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# Noncommutative Structure of Massive Bosonic Strings

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**Abstract:** In this paper we will apply the Faddeev-Jackiw quantization methodology to the massive open strings in the D-brane background with a non-vanishing constant B-field.We shall work in the discrete version, and the reduced phase space is obtained directly by solving the mixed boundary conditions. The non-commutativity is extended to the noncommutativity of pace coordinates and momentum coordinates along the D-brane is reproduced in easy way.We feel that this method of obtaining the noncommutativity in string theory in the massive case is more elegant than previous approaches discussed in the literature.


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One of the interesting develoment in string theory is the realisation of noncommutative geometry [1] and this appears in the work of Witten on open string field theory [2, 3]. Noncommutative geometry arise in D-branes on constant antisymmetric tensor field [4].The main goal of this paper is giving the essential aspects of the symplectic Faddeev Jackiw Quantization and applying it for quantization of massive bosonic open string in the presence of antisymmetric tensor B-field, we find that phase space coordinates of the massive open string end-points become totally noncommutative. The Faddeev-Jackiw formulation and its equivalence with the Dirac approach is studied in detail by several authors. The fundamental commutators relations of open string coordinates $x^i$ on the branes are represented by noncommutativity ,$[x^i, x^j] = i\theta^{ij}$ where $\theta^{ij}$ is an antisymmetric constant tensor deponding on the background constant fields [7, 8, 9, 10, 11, 12, 13,

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14]. Strings attached to branes involve mixed (combination of Dirichlet and Neumann) boundary conditions. This makes the quantization procedure more subtle. The standard canonical commutation relations can not be imposed as quantum commutators as they are not consistent with the boundary conditions. This situation is analogous to that of systems with constraints, where one must build up commutators that are consistent with them. The difference is that the boundary conditions, in the form that comes from the functional variation of the string action, involve velocities. So, they do not correspond to standard Dirac constraints. If we apply directly standard Dirac procedure to the string action we would not find the boundary conditions as constraints. Nevertheless, it has been recently shown in refs.that it is possible to use the Dirac procedure as long as one rewrites the boundary conditions in terms of phase space variables and introduces them as constraints. Then the interesting result of the non commutativity of the phase space coordinates of the string end points emerges. We will see that with the symplectic quantization procedure the boundary conditions arise directly as constraints by means of a discretization of the string worldsheet spatial coordinate.We will show how to find a particular choice of symplectic variables such that the boundary conditions show up naturally as constraints[15].This way we find a straightforward way of building up the phase space coordinates commutators consistent with the mixed boundary conditions and thus reproduce the totaly noncommutativity at the string end points[17,18,19].

## 1 A brief review of the Faddeev-Jackiw method

In this section we discuss the methodology of FJ approach to quantize the singular systems. In this formalism, we first write the Lagrangian of a singular system into the first-order form as follows:).

$$L^0 = a_k^0(\xi)\partial_\tau \xi^k - V(\xi) \tag{1}$$

Where $\xi^i$ is called the symplectic variable, $V(\xi)$ is called the symplectic potential. The first-order form can be implemented by introducing some auxiliary variables $(a_k)$ such as the canonical momentum. The Euler-Lagrange equations of motion for Lagrangian (15) can be written as

$$f_{ij}^0(\xi)\dot{\xi}^j = \frac{\partial V(\xi)}{\partial \xi^i} \qquad (i = 1, 2, 3, ..., n), \tag{2}$$

where $f_{ij}$ is so-called symplectic matrix with following explicit form:

$$f_{ij}^0 = \frac{\partial a_j^0}{\partial \xi^i} - \frac{\partial a_i^0}{\partial \xi^j} \tag{3}$$

If it is non singular we define the commutators of the quantum theory (if there is no ordering problem for the corresponding quantum operators) as

$$[A(\xi), B(\xi)] = \frac{\partial A}{\partial \xi_i}(f^0)^{-1}_{ij}\frac{\partial B}{\partial \xi_j} \tag{4}$$

If the matrix (17) is singular we find the zero modes that satisfy $f^0_{ij}v^\alpha_j = 0$ and the corresponding constraints:

$$\Omega^\alpha = v^\alpha_l \frac{\partial V}{\partial \xi_l} = 0 \tag{5}$$

Now, we modify our original Lagrangian by introducing the constraint term multiplied with some Lagrange multipliers $\lambda^\alpha$ as

$$L^1 = a^0_k(\xi)\dot{\xi}_k + \dot{\lambda}^\alpha \Omega^\alpha - V(\xi) = a^1_r(\tilde{\xi})\dot{\tilde{\xi}} - V(\xi) \tag{6}$$

where we introduced the new notation for the extended variables: $\tilde{\xi}^r = (\xi^k, \lambda^\alpha)$We find now the new matrix $f^1_{ij}$

$$f^1_{ij} = \frac{\partial a^1_j}{\partial \tilde{\xi}^i} - \frac{\partial a^1_i}{\partial \tilde{\xi}^j} \tag{7}$$

If $f^1$ is not singular we define the quantum commutators as

$$[A(\tilde{\xi}), B(\tilde{\xi})] = \frac{\partial A}{\partial \tilde{\xi}_i}(f^1)^{-1}_{ij}\frac{\partial B}{\partial \tilde{\xi}_j} \tag{8}$$

This process of incorporating the constraints in the Lagrangian is repeated until a non singular matrix is found.

## 2 Review of bosonic open string in a background B-field

the electromagnetic field couples with charged open string's endpoints, as the endpoints behave like point particles. this interaction is given by :

$$\frac{1}{2\pi\alpha'}\oint_{\partial\Sigma} d\tau A_i(X)\partial_\tau X^i \tag{9}$$

Where $\partial\Sigma$ is the boundary of the string worldsheet $\Sigma$. In this case $\partial\Sigma$ represents string endpoints at any time. we have put the endpoint charge $q = 1$ at $\sigma = \pi$ and $q = -1$ at $\sigma = 0$ .

The dynamics of strings ending on a p-brane in the background of the antisymmetric field ,$B_{\mu\nu}$ is [1] :

$$S = \frac{1}{4\pi\alpha'}\int d\sigma^2[\eta_{\mu\nu}g^{ab}\partial_a X^\mu \partial_b X^\nu + \epsilon^{ab}B_{\mu\nu}\partial_a X^\mu \partial^a X^\nu] \tag{10}$$
$$+\frac{1}{2\pi\alpha'}\oint_{\partial\Sigma} d\tau A_i(X)\partial_\tau X^i$$

Where $A_i, i = 0, 1, p$ is the $U(1)$ gauge field living on the D-brane, $\sigma^\alpha = (\tau, \sigma)$ are the world-sheet coordinates ,$g_{\alpha\beta} = diag(-,+)$, $\epsilon^{01} = -\epsilon^{01} = 1$ is the anti-symmetric symbol in two dimensions,$B_{\mu\nu} = -B_{\nu\mu}$, $\eta_{\mu\nu} = diag(-,+,...,+)$, and the length of the string is $\pi$.

There is an important relation between B-field and electromagnetic potential $A_i(X)$. Let $F_{ij} = \partial_i A_j(X) - \partial_j A_i(X)$ be electromagnetic field strength. Consider

$$\begin{aligned}
&\frac{1}{4\pi\alpha'}\int_\Sigma d\sigma^2 \epsilon^{\alpha\beta} F_{ij}(X)\partial_\alpha X^i \partial_\beta X^j \\
&= \frac{1}{4\pi\alpha'}\int_\Sigma d\sigma^2 \epsilon^{\alpha\beta}(\partial_i A_j(X) - \partial_j A_i(X))\partial_\alpha X^i \partial_\beta X^j \\
&= \frac{2}{4\pi\alpha'}\int_\Sigma d\sigma^2 \epsilon^{\alpha\beta}\partial_i A_j(X)\partial_\alpha X^i \partial_\beta X^j \\
&= \frac{2}{4\pi\alpha'}\int_\Sigma d\sigma^2 \epsilon^{\alpha\beta}\frac{\partial A_j(X)}{\partial X^\alpha}.\frac{\partial X^\alpha}{\partial X^i}\partial_\alpha X^i \partial_\beta X^j \\
&= \frac{2}{4\pi\alpha'}\int_\Sigma d\sigma^2 \epsilon^{\alpha\beta}\partial_\alpha A_j(X)\partial_\beta X^j \\
&= \frac{2}{4\pi\alpha'}\int_\Sigma d\sigma^2 [\partial_\tau A_j(X)\partial_\sigma X^j - \partial_\sigma A_j(X)\partial_\tau X^j] \\
&= \frac{2}{4\pi\alpha'}\int_\Sigma d\sigma^2 [\partial_\tau (A_j(X)\partial_\sigma X^j) - \partial_\sigma (A_j(X)\partial_\tau X^j)] \\
&= 0 - \frac{2}{4\pi\alpha'}\int_{\partial\Sigma} d\tau A_j(X)\partial_\tau X^j
\end{aligned} \tag{11}$$

Where in the final step we assume that $A_j(X)$ vanishes at initial and final time $\tau$.
So the action becomes :

$$\begin{aligned}
S = -\frac{1}{4\pi\alpha'}\int_\Sigma d\sigma^2 (\eta^{\alpha\beta}\eta_{\mu\nu} + \epsilon^{\alpha\beta}B_{\mu\nu}(X))\partial_\alpha X^\mu \partial_\beta X^\nu \\
-\frac{1}{4\pi\alpha'}\int_\Sigma d\sigma^2 \epsilon^{\alpha\beta} F_{ij}(X)\partial_\alpha X^i \partial_\beta X^j
\end{aligned} \tag{12}$$

If we set the component of B-field to be parallel to the D-brane :

$$\epsilon^{\alpha\beta}B_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu = \epsilon^{\alpha\beta}B_{ij}(X)\partial_\alpha X^i \partial_\beta X^j \tag{13}$$

we call the string coordinates along the brane $X^i$ and the string coordinates normal to the brane $X^a$.
The action becomes :

$$S = -\frac{1}{4\pi\alpha'}\int_\Sigma d\sigma^2 (\eta^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu + \epsilon^{\alpha\beta}\mathbf{F}_{ij}\partial_\alpha X^i \partial_\beta X^j) \tag{14}$$

Where $\mathbf{F}_{ij} = F_{ij} + B_{ij}$.
The resulting field $\mathbf{F}_{ij}$ is invariant under gaug transformation of potential $A_i$ :
$A_i \to A_i + \partial K$,
$B_{ij} \to B_{ij}$.
And under gauge transformation of B-field $A_i \to A_i - \Lambda_i, B_{ij} \to B_{ij} + \partial_i \Lambda_j - \partial_j \Lambda_i$ Let us

now vary the action. The variation of the first part is :

$$\delta(\frac{-1}{4\pi\alpha'}\int_{\Sigma}^{2} d\sigma^2\eta^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu) \tag{15}$$
$$= \frac{-1}{2\pi\alpha'}\int_{\Sigma} d\sigma^2[-\partial_\tau X^\mu\partial_\tau\delta X_\mu + \partial_\sigma X^\mu\partial_\sigma\delta X_\mu]$$
$$= \frac{-1}{2\pi\alpha'}[\int_{\Sigma} d\sigma^2(-\partial_\tau X^\mu\delta X_\mu)\mid_\tau + \int_{\Sigma} d\sigma^2(\partial_\tau^2 - \partial_\sigma^2)X^\mu\delta X_\mu + \int_{\delta\Sigma} d\tau\partial_\sigma X^\mu\delta X_\mu]$$

The first term vanishes due to the requirement that $\delta X^\mu = 0$ at initial and final time $\tau$ . Now we vary the B-field part of the action. Inspired by the relation (3) we have :

$$\frac{-1}{4\pi\alpha'}\int_{\Sigma}\epsilon^{\alpha\beta}\mathbf{F}_{ij}\partial_\alpha X^i\partial_\alpha X^j \tag{16}$$
$$= \frac{1}{4\pi\alpha'}\int_{\Sigma} d\tau\omega_i(X)\partial_\tau X^i$$

We instead vary : $\frac{1}{4\pi\alpha'}\int_{\Sigma} d\tau\omega_i(X)\partial_\tau X^i$

$$\delta(\frac{1}{2\pi\alpha'}\int_{\partial\Sigma} d\tau\omega_i(X)\partial_\tau X^i) \tag{17}$$
$$= \frac{1}{2\pi\alpha'}\int_{\partial\Sigma} d\tau\delta\omega_i(X)\partial_\tau X^i + \frac{1}{2\pi\alpha'}\int_{\partial\Sigma} d\tau\omega_i(X)\partial_\tau\delta X^i$$
$$= \frac{1}{2\pi\alpha'}\int_{\partial\Sigma} d\tau\frac{\partial\omega_i}{\partial X^j}\delta X^j\partial_\tau X^i + \frac{1}{2\pi\alpha'}\int_{\partial\Sigma} d\tau\partial_\tau(\omega_i(X)\delta X^i) - \frac{1}{2\pi\alpha'}\int_{\partial\Sigma} d\tau\partial_\tau\omega_i(X)\delta X^i$$
$$= \frac{1}{2\pi\alpha'}\int_{\partial\Sigma} d\tau\partial_j\omega_i(X)\delta X^j\partial_\tau X^i + 0 - \frac{1}{2\pi\alpha'}\int_{\partial\Sigma} d\tau\partial_\tau\omega_i(X)\delta X^i$$

$\delta X^i = 0$ at initial and final time

$$= \frac{1}{2\pi\alpha'}\int_{\partial\Sigma} d\tau\partial_j\omega_i(X)\delta X^j\partial_\tau X^i - \frac{1}{2\pi\alpha'}\int_{\partial\Sigma} d\tau\frac{\partial\omega_i}{\partial X^j}\frac{\partial X^j}{\partial\tau}\delta X^i$$
$$= \frac{1}{2\pi\alpha'}\int_{\partial\Sigma} d\tau\partial_j\omega_i(X)\delta X^j\partial_\tau X^i - \frac{1}{2\pi\alpha'}\int_{\partial\Sigma} d\tau\partial_j\omega_i(X)\partial_\tau X^j\delta X^i$$
$$= \frac{1}{2\pi\alpha'}\int \partial\Sigma d\tau\partial_i\omega_j\delta X^i\partial_\tau X^j - \frac{1}{2\pi\alpha'}\int_{\alpha\Sigma} d\tau\partial_j\omega_i\partial_\tau X^j\delta X^i$$

We have $\mathbf{F}_{ij} = \partial_i\omega_j - \partial_j\omega_i$.
Then finally we have the reult :

$$\frac{1}{2\pi\alpha'}\int_{\partial\Sigma} d\tau\mathbf{F}_{ij}(X)\delta X^i\partial_\tau X^j \tag{18}$$

Then the variation of the full action is : (7) + (10)

$$\delta S = \frac{-1}{4\pi\alpha'}\int_{\Sigma} d\sigma^2(\partial_\tau^2 - \partial_\sigma^2)X^\mu \delta X_\mu - \frac{1}{4\pi\alpha'}\int_{\partial\Sigma} d\tau \partial_\sigma X^\mu \delta X_\mu \tag{19}$$

$$+ \frac{1}{2\pi\alpha'}\int_{\partial\sigma} d\tau \mathbf{F}_{ij}(X)\delta X^i \partial_\tau X^j$$

$$= \frac{-1}{4\pi\alpha'}\int_{\Sigma} d\sigma^2(\partial_\tau^2 - \partial_\sigma^2)X^\mu \delta X_\mu - \frac{1}{4\pi\alpha'}\int_{\partial\Sigma} d\tau \partial_\sigma X^i \delta X_i$$

$$- \frac{1}{4\pi\alpha'}\int_{\partial\Sigma} d\tau \partial_\sigma X^a \delta X_a + \frac{1}{2\pi\alpha'}\int_{\partial\sigma} d\tau \mathbf{F}_{ij}(X)\delta X^i \partial_\tau X^j$$

The requirement $\delta S = 0$ give the equation of motion $(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0$ and the boundary conditions at $\sigma = 0, \pi$ for $\mu = a$:

$$\delta X^a = 0, \quad X^a = x_0^a \tag{20}$$

$$\frac{-1}{4\pi\alpha'}\int_{\partial\sigma} d\tau \partial_\sigma X^i \delta X_i + \frac{1}{4\pi\alpha}\int_{\partial\sigma} \mathbf{F}_{ij}(X)\delta X^i \partial_\tau X^j$$

$$= \frac{-1}{4\pi\alpha'}\int_{\partial\sigma} d\tau \partial_\sigma X^i \delta X_i - \frac{1}{4\pi\alpha}\int_{\partial\sigma} \mathbf{F}_{ji}(X)\delta X^i \partial_\tau X^j$$

$$= \frac{-1}{4\pi\alpha'}\int_{\partial\Sigma} d\tau \mathbf{F}^i_j(X)\delta X_i \partial_\tau X^j - \frac{1}{4\pi\alpha'}\int_{\partial\Sigma} d\tau \partial_\tau X^i \delta X_i$$

$$= \frac{-1}{4\pi\alpha'}\int_{\partial\Sigma} d\tau (\mathbf{F}^i_j(X)\partial_\tau X^j + \partial_\sigma X^i)\delta X_i$$

So we have the boundary conditions for $\mu = i$

$$\partial_\alpha X^i + \mathbf{F}^i_j(X)\partial_\tau X^j = 0 \tag{21}$$

Where $\mathbf{F}_j{}^i = B_j{}^i + F_j{}^i$ ,we take $F_j{}^i = 0$
The boundary conditions become:

$$g_{ij}\partial\sigma X^j - B_{ij}\partial_\tau X^j = 0 \tag{22}$$

The effect of electromagnetic field is a change in the boundary conditions.

## 3 Massive Bosonic Open string in the presence of antisymmetric tensor filed and Totally Noncommutative Phase Space

Let us now see how the noncommutative structure of massive bosonic string can be calculated using the symplectic Faddeev Jackiw quantization[16,20], and see how this boundary conditions show up as equations of motion by writing down a discretized version of the lagrangian of massive bosonic string in an external B-field associated with the action gived by:

$$S = -\frac{1}{4\pi\alpha'}\int_\Sigma d\xi^2(g_{ij}\eta^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j + g_{ij}(\partial_\sigma m)^2X^iX^j + \epsilon^{\alpha\beta}B_{ij}\partial_\alpha X^i\partial_\beta X^j) \tag{23}$$

$$S = -\frac{1}{4\pi\alpha'}\int_\Sigma d\sigma^2[(\dot{X}^i)^2 - (X'^i)^2 + m'^2(X^i)^2 - 2B_{ij}\dot{X}^iX'^j]$$

The analysis will show us how to impliment boundary conditions in the symplectic quantization dividing the interval $0 < \sigma < \pi$ in $N$ intervals of length $\epsilon$ and introducing the coordinates of the endpoints of the intervals as $X_0^i, X_1^i, ..., X_N^i$ we find

$$L(\epsilon) = \frac{\epsilon}{4\pi\alpha'}\sum_n[(\dot{X}_n^i)^2 - \frac{(X_{n+1}^i - X_n^i)^2}{\epsilon^2} + \frac{m^2}{\epsilon^2}(X_n^i)^2 - 2B_{ij}\dot{X}_n^i\frac{(X_{n+1}^j - X_n^j)^2}{\epsilon}] \tag{24}$$

Where the overdot means the derivative with respect to the time-like parameter $\tau$.
The massless case ($m = 0$) is studied in different aspects by several authors [7, 8, 9, 10, 11, 12, 13, 14] with the well-known result of non commutativity at the end-points.
We develop the expression then we have :

$$\begin{aligned} L(\epsilon) = \frac{\epsilon}{4\pi\alpha'}\sum_n[(\dot{X}_0^i)^2 + (\dot{X}_1^i)^2 + ... + (\dot{X}_N^i)^2 - \frac{(X_1^i - X_0^i)^2}{\epsilon^2} - \frac{(X_2^i - X_1^i)^2}{\epsilon^2} \\ - ... - \frac{(X_N^i - X_N^i - 1)^2}{\epsilon^2} + \frac{m^2}{\epsilon^2}(X_0^i)^2 + \frac{m^2}{\epsilon^2}(X_1^i)^2 + ... + \frac{m^2}{\epsilon^2}(X_N^i)^2 \\ - 2B_{ij}\dot{X}_0^i\frac{(X_1^j - X_0^j)}{\epsilon} - 2B_{ij}\dot{X}_2^i\frac{(X_1^j - X_1^j)}{\epsilon} - ... - 2B_{ij}\dot{X}_{N-1}^i\frac{(X_N^j - X_{N-1}^j)}{\epsilon}] \end{aligned} \tag{25}$$

And we also discretize the mixed boundary condition as :

$$\frac{g_{ij}}{\epsilon}(X_1^j - X_0^j) - B_{ij}\partial_\tau X_0^j = 0 \tag{26}$$

Using the Euler Lagrange equations $\frac{d}{d\tau}(\frac{\partial L}{\partial \dot{X}^i}) - \frac{\partial L}{\partial X^i} = 0$
We get the equation of motion for respectively $X_0^i, X_n^i, X_N^i$ with $1 \leq n \leq N-1$
Then we have for $X_0^i$ :

$$\begin{aligned} &L(\epsilon) = \frac{1}{4\pi\alpha'}\sum_n[\epsilon(\dot{X}_0^i)^2 - \frac{(X_1^i - X_0^i)^2}{\epsilon} + \frac{m^2}{\epsilon}(X_0^i)^2 - 2B_{ij}\partial_\tau X_0^i(X_1^j - X_0^j)] \\ &\frac{\partial L}{\partial \dot{X}^i} = \frac{1}{2\pi\alpha'}(\epsilon(\partial_\tau X_0^i) - B_{ij}(X_i^j - X_0^j)) \\ &\frac{d}{dt}(\frac{\partial L}{\partial \dot{X}^i}) = \frac{1}{2\pi\alpha'}(\epsilon(\partial_\tau^2 X_0^i) - B_{ij}(\partial_\tau X_1^j) + B_{ij}\partial_\tau X_0^j) \\ &\frac{\partial L}{\partial X_0^i} = \frac{1}{2\pi\alpha}(\frac{(X_1^i - X_0^i)}{\epsilon} + \frac{m^2}{\epsilon}X_0^i + B_{ij}\partial_\tau X_0^i) \\ &\frac{d}{d\tau}(\frac{\partial L}{\partial \dot{X}^i}) - \frac{\partial L}{\partial X^i} = 0 \end{aligned}$$

$\Leftrightarrow$

Finally we have the equation of motion for $X_0^i$:

$$\epsilon(\partial_\tau^2 X_0^i) - \frac{(X_1^i - X_0^i)}{\epsilon} - B_{ij}\partial_\tau X_1^j + 2B_{ij}\partial_\tau X_0^j - \frac{m^2}{\epsilon} X_0^i = 0 \tag{27}$$

We do the same computing for $X_n^i$ and $X_N^i$ then we get :

$$\epsilon\partial_\tau^2 X_n^i - \frac{X_{n+1}^i - X_n^i}{\epsilon} + \frac{X_n^i - X_{n-1}^i}{\epsilon} - B_{ij}\partial_\tau X_{n-1}^j - B_{ij}\partial\tau X_{n+1}^j + 2B_{ij}\partial_\tau X_n^j - \frac{m^2}{\epsilon} X_n^i = 0 \tag{28}$$

$$\epsilon\partial_\tau^2 X_N^i + \frac{X_N^i - X_{N-1}^i}{\epsilon} - B_{ij}\partial_\tau X_{N-1}^i - \frac{m^2}{\epsilon} X_N^i = 0. \tag{29}$$

When we take the limit $\epsilon \to 0$ and consider that $X_1^i \to X_0^i$ and $X_{N-1}^i \to X_N^i$ the equations for $X_0^i$ and $X_N^i$ give the open string boundary conditions of eq.(26)
The equations for points $X_n^i$ give no contribution at order zero in $\epsilon$ but to order one in $\epsilon$ (dividing by $\epsilon$ and then taking the limit $\epsilon \to 0$) they take the form of the standard equation of motion for the string coordinates :

$$\partial_\tau^2 X_n^i - \partial_\sigma^2 X_n^i - m'^2 X_n^i = 0 \tag{30}$$

It is important to remark that if we consider some coordinate $X_n^i$ with fixed $n$ and take the limit $\epsilon \to 0$ this coordinate would tend to the end point $\sigma = 0$ Thus when we want to look at points inside the string (with a finite distance to the boundary) we must look at some $X_n^i$ but increase $n$ when we take the limit $\epsilon \to 0$ in order to look at a fixed point.
This analysis of the equations of motion shows us that in the discretised version both boundary conditions and string equations of motion show up in the set of generalized equations of motion but at different orders in the discretization parameter $\epsilon$.This will help us to find an appropriate definition of the symplectic variables such that the boundary conditions lead to zero modes in the symplectic matrix.
The standard way of writing the Lagrangian $L$ of eq. (25) in a first order form would be to introduce the conjugate momenta $\Pi$ as symplectic variables and eliminate the second order time derivatives. The conjugate momenta associated with the string coordinates are :

$$\Pi_0^i = \frac{1}{2\pi\alpha'}(\epsilon\partial_\tau X_0^i - B_{ij}(X_1^j - X_0^j)), \tag{31}$$

$$\Pi_n^i = \frac{1}{2\pi\alpha'}(\epsilon\partial_\tau X_n^i - B_{ij}(X_{n+1}^j - X_n^j)), \tag{32}$$

$$\Pi_N^i = \frac{1}{2\pi\alpha'}(\epsilon\partial_\tau X_N^i) \tag{33}$$

The symplectic first order Lagrangian is then:
for the first term we have

$$L^0(first\ term) = \frac{\epsilon}{4\pi\alpha'}((\partial_\tau X_0^i)^2 - \frac{(X_1^i - X_0^i)2}{\epsilon^2} - 2B_{ij}\partial_\tau X_0^i \frac{(X_1^j - X_0^j)}{\epsilon} + \frac{m^2}{\epsilon}(X_0^i)^2)$$

We have the first symplectic variable:

$$\Pi_0^i = \frac{1}{2\pi\alpha'}(\epsilon\partial_\tau X_0^i - B_{ij}(X_1^j - X_0^j)),$$
$$2\pi\alpha'\Pi_0^i = \epsilon\partial_\tau X_0^i - B_{ij}(X_1^j - X_0^j)$$
$$L^0(first\ term) = \frac{\epsilon}{4\pi\alpha'}(2\pi\alpha' P_0^i + B_{ij}\frac{(X_1^j - X_0^j)}{\epsilon})^2 - \frac{1}{4\pi\alpha'}\frac{(X_1^i - X_0^i)^2}{\epsilon}$$
$$+\epsilon(P_0^i - \frac{1}{2\pi\alpha'}\partial_\tau X_0^i)\partial_\tau X_0^i + \frac{m^2(X_0^i)^2}{4\pi\alpha'\epsilon}$$
$$= \epsilon\eta_{ij}P_0^i\partial_\tau X_0^j - \frac{\epsilon}{4\pi\alpha'}(2\pi\alpha' P_0^i + B_{ij}\frac{(X_1^j - X_0^j)}{\epsilon})^2 - \frac{1}{4\pi\alpha'}\frac{(X_1^i - X_0^i)^2}{\epsilon} + \frac{m^2(X_0^i)^2}{4\pi\alpha'\epsilon}$$

We do the same computing for the other terms , finally we find the symplectic first order lagrangian :

$$L^0 = \eta_{ij}(\epsilon P_0^i\partial_\tau X_0^j + P_1^i\partial_\tau X_1^j + ... + P_n^i\partial_\tau X_n^j + ... + P_{N-1}^i\partial_\tau X_{N-1}^j + \epsilon P_N^i\partial_\tau X_N^j) + V \tag{34}$$

Where

$$\begin{aligned} V = &-\frac{1}{4\pi\alpha'}(\frac{(X_1^i - X_0^i)^2}{\epsilon} + \frac{(X_2^i - X_1^i)^2}{\epsilon} + ... + \frac{(X_N^i - X_{N-1}^i)^2}{\epsilon}) \\ &- \frac{\epsilon}{4\pi\alpha'}[(2\pi\alpha' P_0^i + B_{ij}\frac{(X_1^j - X_0^j)}{\epsilon})^2 + (\frac{2\pi\alpha'}{\epsilon}P_1^i + B_{ij}\frac{(X_2^j - X_1^j)}{\epsilon})^2 ... \\ &(\frac{2\pi\alpha'}{\epsilon}P_{N-1}^i + B_{ij}\frac{(X_N^j - X_{N-1}^j)}{\epsilon})^2 + (2\pi\alpha' P_N^i)^2 \\ &- \frac{m^2}{\epsilon}(X_0^i)^2 - \frac{m^2}{\epsilon}(X_1^i)^2 - ... - \frac{m^2}{\epsilon}(X_N^i)^2] \end{aligned} \tag{35}$$

We build up the symplectic matrix $f_0$ with our coordinates $X, P$

$$f_0 = \begin{array}{c} \\ X_0^i \\ P_0^i \\ X_1^i \\ P_1^i \\ ... \\ X_n^i \\ P_n^i \\ ... \\ X_{N-1}^i \\ P_{N-1}^i \\ X_N^i \\ P_N^i \end{array} \begin{array}{c} \begin{array}{cccccccccccc} X_0^j & P_0^j & X_1^j & P_1^j & ... & X_n^j & P_n^j & ... & X_{N-1}^j & P_{N-1}^j & X_N^j & P_N^j \end{array} \\ \begin{pmatrix} 0 & -\epsilon\delta^{ij} & 0 & 0 & ... & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ \epsilon\delta^{ij} & 0 & 0 & 0 & ... & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta^{ij} & ... & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta^{ij} & 0 & ... & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & .. & ... \\ 0 & 0 & 0 & 0 & ... & 0 & -\delta^{ij} & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & \delta^{ij} & 0 & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & ... & 0 & -\delta^{ij} & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & ... & \delta^{ij} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & ... & 0 & 0 & 0 & -\delta^{ij} \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & ... & 0 & 0 & \epsilon\delta^{ij} & 0 \end{pmatrix} \end{array}$$

In the limit $\epsilon \to 0$ the string theory is recovered the matrix becomes singular with the zero modes.

$$v_1^i = \begin{pmatrix} u^i \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad v_2^i = \begin{pmatrix} 0 \\ u^i \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad v_3^i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ u^i \\ 0 \end{pmatrix} \quad v_4^i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ u^i \end{pmatrix} \tag{36}$$

For $v_1^i$ the corresponding constraint come from $\frac{\partial V}{\partial X_0^j}$

$$\frac{\partial V}{\partial X_0^j} = \frac{1}{2\pi\alpha'}\frac{(X_1^i - X_0^i)}{\epsilon}\delta_{ij} + \frac{\epsilon}{2\pi\alpha'}(2\pi\alpha' P_0^i + B_{ij}\frac{(X_1^j - X_0^j)}{\epsilon})\frac{B_{ij}}{\epsilon} + \frac{m^2}{2\pi\alpha'\epsilon}X_0^i = 0$$
$$\frac{(X_1^i - X_0^i)}{\epsilon}\delta_{ij} + 2\pi\alpha' B_{ij}P_0^i + B_{ij}\frac{(X_1^j - X_0^j)}{\epsilon}B_{ij} + \frac{m^2}{\epsilon}X_0^i\delta_{ij} = 0$$
$$\frac{(X_1^k - X_0^k)}{\epsilon}(\delta_{kj} - B_{ki}B_{ij}) + \frac{m^2}{\epsilon}X_0^k\delta_{kj} - 2\pi\alpha' B_{jk}P_0^k = 0$$

Then we find the first constraint:

$$\Omega_1^i = \frac{(X_1^k - X_0^k)}{\epsilon} + \frac{m^2}{\epsilon}X_0^k M_{kj}^{-1} - 2\pi\alpha' M_{kj}^{-1}B_{jk}P_0^k = 0 \tag{37}$$

in the $\epsilon \to 0$ limit this gives the finit result

$$(\partial_\sigma X^k - 2\pi_\alpha' M_{kj}^{-1}B_{jk}P_0^k)|_{\sigma=0} = 0 \tag{38}$$

Where $M_{kj} = \delta_{kj} - B_{ki}B_{ij}$.
For $v_2^i$ the corresponding constraint come from $\frac{\partial V}{\partial P_0^j}$,same computing leads to the second constraint:

$$\Omega_2^i = -\epsilon(2\pi\alpha' P_0^i + B_{ij}\frac{(X_1^j - X_0^j)}{\epsilon}) = 0 \tag{39}$$

So,this actually gives no contraint as $\epsilon \to 0$
For $v_3^i$ the corresponding contraint is :

$$\Omega_3^i = \frac{(X_N^k - X_{N-1}^k)}{\epsilon} + \frac{m^2}{\epsilon}X_N^k M_{kj}^{-1} - 2\pi\alpha' M_{kj}^{-1}B_{jk}P_{N-1}^k = 0 \tag{40}$$

That leads, when $\epsilon \to 0$ to the finite result :

$$\partial_\sigma X^i - 2\pi\alpha' M_{kj}^{-1}B_{jk}P_N^k \; |_{\sigma=0} = 0 \tag{41}$$

The last constraint will give no contribution when $\epsilon \to 0$

$$\Omega_4^i = -\epsilon 2\pi\alpha' P_N^i = 0. \tag{42}$$

In order to incorporate the contraints $\Omega_1^i$ and $\Omega_3^i$ into the symplectic formalism we introduce lagrange multplier variables $\lambda_1^i$ and $\lambda_3^i$and add a new term to the lagrangian

$$\begin{aligned} L_1 = L_0 + \dot{\lambda_1^i}(\frac{(X_1^k - X_0^k)}{\epsilon} + \frac{m^2}{\epsilon} X_0^k M_{kj}^{-1} - 2\pi\alpha' M_{kj}^{-1} B_{jk} P_0^k) \\ +\dot{\lambda_3^i}(\frac{(X_N^k - X_{N-1}^k)}{\epsilon} + \frac{m^2}{\epsilon} X_N^k M_{kj}^{-1} - 2\pi\alpha' M_{kj}^{-1} B_{jk} P_{N-1}^k) \end{aligned} \tag{43}$$

Thus, from the symplectic Lagrangian (43) we identify the following set of symplectic variables as $q^i = \{X_0^i, P_0^i, ..., X_n^i, P_n^i, ..., X_N^i, P_N^i, \lambda_1^i, \lambda_3^i\}$ and the components of the symplectic 1-forms are :
$a_i = \{\epsilon P_0^i, 0, P_n^i, 0, ..., \epsilon P_N^i, 0, (\frac{(X_1^k - X_0^k)}{\epsilon} + \frac{m^2}{\epsilon} X_0^k M_{kj}^{-1} - 2\pi\alpha' M_{kj}^{-1} B_{jk} P_0^k)$
$, (\frac{(X_N^k - X_{N-1}^k)}{\epsilon} + \frac{m^2}{\epsilon} X_N^k M_{kj}^{-1} - 2\pi\alpha' M_{kj}^{-1} B_{jk} P_{N-1}^k)\}$

By performing somme technical commputing we get this result :

$$\begin{array}{c|cccccccccc} & X_0^j & P_0^j & ... & X_n^j & P_n^j & ... & X_N^j & P_N^j & \lambda_1^j & \lambda_3^j \\ \hline X_0^i & 0 & -\epsilon\delta^{ij} & ... & 0 & 0 & ... & 0 & 0 & \frac{\delta_{ij}}{\epsilon} + \frac{m^2}{\epsilon} M_{ij}^{-1} & 0 \\ P_0^i & \epsilon\delta^{ij} & 0 & ... & 0 & 0 & ... & 0 & 0 & \Gamma^{ij} & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ X_n^i & 0 & 0 & ... & 0 & -\delta^{ij} & ... & 0 & 0 & 0 & 0 \\ P_n^i & 0 & 0 & ... & \delta^{ij} & 0 & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & & \\ X_N^i & 0 & 0 & ... & 0 & 0 & ... & 0 & -\epsilon\delta^{ij} & 0 & -\frac{\delta_{ij}}{\epsilon} + \frac{m^2}{\epsilon} M_{ij}^{-1} \\ P_N^i & 0 & 0 & ... & 0 & 0 & ... & \epsilon\delta^{ij} & 0 & 0 & \Gamma^{ij} \\ \lambda_1^i & -\frac{\delta_{ij}}{\epsilon} - \frac{m^2}{\epsilon} M_{ij}^{-1} & \Gamma^{ij} & ... & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ \lambda_3^j & 0 & 0 & ... & 0 & 0 & ... & \frac{\delta_{ij}}{\epsilon} - \frac{m^2}{\epsilon} M_{ij}^{-1} & \Gamma^{ij} & 0 & 0 \end{array}$$

Where $\Gamma^{ij} = -2\pi\alpha'(M^{-1}B)_{ij}$

As this symplectic matrix is non singular the invers is

The elements of the inverse will be the corresponding commutators . The relevant ones are :

$$
\begin{aligned}
&[X_0^i, P_0^j] = \frac{\delta^{ij}}{2} \\
&[X_0^i, X_0^j] = \frac{-\Gamma^{ij}}{2(\delta^{ij} + m^2 M_{ij}^{-1})} \\
&[P_0^i, P_0^j] = -\frac{m^2 B_{ij}^{-1}}{4\pi\alpha'} + \frac{(\Gamma^{ij})^{-1}}{2} \\
&[X_n^i, X_n^j] = 0 \\
&[P_n^i, P_n^j] = 0 \\
&[P_0^i, X_0^j] = \frac{-\delta^{ij}}{2} \\
&[P_N^i, P_N^j] = \frac{m^2 B_{ij}^{-1}}{4\pi\alpha'} - \frac{(\Gamma^{ij})^{-1}}{2} \\
&[X_N^i, X_N^j] = \frac{\Gamma^{ij}}{2(\delta^{ij} + m^2 M_{ij}^{-1})}
\end{aligned} \tag{44}
$$

Where $X_n^i$ represents a point that has a finite distance to the end points of the string. We see that on the D-brane the whole phase space becomes noncommutative. We call this kind of noncommutativiy as totally noncommutative phase space.

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