Application of Coadjoint Orbits in the Thermodynamics of Non-Compact Manifolds.

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Abstract: Method of the solution of the main problem of homogeneous spaces thermodynamics for non-compact spaces in the case of non-compact Lie groups is presented in the article. The method is based on the method of coadjoint orbits. The formula that allows efficiently evaluate heat kernel on non-compact spaces is obtained. The method is illustrated by non-trivial example.
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1. Introduction

The main goal of the thermodynamics of non-compact spaces is evaluation of the heat kernel Z_β as a sum over the spectrum of energy operator $H = -\Delta$ (Laplace-Beltrami operator)

$$Z_\beta = \sum_n d_n \exp(-\beta E_n), \tag{1}$$

where d_n is a degeneration of eigenvalue E_n, β - inverse temperature. In case of elliptic operator H on compact space, series (1) is always convergent [1].

Heat kernel (1) can be expressed as a trace of density matrix $\rho_\beta(x, x')$

$$Z_\beta = \int \rho_\beta(x,x) d\mu(x), \quad d\mu(x) = \sqrt{|g|}dx, \tag{2}$$

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which is a solution of Bloch equation

$$\frac{\partial \rho_\beta(x,x')}{\partial \beta} + H(x)\rho_\beta(x,x') = 0, \quad \rho_\beta(x,x')|_{\beta=0} = \delta(x-x')/\sqrt{|g|}. \tag{3}$$

Hence H is invariant operator, that problem is closely connected with the theory of Fourier analysis on non-compact spaces.

Many papers ([2],[3],[4]) were dedicated to the investigation of the properties of Laplace operator and heat equation, especially to high-temperature approximation of density matrix and a lot of important results were obtained there. For instance in case of n-dimensional compact space for Z_β is expresed as series

$$Z_\beta = \frac{Vol_M}{(4\pi\beta)^{n/2}} \sum a_i \beta^i, \tag{4}$$

were coefficients a_i represent spectral invariants, which may be expressed through functions on the manifold, symbol of operator H and it's derivatives.

The fact is that mathematical physics usually considers thermodynamics of compact manifolds or at least non-compact manifolds of finite volume. There is no general algorithm to factorize the volume of the manifold in sum (1) or in integral (2) in order to consider specific (by volume) heat kernel.

That work dealt with thermodynamics of non-compact manifolds (non-compact Lie groups as an example). The method of heat kernel evaluation is suggested here and it's based on the theory of coadjoint orbits. The method allows to factorize divergent volume of manifold so we can easily work with sufficiently finite specific (by volume) heat kernel.

Equation of type (3) are usually to be solved in the framework of the method of separation of variables, which allows to build the basis of solutions. Here we face a drawback connected with the problem of composition of the solution to satisfy stated initial conditions.

We use here method of orbits that efficiently uses the symmetries of the equation and we can remedy the drawback referring initial conditions mentioned above. Also the method allows to find global solutions, avoiding the problem of solution sewing in different coordinate maps even in case when the space cannot be covered with one map. The problem in the framework of the method of K-orbits is driven to the space of orbits with geometry and topology much simpler then the initial space had.

As an example of efficient application of the orbits method to problems of the thermodynamics of non-compact manifolds we thoroughly consider these problems for one non-compact unimodular Lie group. The case of the unimodular Lie groups was chosen as an illustrative example of presented method to avoid non-sufficient technical details which may make the main problem hard to understand.

2. Quantization of transition functions to the canonic coordinates on K-orbits

We introduce below basic constructions necessary for the solution of the problem, stated in the Introduction.

Let G be a real connected n-dimensional Lie group and $\mathcal{G}$ its Lie algebra. Group G acts on coalgebra $\mathcal{G}^*$ with coadjoint representation $\mathrm{Ad}^* : G \times \mathcal{G}^* \to \mathcal{G}^*,\ (g, f) \to \mathrm{Ad}^*_g\ f$ by convention:

$$\langle \mathrm{Ad}^*_g\ f, X\rangle \equiv \langle f, \mathrm{Ad}_{g^{-1}}\ X\rangle; \quad f \in \mathcal{G}^*,\ g \in G,\ X \in \mathcal{G}. \tag{5}$$

Here Ad_g is the linear operator of adjoint representation of group G on Lie algebra $\mathcal{G}$. Coordinates of covector f in the basis dual to Lie algebra $\mathcal{G}$ basis $\{e_i\}$ are denoted as $f_i,\ f = f_i e^i,\ \langle e^i, e_j\rangle = \delta^i_j$. Hence the formula (5) in the coordinates looks as $(\mathrm{Ad}^*_g f)_i = (\mathrm{Ad}_{g^{-1}})^j_i f_j$.

The linear degenerate Poisson bracket is defined on the dual space $\mathcal{G}^*$:

$$\{\varphi, \psi\}(f) \equiv \langle f, [\nabla\varphi(f), \nabla\psi(f)]\rangle; \quad \varphi, \psi \in C^\infty(\mathcal{G}^*), \tag{6}$$

where $\nabla\varphi(f) \equiv e_i \partial\varphi(f)/\partial f_i$.

Let's denote as ω_λ a symplectic 2-form (Kirillov form) on an orbit $\mathcal{O}_\lambda$ which acts on vectors tangent to orbit as follows:

$$\omega_\lambda(a, b) = \langle \lambda, [\alpha, \beta]\rangle, \quad a = ad^*_\alpha\lambda \in T_\lambda\mathcal{O}_\lambda,\ b = ad^*_\beta\lambda \in T_\lambda\mathcal{O}_\lambda,\ \alpha, \beta \in \mathcal{G}. \tag{7}$$

It's obvious that the Poisson bracket determined by Kirillov form ω_λ on an orbit $\mathcal{O}_\lambda$ coincides with the restriction of the Poison bracket (6) to that orbit.

According to the Darboux theorem the local canonical coordinates (the Darboux coordinates) exist on an orbit $\mathcal{O}_\lambda$ in which the form ω_λ looks like

$$\omega_\lambda = \sum_{a=1}^{\frac{1}{2}\dim\mathcal{O}_\lambda} dp_a \wedge dq^a.$$

It's easy to see that the transition to the canonical Darboux coordinates $(f_i) \to (p_a, q^a)$ means to find a set of analytic functions of the variables (p, q): $f_i = f_i(q, p, \lambda)$ that satisfy following conditions

$$f_i(0, 0, \lambda) = \lambda_i; \tag{8}$$

$$\frac{\partial f_i(q,p,\lambda)}{\partial p_a}\frac{\partial f_j(q,p,\lambda)}{\partial q^a} - \frac{\partial f_j(q,p,\lambda)}{\partial p_a}\frac{\partial f_i(q,p,\lambda)}{\partial q^a} = C^k_{ij} f_k(q, p, \lambda); \tag{9}$$

$$K_\mu(f(q, p, \lambda)) = K_\mu(\lambda); \quad \mu = 1, \ldots, \operatorname{codim}\mathcal{O}_\lambda. \tag{10}$$

(Here $K_\mu(f)$ are the Casimir function for orbit $\mathcal{O}_\lambda$, i.e. such functions from the space $C^\infty(\mathcal{G}^*)$ that $\{\varphi(f), K_\mu(f)\}|_{\mathcal{O}_\lambda} = 0,\ \forall\varphi \in C^\infty(\mathcal{G}^*)$).

Let's require the transition to the canonical coordinates (in other words, the qp–transition) to be linear on p_a, i.e.

$$f_i(q,p,\lambda) = \alpha_i^a(q)p_a + \chi_i(q,\lambda); \quad \operatorname{rank}\alpha_i^a(q) = \frac{1}{2}\dim\mathcal{O}_\lambda. \tag{11}$$

In general, the linear transition (11) does not exist but if we assume that $\alpha_i^a(q)$ and $\chi_i(q;\lambda)$ are the holomorphic functions of the complex variables q then the class of a K-orbits and a Lie algebras which might allow the linear transition becomes much wider. (We assume that functionals from $\mathcal{G}^*$ are extended linearly on $\mathcal{G}^c$). The domain of definition Q of variables q and the domain of definition P of variables p is determined from the requirement for the function $f_i(q,p,\lambda)$ to be real-valued.

The following theorem is proven in [5].

Theorem 1. *Linear transition to the canonical coordinates on orbit $\mathcal{O}_\lambda$ exists only when for a linear functional λ the normal polarization does exist, i.e. the subalgebra $\mathcal{H} \subset \mathcal{G}^c$ such as*

$$\dim\mathcal{H} = n - \frac{1}{2}\dim\mathcal{O}_\lambda, \quad \langle\lambda, [\mathcal{H},\mathcal{H}]\rangle = 0, \quad \lambda + \mathcal{H}^\perp \subset \mathcal{O}_\lambda.$$

The linear transition (11) is useful in many aspects. Firstly, the procedure of the Poisson bracket quantization is easily obtained because in this case the problem of regularization of operators does not arise (see below). Secondly, quadratic hamiltonians maps (when the transition (11) applied) into also quadratic hamiltonians in respect to impulses and after the quantization we get second order equations. The linear qp-transition exists in the overwhelming majority of cases because it's known (see, e.g. [6]) that for an arbitrary Lie algebra and for any non-degenerate covector a solvable polarization does exist.

We introduce notation j_μ ($\mu = 1,\ldots,\operatorname{ind}(\mathcal{G})$) for local coordinates in the space of orbits $\mathcal{G}^*/G$. We fix representative of an orbit $\lambda = \lambda(j)$, assuming that λ depends on j linearly. Further we will consider covector λ to be non-degenerate, i.e. $\dim\mathcal{O}_\lambda$ to have maximal value.

Let's define the concept of the K-orbit quantization, see [5]. The quantization must be done separately for each type of orbits. It consists in the correspondence to a type of orbits the special infinite-dimensional Lie algebra representation, so called (λ-*representation*). The condition of *integrality* is imposed on the orbits. We remind [7], that an orbit called integer-valued if the Kirillov form integral taken over any 2-dimensional cycle is integer-valued.

Let's consider the transition functions $f_i(q,p;\lambda)$ as the symbols of operators. They are defined as follows, variables p_a are substituted with the differential operators $p_a \to \hat{p}_a \equiv -i\hbar\partial_{q^a}$. Then the covector's coordinates f_i transform into the linear operators $f_i(q,p;\lambda) \to \hat{f}_i = f_i(q,-i\hbar\partial_q;\lambda + i\hbar\beta))$ (here $\hbar$ is a real positive parameter). Under the quantization an arbitrary function $\varphi(f) \in C^\infty(\mathcal{G}^*)$ is assigned with a symmetrized operator function $\varphi(\hat{f})$ of the operators $\hat{f}_i$. The real vector β is determined from the real-value requirement for the functions:

$$\kappa_\mu(\lambda) = K_\mu(\hat{f}). \tag{12}$$

The procedure of quantization is ambiguous. This ambiguity vanishes if we require the operators $\hat{f}_i$ to satisfy the following commutation rules

$$\frac{i}{\hbar}[\hat{f}_i, \hat{f}_j] = C^k_{ij}\hat{f}_k. \tag{13}$$

It's obvious that if the transition to the canonical coordinates is linear, i.e. for this type of an orbit exists a normal polarization, then

$$\hat{f}_i = -i\hbar\alpha^a_i(q)\frac{\partial}{\partial q^a} + \chi_i(q, \lambda + i\hbar\beta)). \tag{14}$$

3. Quantum equations on K–orbits and heat kernel

We define here special infinite-dimensional irreducible representation of Lie algebra $\mathcal{G}$ which we shall call λ-representation. Operators of λ-representation $l_i(q, \partial_q, \lambda)$ are constructed as follows from the quantized canonic transition functions

$$l_i(q, \partial_q, \lambda) = \frac{i}{\hbar}\hat{f}_i(q, \partial_q, \lambda). \tag{15}$$

Let λ be a non-degenerate covector that belongs to an integer-valued orbit. Here following conventions are assumed, the group G is unimodular; $l_i(q, \partial_q, \lambda)$ is a λ-representation of Lie algebra $\mathcal{G}$; dg is an invariant measure on Lie group G; $d\mu(q)$ is the measure on Q such that operators $l_i(q, \partial_q, \lambda)$ are skew-symmetric with respect to the scalar product in $L_2(Q, d\mu(q))$. Note that the left-invariant vector fields (ξ_i) and the right-invariant vector fields (η_i) on G also are skew-symmetric with respect to the scalar product in $L_2(G, dg)$.

We introduce generalized function $D^\lambda_{qq'}(g)$ as the solution of the overdetermined system

$$[\xi_i(g) + l_i(q, \partial_q, \lambda)]D^\lambda_{qq'}(g) = 0; \ [\eta_i(g) + \bar{l}_i(q', \partial'_q, \lambda)]D^\lambda_{qq'}(g) = 0. \tag{16}$$

(since $D^\lambda_{qq'}(g)$ are the generalized functions then all of the equalities that include them (e.g. equation (16)) must be considered in the weak sense).The system (16) is consistent but global solutions exist only for integer-valued orbits. By definition functions $D^\lambda_{qq'}(g)$ appear to be the eigenfunctions of the Casimir operators and they are determined by the system (16) with an accuracy to a constant factor. This arbitrariness is eliminated by introduction of the normalization requirement $D^\lambda_{qq'}(e) = \delta(q, q')$, where $\delta(q, q')$ is the Delta-function with respect to the measure $d\mu(q)$. Note that the functions $D^\lambda_{qq'}(g)$ are to be found in quadratures because the system (16) is overdetermined.

Functions $D^\lambda_{qq'}(g)$ on unimodular Lie groups have following properties

$$\begin{aligned}
&\int D^j_{qq''}(x)D^j_{q''q'}(x')d\mu(q'') = D^j_{qq'}(x'x), \\
&D^j_{qq'}(x) = \overline{D}^j_{q'q}(x^{-1}), \\
&D^j_{qq'}(e) = \delta(q, q').
\end{aligned} \tag{17}$$

The set of functions $D^{\lambda}_{qq'}(g)$ is full and orthogonal [5] and since that satisfies following equations

$$\int\limits_{G} \overline{D}^{\tilde{j}}_{\tilde{q}\tilde{q}'}(g) D^{j}_{qq'}(g)\, dg = \delta(q,\tilde{q})\delta(q',\tilde{q}')\delta(j,\tilde{j}); \tag{18}$$

$$\int\limits_{Q\times Q\times J} \overline{D}^{j}_{qq'}(\tilde{g}) D^{j}_{qq'}(g)\, d\mu(j) d\mu(q) d\mu(q') = \delta(g,\tilde{g}). \tag{19}$$

Because of that for any function $\varphi(g)$ from the dense kernel subspace in $L_2(G, dg)$ the generalized Fourier transformation and inversion are defined:

$$\hat{\varphi}_{\lambda}(q,q') = \int \varphi(g) D^{\lambda}_{qq'}(g)\, dg \tag{20}$$

$$\varphi(g) = \int \hat{\varphi}_{\lambda}(q,q') \overline{D^{\lambda}_{qq'}(g)}\, d\mu(q) d\mu(q') d\mu(\lambda). \tag{21}$$

Here $d\mu(\lambda)$ is the spectral measure of Casimir operators $K_{\mu}(i\hbar\xi)$ ($\mu = 1, \ldots, r$).

In the conventional method of harmonic analysis the functions that perform Fourier transformation are the eigenfunctions for a full n-dimensional set of commuting operators from an enveloping algebras of right- and left-invariant fields $U(\mathcal{G}^L), U(\mathcal{G}^R)$ [8]. The suggested approach has the significant advantage: if the functions $\varphi(g)$ and $\hat{\varphi}_{\lambda}(q,q')$ are connected by the transformations (20), (21) then the same transformation connects the action of the following operators

$$\xi_i(g)\varphi(g) \iff l_i(q, \partial_q, \lambda)\hat{\varphi}_{\lambda}(q,q'), \quad \eta_i(g)\varphi(g) \iff \bar{l}_i(q', \partial_{q'}, \lambda)\hat{\varphi}_{\lambda}(q,q') \tag{22}$$

Let's put in correspondence to the function $H(f)$ on the coalgebra the symmetric function $H(i\hbar\eta)$ of operators $i\hbar\eta_i$ and consider the following differential equation in $L_2(G, dg)$:

$$H(i\hbar\eta)\varphi(g) = E\varphi(g). \tag{23}$$

The equation (23) is called *quantum equation on the Lie group G* [9].

Using transformation (21) and formula (22) we get the equivalent equation for the function $\hat{\varphi}_{\lambda}(q,q')$

$$H(\hat{f}(q,\lambda))\hat{\varphi}_{\lambda}(q,q') = E\hat{\varphi}_{\lambda}(q,q'). \tag{24}$$

Let's consider operator $H(x)$

$$H(x) = G^{ab}(i\hbar\eta_a)(i\hbar\eta_b), \tag{25}$$

where G^{ab} is arbitrary constant symmetric non-degenerate matrix of positively defined quadratic form, η_a — right-invariant vector fields on Lie group. It's easy to see that in this case operator H is Laplace operator on Lie group with rightinvariant metric

$$g^{ij}(x) = G^{ab}\eta^i_a(x)\eta^j_b(x). \tag{26}$$

Obviously Riemannian measure un Lie group $\sqrt{g}dx$ coincides (to the constant factor) with bi-invariant measure on Lie group.

Since set $D^{\lambda}_{qq'}(g)$ is full, the solution of equation (3) may be expressed as follows

$$\rho_\beta(x,x') = \int \tilde{\rho}_\beta(q,q',j,\tilde{q},\tilde{q}',\tilde{j}) D^{j}_{qq'}(x)\overline{D^{\tilde{j}}_{\tilde{q}\tilde{q}'}}(x') d\mu(q)d\mu(q')d\mu(j)d\mu(\tilde{q})d\mu(\tilde{q}')d\mu(\tilde{j}), \quad (27)$$

where function $\tilde{\rho}_\beta(q,q',j,\tilde{q},\tilde{q}',\tilde{j})$ satisfies equation

$$\begin{aligned} &\frac{\partial \tilde{\rho}_\beta(q,q',j,\tilde{q},\tilde{q}',\tilde{j})}{\partial \beta} + H(\hat{f})\tilde{\rho}_\beta(q,q',j,\tilde{q},\tilde{q}',\tilde{j}) = 0, \\ &\tilde{\rho}_\beta(q,q',j,\tilde{q},\tilde{q}',\tilde{j})|_{\beta=0} = \delta(q,\tilde{q})\delta(q',\tilde{q}')\delta(j,\tilde{j}). \end{aligned} \quad (28)$$

Although $H(f)$ in (24) depends only on q, solution of (28) may me written down as

$$\tilde{\rho}_\beta(q,q',j,\tilde{q},\tilde{q}',\tilde{j}) = \mathcal{R}_\beta(q,\tilde{q},j)\delta(q',\tilde{q}')\delta(j,\tilde{j}) \quad (29)$$

Function $\mathcal{R}_\beta(q,\tilde{q},j)$ also satisfies equation

$$\frac{\partial \mathcal{R}_\beta(q,\tilde{q},j)}{\partial \beta} + H(\hat{f})\mathcal{R}_\beta(q,\tilde{q},j) = 0, \quad \tilde{\mathcal{R}}_\beta(q,\tilde{q},j)|_{\beta=0} = \delta(q,\tilde{q}). \quad (30)$$

Let's substitute (30) in (28) and integrate over $\tilde{q}$ and $\tilde{j}$ taking into account special properties (28) of functions $D^{j}_{qq'}(x)$ for unimodular Lie groups, we will get final expression for density matrix on unimodular Lie group:

$$\rho_\beta(x,x') = \int \mathcal{R}_\beta(q,\tilde{q},j) D^{j}_{q\tilde{q}}(x'^{-1}x) d\mu(q)d\mu(\tilde{q})d\mu(j). \quad (31)$$

One might see, that diagonal elements of density matrix have no dependence on group coordinates x. Heat kernel on Lie group as a trace of density matrix (31) is defined as follows

$$\begin{aligned} Z_\beta &= \int \rho_\beta(x,x)d\mu(x) = \int d\mu(x) \int \tilde{\rho}_\beta(q,q,j)d\mu(q)d\mu(j) = \\ &= Vol_G \int \tilde{\rho}_\beta(q,q,j)d\mu(q)d\mu(j), \end{aligned} \quad (32)$$

where integration is performed over entire volume of the group.

One can easily see that specific (by volume) heat kernel (statistic sum) $z_\beta \equiv Z_\beta/Vol_G$ is sufficiently finite because operator H is elliptic.

As an example we will consider the problem of heat kernel on one non-compact Lie group.

4. Example

As an example of the method we will consider 3-dimensional Lie group $E(2)$. Lie algebra of that group is determined by following commutation rules:

$$[e_1,e_2] = 0, \quad [e_2,e_3] = \varepsilon e_1, \quad [e_1,e_3] = -\varepsilon e_2.$$

Lie group $E(2)$ is popular subject of study in theoretical physics and mathematics but usually parameter ε is taken equal 1. That case corresponds with Lie algebra of group of motions of two-dimensional euclidean plane. Other well known value of ε is 0 which represents three dimensional abelian Lie group. We take in consideration arbitrary parameter ε to be able to perform contraction $\varepsilon \to 0$.

Matrix $G^{ab} = \mathrm{diag}(A, B, C)$ and in given case we assume $\det G^{ab} > 0$.

Left-invariant and right-invariant vector fields on that group are

$$\xi_1 = \cos(\varepsilon x_3)\partial_1 + \sin(\varepsilon x_3)\partial_2, \quad \xi_2 = -\sin(\varepsilon x_3)\partial_1 + \cos(\varepsilon x_3)\partial_2, \quad \xi_3 = \partial_3,$$

$$\eta_1 = -\partial_1, \quad \eta_2 = -\partial_2, \quad \eta_3 = \varepsilon x_2\partial_1 - \varepsilon x_1\partial_2 - \partial_3.$$

According (25) we get operator $H(i\hbar\eta)$

$$H = -\hbar^2(A\eta_1^2 + B\eta_2^2 + C\eta_3^2).$$

Canonic transition functions are:

$$f_1 = -j\cos(\varepsilon q), \quad f_2 = j\sin(\varepsilon q), \quad f_3 = p.$$

Functions on Lagrange submanifold Q to the symplectic sheet, i.e functions of variable q here here are functions on S^1 which are $2\pi/\varepsilon$ periodic.

Set of functions $D^j_{qq'}$ we obtain as a solution of (16) and it satisfies (17), (18) and (19), looking as follows

$$D^j_{qq'} = \exp[\frac{ij}{\hbar}(-x_1\cos(\varepsilon q) + x_2\sin(\varepsilon q))]\delta(x_3 + q - q')$$

and we choose measure $d\mu(j) = \varepsilon j dj/(2\pi\hbar)^2$.

The case we are interested in is bolzmannian gas (gas of non-interacting particles of mass m). Because of that we have following restriction put on matrix G: $A = B = C = 1/2m$ and thus $H(\hat{f})$ is a hamiltonian of a free particle.

Operator $H(\hat{f})$, that is built according (24) from quantized canonic transition functions and enters (30):

$$H(\hat{f}) = \frac{1}{2m}(\hat{f}_1^2 + \hat{f}_2^2 + \hat{f}_3^2). \tag{33}$$

Since $\tilde{\mathcal{R}}_\beta(q, \tilde{q}, j)$ on Lagrange submanifold Q to the orbit $\mathcal{O}_\lambda$ must be $2\pi/\varepsilon$ periodic we must build a solution of the equation (24) as a series using Fourier transformation over variable q.

So we get density matrix $\mathcal{R}_\beta(q, \tilde{q}, j)$ on Lagrange submanifold Q to the orbit $\mathcal{O}_\lambda$, as a solution of (24) with operator (33):

$$\tilde{\mathcal{R}}_\beta(q, \tilde{q}, j) = \frac{\varepsilon}{2\pi}\sum_{n=-\infty}^{\infty}\exp(-\frac{1}{2m}((n\hbar\varepsilon)^2 + j^2)\beta - in\varepsilon(q - \tilde{q})) =$$
$$= \frac{\varepsilon}{2\pi}\exp(-\frac{j^2\beta}{2m})\theta_3(\frac{(q-\tilde{q})\varepsilon}{2}, \exp(-\frac{\beta\hbar^2\varepsilon^2}{2m})).$$

Specific statistic sum obtained from formula (32) is

$$z = (\frac{m}{2\pi\beta\hbar^2})^{3/2}\theta_3(0, \exp(-\frac{2m\pi^2}{\hbar^2\varepsilon^2\beta})) \tag{34}$$

which can be written down for more convenient consideration of thermodynamic properties as follows (θ_3 is θ_3 Jacobi function)

$$z = \left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2} \theta_3(0, \exp(-\frac{T}{T_0(\varepsilon)})), \tag{35}$$

where k_B — Bolzmann constant and $T_0(\varepsilon) = \hbar^2\varepsilon^2/2mk_B\pi^2)$.

Mean energy of the particle in that space is

$$u = k_BT^2\frac{\partial \ln z}{\partial T}$$

and we can get specific heat (for one particle) in given space.

Asymptotic transition (contraction) $\varepsilon \to 0$ corresponds to the specific heat for the bolzmannian gas of free particles in flat space R^3.

Here we give the graph for the specific heat c_v of bolzmannian gas (for one particle) depending on $t = T/T_0(\varepsilon)$.

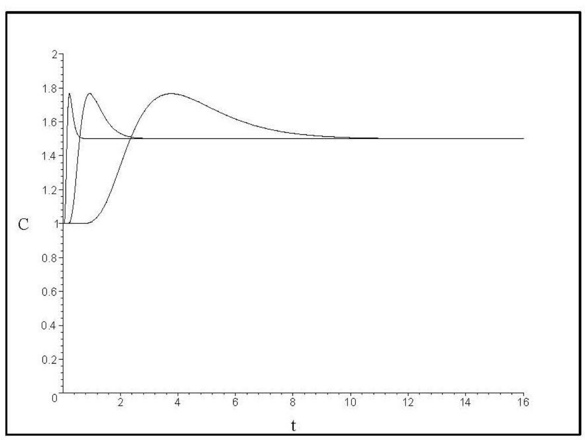

Fig. 1

On the graph one can see that lower temperatures cause elimination of one degree of freedom but at higher temperatures particles behave just like in three dimensional flat space. That abnormal behavior of the specific heat can be explained by the fact that particle "feels" topology of the space. Indeed the group space of $E(2)$ is a cylinder with non-trivial topology which strongly influences thermodynamical properties of bolzmannian gas at low temperatures.

The principal result of the paper is formula (32) that solves the main problem of thermodynamics of homogeneous spaces for non-compact unimodular Lie groups.

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