EJTP 3, No. 10 (2006) 191–209 *Electronic Journal of Theoretical Physics*

Why do Majorana Neutrinos Run Faster than Dirac Neutrinos?

Zhi-zhong Xing * and He Zhang †

Institute of High Energy Physics, Chinese Academy of Sciences, P.O. Box 918, Beijing 100049, China

Received 15 February 2006, Published 28 May 2006

Abstract: The τ-lepton dominance in the one-loop renormalization-group equations (RGEs) of neutrinos sets a cute criterion to parametrize the 3×3 lepton flavor mixing matrix U: its elements U_{3i} (for $i = 1, 2, 3$) should be as simple as possible. Such a novel parametrization is different from the "standard" one used in the literature and can lead to greatly simplified RGEs for three mixing angles and the physical CP-violating phase(s), no matter whether neutrinos are Dirac or Majorana particles. We show that the RGEs of Dirac neutrinos are not identical with those of Majorana neutrinos even if two Majorana CP-violating phases vanish. As the latter can keep vanishing from the electroweak scale to the typical seesaw scale, it makes sense to explore the similarities and differences between the RGE running effects of Dirac and Majorana neutrinos. We conclude that Majorana neutrinos are in general expected to run faster (i.e., more significantly) than Dirac neutrinos.

Keywords: Majorana neutrinos, Dirac neutrinos, neutrino interactions
PACS (2006): 14.60.Pq, 13.15.+g, 25.30.Pt

1. Introduction

The fact that neutrinos have non-vanishing masses is a clean signal of new physics beyond the standard model (SM). To understand the small neutrino mass-squared differences and the large lepton flavor mixing angles observed in solar and atmospheric neutrino oscillation experiments [1, 2, 3, 4], many models based on either new flavor symmetries or some unspecified interactions have been proposed at some superhigh energy scales [5]. Their phenomenological consequences at low energy scales can be confronted with current experimental data, after radiative corrections to those neutrino mixing parameters are

* xingzz@ihep.ac.cn
† zhanghe@ihep.ac.cn

properly taken into account. Such radiative corrections can be very significant in some cases, for instance, when the masses of three light neutrinos are nearly degenerate or the value of $\tan\beta$ is very large in the minimal supersymmetric standard model (MSSM).

An elegant idea to explain the smallness of left-handed neutrino masses is to introduce very heavy right-handed neutrinos and lepton number violation into the SM or MSSM and to make use of the famous seesaw mechanism [6]. Below the seesaw scale, where heavy Majorana neutrinos become decoupled, the effective neutrino coupling matrix κ obeys the following one-loop renormalization-group equation (RGE) [7]:

$$16\pi^2 \frac{\mathrm{d}\kappa}{\mathrm{d}t} = \alpha_{\rm M}\kappa + C\left[\left(Y_l Y_l^\dagger\right)\kappa + \kappa\left(Y_l Y_l^\dagger\right)^T\right] , \tag{1}$$

where $t \equiv \ln(\mu/\Lambda_{\rm SS})$ with μ being an arbitrary renormalization scale between the electroweak scale $\Lambda_{\rm EW} \sim 10^2$ GeV and the typical seesaw scale $\Lambda_{\rm SS} \sim 10^{10\cdots14}$ GeV, and Y_l is the charged-lepton Yukawa coupling matrix. In the SM, $C = -1.5$ and $\alpha_{\rm M} \approx -3g_2^2 + 6y_t^2 + \lambda$; and in the MSSM, $C = 1$ and $\alpha_{\rm M} \approx -1.2g_1^2 - 6g_2^2 + 6y_t^2$, where g_1 and g_2 denote the gauge couplings, y_t stands for the top-quark Yukawa coupling, and λ is the Higgs self-coupling in the SM.

There are also some good reasons to speculate that massive neutrinos might be the Dirac particles [8]. In this case, the Dirac neutrino Yukawa coupling matrix Y_ν must be extremely suppressed in magnitude, so as to reproduce the light neutrino masses of $\mathcal{O}(1)$ eV or smaller at the electroweak scale. Y_ν can run from a superhigh energy scale down to $\Lambda_{\rm EW}$ via the one-loop RGE

$$16\pi^2 \frac{\mathrm{d}\omega}{\mathrm{d}t} = 2\alpha_{\rm D}\omega + C\left[\left(Y_l Y_l^\dagger\right)\omega + \omega\left(Y_l Y_l^\dagger\right)\right] , \tag{2}$$

where $\omega \equiv Y_\nu Y_\nu^\dagger$, $\alpha_{\rm D} \approx -0.45g_1^2 - 2.25g_2^2 + 3y_t^2$ in the SM or $\alpha_{\rm D} \approx -0.6g_1^2 - 3g_2^2 + 3y_t^2$ in the MSSM [8]. In obtaining Eq. (2), we have safely neglected those tiny terms of $\mathcal{O}(\omega^2)$.

Eq. (1) or (2) allows us to derive the explicit RGEs for all neutrino mass and mixing parameters in the flavor basis where Y_l is diagonal and real (positive). In this basis, we have $\kappa = \mathcal{V}_{\rm M}\overline{\kappa}\mathcal{V}_{\rm M}^T$ with $\overline{\kappa} = {\rm Diag}\{\kappa_1, \kappa_2, \kappa_3\}$ for Majorana neutrinos; or $\omega = \mathcal{V}_{\rm D}\overline{\omega}\mathcal{V}_{\rm D}^\dagger$ with $\overline{\omega} = {\rm Diag}\{y_1^2, y_2^2, y_3^2\}$ for Dirac neutrinos. $\mathcal{V}_{\rm M}$ or $\mathcal{V}_{\rm D}$ is just the lepton flavor mixing matrix. At $\Lambda_{\rm EW}$, Majorana neutrino masses are given by $m_i = v^2\kappa_i$ (SM) or $m_i = v^2\kappa_i\sin^2\beta$ (MSSM), while Dirac neutrino masses are given by $m_i = vy_i$ (SM) or $m_i = vy_i\sin\beta$ (MSSM) with $v \approx 174$ GeV.

Note that $\mathcal{V}_{\rm M}$ (or $\mathcal{V}_{\rm D}$) can be parametrized in terms of three mixing angles and a few CP-violating phases. Their RGEs consist of the flavor-dependent contributions from $Y_l Y_l^\dagger$. Because of $y_e^2 \ll y_\mu^2 \ll y_\tau^2$, where y_e, y_μ and y_τ correspond to the electron, muon and tau Yukawa couplings, we only need to take account of the dominant τ-lepton contribution to those one-loop RGEs of neutrino mixing angles and CP-violating phases in an excellent approximation. A careful analysis shows that the τ-dominance is closely associated with the matrix elements $(\mathcal{V}_{\rm M})_{3i}$ or $(\mathcal{V}_{\rm D})_{3i}$ (for $i = 1, 2, 3$). This important observation implies that very concise RGEs can be obtained for those flavor mixing and CP-violating parameters, if $\mathcal{V}_{\rm M}$ (or $\mathcal{V}_{\rm D}$) is parametrized in such a way that its elements

$(\mathcal{V}_{\rm M})_{3i}$ (or $(\mathcal{V}_{\rm D})_{3i}$) are as simple as possible. One may then make use of this criterion to choose the most suitable parametrization of $\mathcal{V}_{\rm M}$ or $\mathcal{V}_{\rm D}$ in deriving the one-loop RGEs.

We find that the so-called "standard" parametrization (advocated by the Particle Data Group [9]), which has extensively been used in describing lepton flavor mixing, does not satisfy the above criterion. Instead, the parametrization recommended in Ref. [10] fulfills our present requirement

$$
\begin{aligned}
U &= \begin{pmatrix} c_l & s_l & 0 \\ -s_l & c_l & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\phi} & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} \begin{pmatrix} c_\nu & -s_\nu & 0 \\ s_\nu & c_\nu & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} s_l s_\nu c + c_l c_\nu e^{-i\phi} & s_l c_\nu c - c_l s_\nu e^{-i\phi} & s_l s \\ c_l s_\nu c - s_l c_\nu e^{-i\phi} & c_l c_\nu c + s_l s_\nu e^{-i\phi} & c_l s \\ -s_\nu s & -c_\nu s & c \end{pmatrix} ,
\end{aligned}
\tag{3}
$$

where $c_l \equiv \cos\theta_l$, $s_l \equiv \sin\theta_l$, $c_\nu \equiv \cos\theta_\nu$, $s_\nu \equiv \sin\theta_\nu$, $c \equiv \cos\theta$ and $s \equiv \sin\theta$. In general, we have $\mathcal{V}_{\rm M} = Q_{\rm M} U P_{\rm M}$ for Majorana neutrinos or $\mathcal{V}_{\rm D} = Q_{\rm D} U P_{\rm D}$ for Dirac neutrinos, where $P_{\rm M}$ (or $P_{\rm D}$) and $Q_{\rm M}$ (or $Q_{\rm D}$) are two diagonal phase matrices. It is clear that U_{3i} (for $i = 1, 2, 3$) shown in Eq. (3) are simple enough to describe the τ-dominant terms in those one-loop RGEs of θ_l, θ_ν, θ and ϕ (as well as two Majorana phases of $\mathcal{V}_{\rm M}$ coming from $P_{\rm M}$). In the approximation that solar and atmospheric neutrino oscillations are nearly decoupled [11], three mixing angles of U can simply be related to those of solar, atmospheric and CHOOZ neutrino oscillations [1, 2, 3]: $\theta_{12} \approx \theta_\nu$, $\theta_{23} \approx \theta$ and $\theta_{13} \approx \theta_l \sin\theta$. Hence our parametrization is also a convenient option to describe current neutrino oscillation data.

One purpose of this paper is to show that Eq. (3) is actually a novel parametrization of τ-dominance in the one-loop RGEs of neutrino mixing angles and CP-violating phases. Compared with the "standard" parametrization used in the literature, Eq. (3) leads to greatly simplified results for relevant RGEs [12]. The latter can therefore allow us to understand the RGE running behaviors of lepton flavor mixing parameters in a much simpler and more transparent way, which is of course useful for model building at a superhigh energy scale to explore possible flavor symmetries or flavor dynamics responsible for the origin of neutrino masses and CP violation.

The other purpose of this paper is to explore the similarities and differences between the RGE running behaviors of Dirac and Majorana neutrinos in the especially interesting case that two Majorana CP-violating phases vanish [13]. We shall show that $\rho = \sigma = 0$ at a specific energy scale leads to $\dot\rho = \dot\sigma = 0$, implying that ρ and σ can keep vanishing at any energy scales between $\Lambda_{\rm EW}$ and $\Lambda_{\rm SS}$. In this case, only three mixing angles $(\theta_l, \theta_\nu, \theta)$ and the so-called Dirac CP-violating phase ϕ undergo the RGE evolution. Note that a kind of underlying flavor symmetry may actually forbid two Majorana phases to take non-zero values in a concrete neutrino model. It is therefore meaningful to ask whether the RGE running behaviors of Majorana neutrinos with $\rho = \sigma = 0$ are identical to those of Dirac neutrinos.

2. RG Equations for Majorana Neutrinos

The general strategy and tactics about how to derive the one-loop RGEs for Majorana neutrino mixing parameters have been outlined in Refs. [14, 15, 16, 17]. To be specific, we take $P_{\rm M} = {\rm Diag}\left\{e^{i\rho}, e^{i\sigma}, 1\right\}$ and $Q_{\rm M} = {\rm Diag}\left\{e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}\right\}$. The phase parameters ρ and σ are physical and referred to as the Majorana phases. The phase parameters ϕ_i (for $i = 1, 2, 3$) are unphysical, but they have their own RGE evolution. Following the procedure described in Ref. [14] and taking the τ-dominance approximation, we obtain the RGEs of κ_i (for $i = 1, 2, 3$) from Eq. (1):

$$\dot{\kappa}_i \; = \; \frac{\kappa_i}{16\pi^2} \left(\alpha_{\rm M} + 2Cy^2_\tau \left|U_{3i}\right|^2\right) \; , \tag{4}$$

where $\dot{\kappa}_i \equiv {\rm d}\kappa_i/{\rm d}t$. In addition, the quantities ρ, σ, ϕ_i and U_{ij} (for $i, j = 1, 2, 3$) satisfy the following equations:

$$\begin{aligned} \sum^3_{j=1} \left[U^*_{j1} \left(i\dot{U}_{j1} - U_{j1}\dot{\phi}_j\right)\right] &= \dot{\rho} \; , \\ \sum^3_{j=1} \left[U^*_{j2} \left(i\dot{U}_{j2} - U_{j2}\dot{\phi}_j\right)\right] &= \dot{\sigma} \; , \\ \sum^3_{j=1} \left[U^*_{j3} \left(i\dot{U}_{j3} - U_{j3}\dot{\phi}_j\right)\right] &= 0 \; ; \end{aligned} \tag{5}$$

and

$$\begin{aligned} \sum^3_{j=1} \left[U^*_{j1} \left(\dot{U}_{j2} + iU_{j2}\dot{\phi}_j\right)\right] &= -\frac{Cy^2_\tau}{16\pi^2} \; e^{i(\rho-\sigma)} \left[\zeta^{-1}_{12} {\rm Re}\left(U^*_{31}U_{32}e^{i(\sigma-\rho)}\right) + i\zeta_{12} {\rm Im}\left(U^*_{31}U_{32}e^{i(\sigma-\rho)}\right)\right] \; , \\ \sum^3_{j=1} \left[U^*_{j1} \left(\dot{U}_{j3} + iU_{j3}\dot{\phi}_j\right)\right] &= -\frac{Cy^2_\tau}{16\pi^2} \; e^{i\rho} \left[\zeta^{-1}_{13} {\rm Re}\left(U^*_{31}U_{33}e^{-i\rho}\right) + i\zeta_{13} {\rm Im}\left(U^*_{31}U_{33}e^{-i\rho}\right)\right] \; , \\ \sum^3_{j=1} \left[U^*_{j2} \left(\dot{U}_{j3} + iU_{j3}\dot{\phi}_j\right)\right] &= -\frac{Cy^2_\tau}{16\pi^2} \; e^{i\sigma} \left[\zeta^{-1}_{23} {\rm Re}\left(U^*_{32}U_{33}e^{-i\sigma}\right) + i\zeta_{23} {\rm Im}\left(U^*_{32}U_{33}e^{-i\sigma}\right)\right] \; , \end{aligned} \tag{6}$$

where $\zeta_{ij} \equiv \left(\kappa_i - \kappa_j\right) / \left(\kappa_i + \kappa_j\right)$. Obviously, those y^2_τ-associated terms only consist of the matrix elements U_{3i} (for $i = 1, 2, 3$). If a parametrization of U assures U_{3i} to be as simple as possible, then the resultant RGEs of relevant neutrino mixing angles and CP-violating phases will be as concise as possible. One can see that the parametrization of U given in Eq. (3) just accords with such a criterion, while the "standard" parametrization advocated in Ref. [9] and used in many papers (e.g., Refs. [14, 15, 16, 17, 18]) does not satisfy this requirement.

Combining Eq. (3) with Eqs. (4), (5) and (6), we arrive at

$$\begin{aligned} \dot{\kappa}_1 &= \frac{\kappa_1}{16\pi^2} \left(\alpha_{\rm M} + 2Cy^2_\tau s^2_\nu s^2\right) \; , \\ \dot{\kappa}_2 &= \frac{\kappa_2}{16\pi^2} \left(\alpha_{\rm M} + 2Cy^2_\tau c^2_\nu s^2\right) \; , \\ \dot{\kappa}_3 &= \frac{\kappa_3}{16\pi^2} \left(\alpha_{\rm M} + 2Cy^2_\tau c^2\right) \; ; \end{aligned} \tag{7}$$

and

$$\begin{aligned}
\dot{\theta}_l &= \frac{Cy_\tau^2}{16\pi^2}\ c_\nu s_\nu c \left[\left(\zeta_{13}^{-1} c_\rho c_{(\rho-\phi)} + \zeta_{13} s_\rho s_{(\rho-\phi)}\right) - \left(\zeta_{23}^{-1} c_\sigma c_{(\sigma-\phi)} + \zeta_{23} s_\sigma s_{(\sigma-\phi)}\right)\right] \ , \\
\dot{\theta}_\nu &= \frac{Cy_\tau^2}{16\pi^2}\ c_\nu s_\nu \left[s^2 \left(\zeta_{12}^{-1} c^2_{(\sigma-\rho)} + \zeta_{12} s^2_{(\sigma-\rho)}\right) + c^2 \left(\zeta_{13}^{-1} c_\rho^2 + \zeta_{13} s_\rho^2\right) - c^2 \left(\zeta_{23}^{-1} c_\sigma^2 + \zeta_{23} s_\sigma^2\right)\right] \ , \\
\dot{\theta} &= \frac{Cy_\tau^2}{16\pi^2}\ cs \left[s_\nu^2 \left(\zeta_{13}^{-1} c_\rho^2 + \zeta_{13} s_\rho^2\right) + c_\nu^2 \left(\zeta_{23}^{-1} c_\sigma^2 + \zeta_{23} s_\sigma^2\right)\right] \ ;
\end{aligned} \tag{8}$$

as well as

$$\begin{aligned}
\dot{\phi} &= \frac{Cy_\tau^2}{16\pi^2} \Big[\left(c_l^2 - s_l^2\right) c_l^{-1} s_l^{-1} c_\nu s_\nu c \left(\zeta_{13}^{-1} c_\rho s_{(\rho-\phi)} - \zeta_{13} s_\rho c_{(\rho-\phi)} - \zeta_{23}^{-1} c_\sigma s_{(\sigma-\phi)} + \zeta_{23} s_\sigma c_{(\sigma-\phi)}\right) \\
&\quad + \widehat{\zeta}_{12} s^2 c_{(\sigma-\rho)} s_{(\sigma-\rho)} + \widehat{\zeta}_{13} \left(s_\nu^2 - c_\nu^2 c^2\right) c_\rho s_\rho + \widehat{\zeta}_{23} \left(c_\nu^2 - s_\nu^2 c^2\right) c_\sigma s_\sigma\Big] \ , \\
\dot{\rho} &= \frac{Cy_\tau^2}{16\pi^2} \left[\widehat{\zeta}_{12} c_\nu^2 s^2 c_{(\sigma-\rho)} s_{(\sigma-\rho)} + \widehat{\zeta}_{13} \left(s_\nu^2 s^2 - c^2\right) c_\rho s_\rho + \widehat{\zeta}_{23} c_\nu^2 s^2 c_\sigma s_\sigma\right] \ , \\
\dot{\sigma} &= \frac{Cy_\tau^2}{16\pi^2} \left[\widehat{\zeta}_{12} s_\nu^2 s^2 c_{(\sigma-\rho)} s_{(\sigma-\rho)} + \widehat{\zeta}_{13} s_\nu^2 s^2 c_\rho s_\rho + \widehat{\zeta}_{23} \left(c_\nu^2 s^2 - c^2\right) c_\sigma s_\sigma\right] \ ,
\end{aligned} \tag{9}$$

where $\widehat{\zeta}_{ij} \equiv \zeta_{ij}^{-1} - \zeta_{ij} = 4\kappa_i\kappa_j/\left(\kappa_i^2 - \kappa_j^2\right)$, $c_a \equiv \cos a$ and $s_a \equiv \sin a$ (for $a = \rho$, σ, $\sigma-\rho$, $\rho-\phi$ or $\sigma-\phi$). Comparing the RGEs of three mixing angles and three CP-violating phases obtained in Eqs. (8) and (9) with their counterparts given in Refs. [14, 15, 16, 17, 18], which were derived by using the "standard" parametrization, we find that great simplification and conciseness have been achieved for our present analytical results.

As a by-product, the RGEs of three unphysical phases ϕ_i are listed below:

$$\begin{aligned}
\dot{\phi}_1 &= +\frac{Cy_\tau^2}{16\pi^2} \Big[c_l s_l^{-1} c_\nu s_\nu c \left(\zeta_{13}^{-1} c_\rho s_{(\rho-\phi)} - \zeta_{13} s_\rho c_{(\rho-\phi)} - \zeta_{23}^{-1} c_\sigma s_{(\sigma-\phi)} + \zeta_{23} s_\sigma c_{(\sigma-\phi)}\right) \\
&\quad + c^2 \left(\widehat{\zeta}_{13} s_\nu^2 c_\rho s_\rho + \widehat{\zeta}_{23} c_\nu^2 c_\sigma s_\sigma\right)\Big] \ , \\
\dot{\phi}_2 &= -\frac{Cy_\tau^2}{16\pi^2} \Big[c_l^{-1} s_l c_\nu s_\nu c \left(\zeta_{13}^{-1} c_\rho s_{(\rho-\phi)} - \zeta_{13} s_\rho c_{(\rho-\phi)} - \zeta_{23}^{-1} c_\sigma s_{(\sigma-\phi)} + \zeta_{23} s_\sigma c_{(\sigma-\phi)}\right) \\
&\quad - c^2 \left(\widehat{\zeta}_{13} s_\nu^2 c_\rho s_\rho + \widehat{\zeta}_{23} c_\nu^2 c_\sigma s_\sigma\right)\Big] \ , \\
\dot{\phi}_3 &= -\frac{Cy_\tau^2}{16\pi^2} \left[s^2 \left(\widehat{\zeta}_{13} s_\nu^2 c_\rho s_\rho + \widehat{\zeta}_{23} c_\nu^2 c_\sigma s_\sigma\right)\right] \ .
\end{aligned} \tag{10}$$

It is easy to check that the relationship $\dot{\phi} = \dot{\rho} + \dot{\sigma} + \dot{\phi}_1 + \dot{\phi}_2 + \dot{\phi}_3$ holds. That is why ϕ_i should not be ignored in deriving the RGEs of other physical parameters, although these three phases can finally be rotated away via rephasing the charged-lepton fields.

Some qualitative comments on the basic features of Eqs. (7)–(10) are in order.

(a) The RGE running behaviors of three neutrino masses m_i (or equivalently κ_i) are essentially identical and determined by $\alpha_{\rm M}$ [15], unless $\tan\beta$ is large enough in the MSSM to make the y_τ^2-associated term is competitive with the $\alpha_{\rm M}$ term. In our phase convention, $\dot{\kappa}_i$ or $\dot{m}_i$ (for $i = 1, 2, 3$) are independent of the CP-violating phase ϕ.

(b) Among three mixing angles, only the derivative of θ_ν contains a term proportional to ζ_{12}^{-1}. Note that $\zeta_{ij}^{-1} = -(m_i + m_j)^2/\Delta m_{ji}^2$ with $\Delta m_{ji}^2 \equiv m_j^2 - m_i^2$ holds, and current solar and atmospheric neutrino oscillation data yield $\Delta m^2_{21} \approx 8\times10^{-5}$ eV^2 and $|\Delta m^2_{32}| \approx |\Delta m^2_{31}| \approx 2.5\times10^{-3}$ eV^2 [11]. Thus θ_ν is in general more sensitive to radiative corrections than θ_l and θ. The RGE running of θ_ν can be suppressed through the fine-tuning of $(\sigma-\rho)$. The smallest mixing angle θ_l may get radiative corrections even if its initial value is zero, thus it can be radiatively generated from other mixing angles and CP-violating phases.

(c) The RGE running behavior of ϕ is quite different from those of ρ and σ, because it includes a peculiar term proportional to s_l^{-1}. This term, which dominates $\dot{\phi}$ when θ_l is sufficiently small, becomes divergent in the limit $\theta_l \to 0$. Indeed, ϕ is not well-defined if θ_l is exactly vanishing. But both θ_l and ϕ can be radiatively generated. We may require that $\dot{\phi}$ should remain finite when θ_l approaches zero, implying that the following necessary condition can be extracted from the expression of $\dot{\phi}$ in Eq. (9):

$$\zeta^{-1}_{13} c_\rho s_{(\rho-\phi)} - \zeta_{13} s_\rho c_{(\rho-\phi)} - \zeta^{-1}_{23} c_\sigma s_{(\sigma-\phi)} + \zeta_{23} s_\sigma c_{(\sigma-\phi)} \;=\; 0 \; . \tag{11}$$

It turns out that

$$\tan\phi = \frac{\widehat{\zeta}_{13}\sin 2\rho - \widehat{\zeta}_{23}\sin 2\sigma}{\left(\zeta^{-1}_{13} + \zeta_{13} + \widehat{\zeta}_{13}\cos 2\rho\right) - \left(\zeta^{-1}_{23} + \zeta_{23} + \widehat{\zeta}_{23}\cos 2\sigma\right)} \tag{12}$$

holds, a result similar to the one obtained in Eq. (25) of Ref. [15]. Note that the initial value of θ_l, if it is exactly zero or extremely small, may immediately drive ϕ to its *quasi-fixed point* (see Ref. [19] for a relevant study of the quasi-fixed point in the "standard" parametrization of lepton flavor mixing). In this interesting case, Eq. (12) can be used to understand the relationship between ϕ and two Majorana phases ρ and σ at the quasi-fixed point.

(d) On the other hand, the RGE running behaviors of ρ and σ are relatively mild in comparison with that of ϕ. A remarkable feature of $\dot{\rho}$ and $\dot{\sigma}$ is that they will vanish, if both ρ and σ are initially vanishing. This observation indicates that ρ and σ cannot simultaneously be generated from ϕ via the one-loop RGE evolution. In contrast, a different conclusion was drawn in Ref. [18], where the "standard" parametrization with a slightly changed phase convention was utilized.

(e) As for three unphysical phases, ϕ_2 and ϕ_3 only have relatively mild RGE running effects, while the running behavior of ϕ_1 may be violent for sufficiently small θ_l. A quasi-fixed point of ϕ_1 is also expected in the limit $\theta_l \to 0$ and under the circumstance given by Eq. (11) or (12).

3. RG Equations for Dirac Neutrinos

Now let us derive the one-loop RGEs for Dirac neutrino mixing parameters. To be specific, we take $P_{\rm D} = {\rm Diag}\left\{e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3}\right\}$ and $Q_{\rm D} = {\rm Diag}\left\{e^{i\alpha}, e^{i\beta}, 1\right\}$. The phase matrix $P_{\rm D}$ can be cancelled in ω, thus it does not take part in the RGE evolution. The phase parameters α and β are also unphysical, but they have their own RGE running behaviors. Following the procedure described in Ref. [8] and taking the τ-dominance approximation, we get the RGEs of y_i (for $i = 1, 2, 3$) from Eq. (2):

$$\dot{y}_i = \frac{y_i}{16\pi^2} \left(\alpha_{\rm D} + C y_\tau^2 \left| U_{3i} \right|^2 \right) , \tag{13}$$

where $\dot{y}_i \equiv {\rm d}y_i/{\rm d}t$. On the other hand, the quantities α, β and U_{ij} (for $i, j = 1, 2, 3$) satisfy the following equations:

$$\begin{aligned}
\sum_{j=1}^{3} \left(U_{j1}^* \dot{U}_{j2} \right) + i \left(\dot{\alpha} U_{11}^* U_{12} + \dot{\beta} U_{21}^* U_{22} \right) &= -\frac{C y_\tau^2}{16\pi^2} \, \xi_{12} U_{31}^* U_{32} , \\
\sum_{j=1}^{3} \left(U_{j1}^* \dot{U}_{j3} \right) + i \left(\dot{\alpha} U_{11}^* U_{13} + \dot{\beta} U_{21}^* U_{23} \right) &= -\frac{C y_\tau^2}{16\pi^2} \, \xi_{13} U_{31}^* U_{33} , \\
\sum_{j=1}^{3} \left(U_{j2}^* \dot{U}_{j3} \right) + i \left(\dot{\alpha} U_{12}^* U_{13} + \dot{\beta} U_{22}^* U_{23} \right) &= -\frac{C y_\tau^2}{16\pi^2} \, \xi_{23} U_{32}^* U_{33} ,
\end{aligned} \tag{14}$$

where $\xi_{ij} \equiv \left(y_i^2 + y_j^2 \right) / \left(y_i^2 - y_j^2 \right)$. Again, the y_τ^2-associated terms in Eqs. (13) and (14) only contain U_{3i} (for $i = 1, 2, 3$). These RGEs can therefore be specified in a relatively concise way, if the parametrization of U shown in Eq. (3) is taken into account.

Explicitly, the Yukawa coupling eigenvalues of three Dirac neutrinos obey the one-loop RGEs

$$\begin{aligned}
\dot{y}_1 &= \frac{y_1}{16\pi^2} \left(\alpha_{\rm D} + C y_\tau^2 s_\nu^2 s^2 \right) , \\
\dot{y}_2 &= \frac{y_2}{16\pi^2} \left(\alpha_{\rm D} + C y_\tau^2 c_\nu^2 s^2 \right) , \\
\dot{y}_3 &= \frac{y_3}{16\pi^2} \left(\alpha_{\rm D} + C y_\tau^2 c^2 \right) .
\end{aligned} \tag{15}$$

The RGEs of three neutrino mixing angles and one (physical) CP-violating phase are given by

$$\begin{aligned}
\dot{\theta}_l &= +\frac{C y_\tau^2}{16\pi^2} \, c_\nu s_\nu c c_\phi \left(\xi_{13} - \xi_{23} \right) , \\
\dot{\theta}_\nu &= +\frac{C y_\tau^2}{16\pi^2} \, c_\nu s_\nu \left[s^2 \xi_{12} + c^2 \left(\xi_{13} - \xi_{23} \right) \right] , \\
\dot{\theta} &= +\frac{C y_\tau^2}{16\pi^2} \, c s \left(s_\nu^2 \xi_{13} + c_\nu^2 \xi_{23} \right) , \\
\dot{\phi} &= -\frac{C y_\tau^2}{16\pi^2} \left(c_l^2 - s_l^2 \right) c_l^{-1} s_l^{-1} c_\nu s_\nu c s_\phi \left(\xi_{13} - \xi_{23} \right) ,
\end{aligned} \tag{16}$$

where $c_\phi \equiv \cos\phi$ and $s_\phi \equiv \sin\phi$. The RGEs of two unphysical phases α and β read

$$\begin{aligned}
\dot{\alpha} &= -\frac{C y_\tau^2}{16\pi^2} \, c_l s_l^{-1} c_\nu s_\nu c s_\phi \left(\xi_{13} - \xi_{23} \right) , \\
\dot{\beta} &= +\frac{C y_\tau^2}{16\pi^2} \, c_l^{-1} s_l c_\nu s_\nu c s_\phi \left(\xi_{13} - \xi_{23} \right) .
\end{aligned} \tag{17}$$

The relationship $\dot{\phi} = \dot{\alpha} + \dot{\beta}$ holds obviously, implying that α and β are not negligible in deriving the RGEs of other physical parameters. One can see that our analytical results are really concise, thanks to the novel parametrization of U that we have taken.

Some qualitative remarks on the main features of Eqs. (15), (16) and (17) are in order.

(1) Like the Majorana case, the RGE running behaviors of three Dirac neutrino masses m_i (or equivalently y_i) are nearly identical and determined by $\alpha_{\rm D}$ [8], unless $\tan\beta$ is sufficiently large in the MSSM. It is also worth mentioning that $\dot{y}_i$ or $\dot{m}_i$ (for $i = 1, 2, 3$) are independent of both the CP-violating phase ϕ and the smallest mixing angle θ_l in our parametrization.

(2) The derivative of θ_ν consists of a term proportional to $\xi_{12} = -(m_1^2 + m_2^2)/\Delta m_{21}^2$. Hence θ_ν is in general more sensitive to radiative corrections than θ_l and θ, whose derivatives are only dependent on $\xi_{13} = -(m_1^2+m_3^2)/\Delta m_{31}^2$ and $\xi_{23} = -(m_2^2+m_3^2)/\Delta m_{32}^2$. Given θ_ν and θ at a specific energy scale, the smallest mixing angle θ_l can be radiatively generated at another energy scale. In this case, however, it is impossible to simultaneously generate the CP-violating phase ϕ (see Ref. [8] for a similar conclusion in the "standard" parametrization of U). The reason is simply that ϕ can always be rotated away when θ_l is exactly vanishing, and the proportionality relationship between $\dot{\phi}$ and $\sin\phi$ forbids ϕ to be generated even when θ_l becomes non-vanishing.

(3) Different from the Majorana case, there is no non-trivial *quasi-fixed point* in the RGE evolution of ϕ for Dirac neutrinos. If $\dot{\phi}$ is required to keep finite when θ_l approaches zero, then ϕ itself must approach zero or π, as indicated by Eq. (16). On the other hand, $\dot{\theta}_l \propto \cos\phi$ implies that the RGE running of θ_l has a turning point characterized by $\phi = \pi/2$ (i.e., $\dot{\theta}_l$ flips its sign at this point). Hence two interesting conclusions analogous to those drawn in Ref. [8] can be achieved: first, θ_l can never cross zero if $\theta_l \neq 0$ and $\sin\phi \neq 0$ hold at a certain energy scale; second, CP will always be a good symmetry if $\theta_l = 0$ or $\sin\phi = 0$ holds at a certain energy scale.

(4) The RGE running behavior of α is quite similar to that of ϕ, because $\dot{\phi} = \dot{\alpha}\,(1 - \tan^2\theta_l)$ holds. In addition, $\dot{\beta} = -\dot{\alpha}\tan^2\theta_l$ holds, implying that β only gets some relatively mild RGE corrections.

Let us remark that the Jarlskog invariant of CP violation [20] takes the same form for Dirac and Majorana neutrinos: $\mathcal{J} = c_l s_l c_\nu s_\nu c s^2 s_\phi$. If neutrinos are Dirac particles, the one-loop RGE of $\mathcal{J}^{\rm D}$ can be expressed as

$$\dot{\mathcal{J}}^{\rm D} \;=\; \frac{C y_\tau^2}{16\pi^2}\, \mathcal{J}_{\rm D} \left[\left(c_\nu^2 - s_\nu^2\right) s^2 \xi_{12} + \left(c^2 - s_\nu^2 s^2\right) \xi_{13} + \left(c^2 - c_\nu^2 s^2\right) \xi_{23} \right] . \tag{18}$$

It becomes obvious that $\mathcal{J}^{\rm D} = 0$ will be a stable result independent of the renormalization scales, provided θ_l or $\sin\phi$ initially vanishes at a given scale. In comparison, we have

$$\begin{aligned} \dot{\mathcal{J}}^{\rm M} = \frac{C y_\tau^2}{16\pi^2} \Big\{ \mathcal{J}_{\rm M} \Big[\left(c_\nu^2 - s_\nu^2\right) s^2 \left(\zeta_{12}^{-1} c_{(\sigma-\rho)}^2 + \zeta_{12} s_{(\sigma-\rho)}^2\right) + \left(c^2 - s_\nu^2 s^2\right) \left(\zeta_{13}^{-1} c_\rho^2 + \zeta_{13} s_\rho^2\right) \\ + \left(c^2 - c_\nu^2 s^2\right) \left(\zeta_{23}^{-1} c_\sigma^2 + \zeta_{23} s_\sigma^2\right) \Big] + c_\nu s_\nu c s^2 \left(C_{12} \widehat{\zeta}_{12} + C_{13} \widehat{\zeta}_{13} + C_{23} \widehat{\zeta}_{23} \right) \Big\} \end{aligned} \tag{19}$$

for Majorana neutrinos, where

$$\begin{aligned} C_{12} &= c_l s_l s^2 c_\phi c_{(\sigma-\rho)} s_{(\sigma-\rho)} \; , \\ C_{13} &= \left[c_l s_l c_\phi \left(s_\nu^2 - c_\nu^2 c^2\right) + \left(c_l^2 - s_l^2\right) c_\nu s_\nu c \right] c_\rho s_\rho \; , \\ C_{23} &= \left[c_l s_l c_\phi \left(c_\nu^2 - s_\nu^2 c^2\right) - \left(c_l^2 - s_l^2\right) c_\nu s_\nu c \right] c_\sigma s_\sigma \; . \end{aligned} \tag{20}$$

One can see that $\mathcal{J}^{\rm M}$ can be radiatively generated from two non-trivial Majorana phases ρ and σ, even if it is initially vanishing at a specific scale. Taking $\rho = \sigma = 0$, we arrive at $C_{12} = C_{13} = C_{23} = 0$ as well as $\dot{\rho} = \dot{\sigma} = 0$. But it is impossible to obtain the equality $\dot{\mathcal{J}}^{\rm M}(\rho = \sigma = 0) = \dot{\mathcal{J}}^{\rm D}$, because $\zeta_{12}^{-1} = \xi_{12}$, $\zeta_{13}^{-1} = \xi_{13}$ and $\zeta_{23}^{-1} = \xi_{23}$ (or equivalently $m_1 m_2 = m_1 m_3 = m_2 m_3 = 0$) cannot simultaneously hold. This observation demonstrates again that the RGE running behavior of $\mathcal{J}^{\rm M}$ is essentially different from that of $\mathcal{J}^{\rm D}$.

4. Comparison Between Dirac and Majorana Neutrinos

The one-loop RGEs for three Yukawa coupling eigenvalues of Dirac neutrinos (y_i with $i = 1, 2, 3$) and their four flavor mixing parameters (θ_l, θ_ν, θ and ϕ) have been derived above. Here we replace y_i by m_i. The RGEs of three neutrino masses, three mixing angles and one CP-violating phase can then be written as

$$\begin{aligned}
\dot{m}_1 &= \frac{m_1}{16\pi^2}\left(\alpha_{\rm D} + C y_\tau^2 s_\nu^2 s^2\right) , \\
\dot{m}_2 &= \frac{m_2}{16\pi^2}\left(\alpha_{\rm D} + C y_\tau^2 c_\nu^2 s^2\right) , \\
\dot{m}_3 &= \frac{m_3}{16\pi^2}\left(\alpha_{\rm D} + C y_\tau^2 c^2\right) ;
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
\dot{\theta}_l &= +\frac{C y_\tau^2}{8\pi^2}\, c_\nu s_\nu c c_\phi \frac{m_3^2\left(m_2^2 - m_1^2\right)}{\left(m_3^2 - m_1^2\right)\left(m_3^2 - m_2^2\right)} , \\
\dot{\theta}_\nu &= -\frac{C y_\tau^2}{16\pi^2}\, c_\nu s_\nu \left[s^2 \frac{m_2^2 + m_1^2}{m_2^2 - m_1^2} - c^2 \frac{2 m_3^2\left(m_2^2 - m_1^2\right)}{\left(m_3^2 - m_1^2\right)\left(m_3^2 - m_2^2\right)}\right] , \\
\dot{\theta} &= -\frac{C y_\tau^2}{16\pi^2}\, c s \left(s_\nu^2 \frac{m_3^2 + m_1^2}{m_3^2 - m_1^2} + c_\nu^2 \frac{m_3^2 + m_2^2}{m_3^2 - m_2^2}\right) , \\
\dot{\phi} &= -\frac{C y_\tau^2}{8\pi^2}\left(c_l^2 - s_l^2\right) c_l^{-1} s_l^{-1} c_\nu s_\nu c s_\phi \frac{m_3^2\left(m_2^2 - m_1^2\right)}{\left(m_3^2 - m_1^2\right)\left(m_3^2 - m_2^2\right)} .
\end{aligned} \tag{22}$$

Note that the neutrino mass-squared differences Δm_{31}^2 and Δm_{32}^2 are much larger in magnitude than Δm_{21}^2, as indicated by current experimental data. Typically, $\Delta m_{21}^2 \approx 8.0 \times 10^{-5}$ eV^2 and $|\Delta m_{31}^2| \approx |\Delta m_{32}^2| \approx 2.5 \times 10^{-3}$ eV^2 [11]. Among three neutrino mixing angles, the RGE running of θ_ν is expected to be most significant. The CP-violating phase ϕ may significantly evolve from one energy scale to another, if θ_l takes sufficiently small values. These qualitative features will become clearer in our subsequent numerical calculations.

The one-loop RGEs for three effective coupling eigenvalues of Majorana neutrinos (κ_i with $i = 1, 2, 3$) and their six flavor mixing parameters (θ_l, θ_ν, θ, ϕ, ρ and σ) have been presented in section II. Here we replace κ_i by m_i and take $\rho = \sigma = 0$ at either $\Lambda_{\rm EW}$ or $\Lambda_{\rm SS}$. As one can see from Eq. (9), $\rho = \sigma = 0$ leads to $\dot{\rho} = \dot{\sigma} = 0$. In other words, two Majorana phases ρ and σ keep vanishing at any energy scales between $\Lambda_{\rm EW}$ and $\Lambda_{\rm SS}$. One may safely simplify the RGEs of θ_l, θ_ν, θ and ϕ obtained in Eqs. (8) and (9) by setting

$\rho = \sigma = 0$, and then compare them with their Dirac counterparts on the same footing. In this case, we arrive at

$$\begin{aligned} \dot{m}_1 &= \frac{m_1}{16\pi^2}\left(\alpha_{\rm M} + 2Cy_\tau^2 s_\nu^2 s^2\right) \ , \\ \dot{m}_2 &= \frac{m_2}{16\pi^2}\left(\alpha_{\rm M} + 2Cy_\tau^2 c_\nu^2 s^2\right) \ , \\ \dot{m}_3 &= \frac{m_3}{16\pi^2}\left(\alpha_{\rm M} + 2Cy_\tau^2 c^2\right) \ ; \end{aligned} \tag{23}$$

and

$$\begin{aligned} \dot{\theta}_l &= +\frac{Cy_\tau^2}{8\pi^2}\, c_\nu s_\nu c c_\phi \frac{m_3(m_2 - m_1)}{(m_3 - m_1)(m_3 - m_2)} \ , \\ \dot{\theta}_\nu &= -\frac{Cy_\tau^2}{16\pi^2}\, c_\nu s_\nu \left[s^2 \frac{m_2 + m_1}{m_2 - m_1} - c^2 \frac{2m_3 (m_2 - m_1)}{(m_3 - m_1)(m_3 - m_2)} \right] \ , \\ \dot{\theta} &= -\frac{Cy_\tau^2}{16\pi^2}\, cs \left(s_\nu^2 \frac{m_3 + m_1}{m_3 - m_1} + c_\nu^2 \frac{m_3 + m_2}{m_3 - m_2} \right) \ , \\ \dot{\phi} &= -\frac{Cy_\tau^2}{8\pi^2}\left(c_l^2 - s_l^2\right) c_l^{-1} s_l^{-1} c_\nu s_\nu c s_\phi \frac{m_3 (m_2 - m_1)}{(m_3 - m_1)(m_3 - m_2)} \ . \end{aligned} \tag{24}$$

As a consequence of $\Delta m_{21}^2 \ll |\Delta m_{31}^2| \approx |\Delta m_{32}^2|$, the mixing angle θ_ν is most sensitive to radiative corrections. The RGE evolution of the CP-violating phase ϕ depends strongly on the smallness of θ_l, on the other hand. These qualitative features are essentially analogous to what we have pointed out for Dirac neutrinos.

It is interesting to note that Eq. (23) can actually be obtained from Eq. (21) with the replacements $\alpha_{\rm D} \Longrightarrow \alpha_{\rm M}$ and $C \Longrightarrow 2C$, while Eq. (24) can be achieved from Eq. (22) with the replacements $m_i^2 \Longrightarrow m_i$ (for $i = 1, 2, 3$). These similarities and differences imply that it is very non-trivial to distinguish between the RGE running behaviors of Dirac neutrinos and Majorana neutrinos with vanishing Majorana CP-violating phases.

Taking $\rho = \sigma = 0$ in the Majorana case, we obtain a simplified expression of $\dot{\mathcal{J}}^{\rm M}$,

$$\dot{\mathcal{J}}^{\rm M} = \frac{Cy_\tau^2}{16\pi^2}\, \mathcal{J}^{\rm M} \left[\left(c_\nu^2 - s_\nu^2\right) s^2 \frac{m_2 + m_1}{m_2 - m_1} + \left(c^2 - s_\nu^2 s^2\right) \frac{m_3 + m_1}{m_3 - m_1} + \left(c^2 - c_\nu^2 s^2\right) \frac{m_3 + m_2}{m_3 - m_2} \right] \ , \tag{25}$$

which is very analogous to $\dot{\mathcal{J}}^{\rm D}$ of Dirac neutrinos,

$$\dot{\mathcal{J}}^{\rm D} = \frac{Cy_\tau^2}{16\pi^2}\, \mathcal{J}^{\rm D} \left[\left(c_\nu^2 - s_\nu^2\right) s^2 \frac{m_2^2 + m_1^2}{m_2^2 - m_1^2} + \left(c^2 - s_\nu^2 s^2\right) \frac{m_3^2 + m_1^2}{m_3^2 - m_1^2} + \left(c^2 - c_\nu^2 s^2\right) \frac{m_3^2 + m_2^2}{m_3^2 - m_2^2} \right] \ . \tag{26}$$

It is obvious that Eq. (26) can be obtained from Eq. (25) with the replacements $m_i \Longrightarrow m_i^2$ (for $i = 1, 2, 3$). Note that $\dot{\mathcal{J}}^{\rm D} \propto \mathcal{J}^{\rm D}$ (or $\dot{\mathcal{J}}^{\rm M} \propto \mathcal{J}^{\rm M}$) holds. This result implies that the Jarlskog parameter will keep vanishing at any energy scales between $\Lambda_{\rm EW}$ and $\Lambda_{\rm SS}$, if it initially vanishes at either $\Lambda_{\rm EW}$ or $\Lambda_{\rm SS}$.

Because $\alpha_{\rm M} > \alpha_{\rm D}$ and $(m_j + m_i)/(m_j - m_i) > (m_j^2 + m_i^2)/(m_j^2 - m_i^2)$ hold (for $m_j > m_i$), the RGE running of each Majorana neutrino parameter is in general expected to be faster (i.e., more significant) than the RGE running of the corresponding Dirac neutrino parameter.

5. Numerical Results

In view of the fact that the absolute mass scale of three light neutrinos and the sign of Δm^2_{32} remain unknown at present, let us consider four typical patterns of the neutrino mass spectrum:

- Normal hierarchy (NH): $m_1 \ll m_2 \ll m_3$. For simplicity, we typically take $m_1 = 0$ at $\Lambda_{\rm EW}$ in our numerical calculations. Then $m_2 = \sqrt{\Delta m^2_{21}}$ and $m_3 = \sqrt{|\Delta m^2_{32}| + \Delta m^2_{21}}$ can be determined from current experimental data.
- Inverted hierarchy (IH): $m_3 \ll m_1 \lesssim m_2$. For simplicity, w typically take $m_3 = 0$ at $\Lambda_{\rm EW}$ in our numerical calculations. Then $m_2 = \sqrt{|\Delta m^2_{32}|}$ and $m_1 = \sqrt{|\Delta m^2_{32}| - \Delta m^2_{21}}$ can be determined from current experimental data.
- Near degeneracy (ND) with $\Delta m^2_{32} > 0$: $m_1 \lesssim m_2 \lesssim m_3$. For simplicity, we typically take $m_1 = 0.2$ eV at $\Lambda_{\rm EW}$ in our numerical calculations.
- Near degeneracy (ND) with $\Delta m^2_{32} < 0$: $m_3 \lesssim m_1 \lesssim m_2$. For simplicity, we typically take $m_1 = 0.2$ eV at $\Lambda_{\rm EW}$ in our numerical calculations.

In addition, we take $\Delta m^2_{21} = 8.0 \times 10^{-5}$ eV^2, $|\Delta m^2_{32}| = 2.5 \times 10^{-3}$ eV^2, $\theta_\nu = 33.8^\circ$, $\theta = 45^\circ$, $\theta_l = 0.5^\circ$ and $\phi = 90^\circ$ as typical inputs at $\Lambda_{\rm EW} \sim M_Z$ in our numerical calculations [13].

5.1 Neutrino Masses

In either the SM or the MSSM with small $\tan\beta$, the RGE running behaviors of three neutrino masses are dominated by $\alpha_{\rm D}$ or $\alpha_{\rm M}$. The y^2_τ-associated term of $\dot{m}^2_i$ in Eq. (21) or (23) becomes important only when $\tan\beta$ takes sufficiently large values in the MSSM [14, 15, 16, 17]. Note that $\alpha_{\rm M} = 2\alpha_{\rm D}$ holds in the MSSM, in which the running effects of m_i for Majorana neutrinos are twice as large as those for Dirac neutrinos.

The first plot in Fig. 1 illustrates the ratios $R_i \equiv m_i(\mu)/m_i(M_Z)$ changing with the energy scale μ in the SM for Dirac and Majorana neutrinos, where $m_1(M_Z) = 0.2$ eV and $M_H = 180$ GeV (the Higgs mass) have typically been input. Since the running of m_i is governed by $\alpha_{\rm D}$ or $\alpha_{\rm M}$, $R_1 \approx R_2 \approx R_3$ holds to a high degree of accuracy. Furthermore, the behaviors of R_i are actually independent of the initial value of m_1 and possible patterns of the neutrino mass spectrum. We observe that R_i in the Majorana case is always larger than R_i in the Dirac case, and their discrepancy can be as large as 0.7 at $\mu \sim \Lambda_{\rm SS} \sim 10^{14}$ GeV.

The relation $R_1 \approx R_2 \approx R_3$ is also a very good approximation in the MSSM with small $\tan\beta$, as shown by the second plot in Fig. 1, where $\tan\beta = 10$ has been input. It is clear that R_i in the Dirac case is numerically distinguishable from R_i in the Majorana case, in particular when the energy scale μ far exceeds M_Z.

If $\tan\beta$ is sufficiently large, the common scaling of three neutrino masses in the RGE evolution will fail [15]. The splitting of R_1, R_2 and R_3, which increases with the energy scale μ, is illustrated by the third plot in Fig. 1 with the input $\tan\beta = 50$. One can see that R_i in the Dirac case is always smaller than R_i in the Majorana case, and their discrepancy is distinguishable at the scales $\mu \gg M_Z$.

5.2 Neutrino Mixing Parameters

Radiative corrections to three neutrino mixing angles, the Dirac CP-violating phase and the Jarlskog parameter are all controlled by the τ-lepton Yukawa coupling eigenvalue y_τ. Because of $y_\tau^2/(8\pi^2) \approx 1.3 \times 10^{-6}$ (SM) or $y_\tau^2/(8\pi^2) \approx 1.3 \times 10^{-6}\,(1+\tan^2\beta)$ (MSSM) at M_Z, significant RGE running effects are expected to appear in the MSSM case when $\tan\beta$ is sufficiently large. To illustrate, here we simply concentrate on the MSSM with $\tan\beta = 50$ and consider four typical patterns of the neutrino mass spectrum in our subsequent discussions and calculations.

(1) In the NH case with $m_1 = 0$, the RGEs of θ_l, θ_ν, θ, ϕ and $\mathcal{J}$ can be simplified as

$$\begin{aligned} \dot{\theta}_l &= +\frac{Cy_\tau^2}{16\pi^2}\, c_\nu s_\nu c c_\phi r \;, \qquad & \dot{\theta}_\nu &= -\frac{Cy_\tau^2}{16\pi^2}\, c_\nu s_\nu \left(s^2 - c^2 r\right) \;, \\ \dot{\theta} &= -\frac{Cy_\tau^2}{16\pi^2}\, cs\left(1+r\right) \;, \qquad & \dot{\phi} &= -\frac{Cy_\tau^2}{16\pi^2}\left(c_l^2 - s_l^2\right) c_l^{-1} s_l^{-1} c_\nu s_\nu c s_\phi r \;, \\ \dot{\mathcal{J}} &= \frac{Cy_\tau^2}{16\pi^2}\,\mathcal{J}\left[2\left(s_\nu^2 s^2 - c^2\right) - \left(c^2 - c_\nu^2 s^2\right) r\right] \;, \end{aligned} \tag{27}$$

in which $r = 2m_2^2/(m_3^2 - m_2^2)$ for Dirac neutrinos or $r = 2m_2/(m_3 - m_2)$ for Majorana neutrinos. Current experimental data yield $r_{\rm D} \approx 0.06$ and $r_{\rm M} \approx 0.4$. Both of them are too small to compensate for the strong suppression induced by y_τ^2 in Eq. (27). Thus the RGE corrections to those flavor mixing and CP-violating parameters are not important in the NH case. Note, however, that the radiative correction to ϕ can be very significant when θ_l is extremely small or becomes vanishing. We find that ϕ quickly approaches its quasi-fixed point $\phi_{\rm QF} = 0$ or π in the $\theta_l \to 0$ limit, an interesting phenomenon which is remarkably different from the non-trivial quasi-fixed point of ϕ discovered in the general $(\rho \neq \sigma \neq 0)$ case for Majorana neutrinos [19]. One can also see that both $\mathcal{J} = 0$ and $\dot{\mathcal{J}} = 0$ hold when θ_l vanishes; i.e., CP is a good symmetry in this limit.

(2) In the IH case with $m_3 = 0$, we arrive at

$$\begin{aligned} \dot{\theta}_l = \dot{\phi} = 0, \qquad \dot{\theta}_\nu = -\frac{Cy_\tau^2}{16\pi^2}\, c_\nu s_\nu s^2 r' \;, \qquad \dot{\theta} = +\frac{Cy_\tau^2}{16\pi^2}\, cs \;, \\ \dot{\mathcal{J}} = \frac{Cy_\tau^2}{16\pi^2}\,\mathcal{J}\left[3c^2 - 1 + \left(s_\nu^2 - c_\nu^2\right) s^2 r'\right] \;, \end{aligned} \tag{28}$$

where $r' = (m_2^2 + m_1^2)/(m_2^2 - m_1^2)$ for Dirac neutrinos or $r' = (m_2 + m_1)/(m_2 - m_1)$ for Majorana neutrinos. We observe that radiative corrections to θ_l and ϕ are vanishingly small, and the correction to θ is also insignificant. Nevertheless, the RGE running effects of θ_ν and $\mathcal{J}$ may get enhanced by r', whose typical value reads $r'_{\rm D} \approx 60$ or $r'_{\rm M} \approx 120$ at M_Z. Fig. 2 illustrates the evolution of θ_ν and $\mathcal{J}$ in the IH case. The discrepancy between Dirac and Majorana cases is obviously distinguishable for both parameters, when the energy scale is much higher than M_Z. In particular, $\mathcal{J}^{\rm D} \sim 2\mathcal{J}^{\rm M}$ holds at $\mu \sim \Lambda_{\rm SS}$, because the corresponding value of θ_ν for Majorana neutrinos is roughly half of that for Dirac neutrinos.

(3) In the ND case with $\Delta m^2_{32} > 0$ and $m_1 = 0.2$ eV, the RGE corrections to those neutrino mixing parameters can significantly be enhanced by the ratios $(m_i^2 + m_j^2)/(m_i^2 - m_j^2)$ in Eqs. (22) and (26) for Dirac neutrinos, or by the ratios $(m_i + m_j)/(m_i - m_j)$ in Eqs. (24) and (9) for Majorana neutrinos. We illustrate the typical evolution behaviors of θ_l, θ_ν, θ and ϕ in Fig. 3. One can see that Majorana neutrinos undergo the RGE corrections more significantly than Dirac neutrinos. The discrepancy between these two cases is about 10° for either θ or ϕ at $\mu \gg M_Z$. It is therefore possible to distinguish the running of Majorana neutrinos from that of Dirac neutrinos. The difference between $\mathcal{J}^{\rm D}$ and $\mathcal{J}^{\rm M}$ is insignificant even at $\mu \sim \Lambda_{\rm SS}$, as shown in Fig. 4, partly because the increase (or decrease) of θ_l can somehow compensate for the decrease (or increase) of θ and ϕ in the Majorana (or Dirac) case.

(4) In the ND case with $\Delta m^2_{32} < 0$ and $m_1 = 0.2$ eV, we get similar enhancements in the RGEs of those neutrino mixing parameters induced by the ratios $(m_i^2+m_j^2)/(m_i^2-m_j^2)$ for Dirac neutrinos, or by $(m_i+m_j)/(m_i-m_j)$ for Majorana neutrinos. However, only the running of θ is sensitive to the sign flip of Δm^2_{32}, as one can see from Eqs. (22) and (24)–(26), in which $\dot{\theta}_\nu$ and $\dot{\mathcal{J}}$ are dominated by the term proportional to $(m_2^2+m_1^2)/(m_2^2-m_1^2)$ (Dirac) or $(m_2 + m_1)/(m_2 - m_1)$ (Majorana). Then the numerical results for θ_l, θ_ν, ϕ and $\mathcal{J}$ in the present case are very similar to those in the ND case with $\Delta m^2_{32} > 0$. For simplicity, we only illustrate the evolution of θ in Fig. 5 by taking $\Delta m^2_{32} < 0$. It is obvious that the running behavior of θ for either Dirac or Majorana neutrinos in Fig. 5 is essentially opposite (or complementary) to that in Fig. 3, just due to the sign flip of Δm^2_{32}.

6. Summery

We have pointed out that the τ-lepton dominance in the one-loop RG equations of relevant neutrino mixing quantities allows us to set a criterion for the choice of the most appropriate parametrization of the lepton flavor mixing matrix U: its elements U_{3i} (for $i = 1, 2, 3$) should be as simple as possible. Such a novel parametrization *does* exist, but it is quite different from the “standard” parametrization advocated by the Particle Data Group and used in the literature. We have shown that this parametrization can lead to greatly simplified RG equations for three mixing angles and the physical CP-violating phase(s), no matter whether neutrinos are Dirac or Majorana particles.

Another goal of this paper is to examine whether the RGE running behaviors of Majorana neutrinos are still different from those of Dirac neutrinos, if two Majorana CP-violating phases vanish at a given energy scale. For this purpose, it is essential to use the afore-mentioned parametrization of the 3×3 lepton flavor mixing matrix, such that its two Majorana phases keep vanishing in the RGE evolution from one scale to another. Taking $\rho = \sigma = 0$ at the electroweak scale, we have carefully compared the similarities and differences between the RGEs of θ_l, θ_ν, θ and ϕ for Majorana neutrinos and those for Dirac neutrinos. Our numerical calculations show that it is possible to distinguish between these two cases in the MSSM with sizable $\tan\beta$, in particular when the masses of three neutrinos are nearly degenerate or have an inverted hierarchy. Furthermore, we conclude that Majorana neutrinos are in general expected to run faster (i.e., more significantly) than Dirac neutrinos from one energy scale to another.

Of course, the numerical examples presented in this work are mainly for the purpose of illustration. The point is that the nature of neutrinos determines their RGE running behaviors, and the latter may be crucial for building a realistic neutrino model. We expect that our analysis can not only complement those previous studies of radiative corrections to the physical parameters of Dirac and Majorana neutrinos, but also help us understand the dynamical role of Majorana phases in a more general picture of flavor physics.

This work is supported in part by the National Nature Science Foundation of China.

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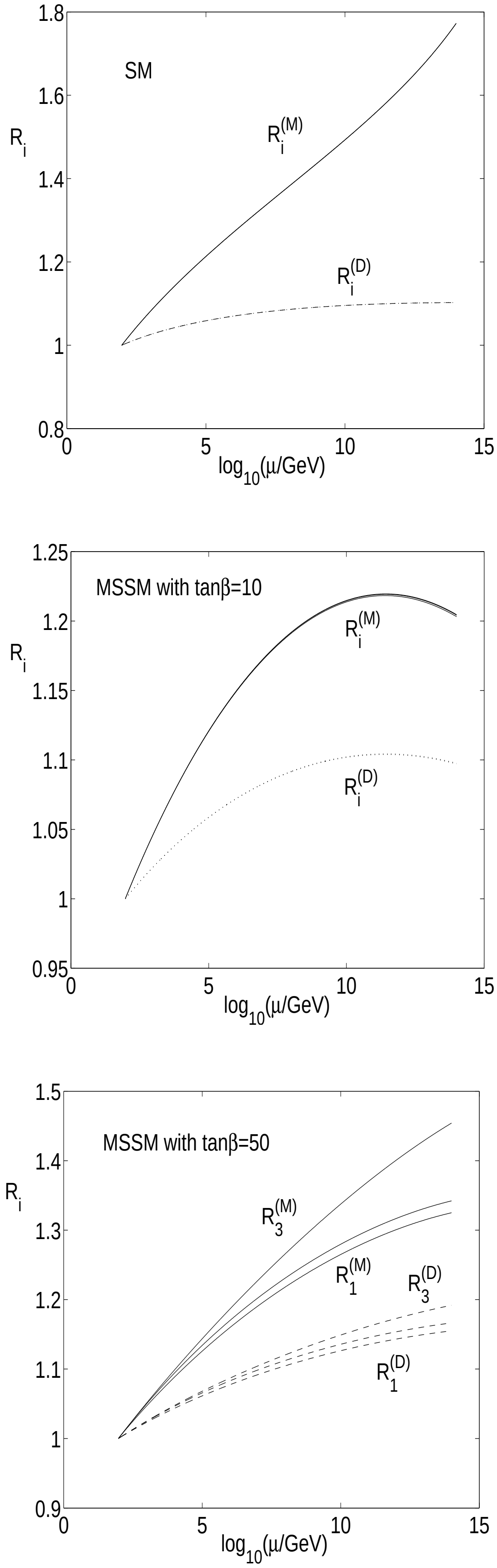

Fig. 1 The running neutrino mass ratios $R_i = m_i(\mu)/m_i(M_Z)$ (for $i = 1, 2, 3$), where the dashed and solid curves stand respectively for the Dirac and Majorana cases.

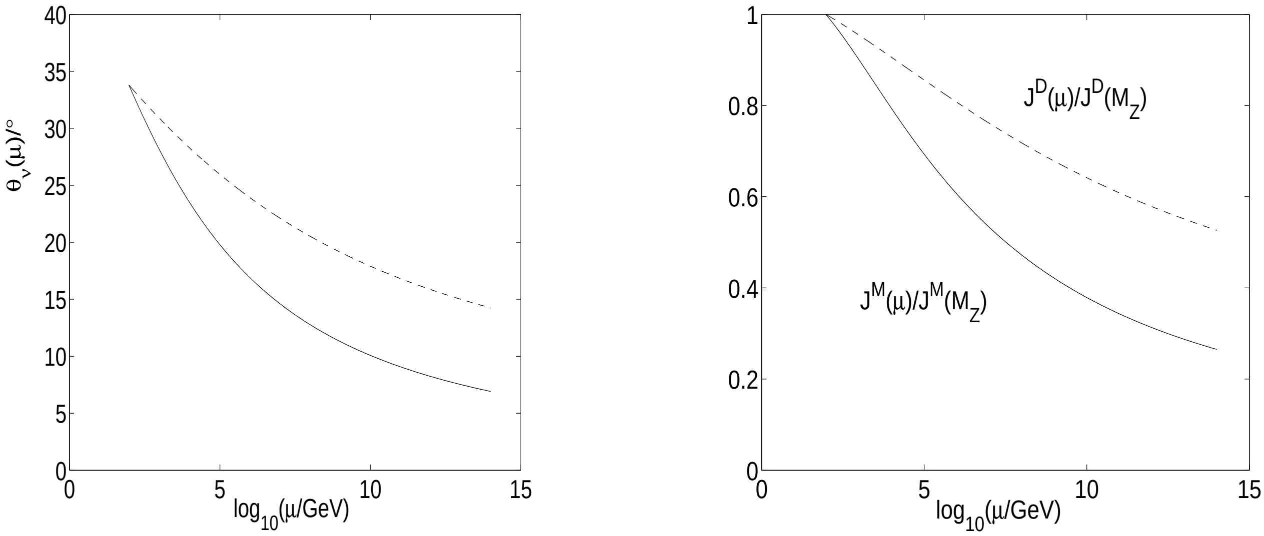

Fig. 2 The running behaviors of θ_ν and $\mathcal{J}$ in the IH case with $\tan\beta = 50$ and $m_3 = 0$ at M_Z within the MSSM, where the dashed and solid curves stand respectively for the Dirac and Majorana cases, and $\mathcal{J}^{\rm D}(M_Z) = \mathcal{J}^{\rm M}(M_Z) \approx 0.0014$.

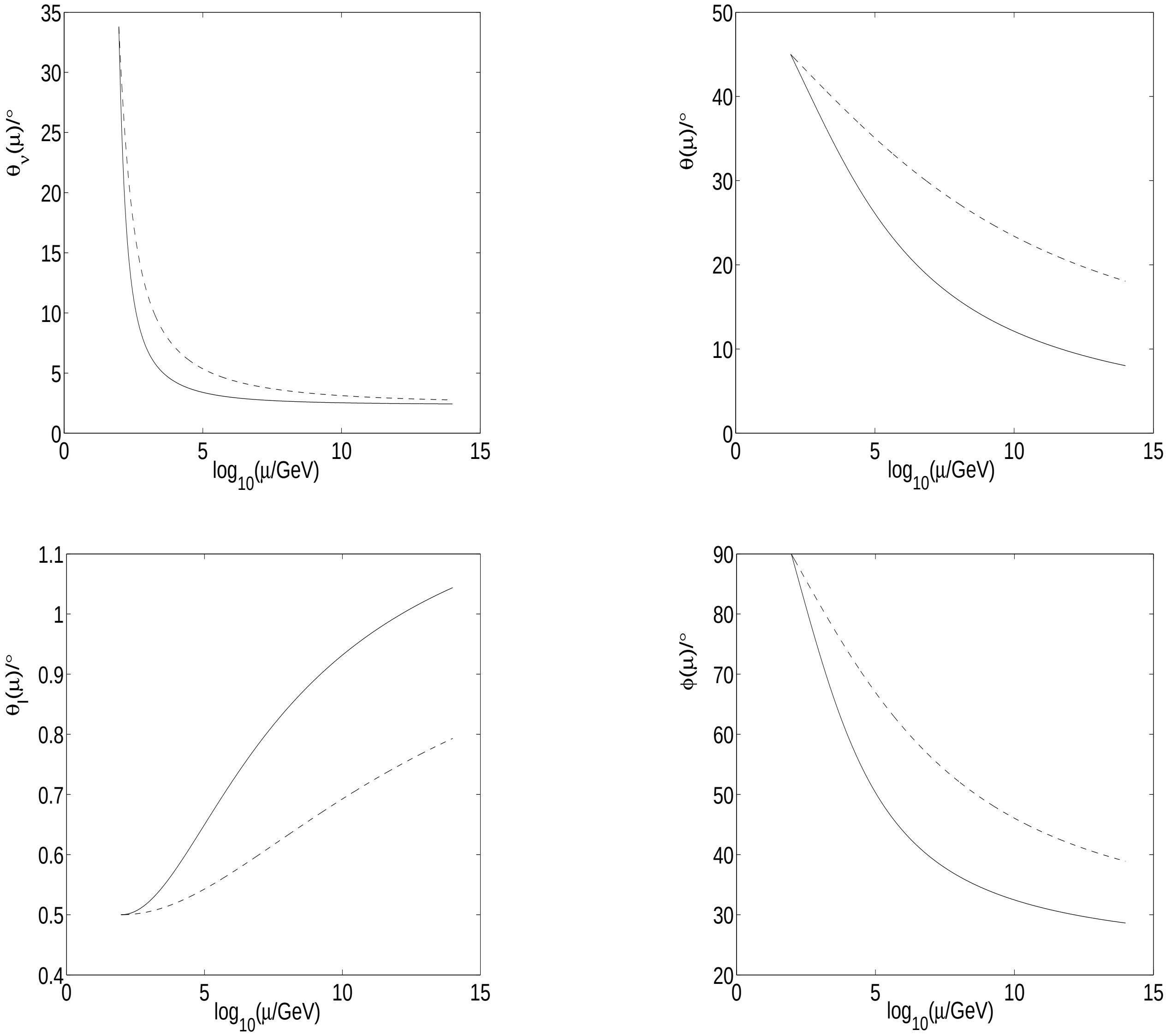

Fig. 3 The running behaviors of θ_l, θ_ν, θ and ϕ in the ND case with $\Delta m^2_{32} > 0$, $\tan\beta = 50$ and $m_1(M_Z) = 0.2$ eV within the MSSM, where the dashed and solid curves stand respectively for the Dirac and Majorana cases.

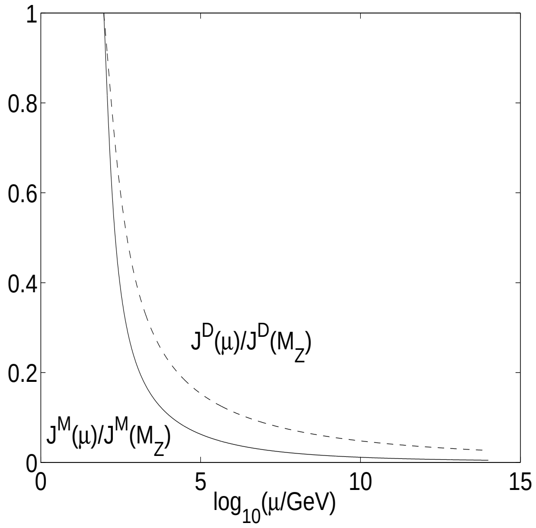

Fig. 4 The running behavior of $\mathcal{J}$ in the ND case with $\Delta m^2_{32} > 0$, $\tan\beta = 50$ and $m_1(M_Z) = 0.2$ eV within the MSSM, where the dashed and solid curves stand respectively for the Dirac and Majorana cases, and $\mathcal{J}^{\rm D}(M_Z) = \mathcal{J}^{\rm M}(M_Z) \approx 0.0014$.

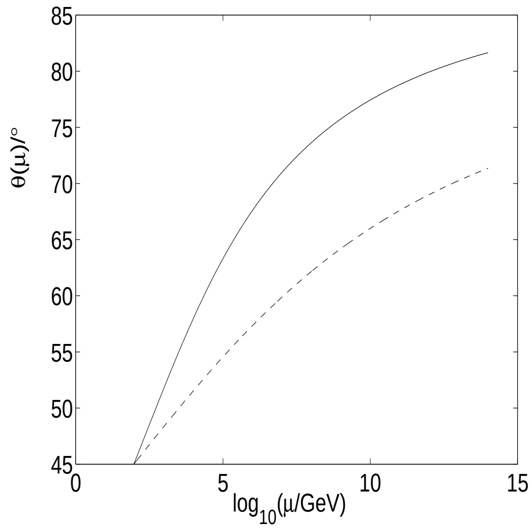

Fig. 5 The running behavior of θ in the ND case with $\Delta m^2_{32} < 0$, $\tan\beta = 50$ and $m_1(M_Z) = 0.2$ eV within the MSSM, where the dashed and solid curves stand respectively for the Dirac and Majorana cases.