The Majorana Oscillator

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Abstract: At present the expression 'Majorana oscillator' does not appear to be in use in theoretical physics. However, the author of this paper heard it in the Seventies, during private conversations with the late Prof. B.Touschek. This little contribution tries to explore the possible meanings of this expression and introduces a new field equation, generalizing the one already introduced by Majorana himself, which could describe a hypothetical 'Majorana oscillator'.

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1. Introduction

The Majorana oscillator came into my life around 1976. At that time I was student at the postgraduate school in Physics of the University of Rome, Italy. In this regard I need to remind younger physicists of the lack, within the university system holding in Italy at that time, of the PhD degree. The highest attainable degree was the one given by postgraduate schools, typically after two years of hard work and the discussion of a final thesis. In Italy, however, there were very few schools of this kind, and limited to a small number of disciplinary domains. Fortunately the latter included Physics and such a circumstance had given me the opportunity of attending the courses of postgraduate school existing in the University of Rome (named 'Scuola di Perfezionamento in Fisica'). The scientific level of the latter was very high, more or less comparable to the one of present PhD schools. From 1974 to 1976 I worked at my final thesis, dealing with the process of particle creation by a quantum field embedded in a Riemannian space, under the supervision of the late Prof. Bruno Touschek. His name being widely known among

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physicists, I don't need an illustration of his outstanding contributions to the development of Physics (the introduction of storage rings and the theory of neutrino would be enough), which are not the subject of this paper. I will limit myself to mention the fact that throughout these two years (and even later) I was engaged in frequent and long discussions with Prof. Touschek about the subject of my thesis and, chiefly, about the very foundations of Quantum Field Theory. At the beginning of 1976 I had completed my work and finished all computations. However, I was not satisfied with my results, which I considered only as approximate, owing to the fact that I had realized a deep conceptual incompleteness of Quantum Field Theory itself. The latter was the subject of a number of discussions with Prof. Touschek, which took place up to the Avril 1976 (when I discussed my thesis), in which, typically, I started by criticizing specific aspects of Quantum Field Theory, so as to stimulate him to answering me, or by showing the inconsistence of my arguments or by making new proposals for overcoming the limits of theoretical schemata used at that time. In particular, I remember that one day (I forgot the exact date) I started by claiming that Quantum Field Theory could not be considered as consistent with Special Relativity, as it was heavily based on a concept of 'oscillator' entirely drawn from traditional classical mechanics. To be more coherent with Special Relativity – I maintained – one should start by relying on a theory of special-relativistic oscillators and then shaping on it the whole formalism. It was during the ensuing discussion that Prof. Touschek used the term 'Majorana oscillator'. At that time I was young and – as it is often the case – very ignorant. Of course, I knew very well the name of Majorana, but not his theory of neutrino. So I thought that the expression 'Majorana oscillator' were denoting a concept widely known among theoretical physicists, but unknown to me (a circumstance, the latter, which it was better not to underline), and in this way I missed the occasion for asking Prof. Touschek about this subject. Long after, when talking with other physicists, I was surprised in realizing that nobody knew the meaning of the expression 'Majorana oscillator'. On the contrary, my colleagues asked me for more information about this concept. When I undertook a search, on books, journals and even on the Web, about Majorana oscillator, nothing emerged. Unfortunately, the untimely death of Prof. Touschek in 1978 prevented me from asking him more details about this subject.

This long introduction explains why now the only possibility I have for clarifying what is a Majorana oscillator is to speculate about the possible meaning of the expression used by Prof. Touschek, taking into account his knowledge, his theoretical purposes and schemata (as emerging from our discussions), as well as the ideas put forward by Majorana itself. This modest contribution contains the outcomes of this speculation, heavily relying on the findings of a number of other authors which, after the publication of the last paper of Ettore Majorana in 1937, tried to draw the conclusions ensuing from his hypotheses. For this reason the second section of the present paper will be devoted to some considerations about Majorana equation and its usefulness in describing some kind of 'oscillator'. Within the third section, on the contrary, we will introduce a new field equation, generalizing the usual Majorana one, designed in an explicit way so as to describe a new kind of oscillator, which could tentatively be called 'Majorana oscillator'. The temporary conclusions of this paper, contained in the fourth section, point that probably the expression used by Prof. Touschek was introduced to denote the solutions of Majorana equation. On the contrary, the generalization of the latter introduced in the third section, while being worthy of exploration, raises a number of problems still difficult to solve.

2. The Majorana Equation

As it is well known, the so-called Majorana equation was introduced by Ettore Majorana in his last paper, entitled "Teoria simmetrica dell'elettrone e del positrone", written in italian language and published in 1937 on the journal "Il Nuovo Cimento" (Majorana, 1937). In it Majorana showed that fermions obeying his equation were undistinguishable from antifermions, and suggested that it was useful to describe neutrinos, by assuming the latter were identical to antineutrinos. Such a 'Majorana neutrino theory' led to a large number of experimental and theoretical investigations, whose temporal trend appears to be characterized by two pronounced peaks (at least in terms of the number of published papers), one more or less in correspondence to the middle Fifties (the epoch of discovery of parity nonconservation), and another corresponding to the last then years (in concomitance with the development of new measuring devices). However, while stressing the importance of the consequences of Majorana hypotheses for elementary particle physics, astrophysics and cosmology, we will fully neglect this subject, by limiting ourselves to quote very few references (Mohapatra & Pal, 1991; Fukugita & Yanagida, 2003; Kang & Kim, 2004; Mohapatra *et al.*, 2005). We will instead focus our attention on some formal properties of Majorana equation.

The departure point is constituted by the well known fact that Majorana equation derives from a suitable representation of matrices used in writing Dirac equation, a representation often denoted as *Majorana representation.* In this regard, we remark that different authors use different conventions and different measure units. Within this paper the generic form of Dirac equation will be written as:

$$i\hbar\partial\psi/\partial t = c\underline{\alpha}\underline{p}\psi + mc^2\beta\psi \tag{1}$$

where ψ is a 4-component spinor, $\underline{\alpha}$ denotes a 3-component vector, each element of which consists in a 4x4 matrix α_k (k = 1, 2, 3), $\underline{p}$ is another 3-component vector, each element of which consists in an operator p_k (k = 1, 2, 3) given by:

$$p_k = (\hbar/i)\partial/\partial x_k \tag{2}$$

and finally β is a 4x4 matrix. Moreover, we will use a space-time metric with signature (1, -1, -1, -1). The α_k and β are anticommuting matrices whose square is the unit matrix and satisfy a set of algebraic relationships which will not be listed here, being widely known among physicists.

The well known consequences of Dirac equation, extensively described in every textbook on relativistic quantum mechanics or on Quantum Field Theory, stem from a particular choice of the form of matrices α_k and β, denoted as *Dirac representation*. However, there exist many other different representations of these matrices, keeping invariant the form of their algebraic relationships. Among them, the most interesting for our purposes are the *Majorana representations*. Here we speak of 'representations' instead of 'representation' because there many different ways, including the one introduced by Majorana in his original paper, for choosing matrices such that all coefficients of Dirac equation (1) become *real numbers*. In the following, in agreement with the previous choice of metric signature, we will adopt the particular choice (see, for instance, Itzykson & Zuber, 1985, p. 694):

$$\beta = \begin{vmatrix} 0 & \sigma 2 \\ \sigma 2 & 0 \end{vmatrix} \quad \alpha 1 = \begin{vmatrix} 0 & -\sigma 1 \\ -\sigma 2 & 0 \end{vmatrix} \quad \alpha 2 = \begin{vmatrix} I & 0 \\ 0 & -I \end{vmatrix} \quad \alpha 3 = \begin{vmatrix} 0 & -\sigma 3 \\ -\sigma 3 & 0 \end{vmatrix} \tag{3}$$

where **I** and **0** are, respectively, the unit and zero 2x2 matrices, while σ_k $(k = 1, 2, 3)$ are the usual 2x2 Pauli matrices.

By substituting (3) and (2) into (1) we obtain an explicit form of Majorana equation. If we denote by ψ_s $(s = 1, ..., 4)$ the components of the 4-spinor ψ, straightforward computations show that these latter obey the following system of 4 partial differential equations, whose coefficients are all real numbers:

$$\partial\psi_1/\partial t = c[(\partial\psi_4/\partial x_1) - (\partial\psi_1/\partial x_2) + (\partial\psi_3/\partial x_3)] - (mc^2/\hbar)\psi_4 \tag{4a}$$

$$\partial\psi_2/\partial t = c[(\partial\psi_3/\partial x_1) - (\partial\psi_2/\partial x_2) - (\partial\psi_4/\partial x_3)] + (mc^2/\hbar)\psi_3 \tag{4b}$$

$$\partial\psi_3/\partial t = c[(\partial\psi_2/\partial x_1) + (\partial\psi_3/\partial x_2) + (\partial \quad \psi_1/\partial x_3)] - (mc^2/\hbar)\psi_2 \tag{4c}$$

$$\partial\psi_4/\partial t = c[(\partial\psi_1/\partial x_1) + (\partial\psi_4/\partial x_2) - (\partial\psi_2/\partial x_3)] + (mc^2/\hbar)\psi_1 \tag{4d}$$

An inspection of these equations lets us understand that they don't allow an easy separation of dependent variables into small and large components, as occurring when choosing Dirac representation (a proof of this fact, though based on a choice of a Majorana representation differing from the one depicted in (3), has been given by Weaver, 1976). On the other hand, the whole system appears as naturally composed of two interacting subsystems: the first including the equations (4.a) and (4.d), and the second including the equations (4.b) and (4.c). Such a subdivision has been already noticed in the past by other authors (see, for instance, McLennan, 1957). It becomes very useful if we go to the non-relativistic limit of (4.a)-(4.d). Namely in this case all terms containing spatial derivatives can be neglected with respect to the ones associated to the coefficients $mc^2/\hbar$, so that we obtain the simple non-interacting subsystems:

$$\partial\psi_1/\partial t = -(mc^2/\hbar)\psi_4, \quad \partial\psi_4/\partial t = (mc^2/\hbar)\psi_1 \tag{5a}$$

$$\partial\psi_2/\partial t = (mc^2/\hbar)\psi_3, \quad \partial\psi_3/\partial t = -(mc^2/\hbar)\psi_2 \tag{5b}$$

each one of which can be immediately recognized as describing an harmonic oscillator with frequency $\omega = mc^2/\hbar$.

Could such a simple circumstance be the origin for the expression 'Majorana oscillator', introduced by Prof. Touschek? Of course, nobody can answer this question, even if two facts induce to think that the answer to it could be positive. First of all, he was deeply involved in investigations about the theory of neutrino (as evidenced by his important contributions to this subject; see, for instance, Radicati & Touschek, 1957; Cini & Touschek, 1958). This implies that he was well-informed about all papers appeared in the Fifties and Sixties dealing with Majorana equation and the underlying mathematics. In second place I remember that in most cases our discussions ended in examples drawn from theory of neutrino, and that in these circumstances he showed off a considerable mathematical ability, producing then and there results and formulae not present in standard textbooks and (perhaps) never proved before.

3. The Majorana Oscillator

Despite the previous arguments, another possible source of inspiration for the concept of Majorana oscillator should be taken into account. It is connected to the publication of a paper by Itô, Mori and Carrieri, appeared in 1967 on *Il Nuovo Cimento*, in which the authors introduced a generalization of usual Dirac equation, describing a system denoted as *Dirac oscillator* (see Itô, Mori & Carrieri, 1967). Such a paper was followed by a number of subsequent papers (for a complete list see Benítez *et al.*, 1990b), all dealing with the same subject and appeared from 1969 to 1986. It can reasonably be supposed that Prof. Touschek knew some of them, though I have no proof of this fact. This topic did not attract much attention, and, as a matter of fact, Moshinsky and Szczepaniak rediscovered it in 1989 (see Moshinsky & Szczepaniak, 1989; further papers are, among others, Benítez *et al.*, 1990a, Szmytkowski & Gruchowski, 2001).

The key idea underlying the introduction of Dirac oscillator consists in adding to the usual Dirac equation a new term whose form be such that, squaring the new Dirac Hamiltonian (remember that, in a sense, Dirac equation was obtained by taking the 'square root' of usual Hamiltonian), or iterating the Dirac operator, one can obtain an expression containing terms like the ones appearing in harmonic oscillator Hamiltonian, that is having the form $p^2 + m^2\ \Omega^2\ \ r^2$, where Ω is a suitable frequency. In order to reach this goal, all authors previously quoted modify the Dirac equation by introducing the transformation:

$$p \rightarrow p - im\Omega r\beta \tag{6}$$

where the components of 3-vector $\boldsymbol{r}$ are given by x_k ($k = 1, 2, 3$), and β is the 4x4 matrix already defined in (3). However, when taking into consideration the Majorana equation, the transformation (7) could give rise to troubles, as it introduces, within a system of equations whose coefficients are all real numbers, new terms containing coefficients which are pure imaginary. Therefore, it seems that a better proposal, in the case of Majorana equation, would consist in introducing the transformation:

$$p \to p - m\Omega r\beta \tag{7}$$

which gives rise to a generalized Majorana equation in which all coefficients are still real numbers.

If we substitute (7) into (1), with the aid of (2) and (3), elementary computations show that the explicit form of the generalized equations describing what we could denote as a *Majorana oscillator*, is:

$$\begin{aligned}\partial\psi_1/\partial t = c[(\partial\psi_4/\partial x_1) - (\partial\psi_1/\partial x_2) + (\partial\psi_3/\partial x_3)] \\ -(mc/\hbar)\Omega(-x_1\psi_1 - x_2\psi_4 - x_3\psi_2) - (mc^2/\hbar)\psi_4\end{aligned} \tag{8a}$$

$$\begin{aligned}\partial\psi_2/\partial t = c[(\partial\psi_3/\partial x_1) - (\partial\psi_2/\partial x_2) - (\partial\psi_4/\partial x_3)] \\ -(mc/\hbar)\Omega(x_1\psi_2 + x_2\psi_3 + x_3\psi_1) + (mc^2/\hbar)\psi_3\end{aligned} \tag{8b}$$

$$\begin{aligned}\partial\psi_3/\partial t = c[(\partial\psi_2/\partial x_1) + (\partial\psi_3/\partial x_2) + (\partial\psi_1/\partial x_3)] \\ -(mc/\hbar)\Omega(-x_1\psi_3 + x_2\psi_2 + x_3\psi_4) - (mc^2/\hbar)\psi_2\end{aligned} \tag{8c}$$

$$\begin{aligned}\partial\psi_4/\partial t = c[(\partial\psi_1/\partial x_1) + (\partial\psi_4/\partial x_2) - (\partial\psi_2/\partial x_3)] \\ -(mc/\hbar)\Omega(x_1\psi_4 - x_2\psi_1 + x_3\psi_3) + (mc^2/\hbar)\psi_1\end{aligned} \tag{8d}$$

The search for solutions of (8.a)-(8.d) is a very difficult affair. Namely it can be shown that these equations do not allow plane wave solutions. To this end it is enough to make an ansatz of the form:

$$\psi_s = a_s F(pr - \omega t) \tag{9}$$

where F is a periodic function and a_s are suitable constant coefficients. By substituting (9) into (8.a)-(8.d), one obtains or a single homogeneous linear system of algebraic equations in the unknown coefficients a_s or two systems of this kind, according to the choice made for F (the last alternative occurs, for instance, when Fcoincides with simple trigonometric functions like *sin* or *cos*). In order to have an infinite number of nonzero solutions, the determinant of this system (or the determinants of these systems) must vanish. In turn, this condition gives rise to a number of relationships to be fulfilled by $\boldsymbol{p}$, $\boldsymbol{r}$ and ω, as well as by oscillator frequency Ω. In all cases these relationships lead to mathematical or physical inconsistencies. To make an example, we will show here the explicit form of the two relationships obtained, when F coincides with the function *sin*. If we put:

$$e = \omega^2 = E^2/\hbar^2, \quad x = c^2(p_1)^2/\hbar^2, \quad y = c^2(p_2)^2/\hbar^2, \quad z = c^2(p_3)^2/\hbar^2, \tag{10}$$

then the two obtained relationships are:

$$(e - x - y - z)^2 - 2z(e - x - y + z) = 0, \quad x = \Omega^2[(x_1)^2 + (x_2)^2 + (x_3)^2] - c^2 = 0 \tag{11}$$

The second of (11) is clearly inconsistent with the assumption that Ω be a constant quantity. As regards the first of (11), it is satisfied when $p_3 = 0$ (a very strange requirement!), in which case the relationship itself takes the well known form $E^2 = c^2p^2$, characterizing ultrarelativistic particles of zero mass (like the neutrino). However, this implies, in turn, that the contribution to (8.a)-(8.d) of the new term proportional to $\boldsymbol{r}$ be vanishing (so that the Majorana oscillator is destroyed).

Without pursuing further this topic, which, in any case, it is worth exploring, we remark that the above considerations tend to exclude the possibility that Prof. Touschek, when speaking of Majorana oscillator, could have in mind a system like the one introduced in this section.

4. Conclusion

The simple arguments introduced in this paper propose two possible ways for defining the Majorana oscillator: either an oscillatory behavior of solutions of Majorana equation in the nonrelativistic case, or a new equation, obtained through the introduction of a suitable (squared root) oscillatory potential, and generalizing the usual Majorana equation. While the second alternative could become *per se* a subject for future mathematical and physical investigations, it seems that the expression 'Majorana oscillator', when introduced by Prof. Touschek, were rather connected to the first alternative. In any case the considerations made in this contribution evidence how much the world of Majorana representations (and of their physical content) be still widely unexplored. It is to be hoped that future investigations will help to fill this gap.

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