin{cases} \alpha\, \mathrm{e}^{iE_0 t/\hbar}\, \mathrm{e}^{-imv(x-R)/\hbar}\, \mathrm{e}^{-t/2T}\, \mathrm{e}^{(x-R)/(2vT)}, \\ \\ 0, \end{cases} \tag{34}$$

for $vt-(x-R) > 0$ and $vt-(x-R) < 0$, respectively. Let us now assume that the nucleus has lost the α particle; this means that, initially, it is $u_0 = 0$ near the nucleus. We now evaluate the probability that such a nucleus will re-absorb an α particle when bombarded with a parallel beam of particles. To characterize the beam we'll have to give the intensity per unit area, the energy per particle, and the duration of the bombardment. The only particles with a high absorption probability are those having energy close to E_0, with an uncertainty of the order h/T. On the other hand, in order to make clear the interpretation of the results, the duration τ of the bombardment must be small compared to T. Then it follows that, from the uncertainty relations, the energy of the incident particles will be determined with an error greater than h/T. Thus, instead of fixing the intensity per unit area, it is more appropriate to give the intensity per unit area and unit energy close to E_0; so, let N be the total number of particles incident on the nucleus during the entire duration of the bombardment, per unit area and unit energy.

Suppose that, initially, the incident plane wave is confined between two parallel planes at distance d_1 and $d_2 = d_1 + \ell$ from the nucleus, respectively. Since we have assumed that the initial wave is a plane wave, it will be

$$u_0 \;=\; u_0(\xi), \tag{35}$$

ξ being the abscissa (distance from the nucleus) of a generic plane that is parallel to the other two. Then, for $\xi < d_1$ or $\xi > d_2$, it is $u_0 = 0$. Furthermore, we'll suppose $d_1 > R$ and, without introducing any further constraint,

$$\ell \;=\; \frac{h\rho}{m\sqrt{2E_0/m}} \;=\; \frac{h\rho}{mv} \;=\; \rho\,\lambda, \tag{36}$$

with ρ an integer number and λ the wavelength of the emitted α particle. We can now expand ψ_0 between d_1 and d_2 in a Fourier series and thus as a sum of terms of the kind

$$k_\sigma \, \mathrm{e}^{\sigma 2\pi i(\xi - d_1)/\ell}, \tag{37}$$

with integer σ. The terms with negative σ roughly represent outgoing particles, and thus we can assume them to be zero. Let us concentrate on the term

$$k_\rho \, \mathrm{e}^{\rho 2\pi i(\xi - d_1)/\ell} \;=\; k_\rho \, \mathrm{e}^{imv(\xi - d_1)/\hbar} \tag{38}$$

and let us set[3]

$$u_0 \;=\; \psi_0 \;+\; k_\rho \, \mathrm{e}^{imv(\xi - d_1)/\hbar}. \tag{39}$$

The eigenfunctions of a free particle moving perpendicularly to the incoming wave, normalized with respect to $\mathrm{d}E$, are

$$\psi_\sigma \;=\; \frac{1}{\sqrt{2hE/m}} \, \mathrm{e}^{i\sqrt{2mE}(\xi - d_1)/\hbar}. \tag{40}$$

Note that the square root at the exponent must be considered once with the positive sign and once with the negative sign, and E runs twice between 0 and ∞. However, only the eigenfunctions with the positive square root sign are of interest to us, since they represent the particles moving in the direction of decreasing ξ. We can set

$$\psi_0 \;=\; \int_0^\infty c_E \, \psi_\rho \, \mathrm{d}E, \tag{41}$$

wherein

$$c_E \;=\; \int_{d_1}^{d_2} \psi_0 \, \psi_\rho^* \, \mathrm{d}\xi. \tag{42}$$

In particular, we put

$$c_{E_0} \;=\; \int_{d_1}^{d_2} \psi_0 \, \frac{1}{\sqrt{hv}} \, \mathrm{e}^{-imv(\xi - d_1)/\hbar} \, \mathrm{d}\xi \;=\; \frac{k_\rho \ell}{\sqrt{hv}}. \tag{43}$$

Since, evidently,

$$N \;=\; c_{E_0}^2, \tag{44}$$

one finds

$$N \;=\; \frac{k_\rho^2 \ell^2}{hv}. \tag{45}$$

Let us now expand u_0 in terms of the eigenfunctions associated with the central field produced by the remaining nuclear constituents. Since only the spherically symmetric eigenfunctions having eigenvalues very close to E_0 are significantly different from zero near the nucleus, we shall concentrate only on these. For $x > R$, the expression of these

[3] Note that the author split the wavefunction of the incident particles into a term related to the principal energy E_0 (the second term in Eq. (39)) plus another term which will be expanded according to Eq. (41). [N.d.T]

eigenfunctions is given in Eqs. (16), (26), (27). Actually, the η_W given by Eq. (16) are the eigenfunctions relative to the problem reduced to one dimension. In order to have the spatial eigenfunctions, normalized with respect to γ, we must consider

$$g_W \;=\; \frac{\eta_W}{\sqrt{4\pi x}}. \tag{46}$$

In this way we will set

$$\psi_0 \;=\; \int_0^\infty p_\gamma\, g_W\, \mathrm{d}\gamma \;+\; \ldots, \tag{47}$$

wherein

$$p_\gamma \;=\; \iiint \mathrm{d}S\, g_W\, \psi_0 \;=\; \int_{d_1}^{d_2} 2\pi\, x\, g_W\, \mathrm{d}x \int_{d_1}^{x} \psi_0\, \mathrm{d}\xi. \tag{48}$$

We can set

$$g_W \;=\; \frac{1}{\sqrt{4\pi x}} \left[A_\gamma\, \mathrm{e}^{i(C+2\pi\gamma)(x-d_1)} \;+\; B_\gamma\, \mathrm{e}^{-i(C+2\pi\gamma)(x-d_1)}\right], \tag{49}$$

and, from Eq. (16),

$$A_\gamma = \frac{\alpha - i\beta'\gamma}{\sqrt{\alpha^2 + \beta'^2\gamma^2}}\, \mathrm{e}^{i(C+2\pi\gamma)(d_1-R)},$$
$$\tag{50}$$
$$B_\gamma = \frac{\alpha + i\beta'\gamma}{\sqrt{\alpha^2 + \beta'^2\gamma^2}}\, \mathrm{e}^{-i(C+2\pi\gamma)(d_1-R)}.$$

We can now assume that d_1, and thus d_2, is arbitrarily large; but $\ell = d_2 - d_1$ has to be small because the duration of the bombardment, which is of the order ℓ/v, must be negligible with respect to T. Since $2\pi\gamma$ is of the same order as α^2, that is to say, of the same order as $1/vT$ (see Eq. (25)), $2\pi\gamma\ell$ is absolutely negligible. For $d_1 < x < d_2$ it is then possible to rewrite Eq. (49) as

$$g^W \;=\; \frac{1}{\sqrt{4\pi x}} \left[A_\gamma\, \mathrm{e}^{imv(x-d_1)/\hbar} \;+\; B_\gamma\, \mathrm{e}^{-imv(x-d_1)/\hbar}\right], \tag{51}$$

given Eqs. (50).

Let us now substitute this into Eq. (48), taking into account Eqs. (39) and (45). We'll simply have

$$\begin{aligned} p_\gamma &= \frac{2\pi B_\gamma}{\sqrt{4\pi}} \int_{d_1}^{d_2} \mathrm{e}^{-imv(x-d_1)/\hbar}\, \mathrm{d}x \int_{d_1}^{x} \mathrm{e}^{imv(\xi-d_1)/\hbar}\, \mathrm{d}\xi \\ &= \frac{hB_\gamma k_\rho \ell}{i\sqrt{4\pi}\, m\, v} \;=\; \frac{B_\gamma h^{3/2}\sqrt{N}}{i\sqrt{4\pi}\, m\, \sqrt{v}} \;=\; q\, B_\gamma, \end{aligned} \tag{52}$$

with

$$q \;=\; \frac{h^{3/2} N^{1/2}}{i\, m\, v^{1/2}\sqrt{4\pi}}. \tag{53}$$

Substituting into Eq. (47), one gets

$$\psi_0 \;=\; q \int_0^\infty B_\gamma g_W\, \mathrm{d}\gamma \;+\; \ldots \tag{54}$$

and, at an arbitrary time,

$$\psi \;=\; \mathrm{e}^{iE_0t/\hbar}\, q \int_0^\infty B_\gamma\, g_W\, \mathrm{e}^{2\pi i v\gamma t}\, \mathrm{d}\gamma \;+\; \ldots, \tag{55}$$

or, taking into account Eqs. (46) and (16),

$$\psi \;=\; \mathrm{e}^{iE_0t/\hbar}\, \frac{q}{\sqrt{4\pi}x} \int_0^\infty \frac{2B_\gamma}{\sqrt{\alpha^2+\beta'^2\gamma^2}}\, \chi_W\, \mathrm{e}^{2\pi i v\gamma t}\, \mathrm{d}\gamma \;+\; \ldots. \tag{56}$$

We now want to investigate the behavior of ψ near the nucleus. There, assuming that other quasi-stationary state different from the one we are considering do not exist, the terms we have not written down in the expansion of ψ can contribute significantly only during a short time interval after the scattering of the wave. If this is the case, ψ will have spherical symmetry near the nucleus. We set

$$\psi \;=\; \frac{u}{\sqrt{4\pi}x}, \tag{57}$$

so that the number of particles that will eventually be captured is

$$\int |u^2|\, \mathrm{d}x \tag{58}$$

(the integration range should extend up to a reasonable distance, for example up to R). Substituting into Eq. (56), and noting that for small values of x we approximately have $\chi_W = \chi_0$, one obtains

$$u \;=\; q\,\chi_0\, \mathrm{e}^{iE_0t/\hbar} \int_0^\infty \frac{2}{\alpha - i\beta'\gamma}\, \mathrm{e}^{2\pi i[vt-(d_1-R)]\gamma}\, \mathrm{d}\gamma. \tag{59}$$

Since, as we already noted, $\alpha\beta' < 0$, and setting $d = d_1 - R$, from Eqs. (26), we find

$$u \;=\; \begin{cases} q\,\alpha\,\chi_0\,\mathrm{e}^{iE_0t/\hbar}\,\mathrm{e}^{-\dfrac{t-d/v}{2T}} = q\,\alpha\,\mathrm{e}^{-iCd}\,\mathrm{e}^{-\dfrac{t-d/v}{2T}}, & \text{for } t > \dfrac{d}{v}, \\[2ex] 0, \quad \text{for } t < \dfrac{d}{v}. & \end{cases} \tag{60}$$

The meaning of these formulae is very clear: The α-particle beam, which by assumption does not last for a long time, reaches the nucleus at the time $t = d/v$, and there is a probability $|q\alpha|^2$ that a particle is captured (obviously, $q^2\alpha^2 \ll 1$). The effect of the beam then ceases and, if a particle has been absorbed, it is re-emitted on the time scale predicted by the laws of radioactive phenomena. If we set $n = |q\alpha|^2$, then from Eqs. (25) and (53) we get

$$n \;=\; \frac{2\pi^2\hbar^3}{m^2v^2T}\, N, \tag{61}$$

which tells us that the absorption probabilities are completely independent of any hypothesis on the form of the potential near the nucleus, and that they only depend on the time-constant T.[4]

Equation (61) has been derived using only mechanical arguments but, as a matter of fact, we can get the same result using thermodynamics. Let us consider one of our radioactive nuclei in a bath of α particles in thermal motion. To the degree of approximation we have treated the problem so far, we can consider the nucleus to be at rest. Due to the assumed spherical symmetry of the system, a particle in contact with the nucleus is in a quantum state with a simple statistical weight. Such a state, of energy E_0, is not strictly stationary, but has a finite half-life; this should be considered, as in all similar cases, as a second-order effect. Assuming that the density and the temperature of the gas of α particles is such that there exist D particles per unit volume and unit energy near E_0, then, in an energy interval $\mathrm{d}E$, we will find

$$D\,\mathrm{d}E \tag{67}$$

particles per unit volume. Let us denote by p the momentum of the particles, so that we have

$$p = \sqrt{2mE_0}, \tag{68}$$

$$\mathrm{d}p = \sqrt{m/2E_0}\,\mathrm{d}E. \tag{69}$$

[4] The original manuscript then continues with two large paragraphs which have however been crossed out by the author. The first one reads as follows:
"Since only the particles with energy near E_0 are absorbed, we can think, with some imagination, that every energy level $E_0 + W$ is associated with a different absorption coefficient ℓ_W, and that such ℓ_W is proportional to the probability that a particle in the quasi-stationary state has energy $E_0 + W$. From (13), (21), (25), and (18), we then have

$$\ell_W = \frac{D}{1 + 4T^2W^2/\hbar^2}. \tag{62}$$

Since the number of incident particles per unit area and unit energy with energy between $(E_0 + W)$ and $(E_0 + W) + \mathrm{d}W$ is $N\mathrm{d}W$, we must have

$$n = N\int_{-\infty}^{\infty} \ell_W\,\mathrm{d}W = N\,D\,\frac{\pi\hbar}{2T}, \tag{63}$$

from which, comparing with (61),

$$D = \frac{1}{\pi}\,\frac{h^2}{m^2v^2} = \frac{\lambda^2}{\pi}. \tag{64}$$

This is a very simple expression for the absorption cross section of particles with energy E_0, i.e., the particles with the greatest absorption coefficient. If we set

$$N' = N\,\frac{\pi\hbar}{2T}, \tag{65}$$

then Eq. (61) becomes

$$n = \frac{\lambda^2}{\pi}\,N', \tag{66}$$

which means that the absorption of N' particles of energy E_0 is equivalent to the absorption of N particles per unit energy." The second paragraph is not reproduced here since it appears to be incomplete. [N.d.T]

E_0 appears instead of E in the previous equations because we are considering particles with energy close to E_0. The $D\mathrm{d}E$ particles fill a unitary volume in ordinary space, and in momentum space they fill the volume between two spheres of radii p and $p + \mathrm{d}p$, respectively. Thus, in phase space they occupy a volume

$$4\pi\, p^2\, \mathrm{d}p \;=\; 4\pi\, m^2\, \sqrt{2E_0/m}\, \mathrm{d}E \;=\; 4\pi\, m^2\, v\, \mathrm{d}E. \tag{70}$$

This volume contains

$$\frac{m^2 v}{2\pi^2 \hbar^3}\, \mathrm{d}E \tag{71}$$

quantum states. Therefore, on average, we have

$$D\, \frac{2\pi^2 \hbar^3}{m^2 v} \tag{72}$$

particles in every quantum state with energy close to E_0. This is also the mean number of particles inside the nucleus, provided that the expression (72) is much smaller than 1, so that we can neglect the interactions between the particles. Since the time-constant ("mean-life") of the particles in the nucleus is T, then

$$n \;=\; \frac{2\pi^2 \hbar^3 D}{m^2 v T} \tag{73}$$

particles will be emitted per unit time and, in order to maintain the equilibrium, the same number of particles will be absorbed. Concerning the collision probability with a nucleus, and then the absorption probability, D particles per unit volume and energy are equivalent to a parallel beam of $N = Dv$ particles per unit area, unit energy and unit time. On substituting, we then find

$$n \;=\; \frac{2\pi^2 \hbar^3}{m^2 v^2 T}\, N, \tag{74}$$

which coincides with Eq. (61).