EJTP **3, No. 10** (2006) 17–31 *Electronic Journal of Theoretical Physics*

Majorana Equation and Exotics: Higher Derivative Models, Anyons and Noncommutative Geometry

Mikhail S. Plyushchay *

Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago 2, Chile

Received 20 April 2006, Published 28 May 2006

Abstract: In 1932 Ettore Majorana proposed an infinite-component relativistic wave equation for particles of arbitrary integer and half-integer spin. In the late 80s and early 90s it was found that the higher-derivative geometric particle models underlie the Majorana equation, and that its (2+1)-dimensional analogue provides with a natural basis for the description of relativistic anyons. We review these aspects and discuss the relationship of the equation to the exotic planar Galilei symmetry and noncommutative geometry. We also point out the relation of some Abelian gauge field theories with Chern-Simons terms to the Landau problem in the noncommutative plane from the perspective of the Majorana equation.

Keywords: Majorana Equation, Dirac Equation, Noncommutative Geometry, Gauge Theories
PACS (2006): 02.40.Gh, 03.65.Pm, 11.15.-q, 11.15.Ex

1. Introduction

Ettore Majorana was the first to study the infinite-component relativistic fields. In the pioneering 1932 paper [1], on the basis of the linear differential wave equation of a Dirac form, he constructed a relativistically invariant theory for arbitrary integer or half-integer spin particles. It was the first recognition, development and application of the infinite-dimensional unitary representations of the Lorentz group. During a long period of time, however, the Majorana results remained practically unknown, and the theory was rediscovered in 1948 by Gel'fand and Yaglom [2] in a more general framework of the group theory representations. In 1966 Fradkin revived the Majorana remarkable work (on the suggestion of Amaldi) by translating it into English and placing it in the context of the later research [3]. In a few years the development of the concept of the infinite-

* mplyushc@lauca.usach.cl

component fields [4]–[8] culminated in the construction of the dual resonance models and the origin of the superstring theory [9]–[15].

After the revival, the Majorana work inspired an interesting line of research based on a peculiar property of his equation: its time-like solutions describe *positive energy* states lying on a Regge type trajectory, but with unusual dependence of the mass, M, on the spin, s, $M_s \propto (const + s)^{-1}$. In 1970, Dirac [16] proposed a covariant *spinor set* of linear differential equations for the infinite-component field, from which the Majorana and Klein-Gordon equations appear in the form of integrability (consistency) conditions. As a result, the new Dirac relativistic equation describes a massive, spin-zero positive-energy particle. Though this line of research [17]–[22] did not find essential development, in particular, due to the problems arising under the attempt to introduce electromagnetic interaction, recently it was pushed [23]–[25] in the unexpected direction related to the anyon theory [26]–[38], exotic Galilei symmetry [39]–[42], and non-commutative geometry [43]–[46].

In pseudoclassical relativistic particle model associated with the quantum Dirac spin-1/2 equation, the spin degrees of freedom are described by the *odd* Grassmann variables [47]. In 1988 it was observed [48] that the (3+1)D particle analogue of the Polyakov string with rigidity [49] possesses the mass spectrum of the *squared* Majorana equation. The model of the particle with rigidity contains, like the string model [49], the higher derivative curvature term in the action. It is this higher derivative term that effectively supplies the system with the *even* spin degrees of freedom of noncompact nature and leads to the infinite-dimensional representations of the Lorentz group. Soon it was found that the quantum theory of another higher derivative model of the (2+1)D relativistic particle with torsion [32], whose Euclidean version underlies the Bose-Fermi transmutation mechanism [50], is described by the *linear* differential infinite-component wave equation of the Majorana form. Unlike the original Majorana equation, its (2+1)D analogue provides with the quantum states of any (real) value of the spin, and so, can serve as a basis for the construction of relativistic anyon theory [31]–[38]. It was shown recently [23, 24] that the application of the special non-relativistic limit ($c \to \infty$, $s \to \infty$, $s/c^2 \to \kappa = const$) [51, 52] to the model of relativistic particle with torsion produces the higher derivative model of a planar particle [40] with associated exotic (two-fold centrally extended) Galilei symmetry [39]. The quantum spectrum of the higher derivative model [40], being unbounded from below, is described by reducible representations of the exotic planar Galilei group. On the other hand, the application of the same limit to the (2+1)D analogue of the Dirac spinor set of anyon equations [38] gives rise to the Majorana-Dirac-Levy-Leblond type infinite-component wave equations [24], which describe irreducible representations of the exotic planar Galilei group corresponding to a free particle with non-commuting coordinates [41].

Here we review the described relations of the Majorana equation to the higher derivative particle models, exotic Galilei symmetry and associated noncommutative structure. We also discuss the relationship of the (2+1)D relativistic Abelian gauge field theories with Chern-Simons terms [55]–[62] to the Landau problem in the noncommutative plane

[25, 41, 53, 54] from the perspective of the Majorana equation.

2. Majorana Equation and Dirac Spinor Set of Equations

Majorana equation [1] is a linear differential equation of the Dirac form,

$$(P^\mu \Gamma_\mu - m)\,\Psi(x) = 0, \tag{2.1}$$

with $P^\mu = i\partial^\mu$ and matrices Γ_μ generating the Lorentz group via the anti-de Sitter SO(3,2) commutation relations similar to those satisfied by the usual γ-matrices[1],

$$[\Gamma_\mu, \Gamma_\nu] = iS_{\mu\nu}, \qquad [S_{\mu\nu}, \Gamma_\lambda] = i(\eta_{\nu\lambda}\Gamma_\mu - \eta_{\mu\lambda}\Gamma_\nu), \tag{2.2}$$

$$[S_{\mu\nu}, S_{\lambda\rho}] = i(\eta_{\mu\rho}S_{\nu\lambda} - \eta_{\mu\lambda}S_{\nu\rho} + \eta_{\nu\lambda}S_{\mu\rho} - \eta_{\nu\rho}S_{\mu\lambda}). \tag{2.3}$$

The original Majorana realization of the Γ_μ corresponds to the infinite-dimensional unitary representation of the Lorentz group in which its Casimir operators C_1 and C_2 and the Lorentz scalar $\Gamma_\mu\Gamma^\mu$ take the values

$$C_1 \equiv \frac{1}{2}S_{\mu\nu}S^{\mu\nu} = -\frac{3}{4}, \qquad C_2 \equiv \epsilon^{\mu\nu\lambda\rho}S_{\mu\nu}S_{\lambda\rho} = 0, \qquad \Gamma_\mu\Gamma^\mu = -\frac{1}{2}. \tag{2.4}$$

A representation space corresponding to (2.4) is a direct sum of the two irreducible SL(2,C) representations characterized by the integer, $j = 0, 1, \ldots$, and half-integer, $j = 1/2, 3/2, \ldots$, values of the SU(2) subalgebra Casimir operator, $M_i^2 = j(j+1)$, $M_i \equiv \frac{1}{2}\epsilon_{ijk}S^{jk}$. In both cases the Majorana equation (2.1) has time-like (massive), space-like (tachyonic) and light-like (massless) solutions. The spectrum in the light-like sector is

$$M_j = \frac{m}{j+\frac{1}{2}}, \qquad j = s+n, \qquad n = 0, 1, \ldots, \qquad s = 0 \quad \text{or} \quad \frac{1}{2}. \tag{2.5}$$

The change $\Gamma_\mu \rightarrow -\Gamma_\mu$ in accordance with (2.2), (2.3) does not effect on representations of the Lorentz group as a subgroup of the SO(3,2). For the Majorana choice with the diagonal generator Γ_0,

$$\Gamma_0 = j + \frac{1}{2}, \tag{2.6}$$

Eq. (2.1) has the time-like ($P^2 < 0$) solutions with positive energy.

In [16], Dirac suggested an interesting modification of the Majorana infinite-component theory that effectively singles out the lowest spin zero time-like state from all the Majorana equation spectrum. The key idea was to generate the Klein-Gordon and Majorana wave equations via the integrability conditions for some covariant set of linear differential equations. Dirac covariant spinor set of (3+1)D equations has the form

$$\mathcal{D}_a\Psi(x,q) = 0, \qquad \mathcal{D}_a = (P^\mu\gamma_\mu + m)_{ab}Q_b, \tag{2.7}$$

[1] We use the metric with signature $(-,+,+,+)$.

where γ-matrices are taken in the Majorana representation, and $Q_a = (q_1, q_2, \pi_1, \pi_2)$ is composed from the mutually commuting dynamical variables q_α, $\alpha = 1, 2$, and commuting conjugate momenta π_α, $[q_\alpha, \pi_\beta] = i\delta_{\alpha\beta}$, while $\Psi(x, q)$ is a single-component wave function. The SO(3,2) generators are realized here as quadratic in Q operators,

$$\Gamma_\mu = \frac{1}{4}\bar{Q}\gamma_\mu Q, \qquad S_{\mu\nu} = \frac{i}{8}\bar{Q}[\gamma_\mu, \gamma_\nu]Q,$$

where $\bar{Q} = Q^t\gamma^0$. The covariance of the set of equations (2.7) follows from the commutation relations $[S_{\mu\nu}, Q] = -\frac{i}{4}[\gamma_\mu, \gamma_\nu]Q$, which mean that the Q_a is transformed as a Lorentz spinor, and so, the set of four equations (2.7) is the spinor set. Note also that $[\Gamma_\mu, Q] = \frac{1}{2}\gamma_\mu Q$, and the Q_a anticommute between themselves for a linear combination of the SO(3,2) generators. This means that the Q_a, Γ_μ and $S_{\mu\nu}$ generate a supersymmetric extension of the anti-de Sitter algebra.

The Klein-Gordon,

$$(P^2 + m^2)\Psi = 0, \tag{2.8}$$

and the Majorana equations (with the parameter m changed in the latter for $\frac{1}{2}m$) are the integrability conditions for the spinor set of equations (2.7) [16]. Taking into account that the $\Gamma_0 = \frac{1}{4}(q_1^2 + q_2^2 + \pi_1^2 + \pi_2^2)$ coincides up to the factor $\frac{1}{2}$ with the Hamiltonian of a planar isotropic oscillator, one finds that the possible eigenvalues of the Γ_0 are given by the sets $j = 0, 1, \ldots$ and $j = 1/2, 3/2, \ldots$ in correspondence with Eq. (2.6). The former case corresponds to the Γ_0 eigenstates given by the even in q_α wave functions, while the latter case corresponds to the odd eigenstates. Having in mind the Majorana equation spectrum (2.5) (with the indicated change of the mass parameter) and Eq. (2.8), one concludes that the spinor set of equations (2.7) describes the positive energy spinless states[2] of the fixed mass.

3. Higher Derivative Relativistic Particle Models

The model of relativistic particle with curvature [48, 63, 64, 65], being an analogue of the model of relativistic string with rigidity [49], is given by the reparametrization invariant action

$$A = -\int (m + \alpha k) ds, \tag{3.1}$$

where $ds^2 = -dx_\mu dx^\mu$, $\alpha > 0$ is a dimensionless parameter[3], and k is the worldline curvature, $k^2 = x''_\mu x''^\mu$, $x'_\mu = dx_\mu/ds$. In a parametrization $x_\mu = x_\mu(\tau)$, Lagrangian of the system is $L = -\sqrt{-\dot{x}^2}(m + k)$, where we assume that the particle moves with the velocity less than the speed of light, $\dot{x}^2 < 0$, $\dot{x}_\mu = dx_\mu/d\tau$, and then $k^2 = (\dot{x}^2\ddot{x}^2 - (\dot{x}\ddot{x}^2)^2)/(\dot{x}^2)^3 \geq 0$

---

[2] Staunton [20] proposed a modification of the Dirac spinor set of equations that describes the spin-1/2 representation of the Poincaré group

[3] For $\alpha < 0$ the equations of motion of the system have the only solutions corresponding to the curvature-free case $\alpha = 0$ of a spinless particle of mass m [48].

[48]. The Lagrangian equations of motion have the form of the conservation law of the energy-momentum vector,

$$\frac{d}{d\tau}P_\mu = 0, \qquad P_\mu = \frac{\partial L}{\partial \dot{x}^\mu} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \ddot{x}^\mu}\right). \tag{3.2}$$

The dependence of the Lagrangian on higher derivatives supplies effectively the system with additional translation invariant degrees of freedom described by the velocity $v_\mu \equiv \dot{x}_\mu$ and conjugate momentum [48]. This higher derivative dependence is responsible for a peculiarity of the system: though the particle velocity is less than the speed of light, the equations of motion (3.2) have the time-like ($P^2 < 0$), the light-like ($P^2 = 0$) and the space-like ($P^2 > 0$) solutions [48], whose explicit form was given in [48, 65]. This indicates on a possible relation of the model (3.1) to the infinite-component field theory associated with the Majorana equation. Unlike the Majorana system, however, the quantum version of the model (3.1) has the states of integer spin only, which lie on the nonlinear Regge trajectory of the form very similar to (2.5) [48],

$$M_l = \frac{m}{\sqrt{1+\alpha^{-2}l(l+1)}}, \qquad l = 0, 1, \ldots . \tag{3.3}$$

The choice of the laboratory time gauge $\tau = x^0$ separates here the positive energy time-like solutions.

Before we pass over to the discussion of a relativistic particle model more closely related to the original (3+1)D Majorana equation from the viewpoint of the structure of the spectrum, but essentially different from it in some important properties, it is worth to note that the higher derivative dependence of the action does not obligatorily lead to the tachyonic states. In Ref. [66] the model given by the action of the form (3.1) with parameter $m = 0$ was suggested. It was shown there that in the case of $\dot{x}^2 < 0$, the model is inconsistent (its equations of the motion have no solutions), but for $\dot{x}^2 > 0$ the model is consistent and describes massless states of the arbitrary, but fixed integer or half-integer helicities $\lambda = \pm j$, whose values are defined by the quantized parameter α, $\alpha^2 = j^2$. The velocity higher than the speed of light in such a model originates from the Zittervewegung associated with nontrivial helicity. System (3.1) with $m = 0$ possesses additional local symmetry [66, 67] (action (3.1) in this case has no scale parameter), and it is such a gauge symmetry that is responsible for separation of the two physical helicity components from the infinite-component Majorana type field (cf. the system given by the Dirac spinor set of equations (2.7)). Recently, the interest to such a higher derivative massless particle system has been revived [68, 69] in the context of the massless higher spin field theories [70, 71].

The (2+1)D relativistic model of the particle with torsion [32] is given by the action

$$A = -\int (m + \alpha\varrho)ds, \qquad \varrho = \epsilon^{\mu\nu\lambda}x'_\mu x''_\nu x'''_\lambda, \tag{3.4}$$

where α is a dimensionless parameter, and ϱ is the particle worldline trajectory torsion. Unlike the model (3.1), here the parameter α can take positive or negative values, and for the sake of definiteness, we assume that $\alpha > 0$. Action (3.4) with $\alpha = 1/2$ appeared originally in the Euclidean version in the context of the Bose-Fermi transmutation mechanism [50, 29]. Like the model of the particle with curvature (3.1), the higher derivative system (3.4) possesses the translation invariant dynamical spin degrees of freedom $J_\mu = -\alpha e_\mu$, $e_\mu = \dot{x}_\mu/\sqrt{-\dot{x}^2}$, as well as the three types of solutions to the classical equations of motion, with $P^2 < 0$, $P^2 = 0$ and $P^2 > 0$ [32]. At the quantum level operators J_μ satisfy the SO(2,1) commutation relations

$$[J_\mu, J_\mu] = -i\epsilon_{\mu\nu\lambda} J^\lambda, \tag{3.5}$$

analogous to those for the (2+1)D γ-matrices. Note that in (2+1)D, there is a duality relation $J_\mu = -\frac{1}{2}\epsilon_{\mu\nu\lambda}S^{\nu\lambda}$ between the (2+1)D vector J_μ and the spin tensor $S_{\mu\nu}$ satisfying the commutation relations of the form (2.3). The parameter α is not quantized here, and it fixes the value of the Casimir operator of the algebra (3.5), $J^2 = -\alpha(\alpha - 1)$ [32]. For the gauge $\tau = x^0$, in representation where the operator J_0 is diagonal, its eigenvalues are $j_0 = \alpha + n$, $n = 0, 1, \ldots$. This means that the spin degrees of freedom of the system realize a bounded from below unitary infinite-dimensional representation D^+_α of the universal covering group of the (2+1)D Lorentz group [72, 73]. The physical states of the system are given by the quantum analogue of the constraint responsible for the reparametrization invariance of the action (3.4) [32],

$$(PJ - \alpha m)\Psi = 0. \tag{3.6}$$

One can treat Eq. (3.6) as a (2+1)D analogue of the original Majorana equation (2.1). The difference of the (2+1)D from the (3+1)D case proceeds from the isomorphism between SO(2,2) and SO(2,1)⊕SO(2,1) algebras, and here the SO(2,1) generators J_μ simultaneously play the role analogous to that played by the SO(3,2) generators Γ_μ satisfying the commutation relations (2.2). In the time-like sector, the solutions of Eq. (3.6) describe the positive energy states of the spin $s_n = \alpha + n$ lying on the Majorana type trajectory [32]

$$M_n = \frac{m}{1 + \alpha^{-1} n}, \qquad n = 0, 1, \ldots. \tag{3.7}$$

4. Fractional Spin Fields

The (2+1)D analogue of the Majorana equation (3.6) being supplied with the Klein-Gordon equation (2.8) describes the fields carrying irreducible representation of the Poincaré ISO(2,1) group of any, but fixed spin $s = \alpha > 0$ [32], and so, can serve as a basis for relativistic anyon theory [26]–[31]. Instead of these two equations, one can obtain the same result starting from the linear differential (2+1)D Majorana-Dirac wave equations suggested in [34][4]. In such a case it is supposed that besides the index n associated with the infinite-dimensional half-bounded unitary representation D^+_α, the infinite-component

[4] Jackiw and Nair [33] proposed an alternative theory based on the (2+1)D Majorana equation supplied with the equation for topologically massive vector gauge field.

field carries in addition a spinor index, and that it satisfies Eq. (3.6) as well as the Dirac equation

$$(P\gamma - m)\Psi = 0. \tag{4.1}$$

As a consequence of Eqs. (3.6), (4.1), the Majorana-Dirac field satisfies not only the Klein-Gordon equation, but also the equations

$$(J\gamma + \alpha)\Psi = 0, \qquad \epsilon_{\mu\nu\lambda}J^\mu\gamma^\nu P^\lambda\Psi = 0, \tag{4.2}$$

and one finds that it describes the positive energy states of the mass m and spin $s = \alpha - \frac{1}{2}$ [34].

The alternative way to describe an anyon field of the fixed mass and spin consists in the construction of the (2+1)D analogue of the Dirac spinor set of equations (2.7) generating the Majorana and Klein-Gordon equations in the form of integrability conditions. The construction needs the application of the so called deformed Heisenberg algebra with reflection intimately related to parabosons [74, 75],

$$[a^-, a^+] = 1 + \nu R, \qquad R^2 = 1, \qquad \{a^\pm, R\} = 0, \tag{4.3}$$

where ν is a real deformation parameter. Here operator $N = \frac{1}{2}\{a^+, a^-\} - \frac{1}{2}(\nu + 1)$ plays the role of a number operator, $[N, a^\pm] = \pm a^\pm$, allowing us to present a reflection operator R in terms of $a^\pm$: $R = (-1)^N = \cos \pi N$. For $\nu > -1$ algebra (4.3) admits infinite-dimensional unitary representations realized on a Fock space[5]. In terms of operators $a^\pm$ the SO(2,1) generators (3.5) are realized in a quadratic form,

$$J_0 = \frac{1}{4}\{a^+, a^-\}, \qquad J_\pm = J_1 \pm iJ_2 = \frac{1}{2}(a^\pm)^2. \tag{4.4}$$

Here $J_\mu J^\mu = -s(s-1)$ with $s = \frac{1}{4}(1\pm\nu)$ on the even/odd eigensubspaces of the reflection operator R, i.e. as in the (3+1)D case we have a direct sum of the two infinite-dimensional irreducible representations of the (2+1)D Lorentz group. These quadratic operators together with linear operators

$$L_1 = \frac{1}{\sqrt{2}}(a^+ + a^-), \qquad L_2 = \frac{i}{\sqrt{2}}(a^+ - a^-), \tag{4.5}$$

extend the SO(2,1) algebra into the OSP(1|2) superalgebra:

$$\{L_\alpha, L_\beta\} = 4i(J\gamma)_{\alpha\beta}, \qquad [J_\mu, L_\alpha] = \frac{1}{2}(\gamma_\mu L)_\alpha, \tag{4.6}$$

where the (2+1)D γ-matrices are taken in the Majorana representation, $(\gamma_0)_\alpha{}^\beta = (\sigma_2)_\alpha{}^\beta$, $(\gamma_1)_\alpha{}^\beta = i(\sigma_1)_\alpha{}^\beta$, $(\gamma_2)_\alpha{}^\beta = i(\sigma_3)_\alpha{}^\beta$, and $(\gamma_\mu)_{\alpha\beta} = (\gamma_\mu)_\alpha{}^\rho\epsilon_{\rho\beta}$. With these ingredients, the (2+1)D analogue of the Dirac spinor set of wave equations (2.7) is [38]

$$\left((P\gamma)_\alpha{}^\beta + m\epsilon_\alpha{}^\beta\right) L_\beta\Psi = 0. \tag{4.7}$$

[5] For negative odd integer values $\nu = -(2k+1)$, $k = 1, 2\ldots$, the algebra has finite, (2k+1)-dimensional nonunitary representations [74].

From these two ($\alpha = 1, 2$) equations the (2+1)D Majorana and Klein-Gordon equations appear in the form of integrability conditions.

The spinor set of equations (4.7) was used, in particular, for investigation of the Lorentz symmetry breaking in the (3+1)D massless theories with fractional helicity states [76].

5. Exotic Galilei Group and Noncommutative Plane

A special non-relativistic limit (c is a speed of light) [51, 52]

$$c \to \infty, \qquad s \to \infty, \qquad \frac{s}{c^2} = \kappa, \tag{5.1}$$

applied to the spinor set of equations (4.7) results in the infinite-component Dirac-Majorana-Lévy-Leblond type wave equations [24]

$$i\partial_t \phi_k + \sqrt{\frac{k+1}{2\theta}}\,\frac{P_+}{m}\,\phi_{k+1} = 0, \tag{5.2}$$

$$P_- \phi_k + \sqrt{\frac{2(k+1)}{\theta}}\,\phi_{k+1} = 0, \tag{5.3}$$

where $k = 0, 1, \ldots$, $P_\pm = P_1 \pm iP_2$, and

$$\theta = \frac{\kappa}{m^2}. \tag{5.4}$$

The first equation (5.2) defines the dynamics. The second equation relates different components of the field allowing us to present them in terms of the lowest component,

$$\phi_k = (-1)^k \left(\frac{\kappa}{2}\right)^{\frac{k}{2}} \left(\frac{P_-}{m}\right)^k \phi_0. \tag{5.5}$$

Though a simple substitution of the second equation into the first one shows that every component ϕ_k satisfies the Shrödinger equation of a free planar particle, the nontrivial nature of the system is encoded in its symmetry. The (2+1)D Poincaré symmetry of the original relativistic system in the limit (5.1) is transformed into the exotic planar Galilei symmetry characterized by the noncommutative boosts [39, 41],

$$[\mathcal{K}_1, \mathcal{K}_2] = -i\kappa. \tag{5.6}$$

The system of the two infinite-component equations (5.2), (5.3) can be presented in the equivalent form

$$i\partial_t \phi = H\phi, \qquad V_- \phi = 0, \tag{5.7}$$

with

$$H = P_i v_i - \frac{1}{2} m v_+ v_-\,, \qquad V_- = v_- - \frac{P_-}{m}\,. \tag{5.8}$$

The translation invariant operators $v_\pm = v_1 \pm i v_2$, $[v_i, v_j] = -i\kappa^{-1}\epsilon_{ij}$, is the non-relativistic limit (5.1) of the noncompact Lorentz generators, $-(c/s)J_\pm \to v_\pm$. The symmetry of the quantum mechanical system (5.7) is given by the Hamiltonian H, the space translation generators P_i, and by the rotation and boost generators,

$$\mathcal{J} = \epsilon_{ij} x_i P_j + \frac{1}{2}\kappa v_+ v_-, \qquad \mathcal{K}_i = m x_i - t P_i + \kappa \epsilon_{ij} v_j. \tag{5.9}$$

These integrals generate the algebra of the two-fold centrally extended planar Galilei group [39, 41] characterized by the non-commutativity of the boosts (5.6).

The first equation from (5.7) is nothing else as a non-relativistic limit of the (2+1)D Majorana equation (3.6) [24]. The system described by it (without the second equation from (5.7)) corresponds to the classical system given by the higher derivative Lagrangian

$$L = \frac{1}{2} m \dot{x}_i^{\,2} + \kappa \epsilon_{ij} \dot{x}_i \ddot{x}_j, \tag{5.10}$$

which, in its turn, corresponds to the non-relativistic limit (5.1) of the relativistic model of the particle with torsion (3.4) [23]. It is interesting to note that the system (5.10) (for the first time considered by Lukierski, Stichel and Zakrzewski [40], in ignorance of its relation to the relativistic higher derivative model (3.4)), reveals the same dynamics as a charged non-relativistic planar particle in external homogeneous magnetic and electric fields [77]. The spectrum of the Hamiltonian (5.8),

$$E_n(P) = \frac{1}{2m} P_i^2 - m\kappa^{-1} n, \qquad n = 0, 1, \ldots, \tag{5.11}$$

is not restricted from below, and the system (5.10), similarly to its relativistic analogue (3.4), describes a reducible representation of the exotic Galilei group. The role of the second equation from (5.7), whose component form is given by Eq. (5.3), consists in singling out the highest (at fixed P_i^2) energy state from (5.11) with $n = 0$, and fixing an irreducible infinite-dimensional unitary representation of the exotic planar Galilei group [24, 77]. The system being reduced to the surface given by this second equation (classically equivalent to the set of the two second class constraints $V_i = 0$, $i = 1, 2$) corresponds to the exotic planar particle considered by Duval and Horvathy [41, 79], which is described by the free particle Hamiltonian and an exotic symplectic two-form,

$$H = \frac{1}{2m} P_i^2, \qquad \omega = dP_i \wedge dx_i + \frac{1}{2}\theta \epsilon_{ij} dP_i \wedge dP_j. \tag{5.12}$$

The system (5.12) reveals a noncommutative structure encoded in the nontrivial commutation relations of the particle coordinates,

$$[x_i, x_j] = i\theta \epsilon_{ij}. \tag{5.13}$$

This noncommutative structure is the non-relativistic limit (5.1) [51] of the commutation relations

$$[x_\mu, x_\nu] = -is\epsilon_{\mu\nu\lambda} \frac{P^\lambda}{(-P^2)^{3/2}} \tag{5.14}$$

associated with the minimal canonical approach for relativistic anyon of spin s [78]. Note that as was observed by Schonfeld [55] (see also [80]), the commutation relations (5.14) are dual to the (Euclidean) commutation relations for the mechanical momentum of a charged particle in the magnetic monopole field. The latter system also admits a description by the higher derivative Lagrangian [80],

$$L_{CM} = \frac{1}{2} m\dot{\vec{r}}^2 - eg\frac{|\vec{r}|}{(\vec{r}\times\dot{\vec{r}})^2}(\vec{r}\times\dot{\vec{r}})\cdot\ddot{\vec{r}}. \tag{5.15}$$

There is a close relationship between the charge-monopole non-relativistic system (5.15) and the model of relativistic particle with torsion (3.4). Indeed, in a parametrization $x_\mu = x_\mu(\tau)$, the torsion term from (3.4) takes the (Minkowski) form of the higher derivative charge-monopole coupling term, but in the *velocity* space with $v^\mu \equiv \dot{x}^\mu$,

$$L_{tor} = -\alpha\frac{\sqrt{-v^2}}{(\epsilon_{\gamma\rho\sigma}v^\rho\dot{v}^\sigma)^2}\epsilon_{\mu\nu\lambda}v^\mu\dot{v}^\nu\ddot{v}^\lambda. \tag{5.16}$$

For system (5.15) the relation $\vec{J}\vec{n} + eg = 0$ is the analogue of the (2+1)D Majorana equation (3.6), where $\vec{n} = \vec{r}/|\vec{r}|$ and $\vec{J}$ is the charge-monopole angular momentum.

The exotic planar particle described by the symplectic structure (5.12), or by the Dirac-Majorana-Lévy-Leblond type equations (5.2), (5.3), can be consistently coupled to an arbitrary external electromagnetic field at the *classical* level [41, 25]. However, at the quantum level the Hamiltonian reveals a nonlocal structure in the case of inhomogeneous magnetic field [25]. Another peculiarity reveals even in the case of homogeneous magnetic field corresponding to the Landau problem for a particle in a noncommutative plane [25, 54, 81], where the initial particle mass m is changed for the effective mass [41]

$$m^* = m(1 - eB\theta), \tag{5.17}$$

see below. As a result, the system develops three essentially different phases corresponding to the subcritical, $eB\theta < 1$, critical, $eB\theta = 1$, and overcritical, $eB\theta > 1$, values of the magnetic field [25, 54].

6. Gauge Theories with Chern-Simons Terms and Exotic Particle

In the case of the choice of finite-dimensional non-unitary representations of the deformed Heisenberg algebra with reflection (4.3) corresponding to the negative odd values of the deformation parameter, $\nu = -(2k+1)$, $k = 1, 2, \ldots$, the (2+1)D spinor set of equations (4.7) describes a spin-j field with $j = k/2$ and both signs of the energy [82, 83, 37]. In particular, in the simplest cases of $\nu = -3$ and $\nu = -5$, Eq. (4.7) gives rise, respectively, to the Dirac spin-1/2 particle theory and to the topologically massive electrodynamics [55, 56]. The latter system is described by the Lagrangian

$$L_{TME} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m}{4}\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{6.1}$$

Let us suppress the dependence on the spatial coordinates x_i by making a substitution $A^\mu(x) \rightarrow \sqrt{m}\, r^\mu(t)$. Then (6.1) takes a form of the Lagrangian of a non-relativistic charged particle in the homogeneous magnetic field $B = m^2 e^{-1}$, $L = \frac{1}{2} m\dot{r}_i^2 + \frac{1}{2} eB\epsilon_{ij} r_i \dot{r}_j$, while the variable r^0 disappears[6].

In Ref. [62], Deser and Jackiw proposed an extension of the topologically massive electrodynamics by adding to Lagrangian (6.1) the higher derivative term of the Chern-Simons form,

$$L_{DJ} = L_{TME} + L_{ECS}, \qquad L_{ECS} = \kappa m^{-1} \epsilon^{\mu\nu\lambda} F_{\mu\sigma} \partial^\sigma F_{\nu\lambda}, \tag{6.2}$$

where κ is a dimensionless numerical parameter. Making the same substitution as before, and changing $r_i \to x_i$, we reduce the (2+1)D field Lagrangian (6.2) to the mechanical Lagrangian for a particle in a plane,

$$L = \frac{1}{2} m\dot{x}_i^2 + \frac{1}{2} eB\epsilon_{ij} x_i \dot{x}_j + \kappa \epsilon_{ij} \dot{x}_i \ddot{x}_j, \tag{6.3}$$

that describes the higher derivative model (5.10) coupled to the external homogeneous magnetic field. The system (6.3), like the free higher derivative system (5.10) underlying the special non-relativistic limit (5.1) of the (2+1)D Majorana equation, has a spectrum unbounded from below. This drawback can be removed by supplying the coupled system with the appropriately modified constraint (5.3) [25]. Classically, this is equivalent to the change of the higher derivative Lagrangian (6.3) for the first order exotic Duval-Horvathy Lagrangian [41]

$$L_{ex} = P_i \dot{x}_i - \frac{1}{2m} P_i^2 + \frac{1}{2} \theta \epsilon_{ij} P_i \dot{P}_j + \frac{1}{2} eB\epsilon_{ij} x_i \dot{x}_j, \tag{6.4}$$

corresponding in a free case to the symplectic structure (5.12)[7]. It generates the equations of motion with the effective mass (5.17), $P_i = m^* \dot{x}_i$, $\dot{P}_i = eB\epsilon_{ij} \dot{x}_j$.

The interacting exotic particle system (6.4) can also be obtained by a reduction of another (2+1)-dimensional Abelian gauge field theory given by the Lagrangian with several Chern-Simons terms,

$$L_H = -\epsilon^{\mu\nu\lambda} \Phi_\mu \partial_\nu A_\lambda - \frac{1}{2} \lambda \Phi_\mu \Phi^\mu - \frac{1}{2} \kappa m^{-1} \epsilon^{\mu\nu\lambda} \Phi_\mu \partial_\nu \Phi_\lambda - \frac{1}{2} \beta m \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \tag{6.5}$$

where λ, κ and β are dimensionless parameters. The system with Lagrangian (6.5) was investigated by Hagen [59], see also [60]. Suppressing the dependence of the fields Φ_μ and A_μ on the spatial coordinates by making the substitutions $A^\mu(x) \to \sqrt{\lambda m}\, r^\mu(t)$ and $\Phi^\mu(x) \to \pi^\mu(t)/\sqrt{m\lambda}$ (we assume $\lambda > 0$), and denoting $\lambda \beta m^2 = eB$ and $\kappa/(\lambda m^2) = \theta$, we reduce (6.5) to the first order Lagrangian

$$L = \epsilon_{ij} \pi_i \dot{r}_j - \frac{1}{2m} \pi_i^2 + \frac{1}{2m} \pi_0^2 + \frac{1}{2} \theta \epsilon_{ij} \pi_i \dot{\pi}_j + \frac{1}{2} eB\epsilon_{ij} r_i \dot{r}_j.$$

---

[6] This corresponds to the nature of the A^0 field, which can be removed by imposing the Weyl gauge $A^0 = 0$.

[7] For the system (6.3), one can get rid of the unbounded from below spectrum by changing the sign in the first, kinetic term. In this case the problem reappears at $\kappa = 0$.

Hence, the π_0 plays the role of the auxiliary variable, and can be omitted using its equation of motion $\pi_0 = 0$[8]. Then, changing the notations $r_i \to x_i$ and $\pi_i \to \epsilon_{ij} P_j$, we arrive at the Lagrangian (6.4).

Therefore, the both systems (6.3) and (6.4), corresponding (in a free case) to the special non-relativistic limit (5.1) of the (2+1)D Majorana equation (3.6) and Dirac spinor set of equations (4.7), can be treated as reduced versions of the relativistic Lagrangians (6.2) and (6.5) of the (2+1)D Abelian gauge field theories with Chern-Simons terms.

7. Conclusion

To conclude, we point out two interesting open problems related to the Majorana equation.

It is known that the spin-statistics connection for the infinite-component fields described by the Majorana type equations is absent [5, 7, 8]. On the other hand, the question on such a connection for the fields of fixed mass and spin described by the Dirac covariant set of equations is open. The question on the spin-statistics relation for the fractional spin field theories constructed on the basis of the (2+1)D analogue of the Majorana equation also still waits for the solution.

As we saw, the original (3+1)D Majorana equation and the Dirac spinor set of equations constructed on its basis, as well as their (2+1)D analogues, have a hidden supersymmetric structure encoded in the Majorana spectrum (2.5). Hence, it would be very natural to try to construct a supersymmetric extension of these theories. Such an attempt was undertaken in Ref. [83] for the case of the (2+1)D analogue of the Dirac spinor set of equations. Within the framework of a restricted approach taken there, the supersymmetric extension was obtained only for a few special cases corresponding to finite-dimensional representations of the underlying deformed Heisenberg algebra with reflection (4.3)[9]. A supersymmetric extension could help to resolve the problem of the electromagnetic coupling, including the quantum case of the non-relativistic exotic particle in the noncommutative plane.

Acknowledgements. The work was supported in part by FONDECYT grant 1050001.

[8] Disappearance of π_0 (r_0) is rooted in the independence of Lagrangian (6.5) of the time derivative of Φ^0 (A^0).

[9] See also ref. [35] for the case of $\nu = 0$.

References

[1] E. Majorana, *Teoria relativistica di particelle con momento intrinseco arbitrario*, *Nuovo Cimento* **9** (1932) 335.

[2] I. M. Gel'fand and A. M. Yaglom, *Zh. Eksp. Teor. Fiz.* **18** (1948) 703, 1096, 1105.

[3] M. Fradkin, *Am. J. of Physics* **34** (1966) 314.

[4] C. Fronsdal, *Phys. Rev.* **156** (1967) 1665; *Phys. Rev.* **182** (1969) 1564; C. Fronsdal and R. White, *Phys. Rev.* **163** (1967) 1835.

[5] E. Abers, I. T. Grodsky and R. E. Norton, *Phys. Rev.* **159** (1967) 1222.

[6] Y. Nambu, *Phys. Rev.* **160** (1967) 1171.

[7] D. Tz. Stoyanov and I. T. Todorov, *J. Math. Phys.* **9** (1968) 2146.

[8] N. N. Bogolyubov, A. A. Logunov, A. I. Oksak and I. T. Todorov, *General Principles Of Quantum Field Theory* Dordrecht, Netherlands: Kluwer (1990).

[9] G. Veneziano, *Nuovo Cim. A* **57** (1968) 190.

[10] A. Neveu and J. H. Schwarz, *Nucl. Phys. B* **31** (1971) 86.

[11] P. Ramond, *Phys. Rev. D* **3**, 2415 (1971).

[12] S. Mandelstam, *Phys. Rept.* **13** (1974) 259.

[13] F. Gliozzi, J. Scherk and D. I. Olive, *Phys. Lett. B* **65** (1976) 282.

[14] M. B. Green and J. H. Schwarz, *Nucl. Phys. B* **181** (1981) 502.

[15] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, Cambridge University Press, Cambridge, 1987.

[16] P. A. M. Dirac, *Proc. R. Soc. Lond. Ser. A* **322** (1971) 435, and ibid. **328** (1972) 1.

[17] L. P. Staunton, *Phys. Rev. D* **13** (1976) 3269; L. P. Staunton and S. Browne, *Phys. Rev. D* **12** (1975) 1026.

[18] L. C. Biedenharn and H. Van Dam, *Phys. Rev. D* **9** (1974) 471; **14** (1976) 405.

[19] A. O. Barut and I. H. Duru, *Phys. Rev. D* **10** (1974) 3448.

[20] L. P. Staunton, *Phys. Rev. D* **10** (1974) 1760.

[21] H. Leutwyler and J. Stern, *Annals Phys.* **112** (1978) 94.

[22] N. Mukunda, H. Van Dam and L. C. Biedenharn, *Phys. Rev. D* **22** (1980) 1938.

[23] P. A. Horváthy and M. S. Plyushchay, *JHEP* **0206** (2002) 033 [hep-th/0201228].

[24] P. A. Horvathy and M. S. Plyushchay, *Phys. Lett. B* **595** (2004) 547 [hep-th/0404137].

[25] P. A. Horvathy and M. S. Plyushchay, *Nucl. Phys. B* **714** (2005) 269 [hep-th/0502040].

[26] J. M. Leinaas and J. Myrheim, *Nuovo Cim. B* **37** (1977) 1.

[27] F. Wilczek and A. Zee, *Phys. Rev. Lett.* **51** (1983) 2250.

[28] Y. S. Wu and A. Zee, *Phys. Lett. B* **147** (1984) 325.

[29] R. MacKenzie, F. Wilczek, *Int. J. Mod. Phys. A* **3** (1988) 2827.

[30] D. V. Volkov, *JETP Lett.* **49** (1989) 541.

[31] M. S. Plyushchay, *Phys. Lett. B* **248** (1990) 107.

[32] M. S. Plyushchay, *Phys. Lett. B* **262** (1991) 71; *Nucl. Phys. B* **362** (1991) 54.

[33] R. Jackiw and V. P. Nair, *Phys. Rev. D* **43** (1991) 1933.

[34] M. S. Plyushchay, *Phys. Lett. B* **273** (1991) 250; *Int. J. Mod. Phys. A* **7** (1992) 7045.

[35] D. P. Sorokin and D. V. Volkov, *Nucl. Phys. B* **409** (1993) 547.

[36] Ch. Chou, V. P. Nair and A. P. Polychronakos, *Phys. Lett. B* **304** (1993) 105 [hep-th/9301037].

[37] J. L. Cortes and M. S. Plyushchay, *J. Math. Phys.* **35** (1994) 6049 [hep-th/9405193].

[38] M. S. Plyushchay, *Phys. Lett. B* **320** (1994) 91 [hep-th/9309148] .

[39] J.-M. Lévy-Leblond, *Galilei group and Galilean invariance.* In: *Group Theory and Applications* (Loebl Ed.), **II**, Acad. Press, New York, p. 222 (1972); A. Ballesteros, M. Gadella and M. del Olmo, *Journ. Math. Phys.* **33** (1992) 3379; Y. Brihaye, C. Gonera, S. Giller and P. Kosiński, hep-th/9503046.

[40] J. Lukierski, P. C. Stichel, W. J. Zakrzewski, *Annals of Physics (NY)* **260** (1997) 224 [hep-th/9612017].

[41] C. Duval and P. A. Horváthy, *Phys. Lett. B* **479** (2000) 284 [hep-th/0002233]; *Journ. Phys. A* **34** (2001) 10097 [hep-th/0106089]

[42] P. A. Horvathy, L. Martina and P. C. Stichel, *Phys. Lett. B* **564** (2003) 149 [arXiv:hep-th/0304215].

[43] N. Seiberg and E. Witten, JHEP **9909** (1999) 032 [hep-th/9908142].

[44] D. Bigatti and L. Susskind, *Phys. Rev. D* **62** (2000) 066004 [hep-th/9908056].

[45] M. R. Douglas and N. A. Nekrasov, *Rev. Mod. Phys.* **73** (2001) 977 [hep-th/0106048].

[46] A. Connes, M. R. Douglas and A. S. Schwarz, *JHEP* **9802** (1998) 003 [hep-th/9711162].

[47] F. A. Berezin and M. S. Marinov, *Annals Phys.* **104** (1977) 336.

[48] M. S. Plyushchay, *Mod. Phys. Lett. A* **3** (1988) 1299; *Int. J. Mod. Phys. A* **4** (1989) 3851.

[49] A. M. Polyakov, *Nucl. Phys. B* **268** (1986) 406.

[50] A. M. Polyakov, *Mod. Phys. Lett. A* **3** (1988) 325.

[51] R. Jackiw and V. P. Nair, *Phys. Lett. B* **480** (2000) 237 [hep-th/0003130].

[52] C. Duval and P. A. Horvathy, Phys. Lett. B **547** (2002) 306 [arXiv:hep-th/0209166].

[53] P. A. Horvathy, *Annals Phys.* **299** (2002) 128 [hep-th/0201007].

[54] M. S. Plyushchay, hep-th/0603034.

[55] J. F. Schonfeld, *Nucl. Phys. B* **185** (1981) 157.

[56] S. Deser, R. Jackiw and S. Templeton, *Phys. Rev. Lett.* **48** (1982) 975; *Annals Phys.* **140** (1982) 372.

[57] S. Deser and R. Jackiw, *Phys. Lett. B* **139** (1984) 371.

[58] A. M. Din and W. J. Zakrzewski, *Phys. Lett. B* **146** (1984) 341.

[59] C. R. Hagen, *Phys. Rev. Lett.* **58** (1987) 1074.

[60] S. Deser and R. Jackiw, *Phys. Rev. Lett.* **59** (1987) 1981.

[61] E. Witten, *Commun. Math. Phys.* **121** (1989) 351.

[62] S. Deser and R. Jackiw, *Phys. Lett. B* **451** (1999) 73 [hep-th/9901125].

[63] R. D. Pisarski, *Phys. Rev. D* **34** (1986) 670.

[64] V. V. Nesterenko, *J. Phys. A* **22** (1989) 1673.

[65] Y. A. Kuznetsov and M. S. Plyushchay, *Nucl. Phys. B* **389** (1993) 181.

[66] M. S. Plyushchay, *Mod. Phys. Lett. A* **4** (1989) 837; *Phys. Lett. B* **243** (1990) 383.

[67] E. Ramos and J. Roca, *Nucl. Phys. B* **436** (1995) 529 [hep-th/9408019]; **452** (1995) 705 [hep-th/9504071].

[68] J. Mourad, hep-th/0410009.

[69] L. Edgren, R. Marnelius and P. Salomonson, *JHEP* **0505** (2005) 002 [hep-th/0503136]; L. Edgren and R. Marnelius, hep-th/0602088.

[70] C. Fronsdal, *Phys. Rev. D* **18** (1978) 3624.

[71] M. A. Vasiliev, *Phys. Lett. B* **567** (2003) 139 [hep-th/0304049]; *Comptes Rendus Physique* **5** (2004) 1101 [hep-th/0409260].

[72] V. Bargmann, *Annals Math.* **48** (1947) 568.

[73] M. S. Plyushchay, *J. Math. Phys.* **34** (1993) 3954.

[74] M. S. Plyushchay, *Nucl. Phys. B* **491** [PM] (1997) 619 [hep-th/9701091];

[75] M. Plyushchay, *Int. J. Mod. Phys. A* **15** (2000) 3679 [hep-th/9903130].

[76] S. M. Klishevich, M. S. Plyushchay and M. Rausch de Traubenberg, *Nucl. Phys. B* **616** (2001) 419 [hep-th/0101190].

[77] M. A. del Olmo and M. S. Plyushchay, *Ann. Phys. (NY)* (2006), in press [hep-th/0508020].

[78] J. L. Cortes and M. S. Plyushchay, *Int. J. Mod. Phys. A* **11** (1996) 3331 [hep-th/9505117].

[79] P. A. Horvathy, hep-th/0602133.

[80] M. S. Plyushchay, *Nucl. Phys. B* **589** (2000) 413 [hep-th/0004032].

[81] S. Bellucci, A. Nersessian and C. Sochichiu, *Phys. Lett. B* **522** (2001) 345 [hep-th/0106138].

[82] M. S. Plyushchay, *Ann. Phys. (NY)* **245** (1996) 339 [hep-th/9601116].

[83] M. S. Plyushchay, *Mod. Phys. Lett. A* **12** (1997) 1153 [hep-th/9705034].