

Fractional Unstable Euclidean Universe

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Abstract: Despite common acceptance of Big Bang hypothesis among most cosmologists, nonetheless there are criticisms from a small number of theorists partly supported by astronomy observation suggesting that redshift data could not always be attributed to cosmological expansion. In this paper, a new approach to cosmology “*fractional calculus*” has been developed that we hope will attract attention from astrophysicists and cosmologists because of the way it challenges the conventional big bang framework.

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1. Motivations

Testing the universe expansion scenario starts namely with its major assumption that general relativistic processes operate to expand wavelengths only while light (*photons*) are in motion, that is light travels at photon’s velocity. As a result, one assumes *expansion-stop* effects during *emission/absorption* in order to insure agreement with the astronomical requirement of a fixed emission wavelength. However, a detailed analysis of the many relativistic gravitational experiments performed over the last few decades reveal conflicts with the cosmic expansion paradigm’s basic assumptions, and are completely in accord with the predictions of the static-spacetime which is the first cosmological model developed by Einstein.

Whatever is the situation, it will not be an easy task to talk about a static universe in a research center where all its scientists, in particular, cosmologist, astrophysicists and astronomers, are convinced that the universe is expanding. Let us summarize the

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situation (*we refer to [1] in this summary*):

- In **1916**, in that seminal paper [2] Einstein predicted that gravity should cause a perfect clock to go.
- In **1954**, J. W. Brault confirmed the magnitude of the gravitational redshift predicted by Einstein after performing redshift measurement of the sodium *D* line emanating from the sun's spectrum without clarifying in details and with clarity its origin [3]. Probably Einstein was correct in explaining *that the origin of the gravitational redshift came from the fact that different gravitational potentials at source and observer meant that clocks at these locations should run at intrinsically different rates*. But one may ask differently: *did the measured redshift instead have its origin in photons experiencing an in-flight energy exchange with gravity as they moved in a changing gravitational potential in their transit from a star to the Earth?*
- In **1965**, R.V. Pound and J.L. Sinder performed an experiment predicting a fractional frequency between ^{57}Fe gammas emitted at the top and received at the bottom of a tower of certain height, confirmed precisely the magnitude of the Einstein redshift without justifying its origin [4].
- In **1987**, J.H. Taylor compared atomic clock time with pulsar timing data by taking into account the change of local atomic clock time due to the monthly variation in the sun's gravitational potential at Earth [5].
- In **1997-98**, a new interpretation of the cosmological Big Bang redshift (**NIR**) had been proposed by R. V. Gentry asserted that its origin might be the gravitational potential rather than the cosmic or space-time expansion [6]. After reexamination of general relativistic experimental results, the same author states that *the universe is governed by Einstein's static-spacetime general relativity instead of Friedmann-Lemaitre expanding-spacetime general relativity*. In his conclusion, R. V. Gentry states that the absence of expansion redshifts in his static-spacetime universe model suggests a reevaluation of the present cosmology. In fact, recent reports of cosmological observations strongly suggest the existence of a repulsive force in the outermost reaches of the universe [7,8]. An important question is whether these observations may reasonably be interpreted to be a remarkable confirmation of the prediction that ours is a universe dominated by a repulsive force due to vacuum gravity modeled by the Einstein's cosmological constant [9]. In another work, R. V. Gentry, show how the **NRI** and a static-spacetime Einstein universe lead to new possibilities for quasar redshifts [1]. The latter may be of considerable interest, in particular for those researchers who have long contended that certain quasars provide strong evidence of intrinsic redshifts. The **NRI** framework's can interpret a variety of cosmological observations with Einstein's gravitational and Doppler redshifts, without the addition of any dynamical spacetime and the Cosmological Principle. The end result of R. V. Gentry's model is the fact that the **NRI** framework is an alternate interpretation of the Hubble relation and the 2.7 K Cosmic Blackbody Radiation (**CBR**).

Following these arguments, one can suggest that the big bang theory might not be correct and its foundations must be revisited.

- In **1998-99**, S. Carlip and R. Scranton [10], criticized R. V. Gentry **NRI** model, by showing that although supposedly based on general relativity, is inconsistent with Einstein General Relativity, that is with the Einstein field equations; *that it requires delicate fine tuning of initial conditions; that it is highly unstable, both gravitationally and thermodynamically; and that its predictions disagree clearly with observation.* In addition, the authors state that **NRI** starts from unclear theoretical foundations, requiring delicate fine tuning, including a simultaneous specification of the initial velocity of each galaxy in the universe [10]. In [11], Gentry showed that the **NRI** very definitely encompasses an expanding universe wherein galaxies are undergoing Doppler recession according to the Hubble yielding the correct form of the Hubble magnitude-redshift relation.
- T. Shimizu and K. Watanabe [12], give a relativistic description of Gentry's **NRI**. They show that Gentry's idea can be partially applied, in a relativistic manner, to some class of cosmological models, e.g., de Sitter space-time to obtain the distance-redshift relation again. This later is coordinate-independent, gauge invariant [13], however, depends on both the observer and sources. In particular, they obtain a new expression of the Friedmann-Robertson-Walker (**FRW**) metric, which is an analogue of a static chart of the de Sitter space-time. Two important functions appeared in their cosmological reduced metric model: the mass function and the gravitational potential. They find that, near the coordinate origin, the reduced metric can be approximated in a static form and that the approximated metric function, satisfies the Poisson equation. Moreover, with suitable model parameters, the approximated metric coincides with exact solutions of the Einstein equation with the perfect fluid matter. By solving the radial geodesics on the approximated space-time, they obtain the distance-redshift relation of geodesic sources observed by the comoving observer at the origin. As a result, the redshift is found to be expressed in terms of a peculiar velocity of the source and the metric function, evaluated at the source position, bringing in mind Gentry's mode, thinking that this is a new interpretation of his **NRI**.

All these important remarks and notes, as well as the general interest in stationary universe represent our motivation in this work.

In what follows, we want to present a pedagogical approach to the fractional equations governing the evolution of a stationary universe, namely the fractional Friedmann Equations (**FFE**). In general, the derivation of standard dynamical equations is intrinsically relativistic. Although in Newtonian theory, the universe must be static, the Friedmann equations can be derived from the simpler Newtonian theory. In fact, it may be considered as puzzling that Newtonian theory and general relativity give the same results. In this paper, a Newtonian derivation of the Friedmann equation is done but where the time derivative is replaced by fractional order.

2. The Importance of Fractional Calculus

In recent years, growing attention has been focused on the importance of fractional derivatives and integrals in science. It is well believed today that fractional calculus is a quite irreplaceable mean for description and investigation of classical and quantum complex dynamical system [14,15,16,17]. In simple words, the fractional derivatives and integrals describe more accurately the complex physical systems and at the same time, investigate more about simple dynamical systems. Dealing with fractional derivatives is not more complex than with usual differential operators.

Cosmological applications are discussed for the first time in the works of H.C. Rosu [18,19]. Our approach is totally different from all those approaches modeling inhomogeneous fractal universe or those using self-similar scaling of density perturbations with scale free initial conditions in a spatially flat universe as a theoretical tool to study structure formation.

As we are interested on a Newtonian derivation of the cosmological dynamical equations, we follow [20,21,22] and we consider a sphere of radius σ filled with a pressure fluid of uniform density ρ free-falling under its own gravitational field in an otherwise Euclidean space. The coordinate of any particle of the fluid is written as $\vec{x} = R(t) \vec{r}$ where \vec{r} is a (*r is dimensionless*) constant vector referred to as the *commoving coordinate*, t is the time coordinate and R the scale factor (*having the dimension of a length*). The edge of the sphere is also moving as $\sigma(t) = R(t) \sigma_0$. Assuming that while sitting on a particle labelled i we are observing another one labelled j then the relative velocity is given by $\vec{v}_{ij} = H \vec{x}_{ij}$ where $H \equiv \dot{R}/R$ and $\vec{x}_{ij} \equiv \Delta \vec{r} = \vec{r}_j - \vec{r}_i$, implying the presence of a radial velocity proportional to the particle displacement so that the expansion is isotropic. This does not imply that all the particles are identical [23].

To determine the equation for the evolution of the scale factor R , we first compute the gravitational potential energy (E_G) of a particle of mass m by applying Gauss's law, then evaluate its kinetic energy (E_K). Because the gravitational potential on any particle inside the sphere is proportional to the distance x^2 , any other energy (*or potential*) deriving from a potential proportional to x^2 will mimic a gravitational effect. This energy is $E_\Lambda = -m\Lambda c^2 x^2/6$, where Λ is a constant (*Einstein's cosmological constant having the dimension of an inverse squared length*) and “ c ” being the celerity of light.

The total energy is then given by:

$$E = E_G + E_K + E_\Lambda \quad (1)$$

$$= -\frac{GM(<x)m}{x} + \frac{1}{2}m\dot{x}^2 - \frac{m\Lambda c^2}{6}x^2 \quad (2)$$

where G is Newton's constant and $M(<x)$ is the mass within the sphere of radius x given by $M(<x) = 4/3\pi\rho x^3$. Using the fact that the mass within any commoving volume is constant, that is $\rho(t) \propto R^{-3}(t)$ and the conservation of the total energy, using $\vec{x} = R(t) \vec{r}$ yields after simple manipulation:

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G\rho}{3} - \frac{kc^2}{3R^2} + \frac{\Lambda c^2}{3} \quad (3)$$

Here $k = -6E/mc^2r^2$ a dimensionless constant (*curvature*). In fact, the last term in equation (3) is equivalent to adding a constant amount of energy to the universe. This does not violate energy conservation. According to the Copernican principle [9], equation (3) can be applied to any regions of the universe. It relates and constraints how the scale factor $R(t)$ evolves given the total density, the curvature and the cosmological constant of the universe.

To obtain the acceleration equation or second Friedman equation, we model the matter and radiation of the universe as a perfect fluid. One then can use the first law of thermodynamics $dQ = dU + pdV = TdS$ (T being the temperature and S the entropy) applied to $U = mc^2 = 4/3\pi R^3\rho c^2$ from which one can easily evaluate the change of U in time dt , that is dU/dt . Assuming $dQ = dS = 0$ (*no bulk heat flow (adiabatic)*) and making use of $dV/dt = 4\pi R^2dR/dt$, the first law gives the fluid equation (*energy conservation law*):

$$\dot{\rho} + 3\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) = 0 \quad (4)$$

The first term $\dot{\rho}$ inform us how fast density changes (*dilutes*) and the second term is the lost of kinetic energy from fluid into gravitational potential energy.

If we differentiate the first Friedman equation (3) with respect to time and making use of equation (4), we find the acceleration equation (*assuming Λ constant*):

$$\frac{\ddot{R}}{R} = -\frac{4}{3}\pi G\left(\rho + 3\frac{p}{c^2}\right) + \frac{\Lambda c^2}{3} \quad (5)$$

The curvature k cancelled out (*a good feature*), so that we can use this equation regardless the geometry of the universe. In fact, this equation tells us how rate of expansion of the universe changes (*slowing down or speeding up*).

3. Fractional Friedman Dynamical Equations

Let us retreat now the same model but where the derivative operators are replaced with fractional one. There are about two dozens of different definitions for fractional derivatives that are in one way or another adapted to various features of classes of functions for which they are defined. The most comprehensive description of the mathematical aspects of the issue is given in the monograph [24]. In general, fractional calculus is a generalization of integration and differentiation to non-integer order fundamental operator ${}_aD_t^\alpha$ where a and t are the limits of the operation. The continuous integro-differential operator is defined as

$${}_aD_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha}, \Re(\alpha) > 0 \\ 1, \Re(\alpha) = 0 \\ \int_a^t (d\tau)^{-\alpha}, \Re(\alpha) < 0 \end{cases} \quad (6)$$

where $\Re(\alpha)$ is the real part of α . The two definitions for the general fractional differ-integral are the Grünwald-Letnikov (**GL**) definition and the Riemann-Liouville (**RL**)

definition [25]. The **GL** is given by

$${}_a D_t^\alpha f(t) = \lim_{h \rightarrow 0} \sum_{j=0}^{[t-a/n]} (-1)^j \binom{\alpha}{j} f(t-jh) \quad (7)$$

where $[x]$ means the integer part of x .

In what follows, as fractional operators, we use the Riemann-Liouville fractional derivative and integrals defined as [24,25]

$${}_a D_t^\alpha R(t) \equiv \frac{d^\alpha R(t)}{dt^\alpha} = \frac{d^n}{dt^n} {}_a I_t^{n-\alpha} R(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{R(t')}{(t-t')^{1+\alpha-n}} dt', n-1 \leq \alpha < n \quad (8)$$

$${}_a I_t^\alpha R(t) = \frac{1}{\Gamma(\alpha)} \int_a^t dt' \frac{R(t')}{(t-t')^{1-\alpha}} \quad (9)$$

$\Gamma(\cdot)$ is the *Gamma* function. An interesting consequence of this definition is the non-vanishing fractional differentiation of a constant:

$$\frac{d^\alpha R_0}{dt^\alpha} = \frac{R_0 t^{-\alpha}}{\Gamma(1-\alpha)}, 0 < \alpha < 1 \quad (10)$$

where $R_0 = cte$. In general,

$$\lim_{\alpha \rightarrow 1} \frac{t^{-\alpha}}{\Gamma(1-\alpha)} = \delta(t), t \geq 0, 0 < \alpha < 1 \quad (11)$$

In the physical world, equation (11) presents a substantial problem. While today we are well familiar with the interpretation of the physical world in integer order equations, we do not (*currently*) have a practical understanding of the world in a fractional order. Our mathematical tools go beyond the practical limitations of our understanding. In general, equations (6) and (7) could be written in terms of the left-sided Riemann-Liouville fractional integral of order $0 < \alpha \leq 1$:

$${}_0 I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau) (t-\tau)^{\alpha-1} d\tau = \int_0^t f(\tau) dg_t(\tau) \quad (12)$$

where

$$g_t(\tau) = \frac{1}{\Gamma(\alpha+1)} \{t^\alpha - (t-\tau)^\alpha\} \quad (13)$$

with the interesting scaling properties

$$g_{t_1}(\tau_1) = g_{kt}(k\tau) = k^\alpha g_t(\tau) \quad (14)$$

where $t_1 = kt$ and $\tau_1 = k\tau$.

The basic idea now is to replace the normal derivative with a fractional one, that is $\dot{R} \equiv dR/dt \rightarrow D_t^\alpha R \equiv (d^\alpha/dt^\alpha) R$ and $\ddot{R} \equiv d/dt (dR/dt) \rightarrow D_t^\alpha (D_t^\alpha R)$, $0 < \alpha < 1$.

The fractional Friedman equations for zero pressure in the presence of the cosmological constant and zero spatial curvature are written as:

$$(D_t^\alpha R)^2 = (A_1 (G\rho)^\alpha + B_1 (\Lambda c^2)^\alpha) R^2 \quad (15)$$

$$D_t^\alpha (D_t^\alpha R) = - (A_2 (G\rho)^\alpha - B_2 (\Lambda c^2)^\alpha) R \quad (16)$$

for solutions with positive, negative and zero spatial curvature, ρ is the density, c the velocity of light and $0 < \alpha < 1$, $A_i, B_i, i = 1, 2$ are constants. In principle, $B_1 = B_2$.

In this work, we are interested in the static models, in particular the Einstein one. That is, when $R(t) = R_0 = cte$, equations (8), (9),(10), (15) and (16) gives for zero pressure

$$(G\rho)^\alpha = \frac{C_1}{t^{2\alpha}} \quad (17)$$

$$(\Lambda c^2)^\alpha = \frac{C_2}{t^{2\alpha}} \quad (18)$$

where $C_i, i = 1, 2$ are positive constants depending on α . It is clear in such model, that the density and the cosmological constant decreases as $1/t^2$, whatever $0 < \alpha < 1$, unless G and c are kept constant [26,27]. If the density is assumed not to vary with time, than from equation (17), it is clear that the gravitational constant decreases as $1/t^2$. In contrast, if the cosmological constant is stable with time, then from (18), the velocity of light decreases as $1/t$ (see for example [28-30]). In fact, recent observations of the cosmological effects of including a time-variation of the velocity of light, into the gravitational field equations have revealed a number of important features. If the speed of light falls sufficiently rapidly over an interval of time then it is possible to solve the **FRW** horizon and flatness problems in a way that differs from the inflationary universe [31,32]. By adopting the variation of the speed of light (**VSL**) frame, the redshift $z = (\lambda_0 - \lambda_1)/\lambda_1$ is given by:

$$z = \frac{c(t_1) R(t_0)}{c(t_0) R(t_1)} - 1 \quad (19)$$

where t_1 denotes the time a light wave leaves a galaxy and t_0 denotes the time when it reaches us on earth [33]. In our model, $R(t_0) = R(t_1)$, so that $z = (c(t_1)/c(t_0)) - 1$. If the velocity of light decreases as $1/t$, then the redshift is $z = t_0/t_1 - 1$. This means that the observed redshift will appear to be larger due to the speed of light in the past being bigger than the presently observed speed, all this in a static universe, reviving again Gentry's **NRI**. In summary, we have developed within the framework of fractional calculus, a homogenous, isotropic, *static*, but *unstable* Euclidean universe with vacuum energy, decreasing light velocity and presence of Redshift. The static Euclidean model describes in this paper have important feature. It differs from the static Einstein elliptic universe where the density and the cosmological constant are constants. In fact, the possibility of variation of the fundamental constants of physics in the static universe was discussed in [34,35]. It was shown when the velocity of light increases/decreases,

the Planck's constant increases/decreases and mass of bodies decreases/increases. In order to keep the density and Λ constants, the gravitation G and the velocity of light c must decrease with time, as it is clear from equations (17) and (18) [36,37,38]. The Hubble constant in our model is defined as $H^\alpha \equiv \dot{R}/R$ or $Ht = (1/\Gamma(1-\alpha))^{1/\alpha}$. The deceleration parameter is defined by $q = -1 - \dot{H}/H^2 = -1 + (\Gamma(1-\alpha))^{-1/\alpha} > -1$ in accord with observations. One can fix the value of α from observations (*not an easy task*). This is another important feature in the model, not existing in Einstein static universe. Roughly speaking, the behavior of the deceleration equation for $0 < \alpha < 1$ described in our model is shown in the following figure:

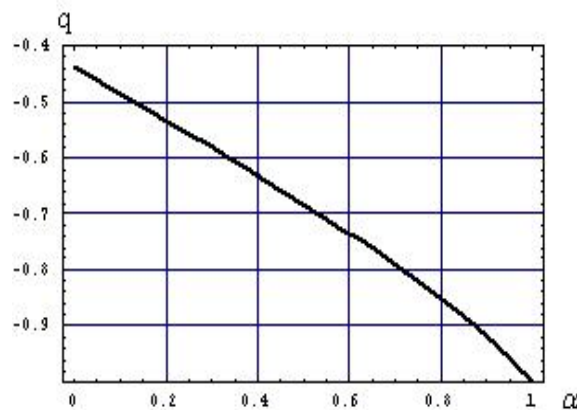


Fig. 1 The behavior of the deceleration parameter for $0 < \alpha < 1$

There exists also another important feature concerning the applications of fractional calculus on cosmology. We also mention it here only to show the reader the benefit of the fractional calculus. The first of the fractional Friedman equation (15) represents a fractional differential equation when the scale radius is considered depending on time, that is not constant.

Applying the fractional integral operator to both left-hand and right-hand sides of equation (3), and solving, one finds:

$$R(t) = R_0 E_{1,\alpha}(Ct^\alpha) \quad (20)$$

where

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha + \beta k)} \quad (21)$$

is so called Mittag-Leffler function [21] and C is a constant depending on G, ρ, c if these later are assumed to be constants with time.

Note that at the origin of time, the universe in this case, have no singularity, that is $R(t) = R_0$ when $t = 0$. In addition, for $t > 0$, the scale factor increase with time, a characteristic that pleases those who favor an accelerated universe, (*of course if we are not in favor with a static universe*) [39].

If $\alpha = 1$, then the solution is given by $R(t) = R_0 e^{Ct}$ corresponding to the standard inflation case [32].

4. Conclusions

In conclusion, we have tried in this work to show the reader the importance of fractional calculus and its potential application in cosmology, in particular the Gentry's **NRI** of a static universe. The static universe provides an interesting candidate to explore whether it could play an important role in the evolution of the universe (*on the stability of Einstein static universe, see [39,40]*). We have employed a new approach to cosmology "*fractional calculus*" that we hope will certainly attract attention from astrophysicists and cosmologists because of the way it challenges the conventional big bang framework. We would like to prove in a future work definitely that our fractional cosmological differential equations agree completely with redshift data to support static universe model. In addition, some important questions arise:

- (1) Does "*homogeneous*" remains a required condition?
- (2) Furthermore, fractional differential equations imply fractality, so doesn't it mean that Einstein's cosmological principle cannot be applicable anymore?
- (3) How the redshift could be measured in terms of gravitational redshift, instead of "*cosmic*" origin?
- (4) How this "*gravitational shift*" arguments could explain "*Hubble constant*" without cosmic expansion, which is indeed the crux of the issue concerning Big Bang [41].
- (5) If we indeed explain that redshift comes from (*Einstein's*) gravitation redshift instead of cosmic expansion, it does not mean to reject cosmic expansion itself automatically. Could it be that both types of redshift (*gravitational and cosmic expansion*) co-exist or neatly linked?
- (6) If solar system expands, then it could be that galaxies are also expanding, albeit not exactly in the same way that Big Bang theory teaches. In this regard, we will explore in the future the possibility of "*expanding solar system*".
- (7) What about inflation, cosmic microwave background radiation (**CMBR**) and its temperature dependence, primordial nucleosynthesis [42,43] and how the present universe is brought?
- (8) Is there any simple variation of the full Einstein field equations with fractional derivatives? Due to the non-locality property of fractional derivatives, one would probably need to replace the Einstein-Hilbert action by some sort of complicated nonlocal expression.
- (9) What about the relation of distance and surface brightness [44] and the time dilation of supernova light curves [45]?

However, further analysis and implications of all these are required in order to make any definite statement.

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