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Relativistic Causality and Quasi–Orthomodular Algebras

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Abstract: The concept of *fractionability* or *decomposability* in parts of a physical system has its mathematical counterpart in the lattice–theoretic concept of *orthomodularity.* Systems with a finite number of degrees of freedom can be decomposed in different ways, corresponding to different groupings of the degrees of freedom. The orthomodular structure of these simple systems is trivially manifest. The problem then arises as to whether the same property is shared by physical systems with an infinite number of degrees of freedom, in particular by the quantum relativistic ones. The latter case was approached several years ago by Haag and Schroer (1962; Haag, 1992) who started from noting that the causally complete sets of Minkowski spacetime form an orthomodular lattice and posed the question of whether the subalgebras of local observables, with topological supports on such subsets, form themselves a corresponding orthomodular lattice. Were it so, the way would be paved to interpreting spacetime as an intrinsic property of a local quantum field algebra. Surprisingly enough, however, the hoped property does not hold for local algebras of free fields with superselection rules. The possibility seems to be instead open if the local currents that govern the superselection rules are driven by gauge fields. Thus, in the framework of local quantum physics, the request for algebraic orthomodularity seems to imply physical interactions! Despite its charm, however, such a request appears plagued by ambiguities and criticities that make of it an ill–posed problem. The proposers themselves, indeed, concluded that the orthomodular correspondence hypothesis is too strong for having a chance of being practicable. Thus, neither the idea was taken seriously by the proposers nor further investigated by others up to a reasonable degree of clarification. This paper is an attempt to re–formulate and well–pose the problem. It will be shown that the idea is viable provided that the algebra of local observables: (1) is considered all over the whole range of its irreducible representations; (2) is widened with the addition of the elements of a suitable intertwining group of automorphisms; (3) the orthomodular correspondence requirement is modified to an extent sufficient to impart a natural topological structure to the *intertwined algebra of observables* so obtained. A novel scenario then emerges in which local quantum physics appears to provide a general framework for non–perturbative quantum field dynamics.

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Introduction

A physical system will be called finite or infinite according as the number N of its degrees of freedom is finite or infinite. It is well–known from both classic and quantum mechanics that finite systems can be described in many equivalent ways by suitable rearrangement of their degrees of freedom. In classic physics, this is carried out by canonical transformations that preserve the symplectic structure of phase space. In quantum physics, the corresponding transformations are implemented by unitary automorphisms of the algebra of observables.

The possibility of rearranging and grouping the degrees of freedoms in different ways reflects the fact that the symplectic space of a finite systems can, in general, be decomposed into smaller symplectic spaces and composed of symplectic spaces to form larger symplectic spaces. This property, which is not as trivial as the decomposition and composition of the subsets of a set, characterizes the structure of a finite symplectic space as an orthomodular lattice of subspaces (App.1). Correspondingly, the algebra of conjugate observables of a finite quantum system (with $N > 1$) is characterized as an orthomodular lattice of subalgebras.

The problem now arises quite naturally as to whether the orthomodular property holds also for infinite systems, in particular for the quantum relativistic ones, as an euristic principle of the preservation of formal properties would suggest. From here on, however, only systems with a non–compact continuum of degrees of freedom will be considered.

Unfortunately, in the framework of quantum relativistic theories, both the concept of degree of freedom and that of canonical conjugate quantities become elusive. The first difficulty arises because a continuum of physical quantities is not a simple set but a topological space. The second one because the canonical commutation relations between conjugate quantities are destroyed by physical interactions. We must therefore expect that in this more delicate context the orthomodularity condition needs to be formulated quite differently from the finite case.

Rearranging a continuum of degrees of freedom is not a simple matter of grouping. Transferring degrees of freedom from a region of a continuum to another is not as simple as picking up handfuls of degrees of freedom from one side to put them into the other side. We must instead detach topological sets from one side and merge them into the other, and this cannot be done without creating and dissolving topological boundaries. To put it differently, managing a continuum of degrees of freedom is tantamount to managing topological boundaries.

The topological boundary problem was posed since the very beginning of quantum mechanics. The representation of the quantum mechanical world as a relationship between an observing system and an observed system implies interfacing the two parts across a common boundary, whose location, however, is quite arbitrary (von Neumann, 1932, 1955). In the framework of general quantum physics, decomposing a system in two parts has its algebraic counterpart in factorizing the operator algebra of the physical system into two mutually commuting subalgebras. In the framework of local quantum physics, this process is complicated by the need of partitioning conservative quantities between complementary regions of spacetime. Thus, from a theoretical standpoint, drawing a boundary is ultimately related to the problem of representing the infinite ways of partitioning conservative quantities among regions separated by topological boundaries. Moreover, the reconstruction of the observer–observed system as a whole requires that the entire algebraic structure is recovered, by some algebraic procedure, probably in a framework of thermodynamic limits, from the whole collection of representations corresponding to the different partitions. As will be seen, this is precisely the central point of the topological boundary problem in local quantum physics.

An interesting approach towards the orthomodularization of local quantum physics was proposed by Haag and Schoer (Haag and Schroer, 1962; Haag, 1992), basing on the facts that lattices formed by *causally complete sets*, or *causal completions*, of Minkowski spacetime (App.2) are orthomodular. They considered the possibility of establishing a correspondence between lattices of causally complete sets and subalgebras of local observables with topological supports on such sets. Remarkably enough, the orthomodular property holds also if the spacetime is equipped with a pseudo–Riemannian metric (Cegla and Jadczyk, 1977; Casini, 2002).

For reasons that will be clear in the next, however, it is preferable to restrict our consideration to the orthomodular sublattices of causal completions formed by *causal shadows* of spacelike sets of events (App.2). The main question is then of whether, in the framework of local quantum physics, suitable subalgebras of observables localized in causal shadows can form an orthomodular lattice with respect to the operations of decomposing an algebra into subalgebras and combining subalgebras to generate larger subalgebras.

The correspondence *causal–shadows* $\leftrightarrow$ *subalgebras* would indeed bear an interesting physical meaning. The observables localized in the causal shadow $\langle S\rangle$ of a region S of a spacelike surface Σ are physically protected from the influence of all the events occurring in the orthocomplement $\langle S'\rangle$ of $\langle S\rangle$, i.e. the causal shadow of the region S' external to S on Σ. Here, the observers with their measurement devices are supposed to reside after preparing a physical state in S. The extension of this correspondence over finer and finer causal–shadow decompositions has its algebraic counterpart in the infinite decomposability of observable quantities, ultimately in the infinite fractionability of matter. This is a property that all good quantum field theories are supposed to possess.

Were the correspondence possible, the way would be paved also for regarding spacetime as an intrinsic property of the algebraic structure of local quantum physics. This perspective is very appealing as it is precisely the principle of general relativity in its most abstract form (Rovelli, 2004).

Here we touch the heart of the problem. How can an algebraic lattice be organized to an extent sufficient to incorporate a topological structure? What are the algebraic analogs of closed sets, open sets, boundaries and all the operations that we are supposed to be able to do in general topology?

To proceed along the path here indicated, the consolidated repertoire of algebraic concepts that form the basis of the current approaches to local quantum physics must be suitably reorganized (Doplicher *et al.*, 1969a, 1969b, 1971, 1974). This attempt will be carried out without any pretensions of completeness and rigor in Sec.s 1, 2, 3 and 4.

In Sec.1, the general concepts that seem to be necessary to well–pose the problem will be introduced. In Sec.2, the notion of orthomodular algebra will be presented and the idea of Haag–Schroer briefly illustrated in order to show that its effective implementation requires a substantial widening of the current vistas on the algebras of observables. The concepts of quasi–orthocomplemented and quasi–orthomodular lattices will be introduced in Sec.3 for the purpose of evidencing certain natural topological properties of von Neumann algebras. The physical meaning and the perspectives of the approach will be finally discussed in Sec.4.

1. Looking for Well–Posing the Problem

Since an orthomodular lattice is *orthocomplemented* (App.1), it is worthwhile describing how orthocomplementation can be implemented in an algebra of bounded operators (the reason why we need to ground our analysis on bounded operators is that sums and products of unbounded operators do not exist in general). To avoid terminological misunderstanding, we introduce a few basic concepts on the algebras of quantum mechanical systems.

1.1 C*–algebras

A selfadjoint algebra of bounded linear operators of a Hilbert space $\mathcal{H}$, *including the unit operator*[1] and closed in the topology of the norm (uniform topology), will be simply called a *C*–algebra*.

The basic properties of a C*–algebra are that the eigenvalue spectrum of any element and the inverse of any element with non zero eigenvalues are unambiguously defined. The norm $|X|$ of an element X of the C*–algebra is the least upper bound of the absolute eigenvalues of X. Closure in the norm topology means that, if a sequence of elements $X_1, X_2, \ldots$ exists such that $|X_1 - X|, |X_2 - X|, \ldots$ converges to zero, then X belongs to the algebra.

Since these properties can be formulated in purely algebraic terms, i.e. independently of the fact that the elements of the algebra operate on the vectors of $\mathcal{H}$, the same notions can be ascribed to the abstract counterpart of the algebra (Bratteli and Robinson, 1987). In the following, however, C*–algebras will be understood sometimes as abstract algebras and some other times as representations of abstract algebras. This ambiguity, a little bit confusing though, has the advantage of simplifying greatly the language.

Let $\mathcal{M}$ be the C*–algebra formed by all bounded operators of $\mathcal{H}$. Let $\mathcal{I}$ be the trivial subalgebra of $\mathcal{M}$ formed by the multiples of the identity I. Any self–adjoint subset of

[1] In other contexts the inclusion of the unit is not required.

$\mathcal{M}$ including the unit and closed in the norm topology forms a C*–algebra $\mathcal{A}$. From a lattice–theoretic point of view, $\mathcal{A}$ is an element of a partially ordered set in which $\mathcal{M}$ is the supremum and $\mathcal{I}$ is the infimum, i.e. $\mathcal{I} \leq \mathcal{A} \leq \mathcal{M}$.

The lattice structure of $\mathcal{M}$ is immediately established as soon as we realize that for any two C*–algebras $\mathcal{A}, \mathcal{B} \leq \mathcal{M}$, both the *join* $\mathcal{A}\widetilde{\vee}\mathcal{B}$, i.e. the smallest C*–algebra that contains $\mathcal{A}$ and $\mathcal{B}$, and the *meet* $\mathcal{A}\widetilde{\wedge}\mathcal{B}$, i.e. the greatest C*–algebra including all the operators common to $\mathcal{A}$ and $\mathcal{B}$, exist in $\mathcal{M}$. Here the symbols $\widetilde{\vee}, \widetilde{\wedge}$ are used to indicate that algebraic closure is achieved in the uniform topology. The property $\mathcal{A}\widetilde{\wedge}\mathcal{B} = \mathcal{A} \cap \mathcal{B}$ holds, i.e. $\widetilde{\wedge}$ is equivalent to the set–theoretic intersection of $\mathcal{A}$ and $\mathcal{B}$.

Let $\mathcal{A}'$ be the C*–algebra formed by all bounded operators that commute with all the elements of $\mathcal{A}$, i.e. the *commutant* of $\mathcal{A}$. The relationships $\mathcal{A}\widetilde{\vee}\mathcal{A}' \leq \mathcal{M}$, $\mathcal{A}\widetilde{\wedge}\mathcal{A}' \geq \mathcal{I}$ can be easily proved.

1.2 Von Neumann algebras

A C*–algebra closed in the *weak topology* (Haag, 1992) is called a *von Neumann algebra*. The simplest way to formulate the criterion of weak closeness is the following: An operator sequence $X_1, X_2, \ldots$ converges weakly to X if the sequences $|\langle\alpha, (X_1 - X)\alpha\rangle|, |\langle\alpha, (X_2 - X)\alpha\rangle|, \ldots$ converge to zero for all vectors $|\alpha\rangle \in \mathcal{H}$. Weak closure means that X belongs to the algebra whenever a sequence $X_1, X_2, \ldots$ weakly converging to X exists. Closure in the uniform topology implies weak closure, but the converse is not true.

Note that the identity $4\langle\alpha, A\beta\rangle = \sum_{n=0}^{3} i^{-n}\langle\alpha + i^n\beta, A(\alpha + i^n\beta)\rangle$ assures that the criterion stated above is equivalent to requiring convergence to zero for the sequence $|\langle\alpha, (X_i - X)\beta\rangle|$, $i = 1, 2, \ldots$, for any pair $|\alpha\rangle, |\beta\rangle \in \mathcal{H}$.

From here on, the von Neumann algebra obtained by weak closure of a C*–algebra $\mathcal{A}$ will be denoted by the barred symbol $\bar{\mathcal{A}}$. For terminology simplicity, von Neumann algebras will be called *closed algebras* and the weak closure operation will be called *algebraic closure*.

The lattice structure of closed algebras can be established by defining the weakly closed join $\bar{\mathcal{A}}_1\vee\bar{\mathcal{A}}_2$ of any two closed algebras $\bar{\mathcal{A}}_1$ and $\bar{\mathcal{A}}_2$ as the smallest closed algebra that contains $\bar{\mathcal{A}}_1$ and $\bar{\mathcal{A}}_2$, and, for the sake of completeness, the weakly closed meet $\bar{\mathcal{A}}_1\wedge\bar{\mathcal{A}}_2$ as the greatest closed algebra that includes $\bar{\mathcal{A}}_1$ and $\bar{\mathcal{A}}_2$. The following relationships

$$\bar{\mathcal{A}}_1\widetilde{\vee}\bar{\mathcal{A}}_2 \leq \bar{\mathcal{A}}_1\vee\bar{\mathcal{A}}_2\,, \quad \bar{\mathcal{A}}_1\widetilde{\wedge}\bar{\mathcal{A}}_2 \equiv \bar{\mathcal{A}}_1\wedge\bar{\mathcal{A}}_2 \equiv \bar{\mathcal{A}}_1\cap\bar{\mathcal{A}}_2 \tag{1}$$

mark then the difference between the uniform and the weak closure.

The most relevant fact regarding closed algebras is that also the concept of weak closure is purely algebraic, as the following theorem due to von Neumann states:

Proposition 1..1 (Bicommutant theorem)**.** Let $\mathcal{A}$ be a C*–algebra (with unit), $\mathcal{A}'$ its commutant and $\mathcal{A}'' \equiv (\mathcal{A}')'$ its *bicommutant.* $\mathcal{A}$ is closed in the weak topology if and only if $\mathcal{A} = \mathcal{A}''$ *(Bratteli and Robinson, 1987).*

Since $\mathcal{A}''' \equiv \mathcal{A}'$, the equality $\mathcal{A}' = \bar{\mathcal{A}}'$ follows. Thus, algebraic closure can be equivalently achieved by including into the algebra all bounded operators that can be obtained either spatially, by the weak closure of Hilbert space representations, or algebraically, i.e. abstractly, by bicommutation.

The algebraic closure possesses remarkable properties, the most important of which are the *duality* relationships:

$$(\bar{\mathcal{A}}_1 \wedge \bar{\mathcal{A}}_2)' = \bar{\mathcal{A}}_1' \vee \bar{\mathcal{A}}_2', \ (\bar{\mathcal{A}}_1 \vee \bar{\mathcal{A}}_2)' = \bar{\mathcal{A}}_1' \wedge \bar{\mathcal{A}}_2', \ \mathcal{M}' = \mathcal{I}, \ \mathcal{I}' = \mathcal{M}. \tag{2}$$

Let $\mathcal{I}$ be the trivial subalgebra of $\mathcal{M}$ formed by the multiples of the identity I. Then $\bar{\mathcal{A}} \geq \mathcal{I}$, $\bar{\mathcal{A}}' \geq \mathcal{I}$ and $\mathcal{Z} = \bar{\mathcal{A}} \wedge \bar{\mathcal{A}}' \geq \mathcal{I}$ is in general a non trivial algebra. Since $\mathcal{Z}$ is formed by all the elements of $\mathcal{M}$ that commute with all the elements of both $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}'$, it is an Abelian algebra. It is called the *center* of $\bar{\mathcal{A}}$ (and of $\bar{\mathcal{A}}'$).

1.3 Orthocomplemented and orthomodular algebras

Let us specify the conditions for a lattice of closed algebras to be orthocomplemented. Let $\mathcal{Z}$ be the center of a closed algebra $\bar{\mathcal{A}}$. If $\mathcal{Z} > \mathcal{I}$ (properly), then $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}'$, as Hilbert space representations, are reducible. In this case, the eigenvalues of a suitable set of independent elements of $\mathcal{Z}$ different from I can be used to label all possible representations. It is then clear (App.1) that in order for the lattice to be orthocomplemented the equality $\bar{\mathcal{A}} \wedge \bar{\mathcal{A}}' = \mathcal{I}$, and consequently its dual $\bar{\mathcal{A}} \vee \bar{\mathcal{A}}' = \mathcal{M}$, must be added to Eq.s (2), i.e. the representation must be *irreducible*, in which case $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}'$ are called *factors*.

Now assume by hypothesis that a formal correspondence between an orthomodular lattice $\mathcal{R}$ of causal shadows and an orthomodular lattice of C*–subalgebras of a C*–algebra exists. It is then clear that if the elements $\langle S \rangle \in \mathcal{R}$ are one–to–one with certain closed subalgebras $\bar{\mathcal{A}}_S$, then the lattice formed by these subalgebras is orthomodular by isomorphism. In particular, all $\bar{\mathcal{A}}_S$ are orthocomplemented, i.e. $\bar{\mathcal{A}}_S \wedge \bar{\mathcal{A}}_S' = \mathcal{I}$, implying that all $\bar{\mathcal{A}}_S$ are factors. The orthomodularity property is then assured provided that whenever $\bar{\mathcal{A}}_{S_1} \leq \bar{\mathcal{A}}_{S_2}$ the equality

$$(\bar{\mathcal{A}}_{S_1}' \wedge \bar{\mathcal{A}}_{S_2}) \vee \bar{\mathcal{A}}_{S_1} = \bar{\mathcal{A}}_{S_2} \tag{3}$$

is satisfied, i.e. $\bar{\mathcal{A}}_{S_1}' \wedge \bar{\mathcal{A}}_{S_2}$ is the orthocomplement of $\bar{\mathcal{A}}_{S_1}$ relative to $\bar{\mathcal{A}}_{S_2}$.

These properties are certainly satisfied for a finite system, as in this case the orthomodular property is tantamount to the decomposability of the Hilbert space into a direct product of as many Hilbert spaces as there are degrees of freedom. This is precisely the case we are not interested in.

2. Untenability of the Orthomodular Correspondence

We are now in a position that allows us to realize how and why the orthomodular correspondence hypothesis fails in the framework of local quantum physics. The first trouble comes from quantum free fields.

Consider a quantum theory of local free fields endowed with electrical charge density $\rho(x,t)$ and assume that a certain causal shadow $\langle S \rangle$ of cross–section S corresponds to the subalgebra $\bar{\mathcal{A}}_S$. Integrating $\rho(x,t)$ over S we obtain the observable Q_S that belongs to $\bar{\mathcal{A}}_S$ by construction, but which must be assigned to the orthocomplement $\bar{\mathcal{A}}'_S = \bar{\mathcal{A}}_{S'}$ as it commutes with all the observables of $\bar{\mathcal{A}}_S$. The orthomodular correspondence assumption is thus patently violated.

Haag and Schroer noted, however, that this difficulty can be circumvented if the current is the source of an electromagnetic field, as in this case Q_S can be expressed by Gauss' theorem as a the flux of the electric field across the boundary of S. If S is open, then its boundary is found in S'. Correspondingly, Q_S is found in $\bar{\mathcal{A}}'_S$. The orthomodular requirement is then saved.

Similar considerations hold if S is a ring–shaped spacelike set. The operator that represents a constant electrical current I_S circulating in S commutes with the subalgebra of the observables localized in $\bar{\mathcal{A}}_S$. Stoke's theorem then assures that I_S is proportional to the circulation of the magnetic field around the surface of the ring. If the ring is an open set, I_S can be ascribed to $\bar{\mathcal{A}}'_S$.

These examples seem to indicate that the attempt to establish the orthomodular correspondence forces the observables into a status of physical interaction.

Unfortunately the idea is untenable. If the algebraic lattice is orthomodular, not only $\bar{\mathcal{A}}$ but also any one of its subalgebras $\bar{\mathcal{A}}_S$ is a factor, i.e. $\bar{\mathcal{A}}_S$ is an irreducible representation. This implies that only one electric charge sector must be ascribed to each region of a spacelike surface.

Actually, the irreducibility requirement trivializes the whole question as, if $\bar{\mathcal{A}}_S$ is an irreducible representation, Q_S is a multiple of the identity, i.e. it belongs to both $\bar{\mathcal{A}}_S$ and $\bar{\mathcal{A}}'_S$. The mystery of Q_S ubiquity is thus explained without invoking the intervention of gauge fields. To put it differently, if we want that more eigenvalues of Q_S be associated with S, so as to avoid the trivialization, then $\bar{\mathcal{A}}_S$ must be a reducible representation, hence the lattice cannot be orthomodular.

2.1 Quasi–orthomodularity

It is then definitely clear that a way out from this impasse can be found only if we modify the way of implementing the orthomodular property in systems with a non–compact continuum of degrees of freedom. The problem seems strictly related to the topological character of the continuum. Indeed, passing from the discrete to the continuum, discreteness is replaced by suitable separability properties and sums are replaced by integrals. Thus, in the framework of local quantum physics, operators and subalgebras, rather than being indexed by the points of a set, are expected to be functions of measurable subsets of a topological space.

As noted App.1, the semi–complete distributive lattice formed by the open or the closed sets of a topology is not orthocomplemented simply because the orthocomplement of a closed set is open and that of an open set is closed. Precisely this asymmetry prevents the lattice from being orthomodular. Clearly, the problem does not exist for discrete topologies, since in this case all topological subsets are both open and closed. The basic difference between the degrees of freedom of a finite system and those of an infinite system is precisely this.

Can we now somehow reconcile the structure of a topological space with the orthocomplementation property? A very similar question can be posed in the causal–shadow context. Can we somehow reconcile spacetime topology with the orthomodular properties of a causal–shadow lattice?

As pointed out in App.2, the orthomodular structure of a causal–shadow lattice $\mathcal{R}$ is a direct inheritance of the orthomodular structure of the Boolean lattice formed by their set–theoretic cross–sections. Most properties of the two structures are therefore closely related. For instance, since cross–sections are locally measurable regions of a spacelike surface, and since Boolean lattices are measurable provided that are discrete–complete (Grätzer, 1978), $\mathcal{R}$ too must be discrete–complete. In this view, however, the topological properties of the spacelike surfaces are neglected and it would be certainly preferable if $\mathcal{R}$ inherited also these. Unfortunately, this is incompatible with the orthomodular requirement. It is then clear that the orthomodularization problem must be primarily solved in the topological context.

For the topologies of measurable sets there is a way to recover, at least formally, orthocomplementarity, and orthomodularity with it: ignoring boundaries. Since the boundaries of measurable sets have measure zero, we can neglect these and regard the closed and the open sets as equivalent, *modulo boundaries*. All closed set of measure zero are then collected in the equivalence class of the lattice null element. By means of this peeling procedure, the fundamental asymmetry of the topological lattice is removed. This does not mean, however, that zero–measure sets, in particular topological boundaries, can be ignored at other levels of the analysis.

If we apply this procedure to a topological lattice of causal shadows, we obtain an orthomodular lattice whose elements are the elements of the topological lattice *modulo cross–section boundaries*. In this way, both the orthomodular structure and the good properties of the underlying topology are preserved.

From here on, orthocomplemented and orthomodular lattices, whose elements are equivalent "modulo something", will be respectively called *quasi–orthocomplemented* and *quasi–orthomodular*. Note that, despite their names, they actually are fully orthocomplemented and orthomodular. The prefix *quasi* is here intended as a way of saying that the halo of an underlying topology is maintained on the background.

In the following sections we will study whether an analogous procedure can be applied to impart a quasi orthomodular structure to an algebra of local quantum physics operators.

2.2 Topological lattices

Preliminary to the problem, a brief glossary of topological notions from a lattice–theoretic standpoint is here presented.

General topologies are distributive lattices generated by a join–complete but generally meet–incomplete family $\{S_\alpha\}$ of distinguished sets S_α, called *open*. Here α is an index running over some set. A theorem by Birkhoff (1933) and Stones (1936) assures that S_α can be interpreted as a lattice of subsets of a set T of points (Grätzer, 1978). This makes any S_α inherit the set–theoretic orthocomplementation $S_\alpha \to S'_\alpha$, with the well–known properties $S''_\alpha = S_\alpha$, $S_\alpha \cap S'_\alpha = 0$, $S_\alpha \cup S'_\alpha = T$.

Since orthocomplementation exchanges joins and meets, the orthocomplements $\{S'_\alpha\}$ of $\{S_\alpha\}$ form a meet–complete but generally join–incomplete family of distinguished sets, which are called *closed*. In the following, open sets will be denoted by S_α and closed sets by $\bar{S}_\alpha$. Redudantly though, we will write $\{\bar{S}'_\alpha\}$ instead of $\{S'_\alpha\}$ to make it explicit that we are dealing with closed sets.

The smallest closed set $\bar{S}$ that contains an open set S is called the *closure* of S. The greatest open set S contained in a closed set $\bar{S}$ is called the *interior* of $\bar{S}$. Thus, given any S_α of the open–set lattice we can form both its orthocomplement $\bar{S}'_\alpha$ and its closure $\bar{S}_\alpha$. Coherently with our notations, the interior of $\bar{S}'_\alpha$ will be denoted by S'_α.

The set $\bar{B}_\alpha = \bar{S}'_\alpha \cap \bar{S}_\alpha$, which is manifestly closed, will be called the *boundary* of S_α. Clearly enough, it is also the boundary of $\bar{S}_\alpha$, S'_α and $\bar{S}'_\alpha$. It is also clear that a general topology can be equivalently based on the lattice properties of its closed sets. Correspondingly, the lattice–theoretic structure of a topological space is characterized by the properties

$$\bar{S}_\alpha \cap \bar{S}'_\alpha = \bar{B}_\alpha \,, \tag{4}$$

$$\bar{S}_\alpha \cup \bar{S}'_\alpha = T \,. \tag{5}$$

From $\bar{S}_\alpha \subset \bar{S}_\beta$, the following equalities also follow

$$(\bar{S}'_\alpha \cap \bar{S}_\beta) \cup \bar{S}_\alpha = \bar{S}_\beta \,, \quad (\bar{S}'_\alpha \cap \bar{S}_\beta) \cap \bar{S}_\alpha \equiv \bar{B}_\alpha \cap \bar{S}_\beta = \bar{B}_{\alpha,\beta} \subset \bar{S}_\beta \,.$$

Interpreting $(S'_\alpha)_\beta = \bar{S}'_\alpha \cap \bar{S}_\beta$ as the quasi–orthocomplement of $\bar{S}_\alpha$ relative to $\bar{S}_\beta$ modulo the boundary $\bar{B}_{\alpha,\beta}$, we see that the closed sets of a topological space form a quasi–orthomodular lattice. As it is well–known from general topology, these simple properties are sufficient to define the concepts of *compactness* and *connectedness* (Simmons, 1963).

Let us enrich our vocabulary with a few auxiliary concepts. Any two closed sets will be called *adjacent* if their intersection is a non–empty common subset of their boundaries. Forming the join of adjacent sets makes their common pieces of boundary dissolve, in the sense that these pieces cannot be recovered anymore from the join. A set of adjacent sets will be called a *tiling* if their join covers the whole space T.

Assume as topological space that formed by the closed sets of a spacelike surface. As explained in App.2, the lattice formed by the causal–shadows of such closed sets inherit the quasi–orthomodular structure of the spacelike surface topology. Consequently, it makes sense to speak of compactness, connectedness, adjacency and tiling of causal shadows. Quite differently from the cross–section topology case, however, the merging of pieces of boundaries following the joining of adjacent sets is inherited by the causal–shadows as the disappearance of the merged pieces and the formation of closed sets of events that are substantially greater than their mere set–theoretic unions. We can characterize this fact saying that boundary merging causes causal–shadow expansion.

It is then clear that, if a correspondence between causal shadows and subalgebras of operators is somehow possible, something equivalent is expected to happen on the algebraic side.

2.3 Pseudo–topological properties of closed algebras

Interpreting bicommutation as an analog of the topological closure, closed algebras appear to share some formal properties with the closed sets of a topology. As in the topological case, infinite meets of closed algebras are closed algebras. This happens because meets are equivalent to set theoretic intersections. Finite joins of closed algebras, although they are not set theoretic, they are nevertheless closed by definition. Since infinite joins can be formally defined as the algebras generated by an infinite number of closed subalgebras, they too make sense. But in general their closure is not assured as infinite products of operators are usually plagued by the phenomenon of disjoint representations. The formal analogy can be pushed even further basing on the following definitions

Definition 2..1 (Boundaries and interiors of closed algebras)**.** Let $\bar{\mathcal{A}}$ be a closed algebra. Define $\bar{\mathcal{B}} = \mathcal{Z} - \mathcal{I}$, i.e. the C*–algebra formed by the elements of the center $\mathcal{Z} = \bar{\mathcal{A}} \wedge \bar{\mathcal{A}}'$ modulo multiples of the identity $\mathcal{I}$, as the *boundary* of $\bar{\mathcal{A}}$ (and of $\bar{\mathcal{A}}'$). Define $\mathcal{A} = \bar{\mathcal{A}} - \bar{\mathcal{B}}$, i.e. the C*–algebra obtained by removing from $\bar{\mathcal{A}}$ all the elements belonging to its boundary $\bar{\mathcal{B}}$, as the *interior* of $\bar{\mathcal{A}}$. The algebra $\mathcal{A}$ will be called *open.*

We omit proving that the lattice of open algebras is meet–incomplete but join–complete. $\mathcal{I}$ and $\mathcal{M}$ can be considered both open and closed. Since $\mathcal{Z}$ is closed and $\mathcal{I}$ is both open and closed, $\bar{\mathcal{B}}$ is closed. Since $\mathcal{A}' = \bar{\mathcal{A}}'$, then $\mathcal{A}'' = \bar{\mathcal{A}}$ holds, i.e. $\mathcal{Z}$, hence $\bar{\mathcal{B}}$, can be immediately recovered by bicommutation. Moreover, $\mathcal{A} \wedge \bar{\mathcal{A}} = \mathcal{I}$ and $\mathcal{A}' \wedge \bar{\mathcal{A}}' = \mathcal{I}$, i.e. both $\mathcal{A}$ and $\mathcal{A}'$ are C*–algebras. So, $\mathcal{I} \leq \bar{\mathcal{A}}, \mathcal{A}, \bar{\mathcal{A}}', \mathcal{A}' \leq \mathcal{M}$ is a lattice of closed and open C*–algebras with $\mathcal{I}$ as infimum and $\mathcal{M}$ as supremum. In order for the topological analogy to be meaningful, however, also the analog of Eq.(5) must be algebraically implemented in some way. Here we meet the crucial point.

Assume that $\mathcal{M}$, as the algebra of all bounded operators in $\mathcal{H}$, contains all the irreducible representation of its subalgebras. If $\bar{\mathcal{A}}$ is a subalgebra of $\mathcal{M}$ with a non trivial boundary $\bar{\mathcal{B}}$, then the operators of $\bar{\mathcal{B}}$ label the different irreducible representations $\pi_\alpha(\bar{\mathcal{A}})$ of $\bar{\mathcal{A}}$ within $\mathcal{M}$ (here α represents the label provided by $\bar{\mathcal{B}}$). Thus, the entire system of reducible representations $\Pi(\bar{\mathcal{A}})$ of $\bar{\mathcal{A}}$ and $\Pi(\bar{\mathcal{A}}')$ of $\bar{\mathcal{A}}'$ can be written as a direct sum of the form

$$\Pi(\bar{\mathcal{A}}) = \bigoplus_{\alpha} \pi_\alpha(\bar{\mathcal{A}})\,, \qquad \Pi(\bar{\mathcal{A}}') = \bigoplus_{\alpha} \pi_\alpha(\bar{\mathcal{A}}')\,, \tag{6}$$

where $\bigoplus_\alpha$ is a summation symbol standing, in general, for Stiltjes–Lebegues integration. Since each $\pi_\alpha(\bar{\mathcal{A}})$ separately considered is a factor, the following relationships hold

$$\pi_\alpha(\bar{\mathcal{A}}) \wedge \pi_\alpha(\bar{\mathcal{A}}') = \mathcal{I}_\alpha\,, \qquad \pi_\alpha(\bar{\mathcal{A}}) \vee \pi_\alpha(\bar{\mathcal{A}}') = \pi_\alpha(\mathcal{M})\,,$$

where $\mathcal{I}_\alpha$ are multiples of the identity in the Hilbert spaces $\mathcal{H}_\alpha$ of the representations $\pi_\alpha(\bar{\mathcal{A}})$ and $\pi_\alpha(\mathcal{M})$ are inequivalent representations of $\mathcal{M}$. With a suitable normalization, we can then write

$$\Pi(\bar{\mathcal{A}}) \wedge \Pi(\bar{\mathcal{A}}') = \Pi(\bar{\mathcal{B}}) = \bigoplus_{\alpha} \mathcal{P}_\alpha\,, \tag{7}$$

where $\mathcal{P}_\alpha$ are the projectors of $\mathcal{H}$ on $\mathcal{H}_\alpha$, and

$$\Pi(\bar{\mathcal{A}}) \vee \Pi(\bar{\mathcal{A}}') = \bigoplus_{\alpha} \pi_\alpha(\mathcal{M}) \leq \mathcal{M}\,. \tag{8}$$

Equation (7) can be readily recognized as the analog of Eq. (4), with $\bar{\mathcal{B}}$, the algebra (without unit) of projectors $\mathcal{P}_\alpha$, playing the role of the topological boundary common to $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}'$.

Equation (8), however, is not the analog of Eq.(5). Thus, to complete the analogy we should find an extension $\sqcup$ of the weakly closed join $\vee$ so that the equation $\Pi(\bar{\mathcal{A}}) \sqcup \Pi(\bar{\mathcal{A}}') = \mathcal{M}$ holds instead of Eq.(8). We will focus on this problem in the next section.

2.4 Hyperfactors and intertwined joins

From here on, the direct sums $\Pi(\bar{\mathcal{A}}), \Pi(\bar{\mathcal{A}}')$ will be called *hyperfactors*. For the sake of simplicity, however, they will be denoted by $\bar{\mathcal{A}}$, $\bar{\mathcal{A}}'$. Accordingly, the meet $\Pi(\bar{\mathcal{B}})$ of the two hyperfactors will be simply denoted by $\bar{\mathcal{B}}$. This is consistent with identifying abstract algebras with their widest disjoint representations.

As discussed in the previous section, the problem of establishing a correspondence between topological properties and algebraic properties leads us to the following dilemma: Either $\bar{\mathcal{A}}$ is a factor of $\mathcal{M}$, in which case the analog $\bar{\mathcal{A}} \vee \bar{\mathcal{A}}' = \mathcal{M}$ of Eq.5 holds but the boundary becomes trivial, or the boundary is not trivial, i.e. $\bar{\mathcal{A}} \wedge \bar{\mathcal{A}}' - \mathcal{I} = \bar{\mathcal{B}} > \mathcal{O}$, where $\mathcal{O}$ is the null algebra, in which case the analogy with Eq.5 fails, as the equality $\bar{\mathcal{A}} \vee \bar{\mathcal{A}}' = \bigoplus_\alpha \pi_\alpha(\mathcal{M}) < \mathcal{M}$ instead holds. Since the correspondence works perfectly for boundaries, we need only to focus on the problem of reversing the decomposition $\mathcal{M} \to \bar{\mathcal{A}}, \bar{\mathcal{A}}'$.

Let $\bar{\mathcal{A}}$ be a hyperfactor of $\mathcal{M}$ and $\mathcal{U}_A$ the group of all unitary operators $U \in \mathcal{M}$ such that

$$U\bar{\mathcal{A}}U^\dagger = \bar{\mathcal{A}}\,, \qquad U\bar{\mathcal{A}}'U^\dagger = \bar{\mathcal{A}}'\,.$$

Since $\mathcal{U}_\mathcal{A}$ provides a group of automorphisms for both $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}'$, we simply write

$$\mathcal{U}_\mathcal{A}\bar{\mathcal{A}}\mathcal{U}_\mathcal{A}^\dagger = \bar{\mathcal{A}}\,, \qquad \mathcal{U}_\mathcal{A}\bar{\mathcal{A}}'\mathcal{U}_\mathcal{A}^\dagger = \bar{\mathcal{A}}'\,.$$

The equality $\mathcal{U}_\mathcal{A}\bar{\mathcal{B}}\mathcal{U}_\mathcal{A}^\dagger = \bar{\mathcal{B}}$ is easily proved, meaning that $\mathcal{U}_\mathcal{A}$ provides also a group of automorphisms for $\bar{\mathcal{B}}$. In general for $B \in \bar{\mathcal{B}}$ and $U \in \mathcal{U}_\mathcal{A}$ it is $UBU^\dagger = \hat{B} \in \bar{\mathcal{B}}$ but in general $\hat{B} \neq B$.

Now consider the subgroup $\mathcal{V}_\mathcal{A} \subset \mathcal{U}_\mathcal{A}$ formed by all the unitaries of $\bar{\mathcal{A}}$ and the subgroup $\mathcal{V}'_\mathcal{A} \subset \mathcal{U}_\mathcal{A}$ formed by all the unitaries of $\bar{\mathcal{A}}'$. Clearly, for all $V \in \mathcal{V}_\mathcal{A}$, $V' \in \mathcal{V}'_\mathcal{A}$ and $B \in \bar{\mathcal{B}}$ we have $VBV^\dagger = B$, $V'BV'^\dagger = B$. In other terms, the automorphisms generated by all the elements VV' of the direct product $\mathcal{V}_\mathcal{A} \times \mathcal{V}'_\mathcal{A}$, which is an invariant group of $\mathcal{U}_\mathcal{A}$, leaves the elements of $\bar{\mathcal{B}}$ unchanged.

It is then evident that the automorphism group that acts non trivially on $\bar{\mathcal{B}}$ is the factor group $\mathcal{F}_\mathcal{A} = \mathcal{U}_\mathcal{A}/(\mathcal{V}_\mathcal{A} \times \mathcal{V}'_\mathcal{A})$, i.e. the equivalence class of $\mathcal{U}_\mathcal{A}$ modulo $\mathcal{V}_\mathcal{A} \times \mathcal{V}'_\mathcal{A}$. This means that there are in $\mathcal{M}$ many equivalent groups that provide equivalent automorphisms for $\bar{\mathcal{A}}$, $\bar{\mathcal{A}}'$ and $\bar{\mathcal{B}}$.

Definition 2..2 (Intertwining group)**.** Let $\hat{\mathcal{U}}_\mathcal{A}$ a unitary group of the type just described, i.e. a representative of the equivalence class $\mathcal{F}_\mathcal{A}$, and assume that $\bar{\mathcal{A}} \vee \bar{\mathcal{A}}' \vee \hat{\mathcal{U}}_\mathcal{A} = \mathcal{M}$. Since $\mathcal{V}_\mathcal{A}$ and $\mathcal{V}'_\mathcal{A}$ are already elements of $\bar{\mathcal{A}}$ $\bar{\mathcal{A}}'$, the result does not depend on the particular choice of the representative. $\hat{\mathcal{U}}_\mathcal{A}$ will be called an *intertwining group* of $\bar{\mathcal{A}}$.

Thus, the existence of an intertwining group depends only on the structure of the algebraic automorphisms provided by $\mathcal{F}_\mathcal{A}$.

Here is the definition of extended join that we need to complete the formal analog with topologies.

Definition 2..3 (Intertwined join and meet)**.** Let $\bar{\mathcal{A}}$ any hyperfactor of $\mathcal{M}$ and assume that an intertwining group $\hat{\mathcal{U}}_\mathcal{A}$ exists for $\bar{\mathcal{A}}$, we define the operation

$$\bar{\mathcal{A}} \sqcup \bar{\mathcal{A}}' \equiv \bar{\mathcal{A}} \vee \bar{\mathcal{A}}' \vee \hat{\mathcal{U}}_\mathcal{A} = \mathcal{M}$$

the *intertwined join* of $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}'$. For the sake of completeness we define also, improperly though, their *intertwined meet* as the common boundary of $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}'$:

$$\bar{\mathcal{A}} \sqcap \bar{\mathcal{A}}' \equiv \bar{\mathcal{A}} \wedge \bar{\mathcal{A}}' - \mathcal{I} = \bar{\mathcal{B}}\,.$$

From here on, to simplify the language, whenever an algebra $\bar{\mathcal{A}}$ will be called or understood as a hyperfactor, the existence of an intertwining group $\hat{\mathcal{U}}_\mathcal{A}$ and of the intertwined join $\bar{\mathcal{A}} \sqcup \bar{\mathcal{A}}' = \mathcal{M}$ will be assumed.

Note that the intertwined join expands the algebra generated by $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}'$ while dissolving the common boundary $\bar{\mathcal{B}}$. This provides the algebraic analog of the expansion of two causal shadows $\langle S \rangle$, $\langle S' \rangle$ paired to the dissolution of their common boundary $\langle S \rangle \cap \langle S' \rangle$.

Studying under what conditions intertwined joins exist requires an algebraic analysis that we cannot carry out in this paper. Their existence for suitable algebras is plausible, however, otherwise no implementation of local quantum physics would be possible. In a certain sense, the construction here proposed is a generalization of the quantum coordinatization procedure for elementary quantum systems suggested by Hermann Weyl many years ago, and, ultimately, it might even be very similar, if not equivalent, to an algebra of fields. Here, however, we will not try to investigate any further this problem, but only to spend some more words in order to make it explicit the physical meaning of this approach.

The degrees of freedom of a classic system are primarily described by the coordinates of a configuration space. What makes of a physical quantity a coordinate is not the fact of being a simple set of numerical values, rather of being a set of numerical values ordered by a translation group. In elementary quantum physics, the eigenvalues of an observable Q representing positions on a real axis are translated by the Abelian group generated by the Heisenberg's conjugate operator P.

In *The theory of groups and quantum mechanics* (1931), Weyl attributed particular importance to the fact that Q and P can be though of as the generators of two continuous Abelian groups of unitary operators $U(x) = \exp(iPx/\hbar)$ and $V = \exp(iQy/\hbar)$, each one of which transforms isomorphically the Abelian algebra generated by the other:

$$U(a)QU^\dagger(a) = Q + a\,, \qquad V(b)PV^\dagger(b) = P - b\,.$$

The Hilbert space implementation of these transforms is usually described as a projective representation of the corresponding classical group of canonical transformations.

Generalizing this fact, Weyl proposed that the coordinatization of quantum mechanical systems without classic analogs be accomplished by the inclusion of projective representations of discrete Abelian groups, which are equivalent to discrete Abelian groups equipped with Abelian groups of isomorphisms. He showed that spin and fermion algebras fall within this class.

Dealing with systems formed by identical particles, following Bose, Fermi statistics or some parastatistics, the Abelian group of isomorphisms must be widened so as to account for the permutation invariance of the degrees of freedom. Note that, as accounted for in (Doplicher and Roberts, 1972), parastatistics provide an alternative schema for introducing non–Abelian gauge fields. Thus, in general, the coordinatization of a quantum system is provided by Abelian algebras equipped with automorphisms groups of a more general type. In other terms, the coordinatization is ruled by a group generated by an Abelian group $\mathcal{V}$ and a generally non–Abelian conjugated group $\mathcal{U}$ which transforms $\mathcal{V}$ automorphically. Let us call it *Weyl's coordinatization group.* To evidence how this view is related to our main subject we introduce the following notion:

Definition 2..4 (Conjugate group)**.** Let $\hat{\mathcal{V}}_{\mathcal{B}}$ the group formed by all the unitaries of an algebraic boundary $\bar{\mathcal{B}}$. Clearly $\hat{\mathcal{V}}_{\mathcal{B}}$ is a subset of both $\mathcal{V}_{\mathcal{A}}$ and $\mathcal{V}'_{\mathcal{A}}$, then of $\mathcal{V}_{\mathcal{A}} \times \mathcal{V}'_{\mathcal{A}}$. Since $\hat{\mathcal{U}}^\dagger_{\mathcal{A}}\mathcal{V}_{\mathcal{B}}\hat{\mathcal{U}}_{\mathcal{A}} = \mathcal{V}_{\mathcal{B}}$, the equality $\mathcal{V}_{\mathcal{B}}\hat{\mathcal{U}}_{\mathcal{A}}\mathcal{V}^\dagger_{\mathcal{B}} = \hat{\mathcal{U}}_{\mathcal{A}}$ also holds. $\mathcal{V}_{\mathcal{B}}$ will be called the *conjugate group* of $\hat{\mathcal{U}}_{\mathcal{A}}$.

We cannot avoid noting that $\bar{\mathcal{V}}_{\mathcal{B}}$ and $\hat{\mathcal{U}}_{\mathcal{A}}$, as defined in the context of hyperfactors, agree with the definition of a Weyl's coordinatization group. This suggests that the boundaries of all hyperfactors of an algebra of local observables, together with their respective intertwining groups, provide the coordinatization of local quantum physics by gauge fields. This view agrees with the role played by the intertwiners according to Doplicher *et al.* (1969a, 1969b, 1971, 1974).

3. Quasi–Orthomodular Algebras

The meaning of quasi–orthocomplementation for the hyperfactors of an algebra $\mathcal{M}$ is provided by the following definition.

Definition 3..1 (Quasi–orthocomplementation)**.** Assume first that $\mathcal{M}$ is an irreducible representation of a closed algebra. We will say that $\mathcal{M}$ is *quasi–orthocomplemented* if and only if every closed subalgebra $\bar{\mathcal{A}} \leq \mathcal{M}$ is a hyperfactor. Then assume that $\mathcal{M}$ is an abstract closed algebra. We say that $\mathcal{M}$ is *quasi–orthocomplemented* if every representation of $\mathcal{M}$ is quasi–orthocomplemented.

The collection of the hyperfactors $\bar{\mathcal{A}}$ in all representations of $\mathcal{M}$ will be still called hyperfactor. Thus $\bar{\mathcal{A}}$ must be understood as a hyperfactor in one sense or in the other according as $\mathcal{M}$ is understood as an irreducible representation or as an abstract algebra. In both cases, the orthocomplementation property is expressed by the condition that the relationships

$$\bar{\mathcal{A}} \sqcap \bar{\mathcal{A}}' = \bar{\mathcal{B}}\,, \qquad \bar{\mathcal{A}} \sqcup \bar{\mathcal{A}}' = \mathcal{M} \tag{9}$$

hold for any closed subalgebra $\bar{\mathcal{A}} \leq \mathcal{M}$. The meet symbol $\sqcap$ is here introduced with the meaning of $\wedge$ modulo $\mathcal{I}$. The formal correspondence with Eq.s (4), (5) is thus accomplished.

The quasi–orthomodular property can now be introduced as *relative quasi–orthocomplementation*:

Definition 3..2 (Quasi–orthomodularity)**.** Let $\bar{\mathcal{A}}_{\alpha;\beta} \leq \bar{\mathcal{A}}_{\beta} \leq \mathcal{M}$ and denote by $\bar{\mathcal{A}}'_{\alpha;\beta}$ the quasi–orthocomplement of $\bar{\mathcal{A}}_{\alpha;\beta}$ relative to $\bar{\mathcal{A}}_{\beta}$, i.e. $\bar{\mathcal{A}}'_{\alpha;\beta} \equiv (\bar{\mathcal{A}}_{\alpha;\beta})' \sqcap \bar{\mathcal{A}}_{\beta}$. Let $\bar{\mathcal{B}}_{\alpha;\beta}$ be the boundary of $\bar{\mathcal{A}}_{\alpha;\beta}$. Its portion in $\bar{\mathcal{A}}_{\beta}$, i.e. $\bar{\mathcal{B}}_{\alpha,\beta} = \bar{\mathcal{B}}_{\alpha;\beta} \sqcap \bar{\mathcal{A}}_{\beta}$, will be called the *interface* between $\bar{\mathcal{A}}_{\alpha;\beta}$ and $\bar{\mathcal{A}}'_{\alpha;\beta}$. The lattice formed by the closed subalgebras of $\mathcal{M}$, whether in the sense of an irreducible representation or of an abstract algebra, is called *quasi–orthomodular* provided that the relationships

$$\bar{\mathcal{A}}_{\alpha;\beta} \sqcap \bar{\mathcal{A}}'_{\alpha;\beta} = \bar{\mathcal{B}}_{\alpha,\beta}\,, \qquad \bar{\mathcal{A}}_{\alpha;\beta} \sqcup \bar{\mathcal{A}}'_{\alpha;\beta} = \mathcal{A}_{\beta} \tag{10}$$

hold for all subalgebras $\bar{\mathcal{A}}_{\beta;\alpha} \leq \bar{\mathcal{A}}_{\beta} \leq \mathcal{M}$. Thus all $\bar{\mathcal{A}}_{\alpha;\beta}$ are hyperfactors of $\bar{\mathcal{A}}_{\beta}$.

We are now in a position to draw a correspondence between quasi–orthomodular lattices of causal shadows and quasi–orthomodular lattices of hyperfactors. This is immediately accomplished by making any causal shadow $\langle S \rangle$, modulo its topological boundary

B_S, one–to–one with the closed subalgebra $\bar{\mathcal{A}}_S$, modulo $\bar{\mathcal{B}}_S = \bar{\mathcal{A}}_S \sqcap \bar{\mathcal{A}}'_S$. This map can then be extended to the underlying topology by ascribing $\bar{\mathcal{B}}_S$ to B_S.

The identifications can be further extended by coherence. Let $\{B_S\}$ be the set of boundaries of a closed–set tiling of a spacetime surface Σ. Since the algebraic boundaries $\bar{\mathcal{B}}_S$ corresponding to different B_S commute with each other, the set $\{\bar{\mathcal{B}}_S\}$ has the structure of a Boolean lattice of Abelian subalgebras without unit. It will be called the *boundary lattice*. If B_{S_1} and B_{S_2} are disjoint, i.e. $B_{S_1} \wedge B_{S_2} = 0$, where 0 is the null set, then $\bar{\mathcal{B}}_{S_1} \sqcap \bar{\mathcal{B}}_{S_2} = \mathcal{O}$ (the null algebra).

If B_{S_1} and B_{S_2} are not disjoint, let $\langle S_A \rangle$ and $\langle S_B \rangle$ be any two adjacent causal shadows of the tiling, then $B_{AB} = B_{S_A} \wedge B_{S_B}$. In this case, the *interface* between S_A and S_B, can be identified with the support of the Abelian subalgebra $\bar{\mathcal{B}}_{AB} = \bar{\mathcal{B}}_A \sqcap \bar{\mathcal{B}}_B$. Since $B_{AB} = \overline{S}_A \wedge \overline{S}_B$ corresponds to $\bar{\mathcal{B}}_{AB} = \bar{\mathcal{A}}_{S_A} \sqcap \bar{\mathcal{A}}_{S_B}$, $\bar{\mathcal{B}}_{AB}$ can be defined the *algebraic interface* of $\bar{\mathcal{A}}_{S_1}$ and $\bar{\mathcal{A}}_{S_2}$. The set $\{\bar{\mathcal{B}}_{AB}\}$ forms a Boolean lattice of Abelian subalgebras without unit that contains the boundary lattice. It will be called the *interface lattice*.

The main advantage of these identifications lies in the fact that, so doing, the algebras of observables completed by the intertwining operators inherit quite naturally all the topological properties of causal shadows. Thus, the notions of compactness and connectness for both measurable spacelike regions and boundaries can be transferred to the algebraic side. For instance, we can single out from the topological lattice a closed compact and simply connected set endowed with a simply connected boundary and transfer it with all its general topological properties to the algebraic side.

Due to the quasi–orthomodularity property, each causal shadow can be decomposed in many ways into a Boolean lattice of smaller causal shadows. The cross–sections of the causal shadows in each of these decomposition form a lattice of measurable topological sets on some spacelike surface. On the algebraic side, each closed algebra, whether factor or hyperfactor, can be decomposed in many ways into a Boolean lattice of closed subalgebras modulo boundaries, i.e sub–hyperfactors. The Boolean character of the decomposition is strictly related to the commutativity of the subalgebras.

We meet here the relevant property of our construction. Causal shadows do not form a topological space, but their cross–sections do. Correspondingly, quasi–orthomodular subalgebras do not have the formal properties of topological sets, but the subalgebras of any one of its quasi–orthomodular decomposition do. Indeed, the two lattices, that of cross–sections and that of the algebraic decomposition, correspond to each other.

3.1 Intertwined algebra of local observables

From what has been said so far, it is definitely clear that the idea of representing local quantum physics as a mere algebra of observables conflicts with the quasi–orthomodular requirement, and is therefore untenable. The point is that a quasi–orthomodular algebra cannot be devised as an irreducible or reducible algebra of observables, but as a reducible algebra of observables embedded in larger algebra by a net of intertwining operators. From here on, this extended structure will be called *intertwined algebra of observables*.

The need for an extension of the algebra of observables, however, was already implicit in the analysis carried out by Doplicher, Haag and Roberts (DHR analysis; Doplicher *et al.*, 1979–1974), in which the role of the intertwining operators was extensively described and fully clarified. Since intertwining operators represent in some way actions of unobservable fields, intertwined algebras of observables may appear a sort of quantum field theories. In an intertwined algebra, however, the subalgebra of local observables is a distinguished structure which forms the backbone of a local quantum physics representation. What marks the difference between the two structures, however, is the strength of the quasi–orthomodular requirement, which is likely to restrict considerably the spectrum of intertwined algebra representations. Whether these constraints are so strong to make any concrete representation impossible is not known, although, of course, we hope that this is not the case.

4. A Novel View

Once established the correspondence in this more general sense, Haag and Schroer's proposal, as well as the DHR analysis mentioned above, appears in a new light. Each causal shadow $\langle S\rangle$ of a compact and connected region S of a spacelike surface Σ can be interpreted as the integrity domain of an algebra closed by an Abelian boundary. The latter can be interpreted as the algebraic structure that provides the labels of all possible configurations of conservative quantities contained in S, i.e. of all possible states of the matter contained in S. Correspondingly, the content of matter of the original domain is fractioned according to all possible partitions of the conservative quantities there contained.

The structure appears even richer if we consider that boundary algebras provide more than conservative quantity labels. Indeed, in the framework of Haag–Schroer's interpretation, the Abelian algebra $\bar{\mathcal{B}}_{AB}$ with support on the topological interface B_{AB} of any two adjacent causal shadows $\langle S_A\rangle$ and $\langle S_B\rangle$ must be thought of as formed by the fluxes across B_{AB} of the gauge fields associated with the conservative quantities contained in S_A and S_B. So, if the intertwined algebras of observables is localized within the causal shadows, those of the gauge fields must be localized on the interfaces of adjacent causal–shadows.

In order for all these identifications to be physically meaningful, however, we need a new basic assumption. If matter homogeneity is postulated, the labels of all algebraic boundaries must run over a common set of possible eigenvalues. In order for this property to hold, the algebraic boundaries $\bar{\mathcal{B}}_A$, $\bar{\mathcal{B}}_B$, respectively corresponding to any two homeomorphic causal–shadow boundaries $B_A, B_B \subset \Sigma$, must be intertwined either by a unitary automorphism or, more in general, by an isometric automorphism (isometries preserve eigenvalues as well). Correspondingly, the interface Abelian algebras themselves must be related by homeomorphisms. In conclusion, the whole algebraic lattice must be coherently intertwined not only *vertically*, i.e. all over the irreducible representations of a same hyperfactor, as described in Sec.2.4, but also *horizontally*, i.e. through irreducible representations of different hyperfactors, by suitable intertwining relationships.

APPENDIX 1

A few basic concepts on *lattices*, in particular the *orthocomplemented*, the *Boolean*, the *orthomodular* and the *modular* ones, are here flashed (Birkhoff, 1933; Stone, 1936; Birkhoff and von Neumann, 1936; Piron, 1964; Grätzer, 1978). The theory of lattices forms one of the highest levels of mathematical abstraction, being overcome only by the theory of categories. Its relevance resides in the MacNeille completion theorem: *every partial ordering can be uniquely embedded into a complete lattice up to isomorphisms* (MacNeille, 1937).

Lattices in general. A lattice $\mathcal{L}$ is a set of objects $A, B, C, \ldots$ of comparable magnitude or size, called the elements of $\mathcal{L}$, equipped with a *partial ordering* $\leq$, a *meet* $\vee$ and a *join* $\wedge$. $A \leq B$ (resp. $B \geq A$) means that A is not greater than B (resp. not smaller than B). $A \leq B$ and $B \geq A$ together means $A = B$. $A < B$ (resp. $B > A$) means $A \leq B$ but $A \neq B$. Sometimes the symbol $\leq$ is better interpreted as *being a part of* or *being included in*. Some other times, e.g. in logics, $\leq$ is better interpreted as *implies* and denoted by $\rightarrow$. In all cases it will be used with such particular meanings in place of the usual symbols.

The *join* of the elements $A, B, \ldots, Z \in \mathcal{L}$ is defined as their *least upper bound* (with respect to the partial ordering) and denoted by $A \vee B \vee \cdots \vee Z$. The *meet* of the same elements is defined as their *greatest lower bound* and denoted by $A \wedge B \wedge \cdots \wedge Z$. Joins and meets of any finite set of elements are supposed to exist in $\mathcal{L}$. In other terms, $\mathcal{L}$ is closed with respect to finite joins and meets. Often, also the greatest lower bound O (*infimum*) and the the least upper bound I (*supremum*) of all the elements of $\mathcal{L}$ are supposed to exist. In any case, assuming that O and I exist does not impose any additional constraint to a lattice.

A lattice closed with respect to infinite (countable and uncountable) joins and meets is called *complete*. Such is, for instance, the lattice formed by all the subsets of a set. A lattice closed with respect to infinite joins (meets) but only finite meets (joins) is called *join–complete* (*meet–complete*). Both are called *semi–complete*. The open (closed) sets of a topology form a join–complete (meet–complete) lattice. A lattice closed only with respect to discrete joins and meets will be called *discrete–complete*.

A and B are called *disjoint* if $A \wedge B = O$. If A and B are disjoint and $A \vee B = I$, the two element are called *complements* of each other. An element of a lattice may possess more than one complement. For instance, let $\mathcal{L}$ be a lattice formed by the linear subspaces of a vector plane, then O is the common origin of all vectors and I is the plane. All vectors which are not parallel to a vector A are complements of A.

Sublattices. Given a lattice and two elements $A, B \in \mathcal{L}$, with $A \leq B$, the set of elements $\{A \leq X \leq B, X \in \mathcal{L}\}$ form a *sublattice*, which has A as infimum and B as supremum.

Chains. A *chain* $\cdots < A < B < C < \cdots$ is a totally ordered subset of a lattice.

The elements of a chain can be indexed by ordinal numbers. Thus a chain may be finite, infinite, continuous open, closed, possess a minimum, a maximum etc as is the case for ordinal numbers. A chain C is *maximal* if there is no chain C' containing C as a proper subset. Using the axiom of choice it can be proved that any partial or total ordering contains a maximal chain (Hausdorff maximality theorem).

Orthocomplemented lattices. $\mathcal{L}$ is called *orthocomplemented* if for every $A \in \mathcal{L}$ there is an $A' \in \mathcal{L}$ such that $A \wedge A' = O$, $A \vee A' = I$ (*complementation property*), $A'' = A$ (*involution property*) and $(A \wedge B)' = A' \vee B'$ (*duality property*). A' is called the *orthocomplement* of A. It can be easily proved that A' is unique and that $A \leq B$ implies $A' \geq B'$. The latter property can be postulated in place of the duality property. Typical examples of orthocomplements are the complement of a set in the usual sense and the subspace orthogonal to a closed subspace of a Hilbert space.

Distributive lattices. A lattice is called *distributive* if $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$ or, equivalently by duality, if $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$, holds for all the elements of $\mathcal{L}$. It has been proved (Birkoff, 1933; Stones, 1936) that any distributive lattice can be embedded in the lattice formed by the subsets of a set. Typical examples are the lattices formed by topological sets. In this case, however, the lattice is join–complete if its elements are the open sets of the topology, or it is meet–complete if its elements are the closed sets.

Boolean lattices. A lattice that is both orthocomplemented and distributive is called *Boolean*. A classic theorem by Stones (1936) states that any Boolean lattice is isomorphic to a lattice formed by selected subsets of a set. Typical examples are the lattice of all the subsets of a set, the lattice of subalgebras of an Abelian algebra, fields of measurable sets. In the latter case, however, only joins and meets of countable sets of elements are supposed to exist. This depends on the fact that sums make sense only over countable sets of summands.

Orthomodular lattices. The concept of *orthomodularity* is equivalent to that of *relative orthocomplementarity*. The importance of orthomodular lattices in physics was recognized for the first time by Constantin Piron (1964), who discovered their fundamental role in the partial ordering of quantum logical propositions. Consider a system S that, in some general sense, can be decomposed into disjoint parts $A, B, \ldots$, which in turn can be decomposed into smaller disjoint parts and so on, up to possible terminations at some atomic parcels, or forever. The concept of *relative orthocomplementation* corresponds to the fact that for each $A \leq B$ the element $A'_B \equiv A' \wedge B$ is the orthocomplement of A within the sublattice of $\mathcal{L}$ that has 0 as infimum and B as supremum, i.e. $A'_B \wedge A = 0$ and $A'_B \vee A = B$. In other terms, the class of orthomodular lattices are characterized by the property $(A' \wedge B) \vee A = B$ whenever $A \leq B$. Another characteristic property of orthomodular lattices is that all the elements of a chain generate a Boolean lattice, which can be supported on sets. Taking the infimum and the supremum of the lattice as chain extremals, we obtain many different orthomodular decompositions of the lattice. The subsets of a set and the closed subspaces of a Hilbert space are typical examples of orthomodular lattices. The decomposition of a Hilbert space into a refinable system of orthogonal subspaces is strictly related to the resolution of the identity into a refinable commutative system of projectors.

Modular lattices. A lattice is called *modular* provided that whenever $A \leq B$ holds, the equality $A \vee (B \wedge C) = B \wedge (A \vee C)$ also holds. It can be proved that a lattice is modular if and only if a measure $D(A) > D(O) = 0$ exists, with O the lattice infimum and $A > O$, such that $D(A \vee B) + D(A \wedge B) = D(A) + D(B)$. $D(A)$ is called the *dimension* of A. This property is strictly related to the following theorem: in a modular lattice any two maximal chains $A < \cdots X \cdots < B$, $A < \cdots Y \cdots < B$ with same extremals A and B are isomorphic. Modular lattices are complemented but, in general, not orthocomplemented. Finite orthomodular lattices, however, are modular. The lattice formed by the subspaces of a projective geometry is modular. The invariant subgroups of a group form a modular lattice. The equivalence of the invariant–subgroup decomposition series (Jordan–Hölder theorem) is a consequence of the modular structure.

APPENDIX 2

Causal completions. *Causally complete sets*, or *causal completions*, are defined as follows: Let A_S be any subset of events of Minkowski spacetime M. Let A' be the set of all events that are spacelike to A_S and call it the *causal complement* of A_S. Let A be the set of all events that are spacelike to A' and call it the *causal completion* of A_S. Since $A'' = A$ and $A''' = A'$, A and A' are uniquely determined by A_S. A set will be called *causally complete* if it is the causal completion of a set A_S. For any two causal completions A and B, define $A \wedge B$ as the greatest lower bound of A and B, i.e. the largest causal completion contained in A and B. It coincides with the set–theoretic intersection of A and B. Define $A \vee B$ as the least upper bound of A and B, i.e., the smaller causal completion containing A and B. In general, it is greater than the set–theoretic union of A and B. Since $(A \vee B)' = A' \wedge B'$, the lattice of is orthocomplemented. Also $A \wedge B = 0$, the null set, $A \vee B = M \equiv 1$ hold. Remarkably, the lattice of causal completions is even orthomodular (Haag, 1992), meaning that the relative orthocomplement $A'_B \equiv A' \wedge B$ does exist for any $A \leq B$ and $A'_B \vee A = B$.

Handling with the whole lattice of causal completions is a hard task. Causal completions comprise for instance sets of both zero and non–zero measure, sets of points of light cones, isolated points, etc. The point is that with a pseudo–Euclidean metric, and more in general with a pseudo–Riemannian metric, it is difficult to define unambiguously the concept of closeness of two points and consequently, for instance, the boundaries of the causal completions. We are therefore motivated to look for a better approach.

The lattice of causal shadows. The task simplifies considerably if we limit our considerations to a much simpler orthomodular sublattice of the lattice just described. Taking advantage of the existence of spacelike surfaces, consider the lattice formed by the causal completions of the subsets of a spacelike surface Σ. The causal completion of any $S \subset \Sigma$ is its *causal shadow*, i.e. the set $\langle S \rangle$ of all points that cannot be reached by light rays passing through the points of the set–theoretic complement $S' \subset \Sigma$. The relationships $\langle S\rangle \wedge \langle S'\rangle = 0$ and $\langle S\rangle \vee \langle S'\rangle = M$ are then evident.

The lattices formed in this way are manifestly orthomodular as they inherit directly the orthomodular structure of the subsets of Σ. Since Σ can be spanned by measurable sets, we can restrict our consideration to causal shadows of measurable cross–sections. Thus, on account of the lattice–theoretic properties of measurable sets (Grätzer, 1978), we can characterize the lattice of causal shadows as discrete–complete.

Causal shadows inherit quite naturally also the topological structure of Σ. In fact, a causal completion can be regarded as open or closed according as S is an open or closed set. Hence, both the closure $\overline{\langle S\rangle}$ and the boundary $B_S = \overline{\langle S\rangle} \wedge \overline{\langle S'\rangle}$ of $\langle S\rangle$ can be unambiguously defined. Note that B_S coincides with the boundary of S in Σ.

It is also manifest that the same closed causal completion can be generated by two closed spacelike sets S_1 and S_2 belonging to different surfaces Σ_1 and Σ_2, i.e. $\langle S_1\rangle = \langle S_2\rangle$, provided that S_1 and S_2 share the same boundary. Actually, it can be equivalently generated by an infinity of spacelike surfaces. From here on, any one S_α of such generating surfaces will be called a *cross–section* of $\langle S\rangle$.

It is worth mentioning that also the concepts of *compactness* and *connectness* can be transferred from the topology of spacelike sets to that of causal completions.

We find here a critical point. The lattice formed by the open sets of a topology is not orthocomplemented. Consequently, we cannot equip the lattice of causal shadows with the topology inherited from Σ without losing the orthomodular property. If we do this, we still obtain a lattice, but the orthocomplementation does not exist in it.

The expansion property. An obvious but important property of the causal–shadow lattice is that the partition of Minkowski spacetime into a causal–shadow pair $\langle S\rangle$, $\langle S'\rangle$ is uniquely determined by the boundary B_S. Indeed, $\langle S\rangle$ and $\langle S'\rangle$, as sets of events, include all and only those events that are not within the (double) light cones originating from the points of B_S. Some implications of this statement are worth noticing. Assume that S is a closed sphere in some spacelike surface of the Minkowski spacetime. Its causal shadow $\langle S\rangle$ is the double–cone (spherical diamond) external to the light cones originated from the points of S'. Let O be the center of S, then the causal shadow $\langle S - O\rangle$ differs from $\langle S\rangle$ by much more than one point. It is indeed the *crown* obtained by removing from $\langle S\rangle$ all the point belonging to the double light–cone diverging from O.

The phenomenon just described belongs to a class of facts that can be characterized by the concepts of *disjointness* and *interactivity.* Two causal shadows are called disjoint if their cross–sections are disjoint. Their join is simply the set theoretic union the two causal shadows. This property survives any change of shape or position of the two cross sections on a given spacelike surface provided that these keep disjoint. Something dramatic happens, however, if the two causal shadows come in touch. In this case, with the disappearing of a piece of boundary a wider set of point, which were previously excluded because they were in the light cones of the piece of boundary, are recruited by the join.

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