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Exact Solution of Majorana Equation via Heaviside Operational Ansatz

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Abstract: In context of a transformation between Majorana and Dirac wavefunctions, it suffices to solve the related interactive Dirac problem and then apply the transformation of variables on the Dirac wavefunction in order to obtain the Majorana wavefunction of the given Majorana equation. Clearly, this connection between solutions continues to hold if the free Majorana and Dirac equations are each coupled to an external gauge field[e.g., Electromagnetism] via the minimum coupling prescription. Applying the formal solution scheme Heaviside Operational Ansatz[heretofore, HOA] put forward in [EJTP 1 (2004), 10-16], provides an exact quadrature solution for the massive minimum-coupled Dirac equation, which may then be transformed into the solution of the corresponding massive minimum-coupled Majorana equation.

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1. Motivation and Layout

In this brief note, after a short recap of HOA methods, we shall apply the scheme to obtain quadrature solutions of the Majorana equation[ref.1] minimal-coupled to an external gauge field(e.g., electromagnetism). Crucial to this will be using the HOA methods to solve the related minimal-coupled Dirac Equation. Once the Dirac solutions are known, the Majorana solutions follow directly. The details of transforming of the Dirac wavefunctions into solutions of the Majorana Equation are elaborated in great detail elsewhere[see

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for example the excellent review article by Valle ref.2] and will not be reproduced here.

2. Recap of HOA

To recall the full details of HOA results, see the original work[EJTP 1 (2004), 10-16]. As pointed out therein,

'Notwithstanding its quantum mechanical origins, the HOA scheme takes on a life of its own and transcends the limits of quantum applications to address a wide variety of purely formal mathematical problems as well. Among other things, the result provides a formula for obtaining an exact solution to a wide variety of variable-coefficient integro-differential equations. Since the functional dependence of the Hamiltonian operator as considered is in general arbitrary upon its arguments(i.e., independent variables, derivative operator symbols[including negative powers thereof, thus the possible integral character]), then its multivariable extension can be interpreted as the most general variable coefficient partial differential operator. Moreover, it is not confined to being a scalar or even vector operator, but may be generally construed an arbitrary rank matrix operator. In all cases of course, its rank dictates the matrix rank of the wavefunction solution.'

In the present case of the Majorana equation and related Dirac equation, we shall be dealing with such a 4x4 matrix Hamiltonian structure and the solution wavefunction will be of a 4-dimensional column vector character.

Recalling the fundamental HOA results, we let x, p, t respectively denote the configuration space, momentum and time variables. The $\hat{}$ denotes the operators, with H and Ψ denoting the Hamiltonian and wavefunction of the phase space representation, respectively. Also the α, γ are otherwise free parameters as specified therein the original work.

Hence for a given Hamiltonian and Initial-State, the configuration space solution obtains from the quantum phase space solution as

$$\hat{H}_{\text{configuration space}}\left(x_1, ..., x_n, -i\hbar\partial_{x_1}, ..., -i\hbar\partial_{x_n}, t\right)\Psi_{\text{configuration space}}\left(x_1, ..., x_n, t\right)$$
$$= i\hbar\partial_t \Psi_{\text{configuration space}}\left(x_1, ..., x_n, t\right)$$

$$\Psi_{\text{configuration space}}\left(x_1, ..., x_n, t\right)$$
$$= \int_{-\infty}^{\infty} \frac{e^{\frac{ix_1 p_1}{2\hbar}}}{\sqrt{4\pi\hbar}} .... \int_{-\infty}^{\infty} \frac{e^{\frac{ix_n p_n}{2\hbar}}}{\sqrt{4\pi\hbar}} L^{-1}_{\begin{pmatrix} (\bar{x}_1, ..., \bar{x}_n) \\ \to (x_1, ..., x_n) \end{pmatrix}} \left[L^{-1}_{\begin{pmatrix} (\bar{p}_1, ..., \bar{p}_n) \\ \to (p_1, ..., p_n) \end{pmatrix}} \left[\begin{array}{l} e^{\frac{-i}{\hbar}\int_0^t \hat{H}\begin{pmatrix} i\hbar\bar{p}_1 + \alpha_1 x_1, ..., i\hbar\bar{p}_n + \alpha_n x_n; \\ -i\hbar\bar{x}_1 + \gamma_1 p_1, ..., -i\hbar\bar{x}_n + \gamma_n p_n; u \end{pmatrix} du} \\ \times \\ \breve{\Psi}_0\left(\bar{x}_1, ..., \bar{x}_n; \bar{p}_1, ..., \bar{p}_n; t = 0\right) \end{array} \right] \right] dp_1..dp_n \quad (1)$$

where

$$\hat{H}_{\text{configuration space}}\begin{pmatrix} x_1,....,x_n, \\ -i\hbar\partial_{x_1},....,-i\hbar\partial_{x_n},t \end{pmatrix}_{\substack{(x_1,....,x_n)\mapsto(i\hbar\partial_{p_1}+\alpha_1 x_1,....,i\hbar\partial_{p_n}+\alpha_n x_n) \\ (-i\hbar\partial_{x_1},....,-i\hbar\partial_{x_n})\mapsto(-i\hbar\partial_{x_1}+\gamma_1 p_1,....,-i\hbar\partial_{x_n}+\gamma_n p_n)}}$$

$$\equiv \hat{H}\begin{pmatrix} i\hbar\partial_{p_1}+\alpha_1 x_1,....,i\hbar\partial_{p_n}+\alpha_n x_n; \\ -i\hbar\partial_{x_1}+\gamma_1 p_1,....,-i\hbar\partial_{x_n}+\gamma_n p_n;t \end{pmatrix} \tag{2}$$

To wit, via HOA the configuration space solution becomes

$$\Psi_{\text{configuration space}}(x_1,....,x_n,t)$$

$$= \int_{-\infty}^{\infty}\frac{e^{\frac{ix_1p_1}{2\hbar}}}{\sqrt{4\pi\hbar}}....\int_{-\infty}^{\infty}\frac{e^{\frac{ix_np_n}{2\hbar}}}{\sqrt{4\pi\hbar}}L^{-1}_{\begin{pmatrix} (\bar{x}_1,....,\bar{x}_n;\bar{p}_1,....,\bar{p}_n) \\ \to(x_1,....,x_n;p_1,....,p_n) \end{pmatrix}}\begin{bmatrix} e^{\frac{-i}{\hbar}\int_0^t \hat{H}_{\text{configuration space}}\,du} \\ \times \\ \breve{\Psi}_{0\,\text{configuration space}}(\bar{x}_1,....,\bar{x}_n;;t=0) \end{bmatrix}dp_1..dp_n \tag{3}$$

where

$$\hat{H}_{\text{configuration space}} = \hat{H}_{\text{configuration space}}\begin{pmatrix} x_1,....,x_n, \\ -i\hbar\partial_{x_1},....,-i\hbar\partial_{x_n},u \end{pmatrix}_{\substack{(x_1,....,x_n)\mapsto(i\hbar\bar{p}_1+\alpha_1 x_1,....,i\hbar\bar{p}_n+\alpha_n x_n) \\ (-i\hbar\partial_{x_1},....,-i\hbar\partial_{x_n})\mapsto(-i\hbar\bar{x}_1+\gamma_1 p_1,....,-i\hbar\bar{x}_n+\gamma_n p_n)}}$$

With that said, a relatively simplistic prescription results for actually using the Ansatz to solve the problem,

Given the function $\hat{H}(\hat{x}_1,....,\hat{x}_n;\hat{p}_1,....,\hat{p}_n,t)$
[respectively $\hat{H}(x_1,....,x_n;-i\hbar\partial_{x_1},....,-i\hbar\partial_{x_n},t)$] replace

$(\hat{x}_1,....,\hat{x}_n;\hat{p}_1,....,\hat{p}_n,t)$[respectively $(x_1,....,x_n;-i\hbar\partial_{x_1},....,-i\hbar\partial_{x_n},t)$] with

$(i\hbar\bar{p}_1+\alpha_1 x_1,....,i\hbar\bar{p}_n+\alpha_n x_n;-i\hbar\bar{x}_1+\gamma_1 p_1,....,-i\hbar\bar{x}_n+\gamma_1 p_n,t)$ in equation (3)

The result of course is the quantum phase space[respectively configuration space] wavefunction for the quantum dynamics wave equation. Before addressing the Majorana equation directly, just a comment on the α and γ parameters in the above formulae. From the HOA, they are otherwise arbitrary except for the condition $\alpha+\gamma=1$. This is explained therein as a consequence of the arbitrary phase shift associated with the quantum phase space wavefunction. Further, any choice of the parameters thus constrained yields a Hamiltonian, which is dynamically equivalent[describes the same physics] as any other choice. However, it is shown in therein that the Hamiltonian operator $\hat{H}(i\hbar\partial_p+\alpha x,-i\hbar\partial_x+\gamma p,t)$, $\ni \alpha+\gamma=1$ takes on the symmetric canonical form when $\alpha = \gamma = \frac{1}{2}$ thusly $\hat{H}\left(i\hbar\partial_p + \frac{x}{2}, -i\hbar\partial_x + \frac{p}{2}, t\right)$, $\ni \alpha + \gamma = 1$. Notwithstanding this and with an eye towards computational simplifications for particular classes of applications, it has been found that other choices than $\alpha = \gamma = \frac{1}{2}$ greatly facilitates evaluation of the integral transforms. Unless otherwise directed, the convention for α and γ shall be specified for particular cases, presently and elsewhere.

3. HOA Solution of Majorana Equation with Minimum-Coupled Electromagnetic Gauge Field

First consider the related Dirac equation with minimum-coupled electromagnetic gauge field
$\mathbf{A}(x_1,x_2,x_3,t) = A_1(x_1,x_2,x_3,t)\,\mathbf{e_{x_1}} + A_2(x_1,x_2,x_3,t)\,\mathbf{e_{x_2}} + A_3(x_1,x_2,x_3,t)\,\mathbf{e_{x_3}}$, $A_0(x_1,x_2,x_3,t)$
interaction

$$\mathbf{H}_{\mathrm{Dirac}_{\mathbf{4\times 4}}}\Psi_D = \left(mc^2 a_0 + \sum_{j=1}^{3}\left(a_j\left(p_j - eA_j\right)c + eA_0\right)\right)\Psi_D = i\hbar\partial_t\Psi_D$$

$$\mathbf{A}(x,y,z,t) = A_1(x,y,z,t)\,\mathbf{e_x} + A_2(x,y,z,t)\,\mathbf{e_y} + A_3(x,y,z,t)\,\mathbf{e_z}\,,\ A_0(x,y,z,t)$$

$$\Psi_D \equiv \begin{pmatrix}\Psi_{D1}\\ \Psi_{D2}\\ \Psi_{D3}\\ \Psi_{D4}\end{pmatrix} : \text{ 4-component Dirac wavefunction}$$

$$\Psi_{D_0} \equiv \begin{pmatrix}\Psi_{D1_0}\\ \Psi_{D2_0}\\ \Psi_{D3_0}\\ \Psi_{D4_0}\end{pmatrix} : \text{ 4-component Dirac Initial State} \tag{4}$$

$$a_0 = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix},\ a_1 = \begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}$$

$$a_2 = \begin{bmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{bmatrix},\ a_3 = \begin{bmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{bmatrix}$$

Similarly for the Majorana equation minimal-coupled to the EM gauge field.

$$
\begin{aligned}
&-imc\sigma^2\rho_M^* + \left(i\hbar\sigma^\mu\partial_\mu - eA_\mu\right)\rho_M = 0 \\
&A_\mu \equiv \left(\mathbf{A}\left(x_1,x_2,x_3,t\right) = A_1\left(x_1,x_2,x_3,t\right)\mathbf{e}_{\mathbf{x_1}} + A_2\left(x_1,x_2,x_3,t\right)\mathbf{e}_{\mathbf{x_2}} + A_3\left(x_1,x_2,x_3,t\right)\mathbf{e}_{\mathbf{x_3}}\ ,\ A_0\left(x_1,x_2,x_3,t\right)\right) \\
&\rho_M = \begin{pmatrix} \rho_{M1} \\ \rho_{M2} \end{pmatrix} :\ 2 - \text{component Majorana wavefunction} \\
&\rho_{M_0} = \begin{pmatrix} \rho_{M1_0} \\ \rho_{M2_0} \end{pmatrix} :\ 2 - \text{component Majorana Initial state} \\
&\sigma^\mu : \text{usual } 2\times 2 \text{ Pauli spin matrices } \sigma^{1,2,3}, \sigma^0 = -i\mathbf{I_{2\times 2}}
\end{aligned}
\tag{5}
$$

Now the connection between the Majorana (5) ρ_M and Dirac [4] $\Psi_D$wavefunctions subject to the Majorana self-conjugacy condition $\Psi_D = \Psi_D^c$ is thoroughly discussed in the excellent review article by Valle [ref 2]; only some key relationships between them are reproduced here for convenience

$$
\begin{aligned}
&\Psi_D \equiv \begin{pmatrix} \Psi_{D1} \\ \Psi_{D2} \\ \Psi_{D3} \\ \Psi_{D4} \end{pmatrix} = \begin{pmatrix} \chi_D \\ \sigma_2\phi_D^* \end{pmatrix}\ ,\quad \Psi_D^c = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}\Psi_D^{*Transpose} = \begin{pmatrix} \phi_D \\ \sigma_2\chi_D^* \end{pmatrix}, \\
&\chi_D = \begin{pmatrix} \Psi_{D1} \\ \Psi_{D2} \end{pmatrix},\quad \sigma_2\phi_D^* = \begin{pmatrix} \Psi_{D3} \\ \Psi_{D4} \end{pmatrix}\ ,\ \phi_D = \begin{pmatrix} i\Psi_{D4}^* \\ -i\Psi_{D3}^* \end{pmatrix}, \\
&\text{Majorana Self-Conjugacy condition } \Psi_D = \Psi_D^c \\
&\chi_D = \tfrac{1}{\sqrt{2}}\left(\rho_{M_2} + i\rho_{M_1}\right)\ ,\ \rho_{M_2} = \tfrac{1}{\sqrt{2}}\left(\chi_D + \phi_D\right) = \tfrac{1}{\sqrt{2}}\begin{pmatrix} \Psi_{D1} + i\Psi_{D4}^* \\ \Psi_{D2} - i\Psi_{D3}^* \end{pmatrix} \\
&\phi_D = \tfrac{1}{\sqrt{2}}\left(\rho_{M_2} - i\rho_{M_1}\right)\ ,\ \rho_{M_1} = \tfrac{i}{\sqrt{2}}\left(\chi_D - \phi_D\right) = \tfrac{i}{\sqrt{2}}\begin{pmatrix} \Psi_{D1} - i\Psi_{D4}^* \\ \Psi_{D2} + i\Psi_{D3}^* \end{pmatrix} \\
&\rho_M = \begin{pmatrix} \rho_{M_1} \\ \rho_{M_2} \end{pmatrix}
\end{aligned}
\tag{6}
$$

So by way of (6), given the related Dirac wavefunction and subject to the Majorana self-conjugacy condition $\Psi_D = \Psi_D^c$, the Majorana wavefunction ascends naturally. Moreover, by way of the HOA method, substituting the Dirac Hamiltonian of (4) gives the quantum phase space dynamics of the Dirac system for initial conditions and EM gauge fields of general form.

$$\begin{aligned}
&\hat{\mathbf{H}}_{\mathrm{Dirac}_{\mathbf{4\times 4}}}\begin{pmatrix} i\hbar\partial_{p_1}+\alpha_1 x_1, i\hbar\partial_{p_2}+\alpha_2 x_2, i\hbar\partial_{p_3}+\alpha_3 x_3; \\ -i\hbar\partial_{x_1}+\gamma_1 p_1, -i\hbar\partial_{x_2}+\gamma_2 p_2, -i\hbar\partial_{x_3}+\gamma_3 p_3; t \end{pmatrix}\Psi_D\left(x_1,x_2,x_3;p_1,p_2,p_3;t\right) \\
&= i\hbar\partial_t\Psi_D\left(x_1,x_2,x_3;p_1,p_2,p_3;t\right) \\
&\left(m_c^2 a_0 + \sum_{j=1}^{3}\begin{pmatrix} a_j\left(-i\hbar\partial_{x_j}+\gamma_j p_j - eA_j\left(i\hbar\partial_{p_1}+\alpha_1 x_1, i\hbar\partial_{p_2}+\alpha_2 x_2, i\hbar\partial_{p_3}+\alpha_3 x_3, t\right)\right)c \\ +eA_0\left(i\hbar\partial_{p_1}+\alpha_1 x_1, i\hbar\partial_{p_2}+\alpha_2 x_2, i\hbar\partial_{p_3}+\alpha_3 x_3; , t\right) \end{pmatrix}\right)\Psi_D = i\hbar\partial_t\Psi_D \\
&\mathbf{A}\left(i\hbar\partial_{p_1}+\alpha_1 x_1, i\hbar\partial_{p_2}+\alpha_2 x_2, i\hbar\partial_{p_3}+\alpha_3 x_3, t\right) = \\
&A_1\left(i\hbar\partial_{p_1}+\alpha_1 x_1, i\hbar\partial_{p_2}+\alpha_2 x_2, i\hbar\partial_{p_3}+\alpha_3 x_3, t\right)\mathbf{e_{x_1}} \\
&+A_2\left(i\hbar\partial_{p_1}+\alpha_1 x_1, i\hbar\partial_{p_2}+\alpha_2 x_2, i\hbar\partial_{p_3}+\alpha_3 x_3, t\right)\mathbf{e_{x_2}} \\
&+A_3\left(i\hbar\partial_{p_1}+\alpha_1 x_1, i\hbar\partial_{p_2}+\alpha_2 x_2, i\hbar\partial_{p_3}+\alpha_3 x_3, t\right)\mathbf{e_{x_3}} \\
&, A_0\left(i\hbar\partial_{p_1}+\alpha_1 x_1, i\hbar\partial_{p_2}+\alpha_2 x_2, i\hbar\partial_{p_3}+\alpha_3 x_3, t\right) \\
&\Psi_D \equiv \begin{pmatrix} \Psi_{D1} \\ \Psi_{D2} \\ \Psi_{D3} \\ \Psi_{D4} \end{pmatrix} : \text{ 4-component Dirac wavefunction}
\end{aligned} \tag{7}$$

Hence the configuration space dynamics for the related minimal-coupled Dirac system

$$\begin{aligned}
&\Psi_{\mathrm{D\ configuration\ space}}\left(x_1,x_2,x_3,t\right) = \\
&\begin{pmatrix} \Psi_{D1}\left(x_1,x_2,x_3,t\right) \\ \Psi_{D2}\left(x_1,x_2,x_3,t\right) \\ \Psi_{D3}\left(x_1,x_2,x_3,t\right) \\ \Psi_{D4}\left(x_1,x_2,x_3,t\right) \end{pmatrix}_{\text{configuration space}} = \\
&= \int_{-\infty}^{\infty}\frac{e^{\frac{ix_1p_1}{2\hbar}}}{\sqrt{4\pi\hbar}}\int_{-\infty}^{\infty}\frac{e^{\frac{ix_2p_2}{2\hbar}}}{\sqrt{4\pi\hbar}}\int_{-\infty}^{\infty}\frac{e^{\frac{ix_3p_3}{2\hbar}}}{\sqrt{4\pi\hbar}} L^{-1}_{\left(\begin{pmatrix}\bar{x}_1,\ldots,\bar{x}_n; \\ \bar{p}_1,\ldots,\bar{p}_n\end{pmatrix} \to \begin{pmatrix} x_1,\ldots,x_n; \\ p_1,\ldots,p_n\end{pmatrix}\right)} e^{\left(\frac{-i}{\hbar}\int_0^t \begin{pmatrix} mc^2 a_0 \\ +\sum_{j=1}^{3}\begin{pmatrix} a_j\begin{pmatrix} -i\hbar\bar{x}_j+\gamma_j p_j \\ -eA_j\begin{pmatrix} i\hbar\bar{p}_1+\alpha_1 x_1, \\ i\hbar\bar{p}_2+\alpha_2 x_2, \\ i\hbar\bar{p}_3+\alpha_3 x_3, u\end{pmatrix}\end{pmatrix} c \\ +eA_0\begin{pmatrix} i\hbar\bar{p}_1+\alpha_1 x_1, \\ i\hbar\bar{p}_2+\alpha_2 x_2, \\ i\hbar\bar{p}_3+\alpha_3 x_3, u\end{pmatrix}\end{pmatrix}\end{pmatrix} du\right)} \begin{pmatrix} \breve{\Psi}_{D0_1} \\ \breve{\Psi}_{D0_2} \\ \breve{\Psi}_{D0_3} \\ \breve{\Psi}_{D0_4} \end{pmatrix} dp_1 dp_2 dp_3 \\
&\breve{\Psi}_{D0}\begin{pmatrix} \bar{x}_1,\bar{x}_2,\bar{x}_3; \\ \bar{p}_1,\bar{p}_2,\bar{p}_3; t=0 \end{pmatrix} = \begin{pmatrix} \breve{\Psi}_{D0_1} \\ \breve{\Psi}_{D0_2} \\ \breve{\Psi}_{D0_3} \\ \breve{\Psi}_{D0_4} \end{pmatrix} : \text{ Transformed Initial-condition vector}
\end{aligned} \tag{8}$$

where the explicit form of the $\hat{\mathbf{H}}_{\mathrm{Dirac}_{\mathbf{4\times 4}}}$ in the exponent 4x4 matrix integral is supplied as

$$\begin{pmatrix}
\begin{pmatrix} mc^2 \\ +ecA_0 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1x_1, \\ i\hbar\bar{p}_2+\alpha_2x_2, \\ i\hbar\bar{p}_3+\alpha_3x_3, t \end{pmatrix} \end{pmatrix} & 0 & \begin{pmatrix} c\left(-i\hbar\bar{x}_3+\gamma_3p_3\right) \\ -eA_3 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1x_1, \\ i\hbar\bar{p}_2+\alpha_2x_2, \\ i\hbar\bar{p}_3+\alpha_3x_3, t \end{pmatrix} \end{pmatrix} & \begin{pmatrix} c\left(-i\hbar\bar{x}_1+\gamma_1p_1\right) \\ -eA_1 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1x_1, \\ i\hbar\bar{p}_2+\alpha_2x_2, \\ i\hbar\bar{p}_3+\alpha_3x_3, t \end{pmatrix} \\ -i(c\left(-i\hbar\bar{x}_2+\gamma_2p_2\right) \\ -eA_2 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1x_1, \\ i\hbar\bar{p}_2+\alpha_2x_2, \\ i\hbar\bar{p}_3+\alpha_3x_3, t \end{pmatrix}) \end{pmatrix} \\
0 & \begin{pmatrix} mc^2 \\ +ecA_0 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1x_1, \\ i\hbar\bar{p}_2+\alpha_2x_2, \\ i\hbar\bar{p}_3+\alpha_3x_3, t \end{pmatrix} \end{pmatrix} & \begin{pmatrix} c\left(-i\hbar\bar{x}_1+\gamma_1p_1\right) \\ -eA_1 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1x_1, \\ i\hbar\bar{p}_2+\alpha_2x_2, \\ i\hbar\bar{p}_3+\alpha_3x_3, t \end{pmatrix} \\ +i(c\left(-i\hbar\bar{x}_2+\gamma_2p_2\right) \\ -eA_2 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1x_1, \\ i\hbar\bar{p}_2+\alpha_2x_2, \\ i\hbar\bar{p}_3+\alpha_3x_3, t \end{pmatrix}) \end{pmatrix} & -\begin{pmatrix} c\left(-i\hbar\bar{x}_3+\gamma_3p_3\right) \\ -eA_3 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1x_1, \\ i\hbar\bar{p}_2+\alpha_2x_2, \\ i\hbar\bar{p}_3+\alpha_3x_3, t \end{pmatrix} \end{pmatrix} \\
\begin{pmatrix} c\left(-i\hbar\bar{x}_3+\gamma_3p_3\right) \\ -eA_3 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1x_1, \\ i\hbar\bar{p}_2+\alpha_2x_2, \\ i\hbar\bar{p}_3+\alpha_3x_3, t \end{pmatrix} \end{pmatrix} & \begin{pmatrix} c\left(-i\hbar\bar{x}_1+\gamma_1p_1\right) \\ -eA_1 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1x_1, \\ i\hbar\bar{p}_2+\alpha_2x_2, \\ i\hbar\bar{p}_3+\alpha_3x_3, t \end{pmatrix} \\ -i(c\left(-i\hbar\bar{x}_2+\gamma_2p_2\right) \\ -eA_2 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1x_1, \\ i\hbar\bar{p}_2+\alpha_2x_2, \\ i\hbar\bar{p}_3+\alpha_3x_3, t \end{pmatrix}) \end{pmatrix} & \begin{pmatrix} -mc^2 \\ +ecA_0 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1x_1, \\ i\hbar\bar{p}_2+\alpha_2x_2, \\ i\hbar\bar{p}_3+\alpha_3x_3, t \end{pmatrix} \end{pmatrix} & 0 \\
\begin{pmatrix} c\left(-i\hbar\bar{x}_1+\gamma_1p_1\right) \\ -eA_1 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1x_1, \\ i\hbar\bar{p}_2+\alpha_2x_2, \\ i\hbar\bar{p}_3+\alpha_3x_3, t \end{pmatrix} \\ +i(c\left(-i\hbar\bar{x}_2+\gamma_2p_2\right) \\ -eA_2 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1x_1, \\ i\hbar\bar{p}_2+\alpha_2x_2, \\ i\hbar\bar{p}_3+\alpha_3x_3, t \end{pmatrix}) \end{pmatrix} & -\begin{pmatrix} c\left(-i\hbar\bar{x}_3+\gamma_3p_3\right) \\ -eA_3 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1x_1, \\ i\hbar\bar{p}_2+\alpha_2x_2, \\ i\hbar\bar{p}_3+\alpha_3x_3, t \end{pmatrix} \end{pmatrix} & 0 & \begin{pmatrix} -mc^2 \\ +ecA_0 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1x_1, \\ i\hbar\bar{p}_2+\alpha_2x_2, \\ i\hbar\bar{p}_3+\alpha_3x_3, t \end{pmatrix} \end{pmatrix}
\end{pmatrix} \tag{9}$$

yields by way of (6), the associated Majorana wavefunction for the dynamics of the minimal-coupled system with arbitrary profile EM interaction and initial-conditions, in terms of the quadrature solutions for the related Dirac system just calculated above in (8)

$$\begin{aligned}
\rho_{M_2} &= \tfrac{1}{\sqrt{2}} \begin{pmatrix} \Psi_{D1}\left(x_1,x_2,x_3,t\right) + i\Psi^*_{D4}\left(x_1,x_2,x_3,t\right) \\ \Psi_{D2}\left(x_1,x_2,x_3,t\right) - i\Psi^*_{D3}\left(x_1,x_2,x_3,t\right) \end{pmatrix} \\
\rho_{M_1} &= \tfrac{i}{\sqrt{2}} \begin{pmatrix} \Psi_{D1}\left(x_1,x_2,x_3,t\right) - i\Psi^*_{D4}\left(x_1,x_2,x_3,t\right) \\ \Psi_{D2}\left(x_1,x_2,x_3,t\right) + i\Psi^*_{D3}\left(x_1,x_2,x_3,t\right) \end{pmatrix} \\
\rho_{M} \quad \text{configuration space} &= \begin{pmatrix} \rho_{M_1} \\ \rho_{M_2} \end{pmatrix}_{\text{configuration space}} \\
\rho_{M_0} \quad \text{configuration space} &= \begin{pmatrix} \rho_{M_{10}} \\ \rho_{M_{20}} \end{pmatrix}_{\text{configuration space}} \qquad : \text{ Majorana Initial-State} \\
\rho_{M_{20}} &= \tfrac{1}{\sqrt{2}} \begin{pmatrix} \Psi_{D1_0}\left(x_1,x_2,x_3,t\right) + i\Psi^*_{D4_0}\left(x_1,x_2,x_3,t\right) \\ \Psi_{D2_0}\left(x_1,x_2,x_3,t\right) - i\Psi^*_{D3_0}\left(x_1,x_2,x_3,t\right) \end{pmatrix} \\
\rho_{M_{10}} &= \tfrac{i}{\sqrt{2}} \begin{pmatrix} \Psi_{D1_0}\left(x_1,x_2,x_3,t\right) - i\Psi^*_{D4_0}\left(x_1,x_2,x_3,t\right) \\ \Psi_{D2_0}\left(x_1,x_2,x_3,t\right) + i\Psi^*_{D3_0}\left(x_1,x_2,x_3,t\right) \end{pmatrix}
\end{aligned} \tag{10}$$

References

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