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On the Hamiltonian Form of Generalized Dirac Equation for Fermions with Two Mass States

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Abstract: Dynamical and non-dynamical components of the 20-component wave function are separated in the generalized Dirac equation of the first order, describing fermions with spin 1/2 and two mass states. After the exclusion of the non-dynamical components, we obtain the Hamiltonian Form of equations. Minimal and non-minimal electromagnetic interactions of particles are considered here.
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1. Introduction

We continue to investigate the first order generalized Dirac equation (FOGDE), describing fermions with spin 1/2 and two mass states. This 20-component wave equation was obtained in [1] on the base of Barut's [2] second order equation describing particles with two mass states. Barut suggested the second order wave equation for the unified description of e, μ leptons. He treated this equation as an effective equation for partly "dressed" fermions using the non-perturbative approach to quantum electrodynamics. Some investigations of Barut's second order wave equation and FOGDE were performed in [3], [4], [5], [6], [7].

The purpose of this paper is to obtain the Hamiltonian Form of the 20-component wave equation of the first order.

The paper is organized as follows. In Sec. 2, we introduce the generalized Dirac

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equation of the first order. The dynamical and non-dynamical components of the 20-component wave function are separated, and quantum-mechanical Hamiltonian is derived in Sec. 3. In Sec. 4, we make a conclusion. In Appendix, we give some useful matrices entering the Hamiltonian. The system of units $\hbar = c = 1$ is chosen, Latin letters run 1, 2, 3, and Greek letters run 1, 2, 3, 4, and notations as in [8] are used.

2. Field Equation of the First Order

The Barut second order field equation describing spin-1/2 and two mass states of particles may be rewritten as [1]:

$$\left(\gamma_\mu \partial_\mu - \frac{a}{m}\partial_\mu^2 + m\right)\psi(x) = 0, \tag{1}$$

where $\partial_\mu = \partial/\partial x_\mu = (\partial/\partial x_m, \partial/\partial(it))$, $\psi(x)$ is a Dirac spinor, m is a parameter with the dimension of the mass, and a is a massless parameter. We imply a summation over repeated indices. The commutation relations $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}$ are valid for the Dirac matrices. Masses of fermions are given by

$$m_1 = \pm m\left(\frac{1 - \sqrt{4a+1}}{2a}\right), \quad m_2 = \pm m\left(\frac{1 + \sqrt{4a+1}}{2a}\right). \tag{2}$$

Signs in Eq. (2) should be chosen to have positive values of m_1, m_2.

Eq. (1) can be represented in the first order form [1]:

$$\left(\alpha_\mu \partial_\mu + m\right)\Psi(x) = 0, \tag{3}$$

where the 20-dimensional wave function $\Psi(x)$ and 20×20-matrices α_μ are

$$\Psi(x) = \{\psi_A(x)\} = \begin{pmatrix} \psi(x) \\ \psi_\mu(x) \end{pmatrix} \qquad \left(\psi_\mu(x) = -\frac{1}{m}\partial_\mu\psi(x)\right), \tag{4}$$

$$\alpha_\mu = \left(\varepsilon^{\mu,0} + a\varepsilon^{0,\mu}\right) \otimes I_4 + \varepsilon^{0,0} \otimes \gamma_\mu. \tag{5}$$

The I_4 is a unit 4×4-matrix, and $\otimes$ is a direct product of matrices. The elements of the entire algebra obey equations as follows (see, for example, [9]):

$$\left(\varepsilon^{M,N}\right)_{AB} = \delta_{MA}\delta_{NB}, \qquad \varepsilon^{M,A}\varepsilon^{B,N} = \delta_{AB}\varepsilon^{M,N}, \tag{6}$$

where $A, B, M, N = 0, 1, 2, 3, 4$.

After introducing the minimal electromagnetic interaction by the substitution $\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu$ (A_μ is the four-vector potential of the electromagnetic field), and the non-minimal interaction with the electromagnetic field by adding two parameters κ_1, κ_2, we come [1] to the matrix equation:

$$\left[\alpha_\mu D_\mu + \frac{i}{2}\left(\kappa_0 P_0 + \kappa_1 P_1\right)\alpha_{\mu\nu}\mathcal{F}_{\mu\nu} + m\right]\Psi(x) = 0, \tag{7}$$

where $P_0 = \varepsilon^{0,0} \otimes I_4$, $P_1 = \varepsilon^{\mu,\mu} \otimes I_4$ are the projection operators, $P_0^2 = P_0$, $P_1^2 = P_1$, $P_0 + P_1 = 1$, and $\alpha_{\mu\nu} = \alpha_\mu\alpha_\nu - \alpha_\nu\alpha_\mu$. Parameters κ_0 and κ_1 characterize fermion anomalous electromagnetic interactions.

The tensor form of Eq. (7) is given by

$$(\gamma_\nu D_\nu + i\kappa_0\gamma_\mu\gamma_\nu\mathcal{F}_{\mu\nu} + m)\,\psi(x) + (aD_\mu + i\kappa_0\gamma_\nu\mathcal{F}_{\nu\mu})\,\psi_\mu(x) = 0, \tag{8}$$

$$(D_\mu + i\kappa_1\gamma_\nu\mathcal{F}_{\mu\nu})\,\psi(x) + (m\delta_{\mu\nu} + i\kappa_1 a\mathcal{F}_{\mu\nu})\,\psi_\nu(x) = 0, \tag{9}$$

where $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the strength of the electromagnetic field. Eq. (8), (9) represent the system of equations for Dirac spinor $\psi(x)$ and vector-spinor $\psi_\nu(x)$ interacting with electromagnetic fields.

3. Quantum-Mechanical Hamiltonian

In order to obtain the quantum-mechanical Hamiltonian, we rewrite Eq. (7) as follows:

$$\begin{aligned} i\alpha_4\partial_t\Psi(x) = \Big[&\alpha_a D_a + m + eA_0\alpha_4 + \\ &+\frac{i}{2}\left(\kappa_0 P_0 + \kappa_1 P_1\right)\alpha_{\mu\nu}\mathcal{F}_{\mu\nu}\Big]\Psi(x). \end{aligned} \tag{10}$$

One can verify with the help of Eq. (6) that the matrix α_4 obeys the matrix equation

$$\alpha_4^4 - (1+2a)\alpha_4^2 + a^2\Lambda = 0, \tag{11}$$

where Λ:

$$\Lambda = \left(\varepsilon^{0,0} + \varepsilon^{4,4}\right) \otimes I_4, \tag{12}$$

is the projection operator, $\Lambda^2 = \Lambda$. It should be noted that the matrix Λ can be considered as the unit matrix in the 8-dimensional sub-space of the wave function [1]. The operator Λ, acting on the wave function $\Psi(x)$, extracts the dynamical components $\Phi(x) = \Lambda\Psi(x)$. We may separate[1] the dynamical and non-dynamical components of the wave function $\Psi(x)$ by introducing the second projection operator:

$$\Pi = 1 - \Lambda = \varepsilon^{m,m} \otimes I_4, \tag{13}$$

so that $\Pi^2 = \Pi$. This operator defines non-dynamical components $\Omega = \Pi\Psi(x)$. Multiplying Eq. (10) by the matrix

$$\frac{(1+2a)}{a^2}\alpha_4 - \frac{\alpha_4^3}{a^2} = \left(\varepsilon^{0,4} + \frac{1}{a}\varepsilon^{4,0}\right) \otimes I_4 - \frac{1}{a}\varepsilon^{4,4} \otimes \gamma_4,$$

and taking into consideration Eq. (11), we obtain the equation as follows:

$$i\partial_t\Phi(x) = eA_0\Phi(x) + \left[\frac{(1+2a)}{a^2}\alpha_4 - \frac{\alpha_4^3}{a^2}\right]\Big[\alpha_a D_a + m + K\Big]\Psi(x), \tag{14}$$

[1] In the work [1], the dynamical and non-dynamical components of the wave function were not separated.

where

$$K = \frac{i}{2}\left(\kappa_0 P_0 + \kappa_1 P_1\right)\alpha_{\mu\nu}\mathcal{F}_{\mu\nu}$$
$$= i\mathcal{F}_{\mu\nu}\left[\kappa_0\left(\varepsilon^{0,0}\otimes\gamma_\mu\gamma_\nu + \varepsilon^{0,\nu}\otimes\gamma_\mu\right) + \kappa_1\left(\varepsilon^{\mu,0}\otimes\gamma_\nu + a\varepsilon^{\mu,\nu}\otimes I_4\right)\right]. \quad (15)$$

It should be mentioned that because $\Lambda + \Pi = 1$, the equality $\Psi(x) = \Phi(x) + \Omega(x)$ is valid. To eliminate the non-dynamical components $\Omega(x)$ from Eq. (14), we multiply Eq. (10) by the matrix Π, and using the equality $\Pi\alpha_4 = 0$, we obtain

$$\Pi\left(\alpha_a D_a + K\right)\left(\Phi(x) + \Omega(x)\right) + m\Omega(x) = 0. \quad (16)$$

With the help of equation $\Pi\alpha_a\Pi = 0$, one may find from Eq. (16), the expression as follows:

$$\Omega(x) = -\left(m + \Pi K\right)^{-1}\Pi\left(\alpha_a D_a + K\right)\Phi(x). \quad (17)$$

With the aid of Eq. (17), Eq. (14) takes the form

$$i\partial_t\Phi(x) = \mathcal{H}\Phi(x), \quad (18)$$

$$\mathcal{H} = eA_0 + \left[\frac{(1+2a)}{a^2}\alpha_4 - \frac{\alpha_4^3}{a^2}\right]\left[\alpha_a D_a + m + K\right]$$
$$\times\left[1 - \left(m + \Pi K\right)^{-1}\Pi\left(\alpha_b D_b + K\right)\right], \quad (19)$$

Eq. (18) represents the Hamiltonian form of the equation for 8-component wave function $\Phi(x)$. It is obvious that for the relativistic description of fermionic fields, possessing two mass states, it is necessary to have 8-component wave function (two bispinors). The Hamiltonian (19) can be simplified by using products of matrices given in Appendix.

Now we consider the particular case of fermions minimally interacting with electromagnetic fields, $\kappa_0 = \kappa_1 = 0$, $K = 0$. In this case, Eq. (18) becomes

$$i\partial_t\Phi(x) = \left[eA_0 + \frac{m}{a}\left(a\varepsilon^{0,4}\otimes I_4 + \varepsilon^{4,0}\otimes I_4 - \varepsilon^{4,4}\otimes\gamma_4\right)\right.$$
$$\left. + \frac{1}{a}\left(\varepsilon^{4,0}\otimes\gamma_m\right)D_m - \frac{1}{m}\left(\varepsilon^{4,0}\otimes I_4\right)D_m^2\right]\Phi(x). \quad (20)$$

In component form, Eq. (20) is given by the system of equations

$$i\partial_t\psi(x) = eA_0\psi(x) + m\psi_4(x),$$
$$i\partial_t\psi_4(x) = \left(eA_0 - \frac{m}{a}\gamma_4\right)\psi_4(x) + \left(\frac{m}{a} + \frac{1}{a}\gamma_m D_m - \frac{1}{m}D_m^2\right)\psi(x). \quad (21)$$

Eq. (21) can also be obtained from Eq. (8), (9), at $\kappa_0 = \kappa_1 = 0$, after the exclusion of non-dynamical (auxiliary) components $\psi_m(x) = -(1/m)D_m\psi(x)$. So, only components with time derivatives enter Eq. (21) and Eq. (18).

4. Conclusion

We have analyzed the 20-component matrix equation of the first order, describing fermions with spin 1/2 and two mass states which is convenient for different applications. There are two parameters characterizing non-minimal electromagnetic interactions of fermions including the interaction of the anomalous magnetic moment of particles. The Hamiltonian form of the equation was obtained, and it was shown that the wave function, entering the Hamiltonian equation, contains 8 components what is necessary for describing fermionic field with two mass states in the formalism of the first order. The Hamiltonian (19) can be used for a consideration of the non-relativistic limit which is convenient for the physical interpretation of constants κ_0, κ_1 introduced. This can be done with the help of the Foldy - Wouthuysen procedure [10].

The approach developed may be applied for a consideration of two families of leptons or quarks, but this requires further investigations.

Appendix

With the help of Eq. (6), one can obtain expressions as follows:

$$\left[\frac{(1+2a)}{a^2}\alpha_4 - \frac{\alpha_4^3}{a^2}\right]\alpha_m D_m = \left(\frac{1}{a}\varepsilon^{4,0}\otimes\gamma_m + \varepsilon^{4,m}\otimes I_4\right)D_m, \tag{22}$$

$$\Pi\alpha_m D_m = \left(\varepsilon^{m,0}\otimes I_4\right)D_m, \tag{23}$$

$$\Pi K = i\kappa_1\mathcal{F}_{m\nu}\left(\varepsilon^{m,0}\otimes\gamma_\nu + a\varepsilon^{m,\nu}\otimes I_4\right), \tag{24}$$

$$\begin{gathered}\left[\frac{(1+2a)}{a^2}\alpha_4 - \frac{\alpha_4^3}{a^2}\right]K = i\frac{\kappa_0}{a}\mathcal{F}_{\mu\nu}\left(\varepsilon^{4,0}\otimes\gamma_\mu\gamma_\nu + \varepsilon^{4,\nu}\otimes\gamma_\mu\right)\\ +i\kappa_1\mathcal{F}_{4\nu}\left(\varepsilon^{0,0}\otimes\gamma_\nu - \frac{1}{a}\varepsilon^{4,0}\otimes\gamma_4\gamma_\nu + a\varepsilon^{0,\nu}\otimes I_4 - \varepsilon^{4,\nu}\otimes\gamma_4\right).\end{gathered} \tag{25}$$

One may verify that the equations

$$\mathcal{F}_{nm}\mathcal{F}_{mi} = B_nB_i - B^2\delta_{ni}, \qquad \mathcal{F}_{nm}\mathcal{F}_{mi}\mathcal{F}_{ik} = -B^2\mathcal{F}_{nk} \tag{26}$$

are hold, where $B^2 = B_m^2$, $B_m = (1/2)\epsilon_{mnk}\mathcal{F}_{nk}$ is the strength of the magnetic field. Eq. (26) allow us to obtain the relation for the matrix $\Sigma \equiv m + \Pi K$:

$$\Sigma^4 - 4m\Sigma^3 + \left(6m^2 - b\right)\Sigma^2 + 2m\left(b - 2m^2\right)\Sigma + m^4 - bm^2 = 0, \tag{27}$$

where $b = a^2\kappa_1^2B^2$. From Eq. (27), we find the inverse matrix Σ^{-1}:

$$\begin{aligned}\Sigma^{-1} &= \frac{1}{m^2\left(b-m^2\right)}\left[\Sigma^3 - 4m\Sigma^2 + \left(6m^2-b\right)\Sigma + 2m\left(b-2m^2\right)\right]\\ &= \frac{1}{m} + \frac{1}{m^2\left(b-m^2\right)}\Big[i\kappa_1\left(m^2-b\right)\mathcal{F}_{m\nu} + am\kappa_1^2\mathcal{F}_{mk}\mathcal{F}_{k\nu}\\ &\quad -ia^2\kappa_1^3\mathcal{F}_{mk}\mathcal{F}_{kn}\mathcal{F}_{n\nu}\Big]\left(\varepsilon^{m,0}\otimes\gamma_\nu + a\varepsilon^{m,\nu}\otimes I_4\right).\end{aligned} \tag{28}$$

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