EJTP **3, No. 10** (2006) 157–190 *Electronic Journal of Theoretical Physics*

Stretching the Electron as Far as it Will Go

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Received 8 May 2006, Published 28 May 2006

Abstract: Effects associated with the existence of isolated zero modes of Majorana fermions are discussed. It is argued that the quantization of this system necessarily contains highly extended quantum states and that populating and depopulating such states by interacting with the quantum system leads to long-ranged teleportation-like processes. Also leads to spontaneous violation of fermion parity symmetry. A quasi-realistic model consisting of a quantum wire embedded in a p-wave superconductor is discussed as an explicit example of a physical system with an isolated Majorana zero mode.

Keywords: Majorana Fermions, Teleportation, quantum Wire P-Wave superconductor, Soliton
PACS (2006): 71.10.Pm, 05.30.Fk, 05.45.Yv, 78.67.Lt,85.35.Be, 74.20.-z

1. Introduction

It is a great pleasure to dedicate this article to the 100'th anniversary of the birth of Ettore Majorana. As a testimony to his lasting influence on science, we shall describe how one of his great insights, used in a modern context, can be related to a particular macroscopic quantum phenomenon.

The idea is related to the observation by Majorana that a relativistic fermion such as the electron can be meaningfully decomposed into more basic degrees of freedom, essentially by taking the real and imaginary parts of its wave-function [1]. In relativistic field theory, what one obtains are called Majorana fermions, which have become the basic building blocks of supersymmetric field theories and supply a scenario whereby the

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neutrinos which are observed in nature can have mass. We shall use this idea in a different context. In quantum condensed matter, the appearance of an emergent Majorana fermion would provide an excitation of a system that has minimal degrees of freedom. The wave-function of the single-particle state would obey a Majorana condition, which would forbid quantum fluctuations of its phase. The utility of this fact has already been recognized in the context of quantum computing [2]-[4]. In the present manuscript, we will elaborate on our previous observation [5] that in some cases this can provide isolated states with wave-functions which are peaked at multiple, well separated locations. In a controlled setting, this can be used to create a condensed matter realization of the Einstein-Podolsky-Rosen effect and even a version of teleportation by long-ranged tunnelling.

Majorana's original motivation for inventing the Majorana fermion was to avoid the negative energy states that relativistic particles invariably seem to possess by identifying the negative and positive energy states of a relativistic wave equation as manifestations of the same quantum excitation.

In second quantization, the positive energy state can be occupied by a particle. Filling a positive energy state creates an excited state of the system with positive energy. On the other hand, a negative energy state should be regarded as typically being already filled by a particle. An excitation of the system is then found by emptying the negative energy state, or creating a hole. The system is put in a higher energy state by removing a negative energy particle, equivalently, creating a positive energy hole.

Majorana's idea can be implemented when there is a particle-hole symmetry. Then, for a given particle state, there exists a hole state with the same energy and with a wave-function that is related to the particle wave-function by a simple transformation. Then, by making the appropriate identification, one could indeed identify these as one and the same quantum state. Of course, the resulting system has half as many degrees of freedom.[1]

To illustrate the idea, let us recall the conventional second quantization of complex fermions, which could be either relativistic or non-relativistic. We begin with the assumption that in some approximation it makes sense to discuss a single non-interacting particle whose wave-function obeys the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{x},t) = H_0\Psi(\vec{x},t) \tag{1}$$

where H_0 is the single-particle Hamiltonian operator. Generally, as in the case of the Dirac equation, the Hamiltonian H_0 could be a matrix, as well as a differential operator, and $\Psi(\vec{x},t)$ a column vector whose indices we shall suppress.[2] The second-quantized field

[1] For a comprehensive account of issues to do with positive and negative energy modes of relativistic bosons and fermions, see the series of papers [6]-[11].

[2] An example is the Dirac Hamiltonian in 3+1-dimensions

$$H_0 = i\vec{\alpha}\cdot\vec{\nabla} + \beta m$$

where $\vec{\alpha}$ and β are a set of four Hermitian, anti-commuting 4×4 Dirac matrices. There exists a matrix Γ with the property $\Gamma\vec{\alpha}\,\Gamma = \vec{\alpha}^*$ and $\Gamma\beta\Gamma = -\beta^*$, so that, $\Gamma H_0\Gamma = -H_0^*$ and $\Gamma\psi_E^* = \psi_{-E}$. This is a

operator also typically obeys this wave equation plus the equal-time anti-commutation relation

$$\left\{\Psi(\vec{x},t),\Psi^\dagger(\vec{y},t)\right\}=\delta(\vec{x}-\vec{y}) \tag{2}$$

It is this anti-commutator which defines $\Psi(x,t)$ as an operator. It can further be used to derive the wave equation (1) from the second quantized Hamiltonian,

$$H=\int dx:\Psi^\dagger(x,t)H_0\Psi(x,t): \tag{3}$$

using the Hamilton equation of motion

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t)=[\Psi(x,t),H]$$

We shall assume that H_0 is a Hermitian operator which has eigenfunctions and a spectrum of real eigenvalues

$$H_0\psi_E(x)=E\psi_E(x)$$

The energy E can be both positive and negative, in fact for the relativistic electron, if (1) were the Dirac equation, there are necessarily negative eigenvalues and the spectrum is unbounded below. The eigenfunctions obey the orthogonality and completeness relations

$$\int d\vec{x}\psi_E^\dagger(\vec{x})\psi_{E'}(\vec{x})=\delta_{EE'}\quad,\quad\sum_E\psi_E(\vec{x})\psi_E^\dagger(\vec{y})=\delta(\vec{x}-\vec{y}) \tag{4}$$

The delta function and summation in these formulae should be understood in a generalized sense where they are a Kronecker delta and a sum for discrete components of the spectrum and a Dirac delta function and integral for continuum spectrum.

In this system, one then forms the second quantized field operator by superposing the wave-functions with creation and annihilation operators,

$$\Psi(x,t)=\sum_{E>0}\psi_E(x)e^{-iEt/\hbar}a_E+\sum_{E<0}\psi_E(x)e^{-iEt/\hbar}b_{-E}^\dagger$$

Here, a_E is the annihilation operator for a particle with energy E and $b_{-E}^\dagger$ is the creation operator for a hole with energy $-E$. When they obey the algebra with non-vanishing anti-commutators

$$\left\{a_E,a_{E'}^\dagger\right\}=\delta_{EE'}\quad,\quad\left\{b_{-E},b_{-E'}^\dagger\right\}=\delta_{EE'}$$

the $\Psi(\vec{x},t)$ obeys the anticommutator (2). The completeness condition in Eq. (4) is essential for establishing this.

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one-to-one mapping of positive to negative energy states. Explicitly, if the matrices are represented by $\vec{\alpha}=\begin{pmatrix}\vec{\sigma} & 0\\ 0 & -\vec{\sigma}\end{pmatrix}$ with $\vec{\sigma}$ the Pauli matrices and $\beta=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, then we can form the matrix $\Gamma=\begin{pmatrix}0 & -i\sigma^2\\ i\sigma^2 & 0\end{pmatrix}$. Note that, in this case $\Gamma=\Gamma^*$ and $\Gamma^2=1$. A Majorana fermion obeys the reality condition $\Psi(\vec{x},t)=\Gamma\Psi^*(\vec{x},t)$.

The ground state of the system, $|0>$, is the state where all positive energy levels are empty and where all negative energy levels are filled, or alternatively all hole states are empty. In the second quantized language, it is annihilated by the annihilation operators,

$$a_E|0>=0=b_{-E}|0>$$

Excited states are created by operating on $|0>$ with $a_E^\dagger$ and $b_{-E}^\dagger$. The excitations created by $a_E^\dagger$ are particles, those created by $b_{-E}^\dagger$ are anti-particles, or holes. A typical state is

$$a_{E_1}^\dagger \dots a_{E_m}^\dagger b_{E_1}^\dagger \dots b_{E_n}^\dagger |0>$$

and such states form a basis for the Fock space of the second quantized theory.

One can formulate Majorana fermions for a system of this kind if there exists a particle-hole symmetry, or, in the relativistic context, a charge conjugation symmetry. For example, consider the situation where a matrix Γ exists such that, for eigenstates of H_0,

$$\psi_{-E}(x) = \Gamma\psi_E^*(x) \tag{5}$$

(This implies that $\Gamma^*\Gamma = 1 = \Gamma\Gamma^*$.) Then, the particles and holes have identical spectra. A Majorana fermion is formed by treating the particle and hole with the same energy as a single excitation. The second quantized field operator is

$$\Phi(x,t) = \sum_{E>0}\left(\psi_E(x)e^{-iEt/\hbar}a_E + \Gamma\psi_E^*(x)e^{iEt/\hbar}a_E^\dagger\right)$$

This fermion does not have both particles and anti-particles. The ground state $|0>$ is annihilated by a_E

$$a_E|0>=0 \quad \forall a_E$$

and $a_E^\dagger$ creates particles, so that the excited states of the system are

$$a_{E_1}^\dagger a_{E_2}^\dagger ... a_{E_k}^\dagger |0>$$

The field operator is (pseudo-)real in the sense that it obeys

$$\Phi(x,t) = \Gamma\Phi^*(x,t) \tag{6}$$

It obeys the anti-commutation relation

$$\left\{\Phi(\vec{x},t), \Phi^\dagger(\vec{y},t)\right\} = \delta(\vec{x}-\vec{y}) \tag{7}$$

To be concrete, in a system of complex fermions where the Hamiltonian such that the spectrum has the particle-hole symmetry (5), we could decompose the complex fermion into two Majorana fermions by taking the real and imaginary parts,

$$\Phi_1(x,t) = \frac{1}{\sqrt{2}}\left(\Psi(x,t) + \Gamma\Psi^*(x,t)\right)$$

$$\Phi_2(x,t) = \frac{1}{\sqrt{2}i}\left(\Psi(x,t) - \Gamma\Psi^*(x,t)\right)$$

Then each of $\Phi_1(x,t)$ and $\Phi_2(x,t)$ are a Majorana fermion.

In spite of the beautiful simplicity of this idea, Majorana fermions are not easy to come by in nature. One could, for example, decompose the relativistic electron, whose wave equation does have a charge-conjugation symmetry, into its real and imaginary parts. However, the interaction of the electron with photons is not diagonal in this decomposition. The real and imaginary components would be rapidly re-mixed by electromagnetic interactions, they cannot be stationary states of the full Hamiltonian of quantum electrodynamics.

One place where we might have better luck is to look for emergent Majorana fermions in quantum condensed matter systems. For example, in a superconductor, the electromagnetic interactions are effectively screened. Indeed, the Bogoliubov quasi-electrons in a superconductor behave like neutral particles. However, even there, in an ordinary s-wave superconductor, the anti-particle of a quasi-electron is another quasi-electron with opposite spin. Indeed, the quasi-electron operator in an s-wave superconductor is the two-component object

$$\begin{pmatrix} \psi_\uparrow(x) \\ \psi^*_\downarrow(x) \end{pmatrix}$$

where $(\uparrow,\downarrow)$ denotes spin up and down. It obeys the charge conjugation condition

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_\uparrow(x) \\ \psi^*_\downarrow(x) \end{pmatrix}^* = \begin{pmatrix} \psi_\downarrow(x) \\ \psi^*_\uparrow(x) \end{pmatrix}$$

which is not an analog of the Majorana condition in eqn. (6), since it entails both conjugation and a flip of the spin.

In order to find a medium where the quasi-electron is a Majorana fermion, we need to consider a superconductor where the condensate has Cooper pairs with the same spin, so that the quasi-electron has the form

$$\begin{pmatrix} \psi_\uparrow(x) \\ \psi^*_\uparrow(x) \end{pmatrix}$$

Then, quasi-electron is pseudo-real, complex conjugation of its wave-function is equivalent to multiplying by the matrix $\Gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_\uparrow(x) \\ \psi^*_\downarrow(x) \end{pmatrix}^* = \begin{pmatrix} \psi_\uparrow(x) \\ \psi^*_\uparrow(x) \end{pmatrix}$$

This gives a physical realization of a Majorana fermion. An example of such a superconductor is one with a p-wave condensate, such as Strontium Ruthenate [12]. There, the condensate has the form $\langle\psi_\uparrow(x)\vec{x}\times\vec{\nabla}\psi_\uparrow(x)\rangle$ (and can in principle have an admixture of spin down as well). Thus, we see that, in such a material, the quasi-electron is a two-component object obeying a Majorana condition. We will make use of this example later in this Paper.

Our particular interest in the following will be in situations where the fermion spectrum has mid-gap, or zero energy states. These are well known to lead to interesting phenomena. Already for complex electrons, mid-gap states give rise to fractional quantum numbers [13, 14]. With Majorana fermions, they are known to lead to peculiar representations of the anti-commutator algebra which can violate basic symmetries [15, 16]. Some interesting effects in the context of zero modes on cosmic strings have also been examined [17]-[20].

To illustrate, let us consider the second quantization of a complex fermion whose spectrum has a zero mode,

$$H_0\psi_0(x) = 0$$

The conjugation symmetry implies that

$$\psi_0(x) = \Gamma\psi_0^*(x)$$

If the fermion is complex (not Majorana), the second quantized field has a term with the zero mode wave-function and an operator, the first term in the following expansion:

$$\Psi(x,t) = \psi_0(x)\alpha + \sum_{E>0}\psi_E(x)e^{-iEt/\hbar}a_E + \sum_{E<0}\psi_E(x)e^{-iEt/\hbar}b^\dagger_{-E} \quad .$$

Here, α obeys the algebra

$$\left\{\alpha,\alpha^\dagger\right\} = 1 \tag{8}$$

and it anti-commutes with all of the other creation and annihilation operators. The existence of this zero mode leads to a degeneracy of the fermion spectrum. The vacuum state is annihilated by all of the annihilation operators a_E and b_E. However, it must also carry a representation of the algebra (8). The minimal representation is two-dimensional. There are two vacuum states, $(|\uparrow>, |\downarrow>)$, which obey

$$a_E|\uparrow> = 0 = a_E|\downarrow> \quad , \quad b_E|\uparrow> = 0 = b_E|\downarrow>$$

and

$$\alpha^\dagger|\downarrow> = |\uparrow> \quad , \quad \alpha^\dagger|\uparrow> = 0$$

$$\alpha|\downarrow> = 0 \quad , \quad \alpha|\uparrow> = |\downarrow>$$

The entire spectrum has a 2-fold degeneracy, with two towers of excited states,

$$a^\dagger_{E_1}...a^\dagger_{E_m}b^\dagger_{E_1}...b^\dagger_{E_n}|\uparrow>$$

and

$$a^\dagger_{E_1}...a^\dagger_{E_m}b^\dagger_{E_1}...b^\dagger_{E_n}|\downarrow>$$

having the identical energies $\sum_i E_i$.

This quantization of the zero mode α was argued by Jackiw and Rebbi [13] to lead to states with fractional fermion number. Indeed, the suitably normal ordered second quantized number operator

$$Q = \int d\vec{x}\frac{1}{2}\left[\psi^\dagger(x,t),\psi(\vec{x},t)\right] = \sum_{E>0}\left(a^\dagger_E a_E - b^\dagger_{-E}b_{-E}\right) + \alpha^\dagger\alpha - \frac{1}{2} \tag{9}$$

has fractional eigenvalues, for example

$$Q|\uparrow>=+\frac{1}{2}|\uparrow> \quad , \quad Q|\downarrow>=-\frac{1}{2}|\downarrow>$$

In actuality, the charge operator is defined only up to an overall additive constant. However, there does exist a symmetry of the theory, gotten at the second quantized level by replacing $\Psi(x,t)$ by $\Gamma\Psi^*(x,t)$. This transformation interchanges particles and anti-particles, and is a symmetry of the suitably normal ordered second quantized Hamiltonian. It should also flip the sign of Q. It implies that, if there is an eigenstate of Q in the system with eigenvalue q,

$$Q|q>=q|q>$$

then there must exist another eigenstate $|-q>$ in the spectrum of Q with eigenvalue $-q$:

$$Q|-q>=-q|-q>$$

In addition, it is easy to argue that the eigenvalues of Q are space by integers, i.e. if q_1 and q_2 are any two eigenvalues, then $q_1 - q_2$ =integer. This is essentially because the raising and lowering operators for Q are $\Psi^\dagger$ and Ψ, respectively and they raise and lower in units of integers. In particular, this implies that

$$q-(-q) = 2q = \text{integer}$$

Thus, the only possibilities are that the entire spectrum of states have integer eigenvalues of Q, q =integer, or the entire spectrum of states have half-odd-integer eigenvalues $q = \frac{1}{2}$-odd integer. It is easy to see that the operator Q as written in (9) indeed flips sign if we interchange $a_E \leftrightarrow b_E$ and $\alpha \leftrightarrow \alpha^\dagger$ and the offset of -1/2 that appears explicitly there is essential for this transformation to work. This leads to the conclusion that, with a single fermion zero mode, the fermion number charge is quantized in half-odd-integer units.

Now, consider what happens for a Majorana fermion with a single zero mode. [3] In this case, a charge analogous to Q is not defined, so the issue of fractional charge is not

[3] We will later construct an explicit example where this precisely this situation occurs.

relevant. But the quantization of the system is still interesting. The second quantized operator is

$$\Phi(x,t) = \psi_0(x)\alpha + \sum_{E>0} \psi_E(x) e^{-iEt/\hbar} a_E + \sum_{E<0} \psi_E(x) e^{-iEt/\hbar} a^\dagger_{-E}$$

This fermion contains half of the degrees of freedom of the previous complex one. Here, the b_E are absent and the zero mode operator is real, $\alpha = \alpha^\dagger$.

The creation and annihilation operator algebra is now

$$\left\{ a_E, a^\dagger_{E'} \right\} = \delta_{EE'}$$

as before, and

$$\alpha^2 = 1/2 \quad , \quad \{\alpha, a_E\} = 0 = \left\{ \alpha, a^\dagger_E \right\} \tag{10}$$

A minimal representation can be constructed by defining a vacuum state where

$$a_E|0>=0 \ \text{ for all } E>0$$

Then, we can represent the zero mode by the operator

$$\alpha = \frac{1}{\sqrt{2}} (-1)^{\sum_{E>0} a^\dagger_E a_E} \tag{11}$$

Indeed

$$\alpha = \alpha^\dagger$$

and, since

$$\sum_E a^\dagger_E a_E |0>=0$$

we have

$$\alpha|0>=\frac{1}{\sqrt{2}}|0>$$

The Klein operator, $(-1)^{\sum_{E>0} a^\dagger_E a_E}$, anti-commutes with a_E and $a^\dagger_E$. A basis for the Hilbert space consists of the vacuum and excited states which are obtained from the vacuum by operating creation operators

$$a^\dagger_{E_1} a^\dagger_{E_2} ... a^\dagger_{E_k} |0>$$

These are eigenstates of $\sum_{E>0} a^\dagger_E a_E$ with integer eigenvalues. Thus, in this basis, $\alpha^2 = 1/2$ when operating on each basis vector, and thus the identity operator on the whole space. The operator in (11) thus satisfies the algebra (10).

Another, inequivalent representation can be obtained by starting with

$$\tilde{\alpha} = -\frac{1}{\sqrt{2}} (-1)^{\sum_{E>0} a^\dagger_E a_E} \tag{12}$$

and a similar construction leads to a Hilbert space whose states are orthogonal the one found above. We emphasize here that there are two inequivalent representations of the

anti-commutator algebra, one where the zero mode operator is represented by α in eq. (11) and one where it is represented by $\tilde{\alpha}$ in eq. (12). Both of these give an irreducible representation and the two representations are not related to each other by an internal automorphism.

We observe that these minimal representations of the anti-commutator algebra have the property that they break a symmetry of the fermion theory under $\Phi(\vec{x},t) \to -\Phi(\vec{x},t)$, which we shall call "fermion parity". Fermion parity is a symmetry of the linear wave equation even when $\Phi(\vec{x},t)$ is a Majorana fermion. At the quantum level, fermion parity symmetry leads to a conservation law for the number of fermions modulo 2. By this conservation law, any physical process must entail creation or destruction of an even number of fermions. For example, if a quantum state is initially prepared with an even number of fermions, after any physical process, the number should remain even. In operator language, there should exist an operator $(-1)^F$ which anti-commutes with $\Phi(\vec{x},t)$,

$$(-1)^F \Phi(\vec{x},t) + \Phi(\vec{x},t)(-1)^F = 0$$

and which therefore commutes with the full second quantized Hamiltonian,

$$(-1)^F H = H(-1)^F$$

where

$$H = \int d\vec{x} \frac{1}{2} : \Phi^\dagger(\vec{x},t) H_0 \Phi(\vec{x},t) :$$

However, we see that in the minimal representations of the anti-commutation algebras (10) discussed above, in the first representation (11),

$$< 0|\Phi(x,t)|0 >= +\frac{1}{\sqrt{2}}\psi_0(x)$$

and in the second representation (12)

$$< 0|\Phi(x,t)|0 >= -\frac{1}{\sqrt{2}}\psi_0(x)$$

In both of these representations, neither the vacuum state, nor any of the excited states can be eigenstates of fermion parity, the operator $(-1)^F$. Thus fermion parity symmetry is broken by the minimal quantization of this model.

Fermion parity is a sacred symmetry of physics in four dimensional space-time [21]. All fundamental fermions in nature have half-odd-integer spin. A flip in sign of all fermion operators can then be realized as a rotation by an angle 2π. Nature should be symmetric under a rotation by 2π. This means that, if we superpose a state with even fermion number and a state with odd fermion number,

$$c_1|\text{even} > +c_2|\text{odd} >$$

no experiment should be devisable, even in principle, to measure the relative sign of c_1 and c_2. In the four dimensional world, unless rotation invariance is broken at a the level

of fundamental physics, we should always be free to insist that $(-1)^F$ is a good symmetry and that we can take all physical states as eigenstates. Of course, this applies in four space-time dimensions. The emergent Majorana fermions that we want to consider here are embedded in four space-time dimensions. We therefore feel free to insist on fermion parity.

This brings up a contradiction with the previous discussion, where we found that fermion parity is necessarily broken by the quantization of the zero mode Majorana fermion system. The only way to restore the symmetry is to use a reducible representation of the anti-commutator algebra. The minimal modification of the representation is equivalent to the introduction of another degree of freedom – and subsequent use of irreducible representations. The new degree of freedom acts like a hidden variable. In the anti-commutator algebra it would be another anti-commuting variable β which has identical properties to α,

$$\beta^2 = 1/2$$

and anti-commutes with all other variables. Then the algebra of α and β would have a two dimensional representation which we could find by considering the fermionic oscillators

$$a = \frac{1}{\sqrt{2}}\left(\alpha + i\beta\right) \quad , \quad a^\dagger = \frac{1}{\sqrt{2}}\left(\alpha - i\beta\right) \tag{13}$$

$$\alpha = \frac{1}{\sqrt{2}}\left(a + a^\dagger\right) \quad , \quad \beta = \frac{1}{\sqrt{2}i}\left(a - a^\dagger\right) \tag{14}$$

which obey

$$a^2 = 0 \quad , \quad a^{\dagger 2} = 0 \quad , \quad \left\{a, a^\dagger\right\} = 1$$

We could then find a vacuum state which is annihilated by a, and another state which is created from the vacuum by $a^\dagger$,

$$a|- >= 0 \quad , \quad a^\dagger|- >= |+ >$$

$$a|+ >= |- > \quad , \quad a^\dagger|+ >= 0$$

so that both are eigenstates of $(-1)^F$ and fermion parity is restored. Later we will see that the hidden variable β can have a physical interpretation.

2. Degeneracy, Tunnelling and Teleportation

In this paper, the most speculative use of Majorana fermions that we shall find is for a kind of teleportation by quantum tunnelling. In the context in which quantum tunnelling is normally studied, a classical object can exist in allowed regions. There exist other forbidden regions where it is not allowed to be. Then, quantum tunnelling makes use of the fact that, when the particle is quantum mechanical, its wave-function does not necessarily go to zero in a classically forbidden region, but decays exponentially. That means that it could, in principle, have support on the other side of such a region and

there is some small probability that an object will be found on the other side. This is called tunnelling.

One might try to make use of quantum tunnelling to transport an object through a classically forbidden region. Unfortunately, the exponential decay of the wave-function across any classical barrier of appreciable size renders it too small to be of any practical use in this regard. A more sophisticated approach would be to create a scenario where the wave-function has peaks of appreciable size at spatially separated locations, perhaps with a forbidden region in between. This too will fail, but for a more sophisticated reason which, since it is related to our later use of Majorana fermions, we will outline. Consider, for example the double well potential depicted in Fig. 1. If the locations of the minima

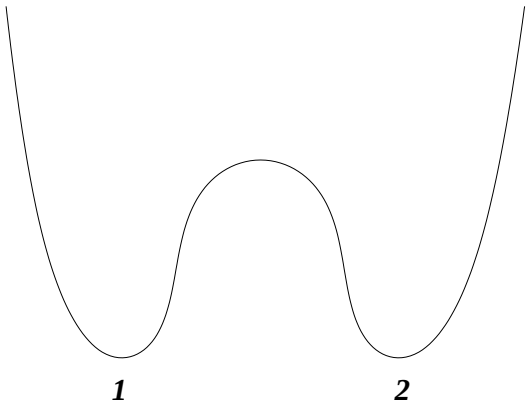


Fig. 1 A double-well potential.

are well separated and the barrier in between them is large, semi-classical reasoning can be applied to this system. Then, the ground state of a particle in this potential should indeed have a peak near each of the minima, and should be approximately symmetric under interchanging the locations of the minima. The typical profile of such a wave-function is drawn in Fig. 2.

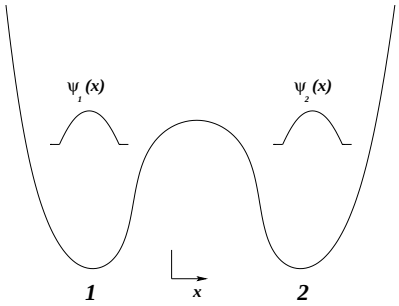


Fig. 2 The ground state of a particle in a double well has two peaks, localized at 1 and 2.

Now, we ask the question. Is this state of use for tunnelling? If this were the energy landscape in which a quantum mechanical particle lived, could we, for example, populate this ground state by interacting with the system in the vicinity of minimum 1 and then depopulate the state by interacting with the system near the other minimum - 2, effectively teleporting the particle from location 1 to location 2?

The answer to this question is 'no'. The reason for this answer is degeneracy, or approximate degeneracy of the quantum state that we are considering. In such a system, when our classical reasoning is good, there must always be a second state, perhaps at slightly higher energy but approximately degenerate with the ground state, whose wave-function is approximately an anti-symmetric function of the positions of the minima. Its typical profile is depicted in Fig. 3. The ground state wave-function has the form $\psi_0(x) = \psi_1(x) + \psi_2(x)$ where $\psi_1(x)$ is localized near minimum 1 and $\psi_2(x)$ is localized near minimum 2. The anti-symmetric state would have the form $\psi_a(x) = \psi_1(x) - \psi_2(x)$.

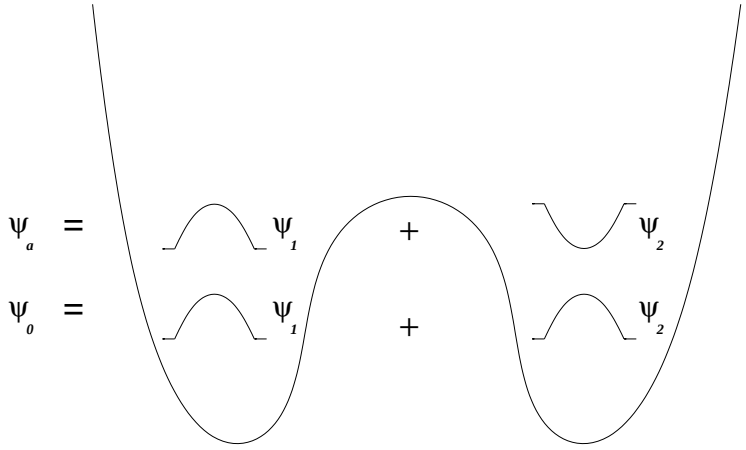


Fig. 3 The almost degenerate state ψ_a also has two peaks but with differing signs.

Now, when we interact with the system near minimum 1, while we overlap the ground state wave-function, $\psi_0(x)$, we also overlap $\psi_a(x)$ by the same amount. Of course, the state that we actually populate is a linear combination of the two,

$$\frac{1}{\sqrt{2}}\left(\psi_0(x) + \psi_a(x)\right) = \sqrt{2}\psi_1(x)$$

whose wave-function is entirely localized at the position of the first minimum. The particle initially has zero probability of appearing near the second minimum. Our attempt at teleportation by tunnelling has been foiled by degeneracy.

Anytime the Schrödinger equation can be analyzed semi-classically in this way, it seems to have a built in protection against the long-ranged behavior that we are looking for.

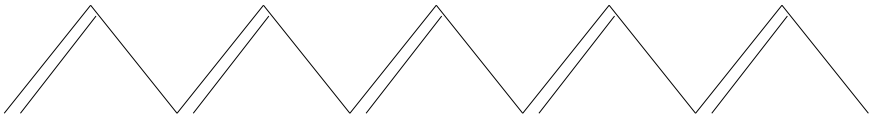
In this argument, because it is a superposition of two stationary states with slightly differing energies, $\psi_1(x)$ is not a stationary state. It should have a small time dependence which eventually mixes it with $\psi_2(x)$. But this time dependence mixes it slowly, in fact its origin is just the conventional tunnelling amplitude for the particle to move from location 1 to location 2 through the barrier in between.

What we need to find is a quantum system where a quantum state which is well isolated from other states in the spectrum can have peaks at different locations. From the argument above, it will be difficult to find states of this kind which obey the regular Schrödinger equation. Where we will look for such states is in quantum condensed matter

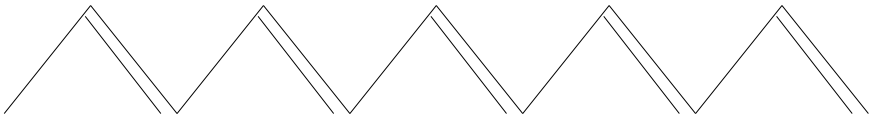
systems, where electrons, or more properly quasi-electrons, can satisfy equations that are very different from the Schrödinger equation. To motivate this, in the next Section, we review some of the pictorial arguments for the appearance of fractionally charged states in polyacetylene. Also, to set the stage for what comes next, we discuss what happens if the fermion spectrum of polyacetylene were Majorana, rather than complex fermions.

3. The Polyacetylene Story

Before we consider a more quantitative model which will illustrate our point, we pause to recall the example of the conducting polymer, polyacetylene. Polyacetylene is a hydrocarbon polymer where each Carbon atom bonds with a Hydrogen atom and as well forms two strong covalent bonds with neighboring Carbon atoms. The fourth valence electron is nominally a conduction electron. However, a Peirls instability localizes it into a charge density wave which is effectively a dimer. The result is a gap in the electron spectrum at the fermi surface and, without impurities or other structures, the material is an insulator. There are two degenerate ground states, depending on the direction chosen by the dimerization. We illustrate these as the A and B phases in the diagram in Fig. 4. In that figure, each line is a covalent bond, using two of the valence electrons of the



Phase A



Phase B

Fig. 4 The two degenerate ground states of polyacetylene.

Carbon atoms.

The conductivity of doped polyacetylene that is seen by experiments is thought to be mostly attributed to the transport of charged solitons along the polyacetylene molecules. A soliton in this system is a defect which interpolates between the two phases. We have depicted a soliton-anti-soliton pair in Fig. 5. Note that it can be obtained from one of the ground states by flipping the direction of the bonds that lie between the locations of the solitons. Also note that the energy of the system could be higher than that of the ground state, since the defects have non-minimal energy configurations, but the energy density should be concentrated in the vicinity of the solitons. Although it will not be an issue for us, since we are interested in other aspects of this system, the solitons turn out to be quite mobile. They also carry electric charge, and can thus account for the high

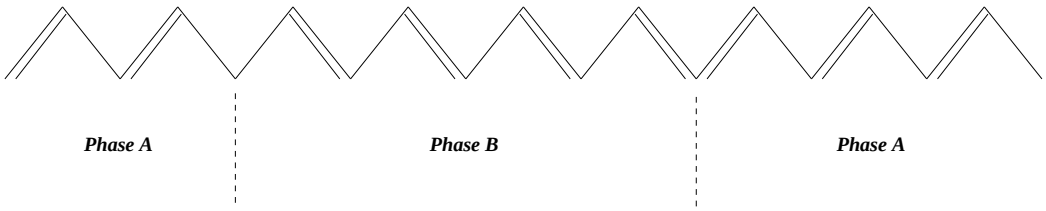


Fig. 5 Solitons form phase boundaries. The soliton anti-soliton pair can be created by flipping the direction of the bonds between them.

conductivity that is attainable in polyacetylene. The density of solitons can be controlled by doping. For some original literature on polyacetylene, see refs. [22]-[29].

There is a simple argument that shows that a soliton of polyacetylene has half of the quantum numbers of an electron [28]. In this argument, we will neglect the spin of the electron. Thus, for the purpose of our arguments, in figures 4 and 5, each bond stands for a single electron, rather than a spin up, spin down pair of electrons. Now, consider what happens when we add an electron to phase A, as in Fig. 6. By flipping the directions

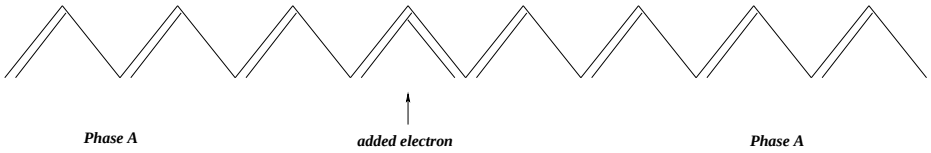


Fig. 6 Phase A with an additional electron.

of some bonds, we can see that we have created a soliton-anti-soliton pair, where each object seems to share half of the added electron. This state is depicted in Fig. 7.

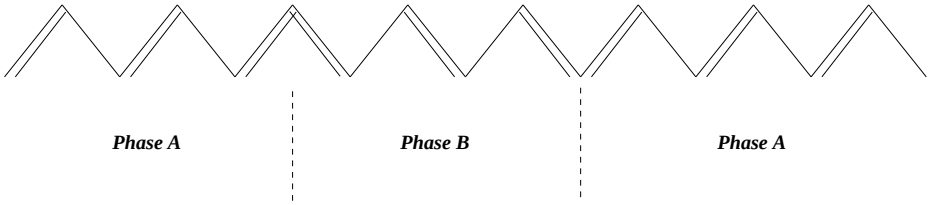


Fig. 7 Beginning with phase A and an additional electron, as shown in Fig. 6, we create a soliton-antisoliton pair which seems to share the electron.

This brings up the question, is the electron really 'split' between the two sites? Or does it exist in an entangled state of some sort which has some probability – $\frac{1}{2}$ – of the "whole" electron being located at either site. This question can be made more precise by asking about measurement of the electron charge, which is a conserved quantum number in this system. If, by further flipping bonds, we separate the solitons to a large distance, and then measure the electron charge in the vicinity of one of the solitons, is the result of the measurement -e/2? Or does this measurement manage to collapse the electron wave-function somehow so that the result is either 0 or -e? In the latter case, the average of many measurements might be -e/2, but any single measurement would either see a whole electron or no electron at all. The answer to this question was found long ago in ref. [30, 31]. The conclusion was that the measurement of the electron charge localized

near one of the solitons should yield -e/2. Put equivalently, the fractional charge of the soliton is a sharp quantum observable. How it manages to do this is interesting, and was discussed in ref. [31]. We shall review it here.[4] This issue has recently been reexamined [48] in conjunction with some ideas about entangled electron states in Helium bubbles [49].

The electron spectrum in polyacetylene has an electron-hole symmetry. We could have created a state with the same energy as the one depicted in Fig. 7 by removing, rather than adding an electron, to give a hole which is apparently split between the soliton and anti-soliton, as shown in Fig. 8.

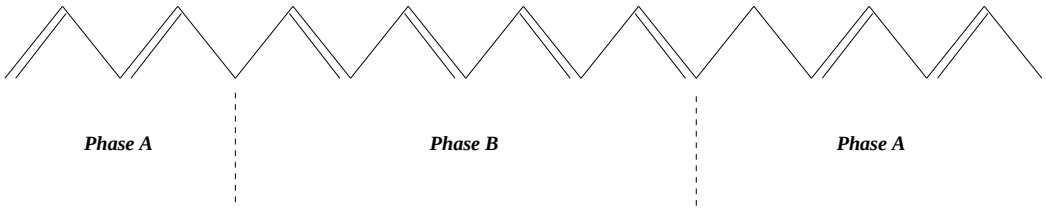


Fig. 8 A soliton-antisoliton pair with a deficit of one electron.

There are apparently four different states of the soliton-anti-soliton system. There are the two overall neutral states, one of which is depicted in Fig. 5 and the other obtained by flipping the intermediate bonds in the opposite direction. We could also go from one of these states to the other by transporting a whole electron from one soliton to the other. The other two states we can obtain by either adding or subtracting an electron from one of the ground states and are those that we have already discussed in Figs. 7 and 8. We can identify these charged soliton states in the low energy electron spectrum. In the single electron spectrum of polyacetylene with a soliton-anti-soliton pair, there are two near-mid-gap states which have small positive and negative energies. Thus the low energy electron spectrum has four states, a ground state, an electron state, a hole state and an electron-hole state.

By their quantum numbers, the electron and hole states can be identified with the configurations in Figs. 7 and 8, respectively. The ground state and the electron-hole state are neutral and must be formed from linear combinations of the two neutral states. Then, in the electron state, the electron wave-function indeed should have two peaks, as depicted in Fig. 9. Similarly the hole wave-function also should have two peaks, as is depicted in Fig. 10. Detailed analysis shows that one is an even and the other is an odd function of relative distance, as shown in the figures.

The electron wave-function has peaks at two locations. So, we could ask the question again: Can we use this system for teleportation? Could we populate the electron state by interacting with one of the solitons and subsequently extract the electron again, and thereby teleport it, by interacting with the other soliton? To understand the answer, which will be 'No!', it is necessary to realize that, once the electric charge is a sharp quantum observable, the electronic states of the solitons are disentangled by a local

[4] For some other literature on this and closely related issues, see refs. [32]-[47].

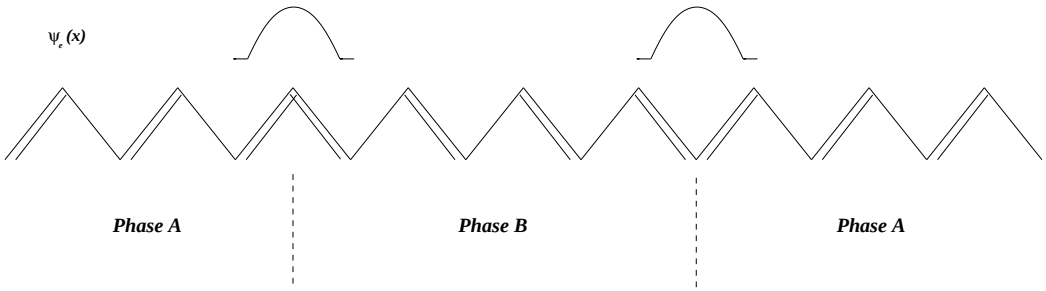


Fig. 9 The electron wave-function.

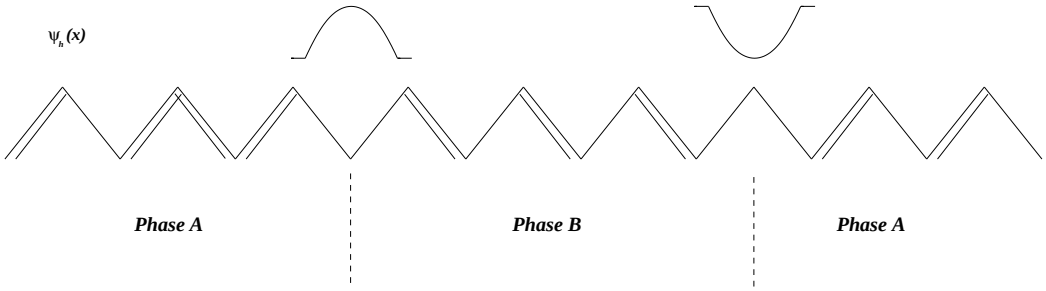


Fig. 10 The hole wave-function.

measurement of an observable such as the charge.

To see how this happens, let us consider a second quantization of this system. The electron operator has the form

$$\psi(x,t) = \psi_e(x)a + \psi_h(x)b^\dagger + ...$$

where we have identified an electron annihilation operator a for the positive energy state and a hole creation operator $b^\dagger$ for the negative energy state. We have neglected the time dependence (the energies of the two states are exponentially small in the soliton separation). We could as well write

$$\psi(x,t) = \psi_1(x)(a + b^\dagger) + \psi_2(x)(a - b^\dagger) + ... \tag{15}$$

where $\psi_1(x) = \frac{1}{\sqrt{2}}(\psi_e(x) + \psi_h(x))$ has support only in the region of the left-hand soliton and $\psi_2(x) = \frac{1}{\sqrt{2}}(\psi_e(x) - \psi_h(x))$ has support only near the right-hand soliton in Figs. 9 and 10.

Now, if we concentrate on the region near the left-hand soliton, $\psi(x,t)$ or $\psi^\dagger(x,t)$ will annihilate or create an excitation using the combination of operators

$$\alpha = \frac{1}{\sqrt{2}}\left(a + b^\dagger\right) \quad , \quad \alpha^\dagger = \frac{1}{\sqrt{2}}\left(a^\dagger + b\right)$$

Similarly, if we concentrate on the region around the right-hand soliton, excitations are created and annihilated using

$$\beta = \frac{1}{\sqrt{2}}\left(a - b^\dagger\right) \quad , \quad \beta^\dagger = \frac{1}{\sqrt{2}}\left(a^\dagger - b\right)$$

The set of operators $(\alpha, \alpha^\dagger, \beta, \beta^\dagger)$ are a Bogoliubov transformation of the creation and annihilation operators $(a, a^\dagger, b, b^\dagger)$. This transformation does not violate fermion number – it superposes operators with the same fermion number. Further, the excitations that the new operators create or annihilate are entirely localized on one or the other of the solitons.

Thus, again, we do not have a process whereby an electron or hole state which has two peaks can be populated by interacting with the system in the vicinity of one of the peaks. We have failed to find teleportation. Instead we have found fractional charge. The charge density integrated over the vicinity of one of the solitons turns out to be

$$Q = -e\left(\alpha^\dagger\alpha - 1/2\right) + \text{ charge of electrons } - \text{ charge of holes}$$

which indeed has half-odd-integer eigenvalues. This Bogoliubov transformation, as a mechanism for disentangling the charge quantum numbers of the solitons was originally found in ref. [31].

In eq. (15), we ignored the small time dependence of the near mid-gap states. At this point, the reader might wonder if the disentanglement of the soliton and anti-soliton charges that we find by the Bogoliubov transformation would not be undone by this time variation. Indeed, it would be, eventually. However the time scale is given by the inverse of the energy gap and is therefore exponentially large in the distance L between the soliton and anti-soliton, $T \sim m^{-1}e^{mL}$, where m is the energy gap. This is roughly the time for quantum mechanical tunnelling between the solitons assuming an energy barrier of height the energy gap extending over distance L. For macroscopic L this time T should be very large.

What has prevented teleportation in this second example is again a degeneracy, this time a slightly more subtle one since, even though the electron and hole state have identical energies, they have opposite signs of charge. Avoiding teleportation has led to fractional charge. It has done this by a hybridization, at the second quantized level, of the propensity of the electron field operator to create an electron and to annihilate a hole in a local state.

Now, imagine that, rather than complex electrons, polyacetylene had Majorana fermions which would be obtained by identifying the particle and hole states as the same excitations. (Here, we are ignoring the obvious disaster that this scenario would lead to in chemistry.) Then, in eqn. (15), we would have to identify $a = b$ and

$$\psi_{\rm Maj}(x,t) = \psi_1(x)(a + a^\dagger) + \psi_2(x)(a - a^\dagger) + ... \tag{16}$$

Now, $a + a^\dagger$ cannot be an annihilation operator, in fact

$$(a + a^\dagger)^2 = 1$$

It is similar to the single zero mode operator "α" that we found for a Majorana fermion in the Eq. (14). In fact, the other combination $\frac{1}{\sqrt{2}i}(a - a^\dagger)$ now plays the role of "β", the "hidden variable". Its purpose in our previous discussion was to provide a quantization

which did not violate fermion parity. Here, this hidden variable is just the fermion zero mode living on the far-away anti-soliton. We could choose the fermion parity conserving quantization by using the states $(|->, |+>)$ defined by

$$a|->=0 \quad , \quad a^\dagger|->=|+>$$

$$a|+>=|-> \quad , \quad a^\dagger|+>=0$$

which can be eigenstates of $(-1)^F$. In these states, the expectation value of the fermion operator vanishes, for example $<0|\psi_{\rm Maj}(x,t)|0>=0$. However, the two solitons are invariably entangled. There is now no conserved fermion number that we can use to measure this entanglement, but there are other effects which we will discuss in later sections once we have made the present reasoning more solid by discussing it in the context of a field theoretical model and them formulated a more realistic model with emergent Majorana fermions.

4. Relativistic Majorana Fermions in a Soliton Background

Single-particle states that are in some sense isolated are well known to occur for Dirac equations, particularly when interacting with various topologically non-trivial background fields such as solitons, monopoles and instantons. The consequences of fermion zero modes such as chiral anomalies [50] and fractional fermion number [13], [14] are well known.

The polyacetylene example, in the context of discussions of fractional charge, that we used in the previous Section is a well-known example of this. In polyacetylene, the low energy electron spectrum can be approximately described by the Dirac equation [22, 51] and the solitons which we discussed using pictures have a mathematical description as soliton-like configurations of a scalar field which couples to the Dirac equation. In this Section, we will make the analysis of the previous Section more quantitative by considering the problem of a 1+1-dimensional relativistic Dirac equation coupled to a soliton background field and a soliton-anti-soliton pair.

Consider, for example, the simple one-dimensional model with Dirac equation

$$\left[i\gamma^\mu\partial_\mu+\phi(x)\right]\psi(x,t)=0 \tag{17}$$

The Dirac gamma-matrices obey the algebra

$$\{\gamma^\mu,\gamma^\mu\}=2g^{\mu\nu}$$

where $g^{\mu\nu}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ is the (inverse of the) metric of two dimensional space-time.

This describes a fermion moving in one dimension and interacting with a scalar field $\phi(x)$ which we shall take to have a soliton-anti-soliton profile. For the purposes of this discussion, we take the ideal case of a step-function soliton located at position $x=0$ and

a step-function anti-soliton located at $x = L$,

$$\phi(x) = \begin{cases} \phi_0 & x < 0 \ , \ x > L \\ -\phi_0 & 0 < x < L \end{cases} \tag{18}$$

We will assume that the solitons are very massive, so they do not recoil when, for example, fermions scatter from them.

If we take

$$\psi(x,t) = \psi_E(x) e^{-iEt}$$

and choose an appropriate basis for the Dirac gamma-matrices, the Dirac equation becomes

$$i \begin{pmatrix} 0 & \frac{d}{dx} + \phi(x) \\ \frac{d}{dx} - \phi(x) & 0 \end{pmatrix} \begin{pmatrix} u_E(x) \\ v_E(x) \end{pmatrix} = E \begin{pmatrix} u_E(x) \\ v_E(x) \end{pmatrix} \tag{19}$$

This equation has a particle-hole symmetry

$$\psi_{-E}(x) = \psi_E^*(x)$$

It is easy to show that it has exactly two bound states. One is a state with small positive energy and the other is the associated hole state with a small negative energy. The wave-functions

$$E_+ \approx +\phi_0 e^{-\phi_0 L} \tag{20}$$

$$\psi_+(x) \approx \sqrt{\phi_0} \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-\phi_0 x} + \mathcal{O}(e^{-\phi_0 L}) & x < 0 \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-\phi_0 x} + \begin{pmatrix} 0 \\ -i \end{pmatrix} e^{\phi_0 (x-L)} + \mathcal{O}(e^{-\phi_0 L}) & 0 < x < L \\ \begin{pmatrix} 0 \\ -i \end{pmatrix} e^{\phi_0 (L-x)} + \mathcal{O}(e^{-\phi_0 L}) & L < x \end{cases} \tag{21}$$

$$E_- \approx -\phi_0 e^{-\phi_0 L} = -E_+ \tag{22}$$

$$\psi_(x) - \approx \sqrt{\phi_0} \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-\phi_0 x} + \mathcal{O}(e^{-\phi_0 L}) & x < 0 \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-\phi_0 x} + \begin{pmatrix} 0 \\ i \end{pmatrix} e^{\phi_0 (x-L)} + \mathcal{O}(e^{-\phi_0 L}) & 0 < x < L \\ \begin{pmatrix} 0 \\ i \end{pmatrix} e^{\phi_0 (L-x)} + \mathcal{O}(e^{-\phi_0 L}) & L < x \end{cases} \tag{23}$$

where, sufficient for our purposes, we give only the large L asymptotics – corrections to all quantities are of order $e^{-\phi_0 L}$. Note that $\psi_-(x)$ is indeed related to $\psi_+(x)$ by $\psi_-(x) = \psi_+^*(x)$.

These states have energy well separated from the rest of the spectrum, which is continuous and begins at $E = \pm\phi_0$. The energies are also exponentially close to zero as the separation L is large. Furthermore, each wave-function has two peaks, one near $x = 0$ and one near $x = L$. They have identical profile near $x = 0$ and they differ by a minus sign near $x = L$. This is the same feature of the electron and hole states that we claimed for the polyacetylene soliton-anti-soliton system in the previous Section.

The second quantized Dirac field now has the form

$$\psi(x,t) = \psi_+(x)e^{-iE_+t}a + \psi_+^*(x)e^{iE_+t}b^\dagger + \dots \tag{24}$$

When L is large, one can consider a second set of almost stationary states which are the superpositions

$$\psi_0(x) = \frac{1}{\sqrt{2}}\left(e^{iE_0t}\psi_+ + e^{-iE_0t}\psi_-\right) \tag{25}$$

$$\approx \sqrt{2\phi_0}\begin{cases} \begin{pmatrix}\cos E_0t\\ 0\end{pmatrix} e^{-\phi_0 x} + \mathcal{O}(e^{-\phi_0 L}) & x < 0 \\ \begin{pmatrix}\cos E_0t\\ 0\end{pmatrix} e^{-\phi_0 x} + \begin{pmatrix}0\\ \sin E_0t\end{pmatrix} e^{\phi_0(x-L)} + \mathcal{O}(e^{-\phi_0 L}) & 0 < x < L \\ \begin{pmatrix}0\\ \sin E_0t\end{pmatrix} e^{\phi_0(L-x)} + \mathcal{O}(e^{-\phi_0 L}) & L < x \end{cases} \tag{26}$$

which has most of its support near $x = 0$ and

$$\psi_L(x) = \frac{1}{\sqrt{2}i}\left(e^{iE_0t}\psi_+ - e^{-iE_0t}\psi_-\right) \tag{27}$$

$$\approx \sqrt{2\phi_0}\begin{cases} \begin{pmatrix}\sin E_0t\\ 0\end{pmatrix} e^{-\phi_0 x} + \mathcal{O}(e^{-\phi_0 L}) & x < 0 \\ \begin{pmatrix}\sin E_0t\\ 0\end{pmatrix} e^{-\phi_0 x} + \begin{pmatrix}0\\ -\cos E_0t\end{pmatrix} e^{\phi_0(x-L)} + \mathcal{O}(e^{-\phi_0 L}) & 0 < x < L \\ \begin{pmatrix}0\\ -\cos E_0t\end{pmatrix} e^{\phi_0(L-x)} + \mathcal{O}(e^{-\phi_0 L}) & L < x \end{cases} \tag{28}$$

which has most of its support near $x = L$.

In terms of these wave-functions, which are localized at the sites of the solitons,

$$\psi(x,t) = \psi_0(x,t)\frac{1}{\sqrt{2}}\left(a + b^\dagger\right) + \psi_L(x,t)\frac{1}{\sqrt{2}i}\left(-a + b^\dagger\right) + \dots \tag{29}$$

We could now consider the creation and annihilation operators

$$\alpha = \frac{1}{\sqrt{2}}\left(a + b^\dagger\right) \quad , \quad \alpha^\dagger = \frac{1}{\sqrt{2}}\left(a^\dagger + b\right)$$

$$\beta = \frac{1}{\sqrt{2}i}\left(a^\dagger - b\right) \quad , \quad \beta^\dagger = \frac{1}{\sqrt{2}i}\left(-a + b^\dagger\right)$$

By interacting with the system at $x = 0$, we could as well be dropping the fermion into the state ψ_0, which is localized there and which has exponentially vanishing probability of occurring at $x = L$ (until $\sin E_0 t$ becomes appreciable, which is just the usual estimate of tunnelling time through a barrier of height ϕ_0 and width L).

It might seem bizarre that, if we begin with the system in its ground state when L is small, then adiabatically increase L that we would not simply end up with the original ground state that has $\psi_-(x)$ populated, $\psi_+(x)$ empty. In fact, this is a possibility. However, as we have argued in the polyacetylene example in the previous Section, as $L \to \infty$, the result is an entangled state of (appropriately defined [30, 31]) fermion number. If we begin with the original ground state, measurement of the fermion number which is localized in the vicinity of one of the solitons will collapse the wave-function to one where the fermion, rather than occupying the negative energy state ψ_-, occupies either the state ψ_0 or the state ψ_L which are localized at $x = 0$ or $x = L$, respectively. As seen from the vicinity of each soliton, these are identical to the Jackiw-Rebbi states [13] of the fermion in a single soliton background, which have fermion number $\pm\frac{1}{2}$. These states are time-dependent, but again, just as in the polyacetylene example, the time scale for charge fluctuations is just the tunnelling time for a particle to go between the locations of the solitons.

What about teleportation? Now, our dumping a fermion into the bound state, if performed near $x = 0$ would populate the state $\psi_0(x)$, rather than $\psi_+(x)$, as all local operators would couple only to this state. It would have appreciable probability of appearing at $x = L$ only after a time over order $E_0^{-1} \sim \phi_0^{-1} e^{\phi_0 L}$.

The situation is somewhat different if we assume that the fermion is a Majorana fermion. The Hamiltonian of a Majorana fermion must have a symmetry which maps positive energy states onto negative energy states. In the case of (19), we have $\psi_{-E}(x) = \psi_E^*(x)$. Then, a fermion and an anti-fermion have the same spectrum, and we can identify them as the same particle.

Now, for the Majorana fermion, the pair of wave-functions $\psi_+(x)$ and $\psi_-(x)$ correspond to the same quantum state which can be either occupied or empty. (We can arbitrarily assign fermion parities $(-1)^F = -1$ for the unoccupied state and $(-1)^F = 1$ for the occupied state, although $+i$ and $-i$ might be more symmetric). In this case, the states ψ_0 and ψ_L are wave-functions for superpositions of the occupied and unoccupied states – they do not have definite fermion parity.

If we begin with the system where the quantum state is an eigenstate of fermion parity and we by some process dump a fermion into the bound state near $x = 0$, its wave-function automatically has a second peak at $x = L$ and it could in principle be extracted there. This defines what we mean by "teleportation".

If we concentrate on the region near $x = 0$ and we are unaware of the region near $x = L$, depending on the quantization, this teleportation will appear as either violation of conservation of fermion number mod 2 or the existence of a hidden variable in the local theory.

5. P-Wave Superconductor Model and Andreev States

Of course, the fermions in polyacetylene are not Majorana, they are electrons with complex wave-functions. The place to look for emergent Majorana fermions in nature is in superconductivity. Here we shall formulate a model whose basic excitations are Majorana fermions. We will do this by using contact with a p-wave superconductor to violate the conservation of total charge, leaving behind conservation of charge modulo 2. In such an environment, the real and imaginary parts of the electron can have different dynamics and the electron is essentially split into two Majorana fermions. They can further be coupled to soliton-like objects, in this case the boundaries of the space, in such a way that only one of the Majorana fermions has zero modes. Then, the scenario that we have been looking for, an isolated single-particle state, can be found.

In these materials, mid-gap bound states, called Andreev states, are a common occurrence. They typically live at surface of the superconductor [52]. In our case, these will be Majorana zero modes.

Majorana zero modes of the type that we are discussing are also known to be bound to vortices in p-wave superconductors where they have the remarkable effect of giving vortices non-Abelian fractional statistics [53],[54]. For concreteness we will consider a slightly simpler model one-dimensional model that was originally discussed by Kitaev [2] in the context of fermionic quantum computation.

We shall consider a quantum wire embedded in a bulk P-wave superconductor as is depicted in Fig. 11.

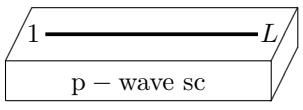


Fig. 11 A quantum wire embedded in a bulk P-wave superconductor.

We shall assume that the wire has a single channel. We shall also assume that the dynamics of electrons in the wire are adequately described by a one-dimensional tight-binding model. We will ignore the spin degree of freedom of the electron. The phenomenon that we will find is to a first approximation spin-independent.

We will assume that the coupling to the neighboring p-wave superconductor is weak and its net effect is to give electrons the possibility of entering and leaving the wire in

pairs by creating or destroying a p-wave cooper pair in the bulk. To describe the electrons, we will use the Hamiltonian

$$H = \sum_{n=1}^{L} \left(\frac{t}{2} a_{n+1}^{\dagger} a_n + \frac{t^*}{2} a_n^{\dagger} a_{n+1} + \frac{\Delta}{2} a_{n+1}^{\dagger} a_n^{\dagger} + \frac{\Delta^*}{2} a_n a_{n+1} + \mu a_n^{\dagger} a_n \right) \tag{30}$$

Sites on the quantum wire are labelled by $n = 1, 2, ..., L$. The operators a_n and $a_n^{\dagger}$ annihilate and create an electron at site n. They obey the anti-commutator algebra

$$\left\{ a_n, a_{n'}^{\dagger} \right\} = \delta_{nn'} \tag{31}$$

The first terms in the Hamiltonian, with coefficients t and t^* are the contribution to the energy of the hopping of electrons between neighboring sites. The second pair of terms, with Δ and Δ^*, arise from the presence of the super-conducting environment. They describe the amplitude for a pair of electrons to leave or enter the wire from the environment. It is assumed that they can do this as a Cooper pair when they are located on neighboring sites. This is effectively an assumption about the size and coherence of the cooper pairs in the superconductor. Even if it were not accurate, the smaller next-to-nearest neighbor, etc. terms that would arise could be taken into account and would not change our result significantly. The last term is the chemical potential, the energy of an electron sitting on a site of the wire. We shall assume a reasonable hierarchy of the parameters, that the amplitude for hopping along the wire is somewhat larger than hopping to and from the bulk, $|t| > |\Delta|$, and that the chemical potential is close enough to zero that the electron band has substantial filling, $|\mu| < |t|$.

5.1 Spectrum of Single-Particle States

Let us discuss the spectrum of the single-particle states in the many-body theory described by the Hamiltonian (30). If $t = |t|e^{i\phi}$ and $\Delta = |\Delta|e^{2i\theta}$, by redefining $a_n \to e^{i(\phi+\theta)} a_n$ for n odd and $a_n \to e^{i(\phi-\theta)} a_n$ for n even, we remove the complex phases of t and Δ, which we can henceforth assume to be positive real numbers. The equation of motion for the fermion wave-function is gotten by taking the commutator of its operator a_k with the Hamiltonian (30),

$$i\hbar \dot{a}_n = [a_n, H]$$

for which we get

$$i\hbar \frac{d}{dt} a_n = \frac{t}{2} \left(a_{n+1} + a_{n-1} \right) - \frac{\Delta}{2} \left(a_{n+1}^{\dagger} - a_{n-1}^{\dagger} \right) + \mu a_n \tag{32}$$

for the sites $n = 2, ..., L-1$.

Because we are using open boundary conditions – the chain simply ends at $n = 1$ and $n = L$, the equations for $\frac{d}{dt} a_1$ and $\frac{d}{dt} a_L$ differ from (32) by missing terms. When we solve (32) as a wave equation, it will be convenient to deal with this by extending the chain

by one site in each direction and then eliminating the extra sites by imposing Dirichlet boundary conditions,

$$a_0(t) = 0 \quad , \quad a_{L+1}(t) = 0$$

With these conditions, (32) describes the dynamics for all $n = 1, 2, ..., L$.

Now, it is most efficient to decompose the electron into real and imaginary parts, $a_n = b_n + ic_n$, and assemble them into a spinor

$$\psi_n = \begin{pmatrix} b_n \\ c_n \end{pmatrix} \tag{33}$$

Note that this spinor obeys the Majorana condition

$$\psi_n = \psi_n^* \tag{34}$$

The equation for the wave-function is

$$\begin{pmatrix} \mu & \hbar\frac{d}{dt} \\ -\hbar\frac{d}{dt} & \mu \end{pmatrix} \psi_n + \begin{pmatrix} \frac{1}{2}(t-\Delta) & 0 \\ 0 & \frac{1}{2}(t+\Delta) \end{pmatrix} \psi_{n+1} + \begin{pmatrix} \frac{1}{2}(t+\Delta) & 0 \\ 0 & \frac{1}{2}(t-\Delta) \end{pmatrix} \psi_{n-1} = 0 \tag{35}$$

In order to solve the equation, we will make the ansatz

$$\psi_n(t) = e^{-i\omega t/\hbar}\psi_n(\omega) \tag{36}$$

The Majorana condition for energy eigenstates is

$$\psi_n^*(\omega) = \psi_n(-\omega)$$

We will normalize the wave-functions with the condition

$$\sum_{n=1}^{L} |\psi_n(\omega)|^2 \;=\; 1 \;=\; \sum_{n=1}^{L} \left(|b_n(\omega)|^2 + |c_n(\omega)|^2\right)$$

Since the equation and boundary conditions are linear, we can further make the ansatz that the wave-functions are superpositions of plane waves,

$$\psi_n(\omega) = \zeta^n \begin{pmatrix} u(\zeta) \\ v(\zeta) \end{pmatrix} \tag{37}$$

Then, the difference equation (35) becomes

The equation for the wave-function is

$$\begin{pmatrix} \frac{1}{2}t(\zeta + 1/\zeta) - \frac{1}{2}\Delta(\zeta - 1/\zeta) + \mu & -i\omega \\ i\omega & \frac{1}{2}t(\zeta + 1/\zeta) - \frac{1}{2}\Delta(\zeta - 1/\zeta) + \mu \end{pmatrix} \begin{pmatrix} u(\zeta) \\ v(\zeta) \end{pmatrix} = 0 \tag{38}$$

which has a solution when the frequencies obey the dispersion relation

$$\omega^2 = \left[\frac{1}{2}t(\zeta + 1/\zeta) + \mu\right]^2 - \left[\frac{1}{2}\Delta(\zeta - 1/\zeta)\right]^2 \tag{39}$$

For a given real value of ω, there are generally four wave-vectors which satisfy this dispersion relation,

$$\zeta_\omega \ , \ 1/\zeta_\omega \ , \ \zeta_\omega^* \ , \ 1/\zeta_\omega^*$$

To find a solution of the wave equation, we must take superpositions of the four solutions of (38) with each of these four wave-vectors. Then we must adjust the four coefficients of the superposition in order to satisfy the four boundary conditions. (Remember that the boundary conditions are for spinors, so there are four boundary conditions in total.) Three of the boundary conditions can be solved by adjusting the coefficients in the superposition. The fourth superposition coefficient can eventually be determined up to phases by normalizing the wave-function. The fourth boundary condition, which has yet to be satisfied, then gives a condition that the wave-vector must obey. Plugging the resulting wave-vector back into the dispersion relation (39) then gives the allowed energy eigenvalue. This gives an algorithm for finding the energies, the allowed wave-vectors (which are $\frac{1}{i}\ln\zeta$ and are generally complex) and the wave-functions.

When L is large, the solutions are of two kinds. One are to a good approximation continuum states, where $\zeta = e^{ik}$ and the continuum spectrum is

$$\omega(k) = \pm\sqrt{\left[t\cos k + \mu\right]^2 + \Delta^2 \sin^2 k}$$

with $k \in (-\pi, \pi]$ (it is quantized approximately as $k = 2\pi \cdot \text{integer}/(L+1)$ which becomes a continuum when $L \to \infty$). This spectrum has an energy gap. The point of closest approach of the positive and negative energy bands occurs when $\cos k = -t\mu/(t^2 - \Delta^2)$ and the gap is $E_{\text{gap}} = 2\Delta\sqrt{\frac{t^2-\Delta^2-\mu^2}{t^2-\Delta^2}}$. We will assume that this gap is significant, so that the mid-gap states that we will discuss next are indeed well isolated.

The other modes in the spectrum are a pair of mid-gap states. When L is large, these states have energies that are exponentially small in L, one is positive, one is negative and they have equal magnitudes. In the following, we will solve for the spectrum of these mid-gap states in the approximation where effects that are exponentially small in L are neglected.

We begin with an un-normalized spinor

$$\zeta^n \begin{pmatrix} i\omega \\ \frac{1}{2}t(\zeta + 1/\zeta) - \frac{1}{2}\Delta(\zeta - 1/\zeta) + \mu \end{pmatrix} + A\zeta^{-n} \begin{pmatrix} i\omega \\ \frac{1}{2}t(\zeta + 1/\zeta) + \frac{1}{2}\Delta(\zeta - 1/\zeta) + \mu \end{pmatrix}$$
$$+B\zeta^{*n} \begin{pmatrix} i\omega \\ \frac{1}{2}t(\zeta^* + 1/\zeta^*) - \frac{1}{2}\Delta(\zeta^* - 1/\zeta^*) + \mu \end{pmatrix} + C\zeta^{*-n} \begin{pmatrix} i\omega \\ \frac{1}{2}t(\zeta^* + 1/\zeta^*) + \frac{1}{2}\Delta(\zeta^* - 1/\zeta^*) + \mu \end{pmatrix} \tag{40}$$

We will solve the boundary condition for the mid-gap state in the limit where L is large. There, we expect the solution to be very close to $\omega = 0$, for which we then need a wave-vector which solves $t(\zeta + 1/\zeta) + 2\mu = -\Delta(\zeta - 1/\zeta)$. Then, to a first approximation, the terms with A and C are absent from (40) and we must choose the $B = -(\zeta - 1/\zeta)/(\zeta^* - 1/\zeta^*)$ in order to satisfy the boundary condition at $n = 0$. Since

$$\zeta = -\frac{\mu}{2(t+\Delta)} + i\sqrt{\frac{t-\Delta}{t+\Delta}}\sqrt{1 - \mu^2/4(t^2-\Delta^2)}$$

so that $\zeta\zeta^* = \frac{t-\Delta}{t+\Delta} < 1$, this gives a wave-function which is maximal at $n = 1$ and which decays exponentially as n increases. This would indeed be the solution for the mid-gap state on the half-line when $L \to \infty$. When L is finite, rather than infinite, in order to satisfy the boundary condition at $n = L+1$ we must include an amplitude for the growing solution. It can be obtained from the decaying one by simply replacing n by $L + 1 - n$ and multiplying the spinor by σ^2. Thus, to a good approximation the mid-gap solution is

$$\psi_n^+ = \sqrt{\frac{\Delta}{2t}\frac{t^2-\mu^2}{t^2-\Delta^2-\mu^2}}\left[\frac{\left(-\mu + i\sqrt{t^2-\Delta^2-\mu^2}\right)^n - \left(-\mu - i\sqrt{t^2-\Delta^2-\mu^2}\right)^n}{(t+\Delta)^n}\begin{pmatrix}0\\ i\end{pmatrix} + \right.$$
$$\left. + \frac{\left(-\mu + i\sqrt{t^2-\Delta^2-\mu^2}\right)^{L+1-n} - \left(-\mu - i\sqrt{t^2-\Delta^2-\mu^2}\right)^{L+1-n}}{(t+\Delta)^{L+1-n}}\right]\begin{pmatrix}1\\ 0\end{pmatrix}$$

This wave-function has infinitesimal positive energy. The wave-function with infinitesimal negative energy is given by

$$\psi_n^- = \sqrt{\frac{\Delta}{2t}\frac{t^2-\mu^2}{t^2-\Delta^2-\mu^2}}\left[\frac{\left(-\mu + i\sqrt{t^2-\Delta^2-\mu^2}\right)^n - \left(-\mu - i\sqrt{t^2-\Delta^2-\mu^2}\right)^n}{(t+\Delta)^n}\begin{pmatrix}0\\ i\end{pmatrix} - \right.$$
$$\left. - \frac{\left(-\mu + i\sqrt{t^2-\Delta^2-\mu^2}\right)^{L+1-n} - \left(-\mu - i\sqrt{t^2-\Delta^2-\mu^2}\right)^{L+1-n}}{(t+\Delta)^{L+1-n}}\right]\begin{pmatrix}1\\ 0\end{pmatrix}$$

We will abbreviate these by naming the function

$$\phi_n = i\sqrt{\frac{\Delta}{2t}\frac{t^2-\mu^2}{t^2-\Delta^2-\mu^2}}\frac{\left(-\mu + i\sqrt{t^2-\Delta^2-\mu^2}\right)^n - \left(-\mu - i\sqrt{t^2-\Delta^2-\mu^2}\right)^n}{(t+\Delta)^n} \tag{41}$$

where $\phi_n = \phi_n^*$ and we have normalized to

$$\sum_n |\phi_n|^2 = \frac{1}{2} \tag{42}$$

The function ϕ_n has maximum magnitude at $n = 1$ and it decays exponentially as n increases. We shall use the notation

$$\psi_n^+ = \phi_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \phi_{L+1-n} \begin{pmatrix} i \\ 0 \end{pmatrix} \tag{43}$$

$$\psi_n^- = \phi_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \phi_{L+1-n} \begin{pmatrix} i \\ 0 \end{pmatrix} \tag{44}$$

We have normalized the spinors so that

$$\sum_n \psi_n^{\pm\dagger} \psi_n^{\pm} = 1 \tag{45}$$

Note that, these wave-functions satisfy the Majorana condition $\psi_n^- = \psi_n^{+*}$. As expected, they have support near $n = 1$ and $n = L$ and are exponentially small in the interior of the quantum wire, far from the boundaries. Further, we have adjusted phases so that the wave-functions are identical in profile in the region near $n = 1$. Then, we expect that they differ in sign in the region near $n = L$ and we confirm from that above that this is so. Also, note that they are complex. To form the real, Majorana spinor, we must superpose them with a creation and annihilation operator,

$$\psi_n(t) = \psi_n^+ e^{-i\omega t} a + \psi_n^- e^{i\omega t} a^\dagger + \text{ non} - \text{zero energy states} \tag{46}$$

Here a and $a^\dagger$ are the annihilation and creation operators for the mid-gap state and ω is their exponentially small energy. Ignoring the energy, we can also write this operator as

$$\psi_n(t) = \phi_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} (a + a^\dagger) + \phi_{L+1-n} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{i}(a - a^\dagger) + \ldots$$

The first term on the right-hand side has support near $n = 0$ and decays exponentially as n increases from 1. The second term has support near $n = L$ and decays exponentially as n decreases from L. They each multiply the operators $\alpha = \frac{1}{\sqrt{2}}\left((a + a^\dagger\right)$ and $\beta = \frac{1}{\sqrt{2}i}\left(a - a^\dagger\right)$, respectively. These are analogous to the operators which we introduced on Section 1. a and $a^\dagger$ must have the anti-commutator

$$\left\{a, a^\dagger\right\} = 1$$

which has a two-dimensional representation, the states $|- >$ and $|+ >$ of Section 1 which we copy here for the reader's convenience,

$$a|- >= 0 \quad , \quad a^\dagger|- >= |+ >$$

$$a|+ >= |- > \quad , \quad a^\dagger|+ >= 0$$

All other excited states of the system are created by operating creation operators for the other, non-zero energy excitations. Remember that it is the states $|+ >$ and $|- >$ which we expect to be eigenstates of fermion parity, $(-1)^F$.

5.2 Second Quantized Electron Operator

Now, we recall that the upper and lower components of the spinor $\psi_n(t)$ that we discussed in the previous subsection are simply the real and imaginary components of the electron field operator, which we can now reconstruct,

$$a_n(t) = \phi_{L+1-n} \frac{1}{i} \left(a - a^\dagger\right) + i\phi_n \left(a + a^\dagger\right) + \dots \tag{47}$$

This is now a complex operator, but its real and imaginary parts have support at opposite ends of the quantum wire. The part of the operator which has not been written, and is indicated by dots in (47), are superpositions of creation and annihilation operators for continuum states. All such states have energies above the gap and extended, plane-wave-like wave-functions. Note that now that the phase symmetry of the system has been broken by coupling to the superconductor, the real and imaginary parts of the electron operator will generally have different properties. This interesting fact will not concern us in the following and we will focus on the mid-gap, or zero mode part of the electron operator.

Note, now, if we operate with any local operator in the vicinity of $n = 1$, the electon operator acts as if it were composed of the combination of zero mode operators $(a + a^\dagger)$. As we have discussed before, this operator squares to a constant. There cannot be any states that it annihilates. Thus, operating it on any state of the system, in the region where the zero mode wavefunction has support, will have an effect. What it does is flip the state from $|- >$ to $|+ >$. Since it is a hermitian operator, it is possible to diagonalize it, the states $\frac{1}{\sqrt{2}}(|- > + |+ >)$ and $\frac{1}{\sqrt{2}}(|- > + |+ >)$ are its eigenvectors. However, these eigenvectors are not eigenstates of fermion parity.

6. Long Ranged Correlations of Electrons

What about teleportation? Let us imagine that we begin with the system in one of its ground states, say $|- >$ and inject an electron so that at time $T = 0$ it is resting at site #1. This means, we being with the state $a_1^\dagger|- >$, where, as we recall, $a_1^\dagger$ is the creation operator for an electron at site #1.

We then ask what is the quantum transition amplitude for the transition, after a time T has elapsed, of this state to one with the electron located at position $\#L$. The final quantum state is $a_L^\dagger|- >$. The amplitude is given by

$$\mathcal{A} =< -|\ a_L\ e^{iHT}\ a_1^\dagger\ |- >= |\phi_1^0|^2 + (T \text{ and } L\text{−dependent}) \tag{48}$$

The T- and L-dependent parts of this matrix element represent the usual propagation via excited quasi-electrons which must travel across the wire. The first term is non-zero and is T and L-independent. By 'teleportation', we are referring to this part of the amplitude. Here, we can evaluate the amplitude explicitly. It is

$$\mathcal{A}_{\text{Tel}} = \left(\frac{2\Delta}{t}\right)\left(\frac{t^2 - \Delta^2 - \mu^2}{(t + \Delta)^2}\right) \tag{49}$$

Which can be appreciable, in the 10-30 percent range, for a surprisingly wide choice of parameters.

However, the teleportation probability is the square of this amplitude, which is somewhat smaller. We could ask a more sophisticated question: What is the probability that the electron, once injected at $n = 1$ appears anywhere within the exponential range of the zero mode wave-function at $n = L$. This probability would be given by

$$\mathcal{P}_{\rm Tel} = \sum_n |\phi_n|^2 |\phi_1^0|^2 = \frac{1}{2}\left(\frac{2\Delta}{t}\right)\left(\frac{t^2 - \Delta^2 - \mu^2}{(t+\Delta)^2}\right) \tag{50}$$

This is what we shall call the "teleportation probability". Again, for a range of parameters t, Δ and μ, it can be appreciable.

7. Discussion

The apparently instantaneous propagation of an electron would seem to be a potential violation of Einstein causality, since in principle a message could be sent at a speed faster than that of light.

Let us review the nature of the system that we have constructed. Once the quantum wire - p-wave superconductor system is prepared, the extended Majorana state of the electron is already there, ready for use. The system has a two-fold degeneracy: at low energy, there are two states $|- >$ and $|+ >$. These are not normal quantum states in that they differ by a quantum number which we would like to preserve, fermion parity $(-1)^F$.

Thus, if we do not allow superpositions of these states, this is effectively a classical bit, like a classical switch that can either be OFF or ON, the wave-function can be in one state or the other.

The system moves from OFF to ON by absorbing or emitting an electron in a way that flips the vacuum from one state to the other. This should occur somewhere in the vicinity of the ends of the wire, where the zero mode wave-functions have support. It can move back from ON to OFF by the identical process, again absorbing or emitting a single electron.

This leads to the rather drastic conclusion that there could be super-luminal transfer of information in this system. One would need only to prepare the system in one of its ground states, with a sender sitting at 1 and a receiver sitting at L. Either ground state is sufficient and neither the sender nor the observer needs to know which it is. All the receiver has to do is wait for an electron to arrive. If it arrives with energy at or above the electron energy gap, he or she can conclude that it propagated normally and was sent at some time in the past. However, if it arrives at very low energy, he or she knows that it tunnelled and that it was sent by the sender at that instant. This is seemingly an instant transfer of information over a finite distance.

There is a obvious way out of this, but it means giving up the fermion parity symmetry that has until now been sacred. If we allow superpositions of the states $|- >$ and

$|+>$ which have even and odd fermion number, then the degenerate ground states are a quantum rather than classical two-level system, there are two states and any superpositions are allowed. Now, in this system, it is easy to prepare states where an electron can spontaneously appear or disappear. Take, for example an eigenstate of the operators that we called α and β. In their eigenstates, $\frac{1}{\sqrt{2}}(|- > \pm|+ >)$, the electron operator has an expectation value $< a_n(t) >= \pm\phi_n \pm \phi_{L+1-n}$. It would thus have an amplitude for simply vanishing or appearing spontaneously.

Then, when the observer at L detects the arrival of a low energy electron he or she cannot distinguish one which was sent from the other side of the wire from one which is spontaneously created. This restores Einstein causality at the expense of our having to admit states onto physics which are not eigenstates of fermion parity. There is the further question of whether such states are consistent with three dimensional physics.

Fermion number mod 2 is an important conservation law in three dimensional physics [55]. Even though the quantum wire that we have discussed is one-dimensional, it is embedded in three dimensional space and the electrons that we are discussing are spin-$\frac{1}{2}$ particles in three dimensional space. This means that their wave-functions individually change sign under a rotation by 2π. More importantly, a state with odd fermion number must change sign under a rotation by 2π whereas a state with even number should remain unchanged. A rotation by angle 2π should not affect physics. Thus, the relative sign of even and odd fermion number states should not have any physical consequences.

If we did allow a superposition of the two states, they would form a single qubit. We could parameterize the state-vector by a point on the Bloch sphere (θ, ϕ) where the state is

$$|\theta, \varphi > \;\; = \;\; \cos\frac{\theta}{2}|- > + e^{i\varphi}\sin\frac{\theta}{2}|+ > \tag{51}$$

Points on the two-dimensional unit sphere are specified by the unit vectors

$$\hat{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

and $0 \leq \theta \leq \pi$, $-\pi < \varphi \leq \pi$. However, as we have argued, the relative sign of the two states should not be an observable. Then the set of "physical states" of the qubit would be the Bloch sphere with a further identification

$$\varphi \sim \varphi + \pi \tag{52}$$

Of course, this identification is allowed only if there are no experiments, even in principle, which could measure the relative sign of the two states in the superposition. Normally, one could measure that sign by an interference experiment.

For example, we could attempt to observe the relative sign by examining interference between the electron which arrives by tunnelling and the one which arrives by conventional transport. However, the teleportation amplitude in the state ϕ

$$< \theta, \varphi|\; a_L\; e^{iHT}\; a_1^\dagger\; |\theta, \varphi >= \cos\theta[\text{teleportation}] + [\text{transport}] \tag{53}$$

The teleportation amplitude is diminished by a factor of $\cos\theta$ whereas the transport amplitude is unchanged. One can make the teleportation amplitude vanish by adjusting

$\theta = \pi/2$. However, the relative amplitude cannot be used to measure the relative sign of the two components of the wave-function.

There is an amplitude for an electron to vanish,

$$< \theta, \varphi|\ e^{iHT}\ a_1^\dagger\ |\theta, \varphi > \sim i \sin\theta \cos\varphi\ \cdot \phi_1 \tag{54}$$

and to appear spontaneously

$$< \theta, \varphi|\ a_L\ e^{iHT}\ \ |\theta, \varphi > \sim \sin\theta \sin\varphi\ \cdot \phi_1 \tag{55}$$

As we expect, the latter two amplitudes change sign when we put $\varphi \to \varphi + \pi$. Actual quantum observables are probabilities which are the modulus squares of amplitudes. They are also insensitive to the relative sign of the two parts of the wave-function.

The above probability amplitudes do not offer a way to distinguish the quantum states with φ and $\varphi + \pi$. At this point, we have not ruled out, but also we have not devised an experiment by which they could be distinguished. Indeed, if there is no such experiment, we are free to cut the Bloch sphere in half by the identification (52) and the ground states would form this peculiar qubit. Teleportation still happens, but so does the spontaneous disappearance or appearance of a single electron and the contradiction with Einstein locality is removed.

We cannot exclude the possibility that the effect that we have been discussing could be interfered with by the superconductor which the quantum wire is in contact with. Here, we have assumed that it acts as a simple bath which supplies and absorbs Cooper pairs but is otherwise innocuous. We cannot rule out that it also has exotic states that should be included in the picture.

Acknowledgements

G.S. thanks Duncan Haldane, Tony Leggett, Girard Milburn and Bill Unruh for discussions and acknowledges the hospitality of the I.H.E.S., Bures-sur-Yvette, the Werner Heisenberg Institute, Munich and the University of Perugia where parts of this work were completed. P.S. thanks M.Rasetti, P. Zanardi and V. Barone for discussions and acknowledges the Pacific Institute for Theoretical Physics, Vancouver and the Center for Theoretical Physics at MIT for hospitality. This work is supported in part by NSERC of Canada and by M.I.U.R. National Project "JOSNET" (grant n.2004027555).

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