

Poisson Bracket and Symplectic Structure of Covariant Canonical Formalism of Fields

Yasuhito Kaminaga*

*School of Mathematics, The University of Edinburgh,
James Clerk Maxwell Building, Peter Guthrie Tait Road,
Edinburgh EH9 3FD, United Kingdom*

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Abstract: The covariant canonical formalism is a covariant extension of the traditional canonical formalism of fields. In contrast to the traditional canonical theory, it has a remarkable feature that canonical equations of gauge theories or gravity are not only manifestly Lorentz covariant but also gauge covariant or diffeomorphism covariant. A mathematical peculiarity of the covariant canonical formalism is that its canonical coordinates are differential forms on a manifold. In the present paper, we find a natural Poisson bracket of this new canonical theory, and study symplectic structure behind it. The phase space of the theory is identified with a ringed space with the structure sheaf of the graded algebra of “differentiable” differential forms on the manifold. The Poisson and the symplectic structure we found can be even or odd, depending on the dimension of the manifold. Our Poisson structure is an example of physical application of Poisson structure defined on the graded algebra of differential forms.

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* Permanent Address: Department of Mathematics, National Institute of Technology, Gunma College, Toriba, Maebashi, Gunma 371–8530, Japan E-mail: kaminaga@nat.gunma-ct.ac.jp

1 Introduction

In the previous paper [1], the present author developed a new analytical mechanics of fields which treats space and time on an equal footing [2–5]. A similar theory has been studied by Théophile De Donder [6], Hermann Weyl [7] and others [8–13], but there is a crucial difference between theirs and ours. That is, they adopt components of tensors (not tensors themselves) as canonical variables while we adopt differential forms themselves (not their components) as canonical variables. This difference causes drastic changes to the resultant theory at least formally. The De Donder-Weyl theory including multisymplectic formalism [6–13] is more similar to the traditional analytical mechanics of fields than ours. For example, in their theory one needs gauge fixing or Dirac bracket or something like that to obtain their canonical equations of gauge theories or those of gravity, as it is the case with the traditional canonical formalism. In our theory, however, one can obtain canonical equations of gauge theories or those of gravity without fixing a gauge nor introducing Dirac bracket nor any other artificial tricks. Our canonical equations are not only manifestly Lorentz covariant but also keep any kind of gauge freedom including diffeomorphism. Our new analytical mechanics of fields is impressive with its simplicity, straightforward nature and mathematical beauty. The purpose of the present paper is to study Poisson structure of our canonical formalism, and to clarify symplectic structure behind it.

This paper is organised as follows. In Section 2 we give a brief review of our covariant canonical formalism [1]. In Section 3, we find a natural definition of our Poisson bracket on the basis of physical motivations and heuristic speculations. In Section 4, we collect important formulae of our Poisson bracket. The heuristic part ends here and purely mathematical part begins at Section 5. In Sections 5 and 6, we develop mathematical tools needed for the study of our symplectic structure. In Section 7, we study symplectic structure of our covariant canonical formalism using tools prepared in Sections 5 and 6, and show that it leads to our Poisson bracket found in the heuristic part.

2 Covariant Canonical Formalism of Fields

Let M be a C^∞ -manifold of dimension n , and let $\Omega(M) = \bigoplus_{r=0}^n \Omega^r(M)$ be the \mathbb{Z}_2 -graded commutative algebra of differential forms on M . For a map $\phi : \Omega^{p_1}(M) \times \cdots \times \Omega^{p_k}(M) \rightarrow \Omega^r(M)$, we put $\beta = \phi(\alpha^1, \dots, \alpha^k) \in \Omega^r(M)$ with $\alpha^i \in \Omega^{p_i}(M)$. If there exists a differential form $\omega_i \in \Omega^{r-p_i}(M)$ such that β behaves

under variations $\delta\alpha^1, \dots, \delta\alpha^k$ as $\delta\beta = \sum_i \delta\alpha^i \wedge \omega_i$, then we say the differential form β is *differentiable* with respect to another differential form α^i , and define the derivative of β by α^i to be $\partial\beta/\partial\alpha^i = \omega^i$. In what follows, we assume sum over repeated indices unless otherwise stated.

Suppose physical fields are represented with even-forms $\varphi^a \in \Omega(M)$ and/or odd-forms $\psi^\alpha \in \Omega(M)$. We define a Lagrangian n -form $L \in \Omega(M)$ so that the covariant Lagrange equations

$$\frac{\partial L}{\partial\varphi^a} - d\frac{\partial L}{\partial d\varphi^a} = 0, \quad \frac{\partial L}{\partial\psi^\alpha} + d\frac{\partial L}{\partial d\psi^\alpha} = 0$$

coincide with field equations. Here, we have implicitly assumed, of course, that L is differentiable with respect to φ^a , $d\varphi^a$, ψ^α and $d\psi^\alpha$. Then, we define the conjugate momentum forms $p_a, \pi_\alpha \in \Omega(M)$ as

$$p_a = \frac{\partial L}{\partial d\varphi^a}, \quad \pi_\alpha = \frac{\partial L}{\partial d\psi^\alpha}, \quad (1)$$

and the Hamiltonian n -form $H \in \Omega(M)$ as

$$H = d\varphi^a \wedge p_a + d\psi^\alpha \wedge \pi_\alpha - L.$$

Suppose that the system is non-singular, that is, (1) can be solved conversely and $d\varphi^a$ and $d\psi^\alpha$ are represented uniquely with canonical variables φ^a , p_a , ψ^α and π_α . Then, H is differentiable with respect to canonical variables, and we get our covariant canonical equations

$$d\varphi^a = -(-1)^n \frac{\partial H}{\partial p_a}, \quad dp_a = -\frac{\partial H}{\partial\varphi^a}, \quad d\psi^\alpha = \frac{\partial H}{\partial\pi_\alpha}, \quad d\pi_\alpha = \frac{\partial H}{\partial\psi^\alpha}. \quad (2)$$

Surprisingly, the non-singular assumption is satisfied not only by gauge theories but also by gravity [1]. Indeed, our canonical equations (2) are manifestly Lorentz covariant as well as gauge/diffeomorphism covariant.

In the following sections, we will denote by $\Lambda^r(M)$ the set of r -forms, on M , which are sufficiently many times differentiable with respect to canonical variables φ^a , p_a , ψ^α and π_α . $\Lambda(M) = \bigoplus_{r=0}^n \Lambda^r(M)$ forms a \mathbb{Z}_2 -graded commutative algebra. Needless to say, it is a subalgebra of $\Omega(M)$.

3 Finding Poisson Bracket

Let us find a natural definition of the Poisson bracket of the covariant canonical formalism. To avoid complexity, we first restrict ourselves to $n = \text{even}$ case only,

where n is the dimension of the manifold M . When n is even, it follows that φ^a is even, p_a is odd, ψ^α is odd, π_α is even and H is even. Our canonical equations (2) reduce to

$$d\varphi^a = -\frac{\partial H}{\partial p_a}, \quad dp_a = -\frac{\partial H}{\partial \varphi^a}, \quad d\psi^\alpha = \frac{\partial H}{\partial \pi_\alpha}, \quad d\pi_\alpha = \frac{\partial H}{\partial \psi^\alpha}. \quad (3)$$

Let $F \in \Lambda(M)$ be an arbitrary, even or odd, differentiable form on the “phase space”. At the moment, we naively consider our phase space to be something like a space of which “coordinates” are differential forms φ^a , p_a , ψ^α and π_α . Its sophisticated definition will be given in Section 5. We then obtain

$$\begin{aligned} dF &= d\varphi^a \wedge \frac{\partial F}{\partial \varphi^a} + dp_a \wedge \frac{\partial F}{\partial p_a} + d\psi^\alpha \wedge \frac{\partial F}{\partial \psi^\alpha} + d\pi_\alpha \wedge \frac{\partial F}{\partial \pi_\alpha} \\ &= -\frac{\partial H}{\partial p_a} \wedge \frac{\partial F}{\partial \varphi^a} - \frac{\partial H}{\partial \varphi^a} \wedge \frac{\partial F}{\partial p_a} + \frac{\partial H}{\partial \pi_\alpha} \wedge \frac{\partial F}{\partial \psi^\alpha} + \frac{\partial H}{\partial \psi^\alpha} \wedge \frac{\partial F}{\partial \pi_\alpha}, \end{aligned}$$

where we have used (3). Now, we postulate

$$dF = -\{H, F\} \quad (4)$$

to define our Poisson bracket as

$$\{H, F\} = \left(\frac{\partial H}{\partial p_a} \wedge \frac{\partial F}{\partial \varphi^a} + \frac{\partial H}{\partial \varphi^a} \wedge \frac{\partial F}{\partial p_a} \right) - \left(\frac{\partial H}{\partial \pi_\alpha} \wedge \frac{\partial F}{\partial \psi^\alpha} + \frac{\partial H}{\partial \psi^\alpha} \wedge \frac{\partial F}{\partial \pi_\alpha} \right).$$

Remembering that H is an even-form, we generalise the definition as

$$\{E_1, F\} = \left(\frac{\partial E_1}{\partial p_a} \wedge \frac{\partial F}{\partial \varphi^a} + \frac{\partial E_1}{\partial \varphi^a} \wedge \frac{\partial F}{\partial p_a} \right) - \left(\frac{\partial E_1}{\partial \pi_\alpha} \wedge \frac{\partial F}{\partial \psi^\alpha} + \frac{\partial E_1}{\partial \psi^\alpha} \wedge \frac{\partial F}{\partial \pi_\alpha} \right). \quad (5)$$

Here, $E_1 \in \Lambda(M)$ denotes an arbitrary differentiable even-form. In what follows, unless otherwise mentioned, $E, E_1, E_2, E_3 \in \Lambda(M)$ are all even-forms and $O, O_1, O_2, O_3 \in \Lambda(M)$ are all odd-forms. Putting $F = E_2, O_2$ in (5), we obtain

$$\{E_1, E_2\} = \left(\frac{\partial E_1}{\partial p_a} \wedge \frac{\partial E_2}{\partial \varphi^a} + \frac{\partial E_1}{\partial \varphi^a} \wedge \frac{\partial E_2}{\partial p_a} \right) - \left(\frac{\partial E_1}{\partial \pi_\alpha} \wedge \frac{\partial E_2}{\partial \psi^\alpha} + \frac{\partial E_1}{\partial \psi^\alpha} \wedge \frac{\partial E_2}{\partial \pi_\alpha} \right), \quad (6)$$

$$\{E_1, O_2\} = \left(\frac{\partial E_1}{\partial p_a} \wedge \frac{\partial O_2}{\partial \varphi^a} + \frac{\partial E_1}{\partial \varphi^a} \wedge \frac{\partial O_2}{\partial p_a} \right) - \left(\frac{\partial E_1}{\partial \pi_\alpha} \wedge \frac{\partial O_2}{\partial \psi^\alpha} + \frac{\partial E_1}{\partial \psi^\alpha} \wedge \frac{\partial O_2}{\partial \pi_\alpha} \right). \quad (7)$$

Next, let us put $E_1 = O_1 \wedge o$ in (6) and (7), in which o is a constant odd-form, an invariant odd-form under variations of canonical variables. Then, we obtain

$$\begin{aligned} \{O_1 \wedge o, E_2\} &= \left[\left(\frac{\partial O_1}{\partial p_a} \wedge \frac{\partial E_2}{\partial \varphi^a} - \frac{\partial O_1}{\partial \varphi^a} \wedge \frac{\partial E_2}{\partial p_a} \right) + \left(\frac{\partial O_1}{\partial \pi_\alpha} \wedge \frac{\partial E_2}{\partial \psi^\alpha} - \frac{\partial O_1}{\partial \psi^\alpha} \wedge \frac{\partial E_2}{\partial \pi_\alpha} \right) \right] \wedge o, \\ \{O_1 \wedge o, O_2\} &= - \left[\left(\frac{\partial O_1}{\partial p_a} \wedge \frac{\partial O_2}{\partial \varphi^a} - \frac{\partial O_1}{\partial \varphi^a} \wedge \frac{\partial O_2}{\partial p_a} \right) + \left(\frac{\partial O_1}{\partial \pi_\alpha} \wedge \frac{\partial O_2}{\partial \psi^\alpha} - \frac{\partial O_1}{\partial \psi^\alpha} \wedge \frac{\partial O_2}{\partial \pi_\alpha} \right) \right] \wedge o. \end{aligned}$$

These results suggest us for putting additional definitions as

$$\{O_1, E_2\} = t \left(\frac{\partial O_1}{\partial p_a} \wedge \frac{\partial E_2}{\partial \varphi^a} - \frac{\partial O_1}{\partial \varphi^a} \wedge \frac{\partial E_2}{\partial p_a} \right) + t \left(\frac{\partial O_1}{\partial \pi_\alpha} \wedge \frac{\partial E_2}{\partial \psi^\alpha} - \frac{\partial O_1}{\partial \psi^\alpha} \wedge \frac{\partial E_2}{\partial \pi_\alpha} \right), \quad (8)$$

$$\{O_1, O_2\} = t' \left(\frac{\partial O_1}{\partial p_a} \wedge \frac{\partial O_2}{\partial \varphi^a} - \frac{\partial O_1}{\partial \varphi^a} \wedge \frac{\partial O_2}{\partial p_a} \right) + t' \left(\frac{\partial O_1}{\partial \pi_\alpha} \wedge \frac{\partial O_2}{\partial \psi^\alpha} - \frac{\partial O_1}{\partial \psi^\alpha} \wedge \frac{\partial O_2}{\partial \pi_\alpha} \right). \quad (9)$$

Here, t and t' are yet unknown real constants, which we assume to be 1 or -1 . Now notice that the Poisson bracket we have just defined in (6)(7)(8)(9) is an *odd-bracket*; that is, $\{E_1, E_2\}$ is odd, $\{E_1, O_2\}$ is even, $\{O_1, E_2\}$ is even, and $\{O_1, O_2\}$ is odd. We can fix t and t' postulating “Leibniz rule” in the form of $\{A, \{B, C\}\} = \{\{A, B\}, C\} \pm \{B, \{A, C\}\}$. Our bracket, for example, satisfies

$$\begin{aligned} \{E, \{E_1, O_2\}\} &= -t' \{\{E, E_1\}, O_2\} - \{E_1, \{E, O_2\}\}, \\ \{E, \{O_1, E_2\}\} &= -t \{\{E, O_1\}, E_2\} + tt' \{O_1, \{E, E_2\}\}. \end{aligned}$$

Hence, we get $t = t' = -1$. Then, our bracket has been totally fixed.

Now, let us turn our attention to $n = \text{odd}$ case. We here outline points only since we can consider it along the same lines in the above. A difference occurs after equation (4) because H is an odd-form when n is odd. That is, when n is odd, the bracket we obtain from (4) is $\{O_1, F\}$, instead of $\{E_1, F\}$ in (5). Other consideration goes well. After lengthy calculation, we finally get a unique Poisson bracket for $n = \text{odd}$ case, too. Again, notice that when n is odd, our bracket becomes an *even-bracket*; that is, $\{E_1, E_2\}$ is even, $\{E_1, O_2\}$ is odd, $\{O_1, E_2\}$ is odd, and $\{O_1, O_2\}$ is even. We will summarise important formulae of our Poisson bracket in the next section.

Before closing this section, note that our Poisson bracket is defined on the space of differential forms. Research on the generalisation of the traditional Poisson bracket to a graded Lie bracket on the space of differential forms has been a long-standing topic of mathematics [14–20]. Our bracket provides a concrete example of such a generalised bracket with physical motivation.

4 List of Poisson Bracket Formulae

In this section, we list, without verbose description, important formulae of our Poisson bracket heuristically found in the previous section. One can prove all

the formulae listed here by straightforward calculation using definitions, though it is tedious. We will give an alternative elegant proof in Section 7 in this paper. Although we have omitted from the following list, the formula (4) holds, of course, by construction; it is the field equation of a differentiable form $F \in \Lambda(M)$ with our Poisson bracket and the Hamiltonian form H .

4.1 Poisson Bracket in Even Dimensions

Definition

$$\begin{aligned}
 \{E_1, E_2\}_{\text{odd}} &= \left(\frac{\partial E_1}{\partial p_a} \wedge \frac{\partial E_2}{\partial \varphi^a} + \frac{\partial E_1}{\partial \varphi^a} \wedge \frac{\partial E_2}{\partial p_a} \right) - \left(\frac{\partial E_1}{\partial \pi_\alpha} \wedge \frac{\partial E_2}{\partial \psi^\alpha} + \frac{\partial E_1}{\partial \psi^\alpha} \wedge \frac{\partial E_2}{\partial \pi_\alpha} \right) \\
 \{E_1, O_2\}_{\text{even}} &= \left(\frac{\partial E_1}{\partial p_a} \wedge \frac{\partial O_2}{\partial \varphi^a} + \frac{\partial E_1}{\partial \varphi^a} \wedge \frac{\partial O_2}{\partial p_a} \right) - \left(\frac{\partial E_1}{\partial \pi_\alpha} \wedge \frac{\partial O_2}{\partial \psi^\alpha} + \frac{\partial E_1}{\partial \psi^\alpha} \wedge \frac{\partial O_2}{\partial \pi_\alpha} \right) \\
 \{O_1, E_2\}_{\text{even}} &= - \left(\frac{\partial O_1}{\partial p_a} \wedge \frac{\partial E_2}{\partial \varphi^a} - \frac{\partial O_1}{\partial \varphi^a} \wedge \frac{\partial E_2}{\partial p_a} \right) - \left(\frac{\partial O_1}{\partial \pi_\alpha} \wedge \frac{\partial E_2}{\partial \psi^\alpha} - \frac{\partial O_1}{\partial \psi^\alpha} \wedge \frac{\partial E_2}{\partial \pi_\alpha} \right) \\
 \{O_1, O_2\}_{\text{odd}} &= - \left(\frac{\partial O_1}{\partial p_a} \wedge \frac{\partial O_2}{\partial \varphi^a} - \frac{\partial O_1}{\partial \varphi^a} \wedge \frac{\partial O_2}{\partial p_a} \right) - \left(\frac{\partial O_1}{\partial \pi_\alpha} \wedge \frac{\partial O_2}{\partial \psi^\alpha} - \frac{\partial O_1}{\partial \psi^\alpha} \wedge \frac{\partial O_2}{\partial \pi_\alpha} \right)
 \end{aligned} \tag{10}$$

Fundamental Brackets

$$\{\varphi^a, p_b\} = \delta_b^a, \quad \{p_b, \varphi^a\} = -\delta_b^a, \quad \{\psi^\alpha, \pi_\beta\} = \delta_\beta^\alpha, \quad \{\pi_\beta, \psi^\alpha\} = -\delta_\beta^\alpha, \quad \text{others} = 0 \tag{11}$$

Symmetry

$$\{E_2, E_1\} = \{E_1, E_2\}, \quad \{O, E\} = -\{E, O\}, \quad \{O_2, O_1\} = -\{O_1, O_2\} \tag{12}$$

Leibniz Rule I

$$\begin{aligned}
 \{E, E_1 \wedge E_2\} &= \{E, E_1\} \wedge E_2 + E_1 \wedge \{E, E_2\} \\
 \{E, E_1 \wedge O_2\} &= \{E, E_1\} \wedge O_2 + E_1 \wedge \{E, O_2\} \\
 \{E, O_1 \wedge E_2\} &= \{E, O_1\} \wedge E_2 - O_1 \wedge \{E, E_2\} \\
 \{E, O_1 \wedge O_2\} &= \{E, O_1\} \wedge O_2 - O_1 \wedge \{E, O_2\} \\
 \{O, E_1 \wedge E_2\} &= \{O, E_1\} \wedge E_2 + E_1 \wedge \{O, E_2\} \\
 \{O, E_1 \wedge O_2\} &= \{O, E_1\} \wedge O_2 + E_1 \wedge \{O, O_2\} \\
 \{O, O_1 \wedge E_2\} &= \{O, O_1\} \wedge E_2 + O_1 \wedge \{O, E_2\} \\
 \{O, O_1 \wedge O_2\} &= \{O, O_1\} \wedge O_2 + O_1 \wedge \{O, O_2\}
 \end{aligned} \tag{13}$$

Jacobi identity

$$\begin{aligned}
 \{E_1, \{E_2, E_3\}\} + \{E_2, \{E_3, E_1\}\} + \{E_3, \{E_1, E_2\}\} &= 0 \\
 \{E_1, \{E_2, O\}\} - \{E_2, \{O, E_1\}\} + \{O, \{E_1, E_2\}\} &= 0 \\
 \{E, \{O_1, O_2\}\} + \{O_1, \{O_2, E\}\} + \{O_2, \{E, O_1\}\} &= 0 \\
 \{O_1, \{O_2, O_3\}\} + \{O_2, \{O_3, O_1\}\} + \{O_3, \{O_1, O_2\}\} &= 0
 \end{aligned} \tag{14}$$

Leibniz Rule II (Another Form of Jacobi Identity)

$$\begin{aligned}
 \{E, \{E_1, E_2\}\} &= \{\{E, E_1\}, E_2\} - \{E_1, \{E, E_2\}\} \\
 \{E, \{E_1, O_2\}\} &= \{\{E, E_1\}, O_2\} - \{E_1, \{E, O_2\}\} \\
 \{E, \{O_1, E_2\}\} &= \{\{E, O_1\}, E_2\} + \{O_1, \{E, E_2\}\} \\
 \{E, \{O_1, O_2\}\} &= \{\{E, O_1\}, O_2\} + \{O_1, \{E, O_2\}\} \\
 \{O, \{E_1, E_2\}\} &= \{\{O, E_1\}, E_2\} + \{E_1, \{O, E_2\}\} \\
 \{O, \{E_1, O_2\}\} &= \{\{O, E_1\}, O_2\} + \{E_1, \{O, O_2\}\} \\
 \{O, \{O_1, E_2\}\} &= \{\{O, O_1\}, E_2\} + \{O_1, \{O, E_2\}\} \\
 \{O, \{O_1, O_2\}\} &= \{\{O, O_1\}, O_2\} + \{O_1, \{O, O_2\}\}
 \end{aligned} \tag{15}$$

4.2 Poisson Bracket in Odd Dimensions

Definition

$$\begin{aligned}
 \{E_1, E_2\}_{\text{even}} &= - \left(\frac{\partial E_1}{\partial p_a} \wedge \frac{\partial E_2}{\partial \varphi^a} - \frac{\partial E_1}{\partial \varphi^a} \wedge \frac{\partial E_2}{\partial p_a} \right) + \left(\frac{\partial E_1}{\partial \pi_\alpha} \wedge \frac{\partial E_2}{\partial \psi^\alpha} + \frac{\partial E_1}{\partial \psi^\alpha} \wedge \frac{\partial E_2}{\partial \pi_\alpha} \right) \\
 \{E_1, O_2\}_{\text{odd}} &= - \left(\frac{\partial E_1}{\partial p_a} \wedge \frac{\partial O_2}{\partial \varphi^a} - \frac{\partial E_1}{\partial \varphi^a} \wedge \frac{\partial O_2}{\partial p_a} \right) + \left(\frac{\partial E_1}{\partial \pi_\alpha} \wedge \frac{\partial O_2}{\partial \psi^\alpha} + \frac{\partial E_1}{\partial \psi^\alpha} \wedge \frac{\partial O_2}{\partial \pi_\alpha} \right) \\
 \{O_1, E_2\}_{\text{odd}} &= - \left(\frac{\partial O_1}{\partial p_a} \wedge \frac{\partial E_2}{\partial \varphi^a} - \frac{\partial O_1}{\partial \varphi^a} \wedge \frac{\partial E_2}{\partial p_a} \right) - \left(\frac{\partial O_1}{\partial \pi_\alpha} \wedge \frac{\partial E_2}{\partial \psi^\alpha} + \frac{\partial O_1}{\partial \psi^\alpha} \wedge \frac{\partial E_2}{\partial \pi_\alpha} \right) \\
 \{O_1, O_2\}_{\text{even}} &= - \left(\frac{\partial O_1}{\partial p_a} \wedge \frac{\partial O_2}{\partial \varphi^a} - \frac{\partial O_1}{\partial \varphi^a} \wedge \frac{\partial O_2}{\partial p_a} \right) - \left(\frac{\partial O_1}{\partial \pi_\alpha} \wedge \frac{\partial O_2}{\partial \psi^\alpha} + \frac{\partial O_1}{\partial \psi^\alpha} \wedge \frac{\partial O_2}{\partial \pi_\alpha} \right)
 \end{aligned} \tag{16}$$

Fundamental Brackets

$$\{\varphi^a, p_b\} = \delta_b^a, \quad \{p_b, \varphi^a\} = -\delta_b^a, \quad \{\psi^\alpha, \pi_\beta\} = -\delta_\beta^\alpha, \quad \{\pi_\beta, \psi^\alpha\} = -\delta_\beta^\alpha, \quad \text{others} = 0 \tag{17}$$

Symmetry

$$\{E_2, E_1\} = -\{E_1, E_2\}, \quad \{O, E\} = -\{E, O\}, \quad \{O_2, O_1\} = \{O_1, O_2\} \tag{18}$$

Leibniz Rule I

$$\begin{aligned}
\{E, E_1 \wedge E_2\} &= \{E, E_1\} \wedge E_2 + E_1 \wedge \{E, E_2\} \\
\{E, E_1 \wedge O_2\} &= \{E, E_1\} \wedge O_2 + E_1 \wedge \{E, O_2\} \\
\{E, O_1 \wedge E_2\} &= \{E, O_1\} \wedge E_2 + O_1 \wedge \{E, E_2\} \\
\{E, O_1 \wedge O_2\} &= \{E, O_1\} \wedge O_2 + O_1 \wedge \{E, O_2\} \\
\{O, E_1 \wedge E_2\} &= \{O, E_1\} \wedge E_2 + E_1 \wedge \{O, E_2\} \\
\{O, E_1 \wedge O_2\} &= \{O, E_1\} \wedge O_2 + E_1 \wedge \{O, O_2\} \\
\{O, O_1 \wedge E_2\} &= \{O, O_1\} \wedge E_2 - O_1 \wedge \{O, E_2\} \\
\{O, O_1 \wedge O_2\} &= \{O, O_1\} \wedge O_2 - O_1 \wedge \{O, O_2\}
\end{aligned} \tag{19}$$

Jacobi identity

$$\begin{aligned}
\{E_1, \{E_2, E_3\}\} + \{E_2, \{E_3, E_1\}\} + \{E_3, \{E_1, E_2\}\} &= 0 \\
\{E_1, \{E_2, O\}\} + \{E_2, \{O, E_1\}\} + \{O, \{E_1, E_2\}\} &= 0 \\
\{E, \{O_1, O_2\}\} + \{O_1, \{O_2, E\}\} - \{O_2, \{E, O_1\}\} &= 0 \\
\{O_1, \{O_2, O_3\}\} + \{O_2, \{O_3, O_1\}\} + \{O_3, \{O_1, O_2\}\} &= 0
\end{aligned} \tag{20}$$

Leibniz Rule II (Another Form of Jacobi Identity)

$$\begin{aligned}
\{E, \{E_1, E_2\}\} &= \{\{E, E_1\}, E_2\} + \{E_1, \{E, E_2\}\} \\
\{E, \{E_1, O_2\}\} &= \{\{E, E_1\}, O_2\} + \{E_1, \{E, O_2\}\} \\
\{E, \{O_1, E_2\}\} &= \{\{E, O_1\}, E_2\} + \{O_1, \{E, E_2\}\} \\
\{E, \{O_1, O_2\}\} &= \{\{E, O_1\}, O_2\} + \{O_1, \{E, O_2\}\} \\
\{O, \{E_1, E_2\}\} &= \{\{O, E_1\}, E_2\} + \{E_1, \{O, E_2\}\} \\
\{O, \{E_1, O_2\}\} &= \{\{O, E_1\}, O_2\} + \{E_1, \{O, O_2\}\} \\
\{O, \{O_1, E_2\}\} &= \{\{O, O_1\}, E_2\} - \{O_1, \{O, E_2\}\} \\
\{O, \{O_1, O_2\}\} &= \{\{O, O_1\}, O_2\} - \{O_1, \{O, O_2\}\}
\end{aligned} \tag{21}$$

5 Phase Space

Let us change our subject slightly. In the preceding sections, we have heuristically introduced the Poisson bracket. In that context, our bracket seems merely an assembly of ad hoc definitions, but nevertheless it has a profound geometrical meaning as is shown below. In the remaining part of the paper, we turn our attention to mathematical structure behind the Poisson bracket. After preparing necessary tools in the present and the next sections, we will get back to the issue of the Poisson bracket in Section 7.

Let us start our consideration from the definition of phase space. The phase space of our canonical theory, intuitively, is a space of which “coordinates” are forms on an n -dimensional C^∞ -manifold M . Obviously, we need a sophisticated definition to it to go further. Remember that for any open subset $U \subset M$, there exists a short exact sequence

$$0 \longrightarrow \Lambda'(U) \longrightarrow \Lambda(U) \longrightarrow C^\infty(U) \longrightarrow 0,$$

where $\Lambda'(U)$ is the ideal of nilpotent elements in $\Lambda(U)$, a \mathbb{Z}_2 -graded commutative algebra of differentiable differential forms on U . Considering this in mind, we identify our phase space with a ringed space (M, \mathcal{O}_M) with the structure sheaf $\mathcal{O}_M : U \mapsto \Lambda(U)$. Notice that (M, \mathcal{O}_M) is a kind of what is called a supermanifold [21–25]. In the following, we think of $x^i \in \Lambda(U)$ as any of canonical variables $\varphi^a, p_a, \psi^\alpha, \pi_\alpha$. We define \mathbb{Z}_2 -graded degree of x^i as $|x^i| = r \pmod{2}$ for $x^i \in \Lambda^r(U)$. In our construction, the set of “functions” on (M, \mathcal{O}_M) is identified with $\Lambda(M)$. A vector field \mathbf{X} on (M, \mathcal{O}_M) is an \mathbb{R} -linear derivation of $\Lambda(U)$

$$\mathbf{X} = X^i \wedge \frac{\partial}{\partial x^i} : \Lambda(U) \rightarrow \Lambda(U)$$

with $X^i \in \Lambda(U)$ such that $\mathbf{X}f = X^i \wedge \partial f / \partial x^i$ for $f \in \Lambda(U)$. The set of all vector fields on (M, \mathcal{O}_M) is denoted $\text{Der}\Lambda(M)$, which is a \mathbb{Z}_2 -graded left $\Lambda(M)$ -module. $|\mathbf{X}|$ implies \mathbb{Z}_2 -graded degree of \mathbf{X} . We define the dual basis $\mathbf{d}x^i \in \text{Hom}_{\Lambda(U)}(\text{Der}\Lambda(U), \Lambda(U))$ of $\partial/\partial x^i \in \text{Der}\Lambda(U)$ as

$$\left\langle \frac{\partial}{\partial x^j}; \mathbf{d}x^i \right\rangle = \frac{\partial x^i}{\partial x^j} = \delta_j^i.$$

Differential forms on (M, \mathcal{O}_M) can be defined in a straightforward manner following generalised techniques of the usual manifold theory so as to accommodate \mathbb{Z}_2 -gradings of the structure sheaf \mathcal{O}_M . The exterior product \wedge on (M, \mathcal{O}_M) , which we write in baldface to distinguish from the exterior product \wedge on M , is defined so as to satisfy on (U, \mathcal{O}_U)

$$\begin{aligned} \mathbf{d}x^i \wedge \mathbf{d}x^j &= \mathbf{d}x^i \otimes \mathbf{d}x^j - (-1)^{x^i x^j} \mathbf{d}x^j \otimes \mathbf{d}x^i, \\ \mathbf{d}x^i \wedge \mathbf{d}x^j \wedge \mathbf{d}x^k &= \mathbf{d}x^i \otimes \mathbf{d}x^j \otimes \mathbf{d}x^k + (-1)^{x^i(x^j+x^k)} \mathbf{d}x^j \otimes \mathbf{d}x^k \otimes \mathbf{d}x^i \\ &\quad + (-1)^{(x^i+x^j)x^k} \mathbf{d}x^k \otimes \mathbf{d}x^i \otimes \mathbf{d}x^j - (-1)^{x^i x^j} \mathbf{d}x^j \otimes \mathbf{d}x^i \otimes \mathbf{d}x^k \\ &\quad - (-1)^{x^j x^k} \mathbf{d}x^i \otimes \mathbf{d}x^k \otimes \mathbf{d}x^j - (-1)^{x^j x^k + x^i(x^j+x^k)} \mathbf{d}x^k \otimes \mathbf{d}x^j \otimes \mathbf{d}x^i, \end{aligned}$$

and so on. Throughout the present paper, we abbreviate the degree symbol $| \quad |$ in the exponent of (-1) for simplicity. As the readers have anticipated, we will use boldfaced \mathbf{d} for the exterior derivative on (M, \mathcal{O}_M) distinguishing from d on M . It is important to notice that differential forms on (M, \mathcal{O}_M) , which we denote by $\Omega(M, \mathcal{O}_M) = \bigoplus_{r=0}^n \Omega^r(M, \mathcal{O}_M)$, form a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ -bigraded right $\Lambda(M)$ -module. We

will say that a differential form $\boldsymbol{\lambda}$ on (M, \mathcal{O}_M) has a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ -bidegree $|\boldsymbol{\lambda}| = \begin{pmatrix} r \\ s \end{pmatrix} \pmod{2}$ if it is an r -form $\boldsymbol{\lambda} \in \Omega^r(M, \mathcal{O}_M)$ which satisfies

$$|\langle \mathbf{X}_1, \dots, \mathbf{X}_r; \boldsymbol{\lambda} \rangle| = \sum_{k=1}^r |\mathbf{X}_k| + s \quad (22)$$

for all vector fields $\mathbf{X}_1, \dots, \mathbf{X}_r \in \text{Der}\Lambda(M)$. Arbitrary forms $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \Omega(M, \mathcal{O}_M)$ with bidegrees $|\boldsymbol{\lambda}_1| = \begin{pmatrix} r_1 \\ s_1 \end{pmatrix}$ and $|\boldsymbol{\lambda}_2| = \begin{pmatrix} r_2 \\ s_2 \end{pmatrix}$ satisfy

$$\boldsymbol{\lambda}_1 \wedge \boldsymbol{\lambda}_2 = (-1)^{r_1 r_2} (-1)^{s_1 s_2} \boldsymbol{\lambda}_2 \wedge \boldsymbol{\lambda}_1.$$

For a differential r -form $\boldsymbol{\lambda} \in \Omega^r(M, \mathcal{O}_M)$ ($r \geq 1$), and a vector field $\mathbf{X} \in \text{Der}\Lambda(M)$, we define the interior product of $\boldsymbol{\lambda}$ and \mathbf{X} , $i_{\mathbf{X}}\boldsymbol{\lambda} \in \Omega^{r-1}(M, \mathcal{O}_M)$, by

$$\langle \mathbf{X}_1, \dots, \mathbf{X}_{r-1}; i_{\mathbf{X}}\boldsymbol{\lambda} \rangle = \langle \mathbf{X}_1, \dots, \mathbf{X}_{r-1}, \mathbf{X}; \boldsymbol{\lambda} \rangle$$

with $\mathbf{X}_1, \dots, \mathbf{X}_{r-1} \in \text{Der}\Lambda(M)$. If $\boldsymbol{\lambda}$ has a bidegree $\begin{pmatrix} r \\ s \end{pmatrix}$, then $i_{\mathbf{X}}\boldsymbol{\lambda}$ has a bidegree

$$\begin{pmatrix} r-1 \\ s + |\mathbf{X}| \end{pmatrix} \pmod{2}.$$

6 Lie Derivatives

The Lie bracket of two vector fields $\mathbf{X}, \mathbf{Y} \in \text{Der}\Lambda(M)$ is the unique vector field defined as

$$[\mathbf{X}, \mathbf{Y}] = \mathbf{X}\mathbf{Y} - (-1)^{XY} \mathbf{Y}\mathbf{X} \in \text{Der}\Lambda(M).$$

The real vector space structure of $\text{Der}\Lambda(M)$ endowed with the Lie bracket $[\cdot, \cdot]$ on $\text{Der}\Lambda(M)$ form a \mathbb{Z}_2 -graded Lie algebra. That is, $[\cdot, \cdot]$ is \mathbb{R} -bilinear and, for any $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \text{Der}\Lambda(M)$, there exist symmetry

$$[\mathbf{Y}, \mathbf{X}] = -(-1)^{\mathbf{X}\mathbf{Y}}[\mathbf{X}, \mathbf{Y}]$$

and Jacobi identity

$$(-1)^{\mathbf{X}\mathbf{Z}}[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + (-1)^{\mathbf{Y}\mathbf{X}}[\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]] + (-1)^{\mathbf{Z}\mathbf{Y}}[\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]] = 0.$$

For a given vector field $\mathbf{X} \in \text{Der}\Lambda(M)$, we define a map $L_{\mathbf{X}} : \text{Der}\Lambda(M) \rightarrow \text{Der}\Lambda(M)$, the Lie derivative on $\text{Der}\Lambda(M)$, as

$$L_{\mathbf{X}}\mathbf{Y} = [\mathbf{X}, \mathbf{Y}] \quad (\mathbf{Y} \in \text{Der}\Lambda(M)). \quad (23)$$

The following are trivial restatements of the properties just mentioned above. \mathbb{R} -bilinearity

$$\begin{aligned} L_{\mathbf{X}}(c_1\mathbf{Y}_1 + c_2\mathbf{Y}_2) &= c_1L_{\mathbf{X}}\mathbf{Y}_1 + c_2L_{\mathbf{X}}\mathbf{Y}_2, \\ L_{c_1\mathbf{X} + c_2\mathbf{X}_2}\mathbf{Y} &= c_1L_{\mathbf{X}_1}\mathbf{Y} + c_2L_{\mathbf{X}_2}\mathbf{Y}, \end{aligned}$$

where $c_1, c_2 \in \mathbb{R}$, symmetry

$$L_{\mathbf{Y}}\mathbf{X} = -(-1)^{\mathbf{X}\mathbf{Y}}L_{\mathbf{X}}\mathbf{Y},$$

and Leibniz rule (Jacobi identity)

$$L_{\mathbf{X}}[\mathbf{Y}, \mathbf{Z}] = [L_{\mathbf{X}}\mathbf{Y}, \mathbf{Z}] + (-1)^{\mathbf{X}\mathbf{Y}}[\mathbf{Y}, L_{\mathbf{X}}\mathbf{Z}].$$

We can naturally generalise the above mentioned Lie derivative $L_{\mathbf{X}}$ on $\text{Der}\Lambda(M)$ to obtain another kind of Lie derivative $L_{\mathbf{X}} : \Lambda(M) \rightarrow \Lambda(M)$, the Lie derivative on $\Lambda(M)$, so that

$$L_{\mathbf{X}}(f \wedge \mathbf{Y}) = L_{\mathbf{X}}f \wedge \mathbf{Y} + (-1)^{\mathbf{X}f}f \wedge L_{\mathbf{X}}\mathbf{Y} \quad (24)$$

for any $f \in \Lambda(M)$ and $\mathbf{X}, \mathbf{Y} \in \text{Der}\Lambda(M)$. The resultant definition of $L_{\mathbf{X}} : \Lambda(M) \rightarrow \Lambda(M)$ is

$$L_{\mathbf{X}}f = \mathbf{X}f \quad (f \in \Lambda(M)). \quad (25)$$

The Lie derivative on $\Lambda(M)$, just defined, is a derivation on the \mathbb{Z}_2 -graded commutative algebra $\Lambda(M)$. That is, it is \mathbb{R} -linear and satisfies Leibniz rule

$$L_{\mathbf{X}}(f \wedge g) = (L_{\mathbf{X}}f) \wedge g + (-1)^{\mathbf{X}f}f \wedge (L_{\mathbf{X}}g) \quad (26)$$

for $f, g \in \Lambda(M)$. The following formula is easily verified.

$$L_{[X, Y]}f = [L_X, L_Y]f \quad (27)$$

Similarly, there is a natural extension of the Lie derivatives L_X on $\text{Der}\Lambda(M)$ and on $\Lambda(M)$ to $L_X : \Omega(M, \mathcal{O}_M) \rightarrow \Omega(M, \mathcal{O}_M)$, the Lie derivative on $\Omega(M, \mathcal{O}_M)$. That is, for $\lambda \in \Omega^r(M, \mathcal{O}_M)$, we define $L_X \lambda \in \Omega^r(M, \mathcal{O}_M)$ so that

$$\begin{aligned} L_X \langle \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_r; \lambda \rangle &= \langle L_X \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_r; \lambda \rangle + (-1)^{X \mathbf{Y}_1} \langle \mathbf{Y}_1, L_X \mathbf{Y}_2, \dots, \mathbf{Y}_r; \lambda \rangle \\ &+ \dots + (-1)^{X(\mathbf{Y}_1 + \mathbf{Y}_2 + \dots + \mathbf{Y}_{r-1})} \langle \mathbf{Y}_1, \mathbf{Y}_2, \dots, L_X \mathbf{Y}_r; \lambda \rangle \\ &+ (-1)^{X(\mathbf{Y}_1 + \mathbf{Y}_2 + \dots + \mathbf{Y}_r)} \langle \mathbf{Y}_1, \mathbf{Y}_1, \dots, \mathbf{Y}_r; L_X \lambda \rangle \end{aligned}$$

for any $\mathbf{Y}_1, \dots, \mathbf{Y}_r \in \text{Der}\Lambda(M)$. The resultant definition of $L_X : \Omega(M, \mathcal{O}_M) \rightarrow \Omega(M, \mathcal{O}_M)$ turns out to be

$$L_X \lambda = (-1)^r (i_X d - d i_X) \lambda \quad (\lambda \in \Omega^r(M, \mathcal{O}_M)) \quad (28)$$

with $r \geq 1$. Note for the unusual sign in this formula. If λ has a bidegree $\binom{r}{s}$, its Lie derivative $L_X \lambda$ has a bidegree $\binom{r}{s + |\mathbf{X}|}$ (mod 2). Lie derivative on $\Omega(M, \mathcal{O}_M)$ satisfies

$$i(L_X \mathbf{Y}) \lambda = (L_X i_{\mathbf{Y}} - (-1)^{X \mathbf{Y}} i_{\mathbf{Y}} L_X) \lambda \quad (29)$$

where $i(\mathbf{Z})\lambda$ implies $i_{\mathbf{Z}}\lambda$.

7 Symplectic Structure

We now define our *symplectic form* $\omega \in \Omega^2(M, \mathcal{O}_M)$ so that

$$\omega = -d\varphi^a \wedge dp_a - d\psi^\alpha \wedge d\pi_\alpha \quad (30)$$

on (U, \mathcal{O}_U) . ω is a closed, non-degenerate 2-form with a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ -bidegree $|\omega| = \binom{2}{n+1}$ (mod 2). Note that when n is even, ω becomes what is called an *odd symplectic form*, and when n is odd, an *even symplectic form* [24].

The symplectic structure ω leads to natural identification between a vector field on (M, \mathcal{O}_M) and a 1-form on (M, \mathcal{O}_M) . Indeed, if we define a map $\flat : \text{Der}\Lambda(M) \rightarrow \Omega^1(M, \mathcal{O}_M)$, $\mathbf{X} \mapsto i_{\mathbf{X}}\omega$, namely

$$\flat(\mathbf{X}) = i_{\mathbf{X}}\omega, \tag{31}$$

then the map \flat gives a linear isomorphism between $\text{Der}\Lambda(M)$ and $\Omega^1(M, \mathcal{O}_M)$. We denote by \sharp the inverse map of \flat .

Definition 1 *The Hamiltonian vector field $\mathbf{X}_f \in \text{Der}\Lambda(M)$ generated by an arbitrary differentiable differential form $f \in \Lambda(M)$ is the unique vector field on (M, \mathcal{O}_M) determined by $\mathbf{X}_f = \sharp df$.*

Note that the Hamiltonian vector field \mathbf{X}_f has a \mathbb{Z}_2 -graded degree $|\mathbf{X}_f| = |f| + n + 1 \pmod{2}$. By definition, it satisfies

$$df = i_{\mathbf{X}_f}\omega. \tag{32}$$

With canonical coordinates $\varphi^a, p_a, \psi^\alpha, \pi_\alpha \in \Lambda(U)$, \mathbf{X}_f can be written as

$$\begin{aligned} \mathbf{X}_f = & (-1)^{(n+1)(f+1)} \frac{\partial f}{\partial p_a} \wedge \frac{\partial}{\partial \varphi^a} - \frac{\partial f}{\partial \varphi^a} \wedge \frac{\partial}{\partial p_a} \\ & + (-1)^{n(f+1)} \frac{\partial f}{\partial \pi_\alpha} \wedge \frac{\partial}{\partial \psi^\alpha} + (-1)^{f+n} \frac{\partial f}{\partial \psi^\alpha} \wedge \frac{\partial}{\partial \pi_\alpha}. \end{aligned} \tag{33}$$

Definition 2 *The Poisson bracket of two differentiable differential forms $f, g \in \Lambda(M)$ is the differentiable differential form $\{f, g\} \in \Lambda(M)$ defined as*

$$\begin{aligned} \{f, g\} := & -\langle \mathbf{X}_f, \mathbf{X}_g; \omega \rangle = -\langle \mathbf{X}_f; i_{\mathbf{X}_g}\omega \rangle = -i_{\mathbf{X}_f}i_{\mathbf{X}_g}\omega \\ = & -i_{\mathbf{X}_f}dg = -\langle \mathbf{X}_f; dg \rangle = -\mathbf{X}_f g = -L_{\mathbf{X}_f}g. \end{aligned} \tag{34}$$

The Poisson bracket $\{f, g\}$ has a \mathbb{Z}_2 -graded degree $|\{f, g\}| = |f| + |g| + n + 1 \pmod{2}$. The real vector space structure of $\Lambda(M)$ endowed with the Poisson bracket $\{ , \}$ form a \mathbb{Z}_2 -graded Lie algebra, as it will be obvious from formulae in what follows. The definition $\{f, g\} = -\mathbf{X}_f g$ of the Poisson bracket together with (33) directly yields

$$\begin{aligned} \{f, g\} = & -(-1)^{(n+1)(f+1)} \frac{\partial f}{\partial p_a} \wedge \frac{\partial g}{\partial \varphi^a} + \frac{\partial f}{\partial \varphi^a} \wedge \frac{\partial g}{\partial p_a} \\ & - (-1)^{n(f+1)} \frac{\partial f}{\partial \pi_\alpha} \wedge \frac{\partial g}{\partial \psi^\alpha} - (-1)^{f+n} \frac{\partial f}{\partial \psi^\alpha} \wedge \frac{\partial g}{\partial \pi_\alpha}. \end{aligned} \tag{35}$$

This result reproduces the heuristic definitions of our Poisson bracket, all of (10) and (16), completely.

Proposition 3 *The assignment $f \mapsto \mathbf{X}_f$ yields a \mathbb{Z}_2 -graded Lie algebra homomorphism from $\Lambda(M)$ to $\text{Der}\Lambda(M)$, that is,*

$$\mathbf{X}_{\{f,g\}} = -[\mathbf{X}_f, \mathbf{X}_g]. \quad (36)$$

Proof. Noticing

$$L_{\mathbf{X}_f} \omega = (i_{\mathbf{X}_f} \mathbf{d} - \mathbf{d}i_{\mathbf{X}_f}) \omega = -\mathbf{d}i_{\mathbf{X}_f} \omega = -\mathbf{d}(\mathbf{d}f) = 0, \quad (37)$$

$$L_{\mathbf{X}} \mathbf{d}g = -(i_{\mathbf{X}} \mathbf{d} - \mathbf{d}i_{\mathbf{X}}) \mathbf{d}g = \mathbf{d}i_{\mathbf{X}} \mathbf{d}g = \mathbf{d}\langle \mathbf{X}; \mathbf{d}g \rangle = \mathbf{d}\mathbf{X}g = \mathbf{d}L_{\mathbf{X}}g, \quad (38)$$

we obtain

$$\begin{aligned} i(-[\mathbf{X}_f, \mathbf{X}_g]) \omega &= -i([\mathbf{X}_f, \mathbf{X}_g]) \omega = -i(L_{\mathbf{X}_f} \mathbf{X}_g) \omega \\ &= -(L_{\mathbf{X}_f} i_{\mathbf{X}_g} - (-1)^{\mathbf{X}_f \mathbf{X}_g} i_{\mathbf{X}_g} L_{\mathbf{X}_f}) \omega = -L_{\mathbf{X}_f} i_{\mathbf{X}_g} \omega \\ &= -L_{\mathbf{X}_f} \mathbf{d}g = -\mathbf{d}L_{\mathbf{X}_f}g = \mathbf{d}\{f, g\} = i(\mathbf{X}_{\{f,g\}}) \omega. \end{aligned}$$

This completes the proof. \square

Now, it is an easy task to give concise proofs to many formulae listed in Section 4.

Symmetry (12)(18):

$$\begin{aligned} \{g, f\} &= -\langle \mathbf{X}_g, \mathbf{X}_f; \omega \rangle \\ &= -(-1)^{(f+n+1)(g+n+1)} \langle \mathbf{X}_f, \mathbf{X}_g; \omega \rangle \\ &= -(-1)^{(f+n+1)(g+n+1)} \{f, g\}. \end{aligned} \quad (39)$$

Leibniz rule I (13)(19):

$$\begin{aligned} \{f, g \wedge h\} &= -L_{\mathbf{X}_f}(g \wedge h) \\ &= (-L_{\mathbf{X}_f}g) \wedge h + (-1)^{(f+n+1)g} g \wedge (-L_{\mathbf{X}_f}h) \\ &= \{f, g\} \wedge h + (-1)^{(f+n+1)g} g \wedge \{f, h\}. \end{aligned} \quad (40)$$

Leibniz rule II (15)(21):

$$\begin{aligned} \{\{f, g\}, h\} &= -\mathbf{X}_{\{f,g\}}h = [\mathbf{X}_f, \mathbf{X}_g]h = \mathbf{X}_f \mathbf{X}_g h - (-1)^{\mathbf{X}_f \mathbf{X}_g} \mathbf{X}_g \mathbf{X}_f h \\ &= -\mathbf{X}_f \{g, h\} + (-1)^{\mathbf{X}_f \mathbf{X}_g} \mathbf{X}_g \{f, h\} = \{f, \{g, h\}\} - (-1)^{\mathbf{X}_f \mathbf{X}_g} \{g, \{f, h\}\}, \end{aligned} \quad (41)$$

from which we obtain Leibniz rule II, that is,

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + (-1)^{(f+n+1)(g+n+1)}\{g, \{f, h\}\}. \quad (42)$$

Putting this formula into a symmetric form, we get

$$\begin{aligned} (-1)^{(f+n+1)(h+n+1)}\{f, \{g, h\}\} + (-1)^{(g+n+1)(f+n+1)}\{g, \{h, f\}\} \\ + (-1)^{(h+n+1)(g+n+1)}\{h, \{f, g\}\} = 0, \end{aligned} \quad (43)$$

which, of course, is *Jacobi identity* (14)(20).

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