

# Parametric Relationships Among Some Phenomenological Non-Relativistic Hadronic Potentials

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**Abstract:** In recent years, parametric relationships between interatomic potential energy functions have been developed in the realm of molecular chemistry and condensed matter physics. However, no parametric relationships have been developed so far among intra-atomic potentials. As an extension of previous works into the realm of intra-atomic potentials, we herein consider the possibility that hadronic potentials can be interrelated via their parameters. Hadronic potentials give quantitative description of interquark energy in terms of interquark distance, hence understanding how each potential function influences the theoretical modeling can be sought via knowledge of interrelationship amongst the potentials' parameters. Phenomenological non-relativistic hadronic potentials are related amongst the mixed-powerlaw potential themselves, and with the Logarithmic potentials using calculus. Exact nonlinear relationships were obtained between the parameters whereby the interquark distance is included as one of the variables. It is also demonstrated that, when the interquark distance in the parametric relationships is assigned a fixed value of unity, the parametric relationships remain valid from the plotted potential energy curves.

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## 1. Introduction

It is well-known that non-relativistic potential models have proven extremely successful for the description of hadrons as bound states of quarks [1], especially if the hamiltonian is interpreted as an effective hamiltonian with the coefficients normalized by the

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relativistic corrections [2]. In this paper, we take interest in the relationship amongst some purely phenomenological potentials and some Quantum Chromodynamics (QCD) motivated phenomenological potentials. The study of QCD is a highly challenging topic in elementary particle physics. As a result of the deficiency of the analytical methods of quantum field theory, numerical methods have been introduced for QCD. Hence the continuum space-time model has been substituted with the 4-D grid points [3]. Interested readers are directed to the recent progress of QCD with the aid of high performance and parallel computing [4,5]. The motivation for establishing relationships between these potentials is to show how these potentials are related to one another since theoretical modeling of interquark potentials is chiefly dependent on the choice of potentials. Relationships between potentials have recently been developed for relating molecular force fields which is of interest to the chemical community [6-14], and for relating many-body interatomic potentials which is of concern to the condensed matter community [15-23], by means of establishing connections between parameters from different interatomic potential energy functions. Unlike the recently completed works, this paper focuses on the “intra-”, rather than the “inter-”, atomic potentials. It is instructive to note that there are two broad methods to mathematically relate the parameters – (a) the series expansion approach [8-12], and (b) the derivative approach [13-23]. In the foregoing analysis, we adopt the latter approach for hadronic potentials due to its versatility.

## 2. Analysis

Limiting our scope of phenomenological potentials to the form

$$V(r) = Ar^a - Br^{-b} - C \quad (1)$$

where  $A, B, C, a, b$  are non-negative constants, we have included two broad categories whereby  $a = b$  [24-29] and  $a \neq b$  [30-34], as listed in Table 1. For convenience, we shall refer to the potential of the form described by Eq.(1) as the mixed-powerlaw potential. Due to its ease of usage, we also consider the Logarithmic potential

$$V_L = A_L \ln r - B_L, \quad (2)$$

which is also a purely phenomenological potential [35-37] whereby  $A_L$  and  $B_L$  are non-negative constants. A relationship between the logarithmic potential and the mixed-powerlaw potential can be obtained by equating these potentials and their first derivatives

$$\frac{d^n V_L}{dr^n} = \frac{d^n V}{dr^n} \quad ; \quad (n = 0, 1) \quad (3)$$

which, upon solving simultaneously, gives

$$A_L = aAr^{+a} + bBr^{-b} \quad (4)$$

and

$$B_L = A(a \ln r - 1)r^{+a} + B(b \ln r + 1)r^{-b} + C. \quad (5)$$

Alternatively, taking

$$\frac{d^n V}{dr^n} = \frac{d^n V_L}{dr^n} \quad ; \quad (n = 0, 1, 2) \quad (6)$$

and solving simultaneously yields

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{A_L}{a+b} \begin{pmatrix} (ar^a/b)^{-1} \\ (ar^b/b)^{+1} \end{pmatrix} \quad (7)$$

and

$$C = B_L - \left( \frac{a-b}{ab} + \ln r \right) A_L. \quad (8)$$

As such, Eqs.(4) and (5) give the exact non-constant coefficients of the Logarithmic potential in terms of mixed-powerlaw parameters and the interquark distance,  $r$ , whilst Eqs.(7) and (8) express the exact non-constant coefficients of the mixed-powerlaw potential in terms of Logarithmic parameters and the interquark distance. For the special case whereby  $a = b \equiv x$  as in references [24-29], Eqs.(7) and (8) simplify to

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{A_L}{2x} \begin{pmatrix} r^{-x} \\ r^{+x} \end{pmatrix} \quad (9)$$

and

$$C = B_L - A_L \ln r \quad (10)$$

respectively. To relate parameters among mixed-powerlaw potentials

$$V_1(r) = A_1 r^{a_1} - B_1 r^{-b_1} - C_1 \quad (11)$$

and

$$V_2(r) = A_2 r^{a_2} - B_2 r^{-b_2} - C_2, \quad (12)$$

we solve the derivatives

$$\frac{d^n V_1}{dr^n} = \frac{d^n V_2}{dr^n} \quad ; \quad (n = 0, 1, 2) \quad (13)$$

simultaneously. This leads to

$$A_1 = \frac{a_2}{a_1} \left( \frac{a_2 + b_1}{a_1 + b_1} \right) A_2 r^{-(a_1 - a_2)} + \frac{b_2}{a_1} \left( \frac{b_1 - b_2}{a_1 + b_1} \right) B_2 r^{-(b_2 + a_1)}, \quad (14)$$

$$B_1 = \frac{b_2}{b_1} \left( \frac{b_2 + a_1}{b_1 + a_1} \right) B_2 r^{+(b_1 - b_2)} + \frac{a_2}{b_1} \left( \frac{a_1 - a_2}{b_1 + a_1} \right) A_2 r^{+(a_2 + b_1)} \quad (15)$$

$$C_1 = \left[ \frac{a_2 b_1 (a_2 + b_1) - a_1 a_2 (a_1 - a_2)}{a_1 b_1 (a_1 + b_1)} - 1 \right] A_2 r^{+a_2} - \left[ \frac{b_2 a_1 (b_2 + a_1) - b_1 b_2 (b_1 - b_2)}{b_1 a_1 (b_1 + a_1)} - 1 \right] B_2 r^{-b_2} + C_2 \quad (16)$$

Equations (14)-(16) show that parameters of  $V_1$  can be exactly expressed in terms of parameters of  $V_2$  and the interquark distance. For the converted parameters to be

constant, there is a need to assign the interquark distance to a fixed value. As a result, potentials related in this manner coincide only at the fixed interquark distance. Selecting  $r = 1$ , we have the Logarithmic constants in terms of mixed-powerlaw parameters

$$\begin{Bmatrix} A_L \\ B_L \end{Bmatrix} = \begin{bmatrix} a & b & 0 \\ -1 & +1 & +1 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix}, \quad (17)$$

and vice versa

$$\begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \frac{1}{(a+b)} \begin{bmatrix} (b/a) & 0 \\ (a/b) & 0 \\ (b/a) - (a/b) & (a+b) \end{bmatrix} \begin{Bmatrix} A_L \\ B_L \end{Bmatrix}, \quad (18)$$

whilst parametric relationships among mixed-powerlaw potentials are

$$\begin{Bmatrix} A_1 \\ B_1 \end{Bmatrix} = \left( \frac{1}{a_1 + b_1} \right) \begin{bmatrix} \left( \frac{a_2 + b_1}{a_1} \right) & \left( \frac{b_1 - b_2}{a_1} \right) \\ \left( \frac{a_1 - a_2}{b_1} \right) & \left( \frac{b_2 + a_1}{b_1} \right) \end{bmatrix} \begin{Bmatrix} a_2 A_2 \\ b_2 B_2 \end{Bmatrix} \quad (19)$$

and

$$C_1 = \left[ \frac{a_2 b_1 (a_2 + b_1) - a_1 a_2 (a_1 - a_2)}{a_1 b_1 (a_1 + b_1)} - 1 \right] A_2 - \left[ \frac{b_2 a_1 (b_2 + a_1) - b_1 b_2 (b_1 - b_2)}{b_1 a_1 (b_1 + a_1)} - 1 \right] B_2 + C_2. \quad (20)$$

As such, the correlation is theoretically valid only for  $r$  close to 1. For the special case whereby  $a = b = x$ , Eq.(18) reduces to

$$\begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \frac{1}{2x} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 2x \end{bmatrix} \begin{Bmatrix} A_L \\ B_L \end{Bmatrix} \quad (21)$$

whilst Eqs.(19) and (20) collapse into

$$\begin{Bmatrix} A_1 \\ B_1 \\ C_1 \end{Bmatrix} = \begin{bmatrix} \frac{1}{2} \frac{x_2}{x_1} \left( 1 + \frac{x_2}{x_1} \right) & \frac{1}{2} \frac{x_2}{x_1} \left( 1 - \frac{x_2}{x_1} \right) & 0 \\ \frac{1}{2} \frac{x_2}{x_1} \left( 1 - \frac{x_2}{x_1} \right) & \frac{1}{2} \frac{x_2}{x_1} \left( 1 + \frac{x_2}{x_1} \right) & 0 \\ \left( \frac{x_2^2}{x_1^2} - 1 \right) & \left( 1 - \frac{x_2^2}{x_1^2} \right) & 1 \end{bmatrix} \begin{Bmatrix} A_2 \\ B_2 \\ C_2 \end{Bmatrix}. \quad (22)$$

### 3. Results and Discussion

While the nonlinear parametric relationships described by Eqs.(4),(5),(7),(8) and (14) to (16) are exact, the assignment of a fixed interquark distance value ( $r = 1$ ) calls for verification to be made to examine the validity of the developed parametric relationship. A relationship between the Logarithmic and the mixed-powerlaw potentials is of interest since the 5-parameter mixed-powerlaw enables better fit while the 2-parameter Logarithmic potential is more convenient as shown in Eq.(17). As such, the precision of mixed-powerlaw can be converted for use in the more easily executable Logarithmic potential. As an illustration, however, we plot the mixed-powerlaw potentials based on Logarithmic parameters  $A_L = 0.733 \text{ GeV}$  and  $B_L = 0.6631 \text{ GeV}$  [35] using Eq.(21) so that the influence of parameters  $a = b = x$  can be observed, as shown in Fig.1. Fig.2 shows the plots of Cornell potential [24-26] and Turin potential [27], and the plot of Cornell function using Turin parameters using Eq.(22). We observe that within the interquark distance of  $0 < r < 2.5$  the simplifying assumption of  $r = 1$  is valid. An understanding of the relation amongst mixed-powerlaw potentials would be useful for plotting convenient potentials (such as the Cornell potential whereby  $a = b = 1$ ) using the more elaborate potential parameters (such as  $a, b \neq 1$ ).

### 4. Conclusion

Two sets of relationship have been developed:

- (1) between the Logarithmic and the mixed-powerlaw potentials, and
- (2) among the mixed-powerlaw potentials.

This was attained by relating their coefficients. Plotted results reveal good agreement, and hence the validity of the currently developed parametric relationships among the presently considered phenomenological non-relativistic hadronic potentials. The parametric relations developed herein reveals the difference between parameters from each potentials, thereby shedding certain insights on how each potential differ from one another in an analytical manner. More importantly, the parametric relationships enable convenient execution of simpler potentials based on parameters corresponding to the more complicated but more accurate potentials.

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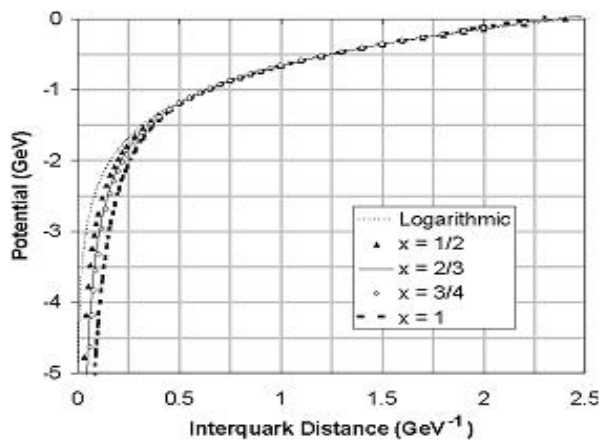
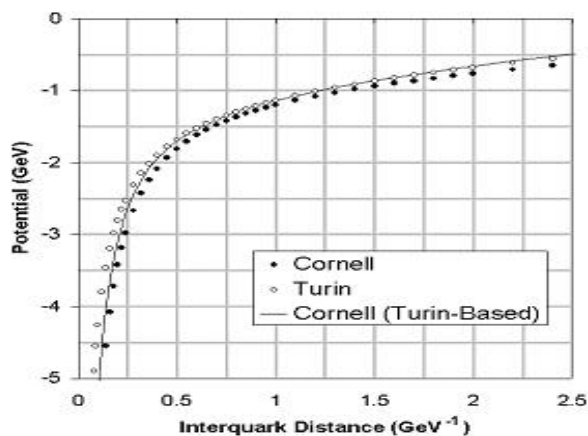
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**Table 1.** Phenomenological non-relativistic potentials of the form

$$V(r) = Ar^a - Br^{-b} - C.$$

Potential	Indices	References
Cornell	$a = b = 1$	[24-26]
Turin	$a = b = \frac{3}{4}$	[27]
Song	$a = b = \frac{2}{3}$	[28]
Song-Lin	$a = b = \frac{1}{2}$	[29]
Heikkila-Tornquist-Ono	$a = \frac{2}{3}$ , $b = 1$	[30]
Martin	$a = \frac{1}{10}$ , $b = 0$	[31-33]
Grant-Rosner-Rynes	$a = 0$ , $b = 0.045$	[34]

**Fig. 1** Plots of mixed-powerlaw potentials using Logarithmic potential parameters [35].**Fig. 2** Plots of Cornell [24-26] and Turin [27] potentials, and the Turin-based Cornell functional.