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Relativistic Klein-Gordan Equation with Position Dependent Mass for q-deformed modified Eckart plus Hylleraas potential

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Abstract: Relativistic Klein-Gordan equation with Position Dependent Mass has been solved analytically for the q-deformed modified Eckart plus Hylleraas potential. A generalised series is used to obtain the bound state solutions of the K-G equation using the Frobenious Method . The one dimensional K-G equation for the mass dependent modified Eckart plus Hylleraas potential in absence of scalar potential are studied in this paper. The exactly normalized bound state wave function and energy expressions are obtained by using N-U method. Also, the bound state solutions are found for the Hulthén and Rosen-Morse potential.

Keywords: Klein-Gordan equation; q-deformed modified Eckart plus Hylleraas potential; position dependent mass; N-U method; Frobenious Method
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1 Introduction

Quantum Mechanical phenomena are described by Schrödinger equation which dictates the dynamics of quantum systems represented by Hamiltonian Operator. Solutions of Klein-Gordan Equation for some physical potential have important applications in Molecular Physics, Quantum Chemistry, Nuclear physics, condensed matter Physics, high energy physics. The study of potentials such as Hulthén [1], Morse [2], Rosen-Morse [3], Pseudo-harmonic [4], Poschl-Teller [5, 6], Kratzer-Fuez [7], generalized Wood Saxon [8], ring-shaped Hartmann [9] and the corresponding wave functions has been performed using various methods.

Recently, there has been renewed interest in solving Quantum Mechanical systems within the frame work of Nikiforov-Uvarov method[10-14].This technique is successfully

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used to solve Schrödinger, Klein-Gordan, Dirac and Duffin-Kemmer-Petieu Equqtions.

In nuclear physics, the shape form of the potential also plays an important role particularly when studying the structure of deformed nuclei or the interaction between them. Therefore , our aim , in the present work is to investigate analytical bound state solutions of the Klein-Gordon equation with q-deformed modified Eckart plus Hylleraas potential[15-19] in the Frobenius method [20] as well as in N-U method. Also, we will show that, when the deformation parameter q takes a particular value ($q = 1$), the obtained results lead to the solutions of the same problem for modified Eckart plus Hylleraas potential.

In recent years , the solutions of the non-relativistic wave equation with position-dependent mass have been a topic of great interest[21-25] , but there are only few papers that give the solution of the relativistic wave equation with position-dependent mass in quantum mechanics. Exact solution of the Dirac equation with position-dependent mass in the Coulomb field [26], Kepler problem in Dirac theory for a particle whose potential and mass are inversely proportional to the distance from the force center [27], the approximate solution of the one-dimensional Dirac equations with spatially dependent mass for the generalized Hulthen potential [28], the exact solution of the one-dimensional K-G equation with spatially dependent mass for the inversely linear potential [29] are some papers on relativistic wave equations with position dependent mass.

Our focus is to study the quantum systems with Position Dependent Effective Mass (PDEM). PDEM Klein-Gordan Equation plays an important role in the study of electronic properties of semi-conductors in homogeneous crystals,quantum dots,He clusters, quantum liquids etc. Exact solutions of effective mass Klein-Gordan Equations are difficult to obtain, as such, approximate numerical techniques are often used. Our work is generalised as follows:- In section 2 we have discussed the NU method and Frobenious method . We give a brief discussion of Klein-Gordan Equation with position-dependent mass in section 3. In section 4 we discuss the solutions of Klein-Gordan Equation by using both the methods and section 5 is left for conclusion.

2. Overview of Nikiforov-Uvarov and Frobenius Method Method

A. Overview of Nikiforov-Uvarov Method

The N-U method is based on solving a second order linear differential equation by reducing it to a generalized hypergeometric type. In both relativistic and non-relativistic quantum mechanics, the wave equation with a given potential can be solved by this method by reducing the one dimensional K-G equation to an equation of the form :

$$\Psi''(x) + \frac{\tilde{\tau}(x)}{\sigma(x)}\Psi'(x) + \frac{\tilde{\sigma}(x)}{\sigma^2(x)}\Psi(x) = 0 \tag{1}$$

Where $\sigma(x)$ and $\tilde{\sigma}(x)$ are polynomials of degree atmost 2 and $\tilde{\tau}(x)$ is a polynomial of degree atmost 1 . In order to find a particular solution to equation(1) , we set the following wave function as a multiple of two independent parts

$$\Psi(x) = \Phi(x)y(x) \tag{2}$$

Thus equation (1) reduces to a hyper-geometric type equation of the form :

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda y(x) = 0$$

Where $\tau(x) = \tilde{\tau}(x) + 2\pi(x)$ satisfies the condition $\tau'(x) < 0$ and $\pi(x)$ is defined as

$$\pi(x) = \frac{\sigma'(x) - \tilde{\tau}(x)}{2} \pm \sqrt{(\frac{\sigma'(x) - \tilde{\tau}(x)}{2})^2 - \tilde{\sigma}(x) + K\sigma(x)} \quad (3)$$

in which K is a parameter . Determining K is the essential point in calculation of $\pi(x)$. Since $\pi(x)$ has to be a polynomial of degree at most one, the expression under the square root sign in Eq. (3) can be put into order to be the square of a polynomial of first degree [10], which is possible only if its discriminant is zero. So, we obtain K by setting the discriminant of the square root equal to zero . Therefore, one gets a general quadratic equation for K . By using

$$\lambda = K + \pi'(x) = -n\tau'(x) - \frac{n(n-1)}{2}\sigma''(x) \quad (4)$$

The values of K can be used for the calculation of energy eigenvalues . Polynomial solutions $y_n(x)$ are given by the Rodrigues relation

$$y_n(x) = \frac{B_n}{\rho(x)}(\frac{d}{dx})^n[\sigma^n(x)\rho(x)] \quad (5)$$

in which B_n is a normalization constant and $\rho(x)$ is the weight function satisfying

$$\rho(x) = \frac{1}{\sigma(x)} exp \int \frac{\tau(x)}{\sigma(x)} dx \quad (6)$$

on the other hand , second part of the wave function $\phi(x)$ in relation (2) is given by

$$\phi(x) = exp \int \frac{\pi(x)}{\sigma(x)} dx \quad (7)$$

B.Overview of Frobenius Method

This method finds the solutions of a differential equation in the form of series,either a whole series,a Laurent series, or even a series involving contribute exhibitors. The difference between these situations is the properties of regularity of the equation coefficients. To do this you must put the equation in the form:

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0 \quad (8)$$

Suppose a regular singular point x_0,singular functions P(x) and Q(x) and using the Fuck's theorem, we can write the solutions of the differential equation in the form:

$$y(x) = \sum_0^\infty a_k(x - x_0)^{k+r} \quad (9)$$

The indicial equation is obtained for

$$r(r-1)+P(0)r+Q(0)=0 \tag{10}$$

For each found values r, we determine the values a_k and then the solutions of the differential equation.

3. Brief discussion of Klein Gordan Equation with position dependent mass

The one dimensional K-G equation for a spinless particle of mass m in the natural units $\hbar = c = 1$ can be expressed

$$\Psi''(x)+[(E-V(x))^2-(m+S(x))^2]\Psi(x)=0 \tag{11}$$

where E , V(x) and S(x) are the relativistic energy of the particle , vector and scalar potentials respectively. Now considering the q-deformed modified Eckart plus Hylleraas Potential of the form:

$$V(x)=\frac{V_0}{b}\left(\frac{a-e^{-2\alpha x}}{1-qe^{-2\alpha x}}\right)-V_1\frac{e^{-2\alpha x}}{1-qe^{-2\alpha x}}+V_2\frac{e^{-2\alpha x}}{(1-qe^{-2\alpha x})^2} \tag{12}$$

Where q is the shape parameter.

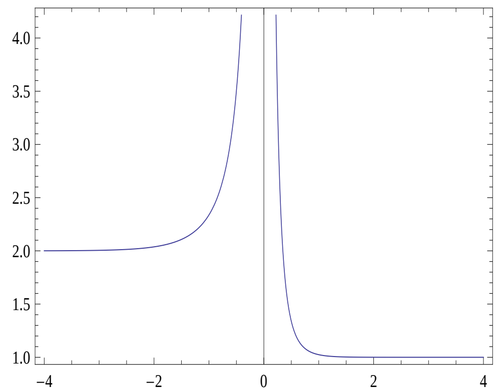

Fig.1.The modified Eckart plus Hylleraas Potential with unit value of α, a, b, q.

We prefer to use the mass function equals to the rest mass along with the vector part of the potential as

$$m(x)=m_0+\frac{V_0}{b}\left(\frac{a-e^{-2\alpha x}}{1-qe^{-2\alpha x}}\right)-V_1\frac{e^{-2\alpha x}}{1-qe^{-2\alpha x}}+V_2\frac{e^{-2\alpha x}}{(1-qe^{-2\alpha x})^2} \tag{13}$$

to obtain an exactly solvable Schrödinger-like equation in absence of scalar potential . The mass function should also be a physical distribution , so we restrict ourself in the range $0 \leq x < \infty$, which gives the finite mass values as follows :

$$m(x) = \begin{cases} m_0 + \frac{V_0}{b}(a-1) - V_1 + V_2 \ (for q \to 0) \ , x \to 0 \\ m_0 + \frac{V_0 a}{b} \qquad , x \to \infty \end{cases}$$

Actually, this distribution corresponds to shifted scalar potential function in the problem. Substituting equation (13) in equation (11) we have

$$\Psi''(x) + \left[(E^2 - m_0^2) - 2(E + m_0) \left\{ \frac{V_0}{b} \left(\frac{a - e^{-2\alpha x}}{1 - qe^{-2\alpha x}} \right) - V_1 \frac{e^{-2\alpha x}}{1 - qe^{-2\alpha x}} + V_2 \frac{e^{-2\alpha x}}{(1 - qe^{-2\alpha x})^2} \right\} \right] \Psi(x) = 0 \qquad (14)$$

4A.Application of Nikiforov-Uvarov Method

Introducing a new variable $s = e^{-2\alpha x}$ it is straight forward to show that (14) takes the form:

$$\Psi''(s) + \frac{1 - qs}{s(1 - qs)} \Psi'(s) + \frac{1}{s^2(1 - qs)^2} \left[s^2 q^2 (\epsilon^2 - \gamma^2 - \zeta^2) + 2qs(\gamma^2 - \epsilon^2) + (\epsilon^2 - \omega^2) \right] \Psi(s) = 0 \qquad (15)$$

Where we use the notations $\frac{E^2 - m_0^2}{4\alpha^2} = \epsilon^2$,$\gamma^2 = \frac{E + m_0}{4\alpha^2 q} \{2V_1 + 2\frac{V_0}{b}(aq + 1) - 2V_2\}$, $\zeta^2 = \frac{E + m_0}{4\alpha^2 q} \{(V_1 + V_2) - \frac{V_0}{b}(aq - 1)\}$ and $2\frac{V_0}{b}\frac{E + m_0}{4\alpha^2} = \omega^2$ comparing equation (15) with equation (1) we have

$$\begin{aligned} \tilde{\tau}(s) &= 1 - qs; \\ \sigma(s) &= s(1 - qs); \\ \tilde{\sigma}(s) &= s^2 q^2 (\epsilon^2 - \gamma^2 - \zeta^2) + 2qs(\gamma^2 - \epsilon^2) + (\epsilon^2 - \omega^2); \end{aligned} \qquad (16)$$

Substituting equation (16) the relation (3) we get

$$\pi(s) = -\frac{qs}{2} \pm \sqrt{q^2 s^2 (\frac{1}{4} + \gamma^2 + \zeta^2 - \epsilon^2 - k_1) + qs(k_1 - 2\gamma^2 + 2\epsilon^2) + (\omega^2 - \epsilon^2)} \qquad (17)$$

where k_1 satisfies the relation $k = k_1 q$ Further the discriminant of the upper expression under the square root has to be set equal to zero. Therefore, we obtain

$$\Delta = q^2 (k_1 + 2\epsilon^2 - 2\gamma^2)^2 - 4q^2 (\frac{1}{4} + \gamma^2 + \zeta^2 - \epsilon^2 - k_1)(\omega^2 - \epsilon^2) \qquad (18)$$

Solving equation (18) for constant k_1 , we obtain the double roots as , $k_1\prime, k_1\prime\prime = 2(\gamma^2 - \omega^2) \pm 2\xi\eta$, where $\xi^2 = \omega^2 - \epsilon^2$ and $\eta^2 = (\frac{1}{4} + \zeta^2 + \omega^2 - \gamma^2)$.

Thus substituting these values for each k_1 into equation (17) , we obtain

$$\pi(s) = -\frac{qs}{2} \pm \begin{cases} (\xi-\eta)qs-\xi;\ for\ \ k_1\prime = 2(\gamma^2-\omega^2)+2\xi\eta \\ (\xi+\eta)qs-\xi;\ for\ k_1\prime\prime = 2(\gamma^2-\omega^2)-2\xi\eta \end{cases} \tag{19}$$

By choosing an appropriate value for k in $\pi(s)$ which satisfies the condition $\tau'(s) < 0$, one gets $\pi(s) = -qs(\xi+\eta+\frac{1}{2})+\xi$ for $k = 2(\gamma^2-\omega^2)-2\xi\eta$; giving the function:

$$\tau(s) = 1-2qs[1+(\xi+\eta)]+2\xi \tag{20}$$

If we consider $\lambda = k+\Pi'$ defined in (4) we obtain

$$\lambda = q[2(\gamma^2-\omega^2)-2\xi\eta-\frac{1}{2}-(\xi+\eta)] \tag{21}$$

Again using equation (4) , we have:

$$\lambda_n = q[n^2+n+2n(\xi+\eta)] \tag{22}$$

Using the condition $\lambda = \lambda_n$ one obtains the eigen values of ϵ from the following equation:

$$\omega^2-\epsilon^2 = \left[\frac{8(\gamma^2-\omega^2)-(2n+1)^2-1-2\eta(2n+1)}{4(2n+1)+2\eta}\right]^2 \tag{23}$$

From (6) it can be shown that the weight function $\rho(s)$ is $\rho(s) = s^{2\xi}(1-qs)^{2\eta}$ and by substituting $\rho(s)$ into the Rodrigues relation (5) one gets

$$y_n(s) = \frac{B_n}{s^{2\xi}(1-qs)^{2\eta}}(\frac{d}{ds})^n[s^n(1-qs)^n s^{2\xi}(1-s)^{2\eta}] = \frac{B_n}{s^{2\xi}(1-qs)^{2\eta}}P_n^{(2\xi,2\eta)}(s) \tag{24}$$

where $P_n^{(2\xi,2\eta)}(s)$ stands for Jacobi polynomial [30] and B_n is the normalizing constant. The other part of the wave function is simply found from (7) as ,

$$\phi(s) = s^{\xi}(1-qs)^{(\frac{1}{2}+\eta)} \tag{25}$$

Finally , the wave function is obtained as follows

$$\psi(s) = B_n s^{-\xi}(1-qs)^{(-\eta+\frac{1}{2})}P_n^{(2\xi,2\eta)}(s) \tag{26}$$

4B.Application of Frobenius Method

consider the same Klein-Gordan equation and the same Eckart plus modified Hylleraas Potential given in section 3. After development , we get the following equation:

$$\psi''(s)+\frac{1}{s}\psi'(s)+\frac{1}{s^2}\left[\epsilon^2-2\beta^2\left\{\frac{V_0}{b}\frac{a-s}{1-qs}-V_1\frac{s}{1-qs}+V_2\frac{s}{(1-qs)^2}\right\}\right]\Psi(s) = 0 \tag{27}$$

where we use the notations $\frac{E^2-m_0^2}{4\alpha^2}=\epsilon^2$ and $\frac{E+m_0}{4\alpha^2}=\beta^2$
Comparing (27) with the equation (8) we have, $P(s)=1$ and
$Q(s)=\left[\epsilon^2-2\beta^2\left\{-V_1\frac{s}{1-qs}+\frac{V_0}{b}\frac{a-s}{1-qs}+V_2\frac{s}{(1-qs)^2}\right\}\right]$
Putting these values the equation (27) becomes ,

$$\psi''(s)+\frac{P(s)}{s}\psi'(s)+\frac{Q(s)}{s^2}\psi(s)=0 \tag{28}$$

By using Fuck's theorem , we can write :

$$\psi(s)=\sum_{k=0}^{\infty}a_k s^{k+r}, \quad with \quad a_0\neq 0 \tag{29}$$

Differentiation gives us:

$$\psi''(s)=\sum_{k=0}^{\infty}(k+r-1)(k+r)a_k s^{k+r-2} \quad and \quad \psi'(s)=\sum_{k=0}^{\infty}(k+r)a_k s^{k+r-1} \tag{30}$$

Putting equation (30) in equation (28) one obtains:

$$\sum_{k=0}^{\infty}a_k s^k\{[(k+r)^2+\epsilon^2-2\frac{V_0}{b}a\beta^2]+s^2[q^2\{(k+r)^2+\epsilon^2\}-2qV_1\beta^2-2q\frac{V_0}{b}\beta^2]$$
$$+s[-2q(k+r)^2-2q\epsilon^2+2V_1\beta^2+2\frac{V_0}{b}(qa+1)\beta^2-2V_2\beta^2]\}=0 \tag{31}$$

By effecting a change of variable we obtain:

$$a_0[(q^2+1)(r^2+\epsilon^2)-2qV_1\beta^2-2\frac{V_0}{b}\beta^2(a+q)]+\sum_{n=1}^{\infty}s^n[a_n\{(q^2+1)\{(n+r)^2+\epsilon^2\}$$
$$-2V_1\beta^2q-2\frac{V_0}{b}\beta^2(a+q)\}+a_{n-1}\{2V_1\beta^2+2\frac{V_0}{b}(aq+1)\beta^2-2V_2\beta^2$$
$$-2q\{(n+r-1)^2+\epsilon^2\}\}]=0 \tag{32}$$

By solving the indicial equation $I=a_0[(q^2+1)(r^2+\epsilon^2)-2qV_1\beta^2-2\frac{V_0}{b}\beta^2(a+q)]$,we obtain

$$(q^2+1)(r^2+\epsilon^2)-2qV_1\beta^2-2\frac{V_0}{b}\beta^2(a+q)=0$$
$$i.e. \quad r=\pm\sqrt{\frac{-\epsilon^2(q^2+1)+2qV_1\beta^2+2\frac{V_0}{b}\beta^2(a+q)}{q^2+1}}=\pm\nu \tag{33}$$

For $r=\nu$ we have:

$$a_n=\prod_{i=1}^{n}\frac{2q\{(i+\nu-1)^2+\epsilon^2\}-2V_1\beta^2-2\frac{V_0}{b}(aq+1)\beta^2+2V_2\beta^2}{(q^2+1)\{(i+\nu)^2+\epsilon^2\}-2V_1q\beta^2-2\frac{V_0}{b}(a+q)\beta^2}a_0 \quad , \quad n=1,2,.... \tag{34}$$

So it gets a representation of the solution

$$a_k = \prod_{i=1}^{k} \frac{2q\{(i+\nu-1)^2+\epsilon^2\} - 2V_1\beta^2 - 2\frac{V_0}{b}(aq+1)\beta^2 + 2V_2\beta^2}{(q^2+1)\{(i+\nu)^2+\epsilon^2\} - 2V_1 q\beta^2 - 2\frac{V_0}{b}(a+q)\beta^2} a_0 \quad , \quad k=1,2,..... \tag{35}$$

Using the relations (33) and (23) , we obtain the energy eigenvalue associated with the wave function. We can express the solutions obtained based on the Jacobi polynomial [31]:this result is more accurate.The coefficients of the solution being assessed explicitly,we seek the bounded solutions.We will only retain the negative value .

5. Discussion

In this subsection we consider some special cases of the potential in consideration: (I) Hulthen Potential:

If we set $V_0 = V_2 = 0$ and $a = 0$ and $b = 1$,the potential in (12) reduces to

$$V(x) = -V_1 \frac{e^{-2\alpha x}}{1 - qe^{-2\alpha x}} \tag{36}$$

which is the Hulthen potential.

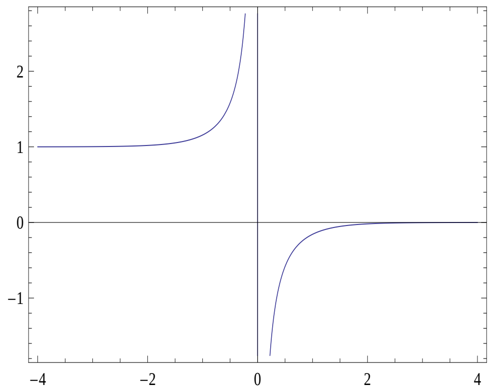

Fig.2.The Hulthén Potential with unit value of α, q.

Furthermore we get the eigen values ϵ from the equation

$$\epsilon^2 = -\left[\frac{8\gamma^2 - (2n+1)^2 - 1 - 2\eta(2n+1)}{4(2n+1)+2\eta}\right]^2 \tag{37}$$

and the eigen function is

$$\psi(s) = B_n s^{-\xi}(1-qs)^{(-\eta+\frac{1}{2})} P_n^{(2\xi,2\eta)}(s) \tag{38}$$

where $\gamma^2 = 2\zeta^2, \omega^2 = 0, \eta^2 = (\frac{1}{4} - \zeta^2), \xi^2 = -\epsilon^2$.

Again, applying Frobenius method we obtain

$$a_k = \prod_{i=1}^{k} \frac{2q\{(i+\nu-1)^2+\epsilon^2\} - 2V_1\beta^2}{(q^2+1)\{(i+\nu)^2+\epsilon^2\} - 2V_1 q\beta^2} a_0 \quad , \quad k=1,2,.... \tag{39}$$

(II) Rosen-Morse Potential:

If we set $V_1 = V_2 = 0$ and $a = -1$ and $b = 1$,the potential in (12) reduces to

$$V(x) = -V_0 \frac{1 + e^{-2\alpha x}}{1 - qe^{-2\alpha x}} \tag{40}$$

which is the Rosen-Morse potential.

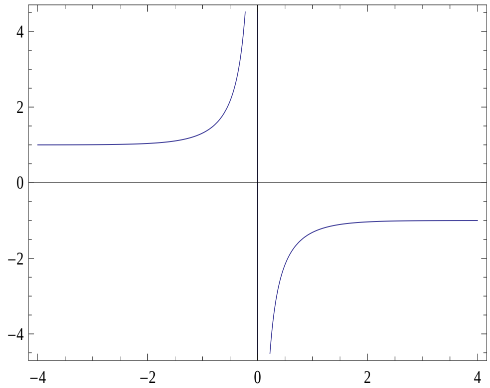

Fig.3.The Rosen-Morse Potential with unit value of α, q.

Furthermore we get the eigen values ϵ from the equation

$$\epsilon^2 = \omega^2 - \left[\frac{8(\gamma^2 - \omega^2) - (2n+1)^2 - 1 - 2\eta(2n+1)}{4(2n+1) + 2\eta}\right]^2 \tag{41}$$

and the eigen function is

$$\psi(s) = B_n s^{-\xi}(1 - qs)^{(-\eta+\frac{1}{2})} P_n^{(2\xi,2\eta)}(s) \tag{42}$$

where $\gamma^2 = 2\frac{V_0(aq+1)}{b}\frac{E+m_0}{4\alpha^2 q}$,$\zeta^2 = \frac{-V_0(aq-1)}{b}\frac{E+m_0}{4\alpha^2 q}$, $\omega^2 = 2\frac{V_0}{b}\frac{E+m_0}{4\alpha^2 q}$, $\eta^2 = (\frac{1}{4} + \omega^2 + \zeta^2 - \gamma^2)$, $\xi^2 = \omega^2 - \epsilon^2$.

Again, applying Frobenius method we obtain

$$a_k = \prod_{i=1}^{k} \frac{2q\{(i+\nu-1)^2 + \epsilon^2\} - 2\frac{V_0}{b}(aq+1)\beta^2}{(q^2+1)\{(i+\nu)^2 + \epsilon^2\} - 2\frac{V_0}{b}(a+q)\beta^2} a_0 \quad , \quad k = 1, 2, ... \tag{43}$$

(III) shape parameter $q = 1$:

For N-U method we have the wave function as

$$\psi(s) = B_n s^{-\xi}(1 - s)^{(-\eta+\frac{1}{2})} P_n^{(2\xi,2\eta)}(s) \tag{44}$$

one obtains the eigen values of ϵ from the following equation:

$$\omega^2 - \epsilon^2 = \left[\frac{8(\gamma^2 - \omega^2) - (2n+1)^2 - 1 - 2\eta(2n+1)}{4(2n+1) + 2\eta}\right]^2 \tag{45}$$

For Frobenius method, we have

$$a_k = \prod_{i=1}^{k} \frac{\{(i+\nu-1)^2 + \epsilon^2\} - V_1\beta^2 - \frac{V_0}{b}(a+1)\beta^2 + V_2\beta^2}{\{(i+\nu)^2 + \epsilon^2\} - V_1\beta^2 - \frac{V_0}{b}(a+1)\beta^2} a_0 \quad , \quad k = 1, 2, ... \tag{46}$$

5. Conclusion

In this article , the exact solution of the effective mass K-G equation for the modified Eckart plus Hylleraas potential in absence of Lorentz scalar potential. The eigen values and eigen functions are obtained using the Frobenius method as well as Nikiforov-Uvarov method. We gave a schematic graphical representation of the modified Eckart plus Hylleraas potential with a shape parameter 'q' and also the graphical representation of Hulthén and Rosen-Morse Potential. The eigen values of the potential reduces to that of well known potentials viz., Hulthén Potential in equation (36) and Rosen-Morse Potential in equatioon (40), when we make appropriate choices of parameter a, b, V_0, V_1, V_2 . Finally we also obtain the wave function which is expressed in terms of the Jacobi Polynomials.

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