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Neimark-Sacker and Closed Invariant Curve Bifurcations of A Two Dimensional Map Used For Cryptography

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Abstract: The purpose of this paper is to study dynamics and bifurcations of a family of two-dimensional noninvertible maps used for cryptosystems. Especially, the bifurcation of Neimark-Sacker is analysed algebraically and illustrated by numerical simulations. Furthermore, global bifurcations caused when a closed invariant curve intersects the critical lines are observed by simulation. On the other hand, several chaotic attractors have been observed in the phase plane for some particular values of the parameters.

Keywords: Discrete dynamical systems; noninvertible maps; critical curves; Neimark-Sacker bifurcation; basins of attraction; global bifurcations; chaotic attractors
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1 Introduction and basic terminologies

Let us consider the recurrence relation: $X_{n+1} = T_\xi(X_n) = F(X_n, \xi)$, where $\xi = (a, b) \in \mathbb{R}^2$ denotes a parameter, $X_n = (x_n, y_n) \in \mathbb{R}^2$ and $F = (f, g)$ is a smooth function. Explicitly, T_ξ has the form:

$$T_\xi : \begin{cases} x_{n+1} = f(x_n, y_n, \xi) \\ y_{n+1} = g(x_n, y_n, \xi) \end{cases} \tag{1}$$

For a given positive integer k, the *image* of rank k of $X \in \mathbb{R}^2$ is, by definition, $X' = T_\xi^k(X)$. Similarly, a *preimage* of rank k of $X' \in \mathbb{R}^2$ is a point $X \in \mathbb{R}^2$ satisfying $T_\xi^k(X) = X'$.

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For a given $k \in \mathbb{N}$ and a given $X' \in \mathbb{R}^2$, we denote by $T_\xi^{-k}(X')$ the set of the preimages of rank k of X'. When the inverse mapping $x = T_\xi^{-1}(x)$ is uniquely determined, we say that T_ξ is *invertible*; otherwise, T_ξ is said to be a *noninvertible* map. If the map T_ξ is noninvertible, we have the notion of *critical variety* (*critical point*, *critical line*, *critical curve*, etc):

1.1 Critical curves

The so-called *critical curves* are specific to noninvertible two-dimensional maps. It was introduced by C. Mira in 1964 (see [5]). If $T = T_\xi$ is noninvertible, We call *critical curve* of T of rank 1, the locus LC (abreviation of critical line) of points having at least two coincident preimages of rank 1. The locus of points which are coincident preimages of points of LC is denoted by LC_{-1} and called the set of *merging preimages*. When especially the map T is differentiable, the set LC_{-1} is simply the set of the points of $\mathbb{R}^2$ in which the jacobian determinant of T vanishes. For $k \in \mathbb{N}$, The critical curve of rank k of T, denoted LC_{k-1}, is defined by $LC_{k-1} := T^k(LC_{-1})$. The critical curve LC can be constituted of one or several branches. These branches separate the plane of phases in opened regions denoted by Z_i; each region Z_i corresponds to the set of points which have exactly i preimages of rank 1 by T. We refer to [5] for more precisions.

1.2 Phase plane of type $(Z_1 - Z_3 - Z_1)$

The so-called two-dimensional $(Z_1 - Z_3 - Z_1)$ maps (see [5], [4]) are such that the plane is divided into three unbounded open regions: a region Z_3, whose points generate three real rank-one preimages, bordered by two regions Z_1, whose points generate only one real rank-one preimage. For such noninvertible maps, the critical curve LC is discontinuous and constituted of two disjoint straight lines L and L' dividing the plane of phases into three regions: Z_1^1, Z_1^2 and Z_3.

1.3 Bifurcation of Neïmark-Sacker

The Neimark-Sacker bifurcation occurs when a stable focus of order k loses its stability as a bifurcation parameter is varied with the consequent birth of a closed invariant curve. The bifurcation can be *supercritical* or *subcritical*, resulting in a stable or unstable (within an invariant two-dimensional manifold) closed invariant curve, respectively (see [6, Chap 4]). Let us consider a as the bifurcation parameter and b fixed; so we have the two following cases:

- In the subcritical case: A repelling closed invariant curve Γ exists surrounding the stable fixed point, for $a < a_0$. As a increases the repelling closed curve decreases in size and shrinks merging with the fixed point at $a = a_0$; leaving a repelling focus.
- In the supercritical case: At $a = a_0$ the fixed point becomes an unstable focus and for $a > a_0$ an attracting closed invariant curve Γ exists, surrounding the unstable

fixed point.
If the smooth function F has the fixed point $x = x_*(a)$ with simple eigenvalues $\lambda_{1,2} = r(a)e^{\pm i\theta_0(a)}$, $0 < \theta_0(a) < \pi$, then the Neïmark-Sacker bifurcation is obtained by solving the equation $r(a) = 1$ and then by verifying that the solution $a = a_c$ satisfies the non-degeneracy conditions:

$$\frac{dr}{da}(a_c) \neq 0 \quad \text{and} \quad e^{ik\theta(a_c)} \neq 1 \text{ for } k = 1, 2, 3, 4$$

(see Theorem A of Section 2.2 below).

1.4 Bifurcation of a closed invariant curve

This bifurcation is generated by the transformation of an invariant closed curve, born from a focus fixed point of a non-invertible map $T = T_\xi$ of the plane via a supercritical Neïmark-Sacker bifurcation, when some parameter is gradually moved away from its bifurcation value. Just after the Neïmark-Sacker bifurcation, an attracting invariant closed curve, say Γ, appears around the unstable focus. While $\Gamma \cap LC_{-1} = \emptyset$, the curve Γ, as well as the area of the phase plane enclosed by Γ, say $\mathscr{A}(\Gamma)$, is both forward invariant (under T) and backward invariant (under T^{-1}, where T^{-1} is an inverse of T). The situation changes when Γ touches the set of merging preimages LC_{-1}, that is when $\Gamma \cap LC_{-1} \neq \emptyset$. As soon as Γ comes into contact with LC_{-1} in a point A_0, that is $\Gamma \cap LC_{-1} = \{A_0\}$, this bifurcation appears for some value a_0 of the bifurcation parameter a. The image A_1 of A_0 by T is a point of contact between Γ and LC. For $a = a_0 + \epsilon$, where $\epsilon > 0$ is sufficiently small, we obtain that $\Gamma \cap LC_{-1} = \{A_0^1, A_0^2\}$ and the area $\mathscr{A}(\Gamma)$ is no longer forward invariant; in addition, we will observe the creation of convolutions of Γ. Those properties are related to the fact that curves crossing LC_{-1} are folded along LC. The iterate of rank n of the point A_0, denoted A_n, is a tangential point of contact between Γ and $T^n(LC_{-1}) = LC_{n-1}$. This bifurcation creates oscillations of Γ along the LC_n's and it is responsible of the changes of the form of Γ. When the bifurcation parameter grows more, another phenomenon can be observed; that is the appearance of knots, or loops or self intersections of the unstable set of the saddle belonging to the closed invariant curve. Then this situation is followed by homoclinic situations (intersections between the stable and unstable sets of the saddle) which leads to a chaotic attractor (see e.g., [2], [1]).

2 A study of a nonlinear two dimensional map used for cryptography

Using nonlinear non-invertible two dimensional maps, it is possible to generate pseudo-random sequences, from which interesting cryptosystems can be born. Such cryptosystems rely on the consideration of chaotic signal properties resulting from those nonlinear maps. The model we present here is proposed in the study of chaotic signals in the area of communications as well as in the signal processing. This model is implemented on a Digital Signal Processor (DSP), which resists all the attacks we have thought of (see [3]); it consists on the dynamic system generated by the cubic map $T_\xi : (x, y) \mapsto (x', y')$, defined by:

$$T_\xi : \begin{cases} x' = y \\ y' = a(x - x^3) + b(y - y^3) \end{cases},$$

(where $\xi = (a, b) \in \mathbb{R}^2$ is a parameter, with $a \neq 0$) .

We begin with the following:

Proposition 2.1. *The map T_ξ is non-invertible of the type $Z_1 - Z_3 - Z_1$. Precisely, we have:*

$$LC_{-1} = \left\{ (x, y) \in \mathbb{R}^2 : \; x = \pm\frac{\sqrt{3}}{3} \right\} \qquad and \qquad LC = L^1 \cup L^2,$$

where L^1 and L^2 are the curves respectively defined by the equations $y = b(x - x^3) + \frac{2\sqrt{3}}{9}a$ and $y = b(x - x^3) - \frac{2\sqrt{3}}{9}a$.

Proof 2.2. The locus LC_{-1} of the coincident preimages of rank 1 of T_ξ is given by

$$LC_{-1} = \left\{ (x, y) \in \mathbb{R}^2 : \; \det(JT_\xi(x, y)) = 0 \right\},$$

where $JT_\xi(x, y)$ denotes the Jacobian matrix of T_ξ at the point (x, y). The calculation gives:

$$JT_\xi(x, y) = \begin{pmatrix} 0 & 1 \\ a(1 - 3x^2) & b(1 - 3y^2) \end{pmatrix}$$

and then

$$\det(JT_\xi(x, y)) = a(3x^2 - 1).$$

Hence:

$$LC_{-1} = \left\{ (x, y) \in \mathbb{R}^2 : \; 3x^2 - 1 = 0 \right\} = \left\{ (x, y) \in \mathbb{R}^2 : x = \pm\frac{\sqrt{3}}{3} \right\}.$$

So, the locus LC_{-1} is constituted of two branches L^1_{-1} and L^2_{-1}, where $L^1_{-1} = \{(x, y) \in \mathbb{R}^2 : \; x = \frac{\sqrt{3}}{3}\}$ and $L^2_{-1} = \{(x, y) \in \mathbb{R}^2 : \; x = -\frac{\sqrt{3}}{3}\}$. Consequently, the critical line $LC = T_\xi(LC_{-1})$ is constituted of the two distinct curves $L^1 := T_\xi(L^1_{-1})$ and $L^2 := T_\xi(L^2_{-1})$, limiting an open region Z_3 for which any point has three preimages of rank 1 and two open regions Z_1 for which any point has a unique preimage of rank 1. The calculation gives:

$$L^1 := T_\xi(L^1_{-1}) = \left\{ (x, y) \in \mathbb{R}^2 : \; y = b(x - x^3) + \frac{2\sqrt{3}}{9}a \right\}$$

$$L^2 := T_\xi(L^2_{-1}) = \left\{ (x, y) \in \mathbb{R}^2 : \; y = b(x - x^3) - \frac{2\sqrt{3}}{9}a \right\}.$$

This completes the proof of the proposition.

Graphical representation of the critical lines

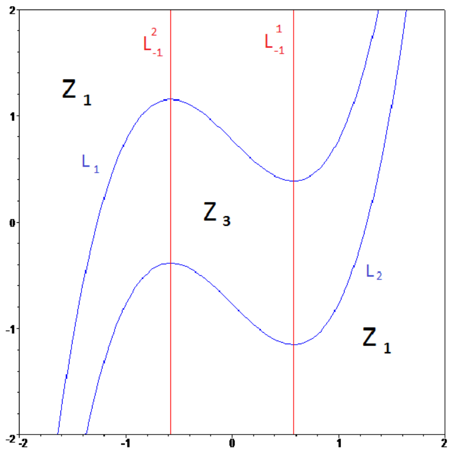


Figure 1 The critical lines of T_ξ for $a = 2$ and $b = -1$

2.1 Existence of fixed points and local stability of the origin

The results about the existence of fixed points of T_ξ are given by the following proposition:
Proposition 2.3.

(i) *If* $a + b \in [0, 1)$*, then the map* T_ξ *has a unique fixed point* $O(0, 0)$*, which is simple.*
(ii) *If* $a + b = 1$*, then the map* T_ξ *has a unique fixed point* $O(0, 0)$*, which is triple.*
(iii) *If* $a+b \in (-\infty, 0) \cup (1, +\infty)$*, then the map* T_ξ *has three simple fixed points:* $O(0, 0)$*,* $P\left(\sqrt{\frac{a+b-1}{a+b}}, \sqrt{\frac{a+b-1}{a+b}}\right)$ *and* $Q\left(-\sqrt{\frac{a+b-1}{a+b}}, -\sqrt{\frac{a+b-1}{a+b}}\right)$.

Proof 2.4. A fixed point $\mathbf{x} = (x, y)$ of T_ξ satisfies $(y, a(x - x^3) + b(y - y^3)) = (x, y)$. Simplifying this, we get that $y = x$ and x satisfies the polynomial equation:

$$x\left[(a + b - 1) - (a + b)x^2\right] = 0.$$

The results of the proposition follow.

2.1.1 Stability of the origin

Now, we will state the topological classification of the fixed point $O(0, 0)$ according to the values of the parameters a and b.

Proposition 2.5. *For the fixed point* $O(0, 0)$*, the following topological classifications hold:*

(1) $O(0, 0)$ *is a saddle if one of these conditions is realized:*

(a) $a < 0$*,* $b \leq -2$ *and* $a > b + 1$*.*
(b) $a < 0$*,* $b > 0$ *and* $a + b > 1$*.*
(c) $a > 0$*,* $b \leq -2$ *and* $a + b < 1$*.*
(d) $a > 0$*,* $-2 < b < 0$*,* $a > b + 1$ *and* $a + b < 1$*.*
(e) $a > 0$*,* $b \geq 2$ *and* $a < b + 1$*.*

(f) $a > 0$, $-2 < b < 0$, $a + b < 1$ *and* $a > b + 1$.
(g) $a > 0$, $0 < b < 2$, $a + b > 1$ *and* $a < b + 1$.

(2) $O(0, 0)$ *is a sink if one of these conditions is realized:*
(a) $b^2 + 4a > 0$, $a < 0$, $-2 < b < 0$ *and* $a < b + 1$.
(b) $b^2 + 4a > 0$, $a < 0$, $0 < b < 2$ *and* $a + b < 1$.
(c) $a > 0$, $-2 < b < 0$ *and* $a < b + 1$.
(d) $a > 0$, $b > 0$ *and* $a + b < 1$.
(e) $b^2 + 4a = 0$ *and* $|b| < 2$.

(3) $O(0, 0)$ *is an unstable node if one of these conditions is realized:*
(a) $b^2 + 4a > 0$, $a < 0$, $b \leq -2$ *and* $a < b + 1$.
(b) $b^2 + 4a > 0$, $a < 0$, $b \geq 2$ *and* $a + b < 1$.
(c) $b \leq 0$ *and* $a + b > 1$.
(d) $b \geq 0$ *and* $a > b + 1$.
(e) $b \leq 0$ *and* $a + b > 1$.
(f) $b^2 + 4a = 0$ *and* $|b| > 2$.

(4) $O(0, 0)$ *is a stable focus if* $b^2 + 4a < 0$ *and* $a > -1$.
(5) $O(0, 0)$ *is an unstable focus if* $b^2 + 4a < 0$ *and* $a < -1$.
(6) $O(0, 0)$ *is non-hyperbolic if* $a + b = 1$ *or* $a - b = 1$ *or* $a = -1$ *and* $b \in [-2, 2]$.

Proof 2.6. The jacobian matrix of T_ξ at $O(0, 0)$ is given by:

$$JT_\xi(0, 0) = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}.$$

The characteristic polynomial of $JT_\xi(0, 0)$ is then given by:

$$P(\lambda) = \lambda^2 - b\lambda - a$$

and has the discriminant $\Delta = b^2 + 4a$. The results follow from the study of the sign of Δ and from the comparison of the modules of the two complexes eigenvalues of $JT_\xi(0, 0)$ to 1.

2.2 Neimark-Sacker bifurcation

We begin by finding the necessary and sufficiently condition for the equilibrium P to be non-hyperbolic. Note that the results obtained below (Proposition 2.7 and Theorem 2.9) become true for the equilibrium Q.

Proposition 2.7. *Let* $(a, b) \in \mathbb{R}^2$ *such that* $a + b \in (-\infty, 0) \cup (1, +\infty)$. *Then the fixed point* $P\left(\sqrt{\frac{a+b-1}{a+b}}, \sqrt{\frac{a+b-1}{a+b}}\right)$ *is non-hyperbolic if and only if one of the two following conditions holds:*

- $a^2 - b^2 - a + 2b = 0$.
- $a \in (-\infty, -1) \cup (1, +\infty)$ *and* $b = \frac{-2a^2+4a}{2a-1}$.

Proof 2.8. The jacobian matrix of T_ξ at P is given by:

$$JT_\xi(P) = \begin{pmatrix} 0 & 1 \\ a\left(\frac{-2a-2b+3}{a+b}\right) & b\left(\frac{-2a-2b+3}{a+b}\right) \end{pmatrix}$$

and has the characteristic polynomial

$$\chi(\lambda) = \lambda^2 + b\left(\frac{2a+2b-3}{a+b}\right)\lambda + a\left(\frac{2a+2b-3}{a+b}\right).$$

Now, we distinguish the three following cases according to the sign of the discriminant Δ of χ.

1^st^ case: (If $\Delta \geq 0$). In this case, the point P is non-hyperbolic if and only if one of the roots of χ is equal to 1 or -1; that is $\chi(-1) = 0$ or $\chi(1) = 0$.
Since $a + b \in (-\infty, 0) \cup (1, +\infty)$, it is easy to see that $\chi(1) \neq 0$. So P is non-hyperbolic if and only if $\chi(-1) = 0$; which gives the condition

$$a^2 - b^2 - a + 2b = 0.$$

2^nd^ case: (If $\Delta < 0$). In this case, χ has two complex conjugate roots λ_1 and λ_2 $(\lambda_2 = \overline{\lambda_1})$. So, P is non-hyperbolic if and only if $|\lambda_1| = 1$. But since

$$|\lambda_1| = 1 \Leftrightarrow |\lambda_1|^2 = 1 \Leftrightarrow \lambda_1\overline{\lambda_1} = 1 \Leftrightarrow \lambda_1\lambda_2 = 1$$

and $\lambda_1\lambda_2 = a\left(\frac{2a+2b-3}{a+b}\right)$, it follows that P is non-hyperbolic if and only if

$$a\left(\frac{2a+2b-3}{a+b}\right) = 1 \tag{1}$$

which gives

$$b = \frac{-2a^2+4a}{2a-1} \tag{2}$$

On the other hand, the condition $\Delta < 0$ is equivalent to:

$$b^2\left(\frac{2a+2b-3}{a+b}\right)^2 - 4a\left(\frac{2a+2b-3}{a+b}\right) < 0.$$

By substituting in this last equation $\frac{2a+2b-3}{a+b}$ by $\frac{1}{a}$ (according to (1)), we get (after simplifying):

$$b^2 - 4a^2 < 0,$$

that is

$$(b+2a)(b-2a) < 0.$$

Next, by substituting in this last inequality b by $\frac{-2a^2+4a}{2a-1}$ (according to (2)), we get

$$\left(\frac{2a^2+2a}{2a-1}\right)\left(\frac{-6a^2+6a}{2a-1}\right) < 0,$$

that is

$$(1+a)(1-a) < 0,$$

which holds if and only if $a \in (-\infty, -1) \cup (1, +\infty)$.
In conclusion, P is non-hyperbolic (in this case) if and only if $a \in (-\infty, -1) \cup (1, +\infty)$ and $b = \frac{-2a^2+4a}{2a-1}$.
This completes the proof of the proposition.

From Proposition 2.7, it is established that $P(\sqrt{\frac{a+b-1}{a+b}}, \sqrt{\frac{a+b-1}{a+b}})$ is non-hyperbolic when $a \in (-\infty, -1) \cup (1, +\infty)$ and $b = \frac{-2a^2+4a}{2a-1}$. Henceforth, we choose a as a bifurcation parameter to study Neimark-Sacker bifurcation of T in the small neighbourhood of P. We obtain the following:

Theorem 2.9. *For any $a_0 \in (-\infty, -1) \cup (1, +\infty) \setminus \{\frac{5}{4}, 2\}$ and $b_0 = \frac{-2a_0^2+4a_0}{2a_0-1}$, we have a Neimark-Sacker bifurcation of $T_{(a_0,b_0)}$ about the equilibrium $P_0(\sqrt{\frac{a_0+b_0-1}{a_0+b_0}}, \sqrt{\frac{a_0+b_0-1}{a_0+b_0}}) = (\sqrt{\frac{a_0+1}{3a_0}}, \sqrt{\frac{a_0+1}{3a_0}})$. In addition, this bifurcation is supercritical if $a_0 \in (-\infty, r_1) \cup (1, \frac{5}{4}) \cup (\frac{5}{4}, 2) \cup (2, +\infty)$ and it is subcritical if $a_0 \in (r_1, -1)$ (where $r_1 \simeq -3.415895$).*

To prove this theorem, we use the following theorem for Neimark-Sacker bifurcation in two dimensions:

Theorem A (Generic Neimark-Sacker bifurcation [6, Chap 4])
For any generic two-dimensional one parameter system

$$X \mapsto F(X, a),$$

having at $a = 0$ the fixed point $X_0 = (0, 0)$ with complex multipliers $\lambda_{1,2} = e^{\pm i\theta_0}$, there is a neighbourhood of X_0 in which a unique closed invariant curve bifurcates from X_0 as a passes through zero.
The system has to satisfy the following genericity conditions:
(1) $r'(0) \neq 0$, where $\lambda_{1,2}(a) = r(a)e^{\pm i\theta(a)}$, $r(0) = 1$, $\theta(0) = \theta_0$.
(2) $e^{\pm ik\theta_0} \neq 1$ for $k = 1, 2, 3, 4$.
(3) $d(0) \neq 0$, where $d(0)$ is the first Lyapunov coefficient which is given by:

$$d(0) = \Re\left(\frac{e^{-i\theta_0} g_{21}}{2}\right) - \Re\left(\frac{(1-2e^{i\theta_0})e^{-2i\theta_0}}{2(1-e^{i\theta_0})} g_{20} g_{11}\right) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2,$$

where g_{ij} $(i, j = 0, 1, 2)$ is the coefficient of $Z^i \overline{Z}^j$ in a specific form of F (see [6, Chap 4] for more details).
In addition, if $d(0) < 0$ then we have a supercritical Neimark-Sacker bifurcation and if $d(0) > 0$, we have a subcritical Neimark-Sacker bifurcation.

Proof 2.10 (Proof of Theorem 2.9)**.** Let a be a parameter which varies in a small neighbourhood of a_0. Let $\lambda_{1,2}(a)$ be the multipliers corresponding to the couple (a, b_0)

and $r(a) := |\lambda_{1,2}(a)|$. So $\lambda_{1,2}(a)$ are the roots of the characteristic polynomial of $JT_\xi(P)$, where $P(\sqrt{\frac{a+b_0-1}{a+b_0}}, \sqrt{\frac{a+b_0-1}{a+b_0}})$. That is $\lambda_{1,2}(a)$ are the roots of

$$\chi(\lambda) = \lambda^2 + b_0 \left(\frac{2a + 2b_0 - 3}{a + b_0}\right) \lambda + a \left(\frac{2a + 2b_0 - 3}{a + b_0}\right).$$

If a is sufficiently close to a_0, this polynomial has a negative discriminant, so $\lambda_{1,2}(a)$ are complex conjugates. Then we get

$$r(a)^2 = \lambda_1(a)\overline{\lambda_1(a)} = \lambda_1(a)\lambda_2(a) = a \left(\frac{2a + 2b_0 - 3}{a + b_0}\right).$$

Thus

$$r(a) = \sqrt{a \left(\frac{2a + 2b_0 - 3}{a + b_0}\right)}.$$

By deriving this with respect to a, we get:

$$r'(a) = \frac{\frac{2a+2b_0-3}{a+b_0} + \frac{3}{(a+b_0)^2} a}{2\sqrt{a\left(\frac{2a+2b_0-3}{a+b_0}\right)}}.$$

Particularly, we have:

$$r'(a_0) = \frac{\frac{2a_0+2b_0-3}{a_0+b_0} + \frac{3}{(a_0+b_0)^2} a_0}{2\sqrt{a_0\left(\frac{2a_0+2b_0-3}{a_0+b_0}\right)}}.$$

Replacing b_0 by $\frac{-2a_0^2+4a_0}{2a_0-1}$, we get:

$$r'(a_0) = \frac{2}{3a_0}\left(a_0^2 - a_0 + 1\right),$$

showing that

$$r'(a_0) \neq 0 \tag{3}$$

Next, solving $\chi(\lambda) = 0$ for $a = a_0$, we obtain the explicit formulas:

$$\lambda_{1,2}(a_0) = \frac{-b_0}{2a_0} \pm i\sqrt{1 - \frac{b_0^2}{4a_0^2}}.$$

Since the discriminant of $\chi(\lambda)$ (for $a = a_0$) is negative, we have $\lambda_{1,2}(a_0) \neq 1$ and $\lambda_{1,2}(a_0)^2 \neq 1$. It remains to show that $\lambda_{1,2}(a_0)^3 \neq 1$ and that $\lambda_{1,2}(a_0)^4 \neq 1$. We have

$$\begin{aligned}\lambda_{1,2}(a_0)^3 = 1 \Longleftrightarrow \lambda_{1,2}(a_0) = e^{\pm\frac{2\pi i}{3}} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \Longleftrightarrow -\frac{b_0}{2a_0} = -\frac{1}{2} \\ \Longleftrightarrow b_0 = a_0 \Longleftrightarrow \frac{-2a_0^2 + 4a_0}{2a_0 - 1} = a_0 \Longleftrightarrow 4a_0^2 - 5a_0 = 0 \\ \Longleftrightarrow a_0 = 0 \text{ or } a_0 = \frac{5}{4}.\end{aligned}$$

Since the values 0 and $\frac{5}{4}$ are excluded for a_0, we have certainly $\lambda_{1,2}(a_0)^3 \neq 1$.
Similarly, we have:

$$\lambda_{1,2}(a_0)^4 = 1 \Longleftrightarrow \lambda_{1,2}(a_0) = \pm i \Longleftrightarrow -\frac{b_0}{2a_0} = 0 \Longleftrightarrow b_0 = 0$$

$$\Longleftrightarrow \frac{-2a_0^2 + 4a_0}{2a_0 - 1} = 0 \Longleftrightarrow a_0 \in \{0, 2\}.$$

Since the values 0 and 2 are excluded for a_0, we have $\lambda_{1,2}(a_0)^4 \neq 1$. In conclusion, we have:

$$\lambda_{1,2}(a_0)^k \neq 1 \text{ for } k = 1, 2, 3, 4 \tag{4}$$

The relations (3) and (4) respectively correspond to the conditions 1. and 2. of Theorem A. Now, we are going to verify the condition 3. of Theorem A. To do so, we first transform the equilibrium $P_0(\sqrt{\frac{a_0+b_0-1}{a_0+b_0}}, \sqrt{\frac{a_0+b_0-1}{a_0+b_0}})$ of $T_{(a_0,b_0)}$ to the origin. Putting $\tau := \sqrt{\frac{a_0+b_0-1}{a_0+b_0}}$, the calculation gives the new system:

$$\begin{cases} x' = y \\ y' = a_0\left[x + \tau - (x+\tau)^3\right] + b_0\left[(y+\tau) - (y+\tau)^3\right] - \tau \end{cases}.$$

Since $1 - 3\tau^2 = -\frac{1}{a_0}$, the last system is equivalent to

$$T' : \begin{cases} x' = y \\ y' = -x - \frac{b_0}{a_0}y - 3a_0\tau x^2 - 3b_0\tau y^2 - a_0x^3 - b_0y^3 \end{cases}.$$

Then, following Kuznetsov [6], we must transform T' by putting

$$\begin{pmatrix} x \\ y \end{pmatrix} = 2\,\Re\left((X + iY)\mathbf{q}\right),$$

where $\mathbf{q}$ is the eigenvector corresponding to the eigenvalue $\lambda_1 = -\frac{b_0}{2a_0} + i\sqrt{1 - \frac{b_0^2}{4a_0^2}}$ of the jacobian matrix of T' at the origin. The calculation gives:

$$\mathbf{q} = \begin{pmatrix} 1 \\ -\frac{b}{2a} + i\sqrt{1 - \frac{b^2}{4a^2}} \end{pmatrix}.$$

So, the transformation we need to use is:

$$\begin{cases} x = 2X \\ y = -\frac{b_0}{a_0}X - 2\sqrt{1 - \frac{b_0^2}{4a_0^2}}Y \end{cases}.$$

Using this last one, T' is transformed to:

$$T'' : \begin{cases} X' = -\frac{b_0}{2a_0}X - \sqrt{1-\frac{b_0^2}{4a_0^2}}Y \\ Y' = \sqrt{1-\dfrac{b_0^2}{4a_0^2}}X - \dfrac{b_0}{2a_0}Y + \displaystyle\sum_{2\leq i+j\leq 3} c_{ij}X^iY^j \end{cases},$$

where the c_{ij}'s are the real numbers given by:

$$\begin{aligned} c_{20} &= \frac{12a_0^3+3b_0^3}{2a_0^2\sqrt{1-\frac{b_0^2}{4a_0^2}}}\tau \ ; \quad c_{11} = \frac{6b_0^2}{a_0}\tau \ ; \quad c_{02} = 6b_0\tau\sqrt{1-\frac{b_0^2}{4a_0^2}} \ ; \\ c_{30} &= \frac{8a_0^4-b_0^4}{2a_0^3\sqrt{1-\frac{b_0^2}{4a_0^2}}} \ ; \quad c_{21} = -3\frac{b_0^3}{a_0^2} \ ; \ c_{12} = -6\frac{b_0^2}{a_0}\sqrt{1-\frac{b_0^2}{4a_0^2}} \ ; \\ c_{03} &= -4b_0\left(1-\frac{b_0^2}{4a_0^2}\right). \end{aligned}$$

Next, by introducing the complex variables $Z = X + iY$ and $Z' = X' + iY'$, the system T'' is transformed to:

$$Z' = \lambda_1 Z + \sum_{2\leq k+\ell\leq 3} g_{k\ell}Z^k\overline{Z}^\ell,$$

where the $g_{k\ell}$'s are the complex numbers given by:

$$\begin{aligned} g_{20} &= \tfrac{i}{4}c_{20} + \tfrac{1}{4}c_{11} - \tfrac{i}{4}c_{02} \ ; \qquad g_{11} = \tfrac{i}{2}c_{20} + \tfrac{i}{2}c_{02} \ ; \\ g_{02} &= \tfrac{i}{4}c_{20} - \tfrac{1}{4}c_{11} - \tfrac{i}{4}c_{02} \ ; \qquad g_{30} = \tfrac{i}{8}c_{30} + \tfrac{1}{8}c_{21} - \tfrac{i}{8}c_{12} - \tfrac{1}{8}c_{03} \ ; \\ g_{21} &= \tfrac{3}{8}ic_{30} + \tfrac{1}{8}c_{21} + \tfrac{i}{8}c_{12} + \tfrac{3}{8}c_{03} \ ; \ g_{12} = \tfrac{3}{8}ic_{30} - \tfrac{1}{8}c_{21} + \tfrac{i}{8}c_{12} - \tfrac{3}{8}c_{03} \ ; \\ g_{03} &= \tfrac{i}{8}c_{30} - \tfrac{1}{8}c_{21} - \tfrac{i}{8}c_{12} + \tfrac{1}{8}c_{03}. \end{aligned}$$

The expressions of g_{20}, g_{11}, g_{02} and g_{21} in terms of a_0 and b_0 are given by:

$$\begin{aligned} g_{20} &= \frac{3b_0^2}{2a_0}\tau + i\frac{12a_0^3+6b_0^3-12a_0^2b_0}{8a_0^2\sqrt{1-\frac{b_0^2}{4a_0^2}}}\tau \ ; \ g_{11} = \frac{3(a_0+b_0)}{\sqrt{1-\frac{b_0^2}{4a_0^2}}}\tau i \ ; \\ g_{02} &= -\frac{3b_0^2}{2a_0}\tau + i\frac{6a_0^3+3b_0^3-6a_0^2b_0}{4a_0^2\sqrt{1-\frac{b_0^2}{4a_0^2}}}\tau \ ; \ g_{21} = -\frac{3}{2}b_0 + i\frac{3(2a_0^2-b_0^2)}{4a_0\sqrt{1-\frac{b_0^2}{4a_0^2}}}. \end{aligned}$$

Then, using those formulas, we find that the first Lyapunov coefficient $d(0)$ is given by:

$$\begin{aligned} d(0) &:= \Re\left(\frac{e^{-i\theta_0}g_{21}}{2}\right) - \Re\left(\frac{(1-2e^{i\theta_0})e^{-2i\theta_0}}{2(1-e^{i\theta_0})}g_{20}g_{11}\right) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2 \\ &= \Re\left(\frac{\lambda_2 g_{21}}{2}\right) - \Re\left(\frac{(1-2\lambda_1)\lambda_2^2}{2(1-\lambda_1)}g_{20}g_{11}\right) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2 \\ &= \frac{3}{4}\frac{a_0}{(4a_0^2-b_0^2)(a_0+b_0)}\left[49a_0^3 + 40a_0^2b_0 - 4a_0b_0^2 - 4b_0^3 - 45a_0^4\right. \\ &\quad \left. -81a_0^3b_0 - 33a_0^2b_0^2 + 6a_0b_0^3 + 3b_0^4\right]. \end{aligned}$$

By substituting b_0 by $\frac{-2a_0^2+4a_0}{2a_0-1}$ in the last expression of $d(0)$, we get:

$$d(0) \;=\; -\frac{3}{16}\frac{a_0(8a_0^3+24a_0^2-14a_0-9)}{(2a_0-1)(a_0-1)} \;=\; -\frac{3}{2}a_0\frac{(a_0-r_1)(a_0-r_2)(a_0-r_3)}{(2a_0-1)(a_0-1)},$$

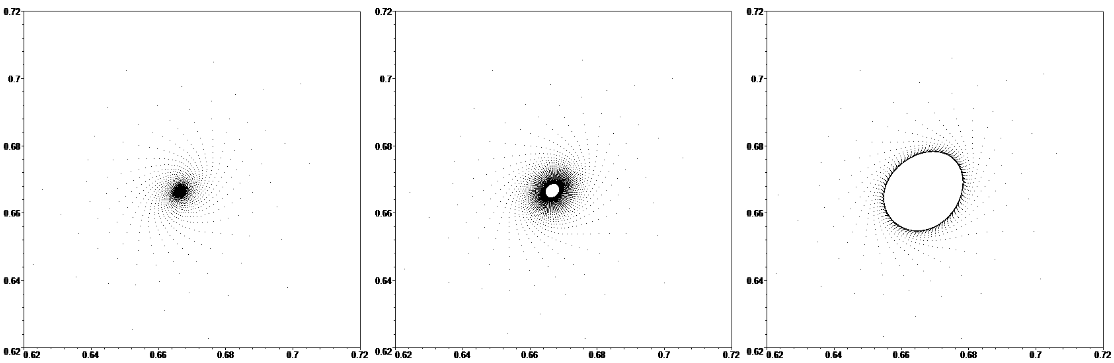
where $r_1 = -3.415895\ldots$, $r_2 = -0.402449\ldots$ and $r_3 = 0.818345\ldots$.
Consequently, we have:

$$\begin{cases} d(0) < 0 & \text{if } a_0 \in (-\infty, r_1) \cup (1, \frac{5}{4}) \cup (\frac{5}{4}, 2) \cup (2, +\infty) \\ d(0) > 0 & \text{if } a_0 \in (r_1, -1) \end{cases},$$

showing (according to Theorem A) that we have a Neimark-Sacker bifurcation of $T_{(a_0,b_0)}$ about the equilibrium $P_0(\sqrt{\frac{a_0+b_0-1}{a_0+b_0}}, \sqrt{\frac{a_0+b_0-1}{a_0+b_0}})$, which is supercritical if $a_0 \in (-\infty, r_1) \cup (1, \frac{5}{4}) \cup (\frac{5}{4}, 2) \cup (2, +\infty)$ and subcritical if $a_0 \in (r_1, -1)$. This completes the proof of the theorem.

2.3 Numerical simulations

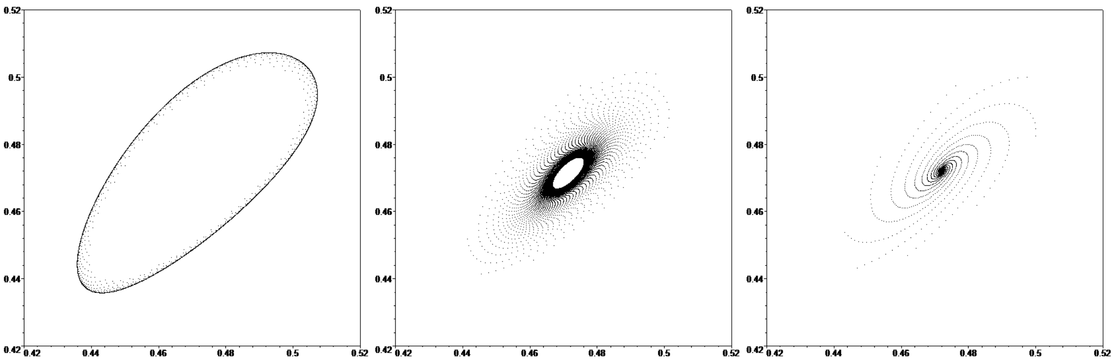
Now, we will give some numerical simulations for the system T_ξ to support our theoretical results. for $a_0 = 3 \in (-\infty, -1) \cup (1, +\infty) \setminus \{\frac{5}{4}, 2\}$, we find $b_0 = \frac{-2a_0^2+4a_0}{2a_0-1} = -1.2$. So, If we fix $b = -1.2$, then $a = 3$ is a bifurcation value. Figure 2 shows that the equilibrium is stable if $a < 3$, loses its stability at $a = 3$ and an attracting invariant closed curve appears from the equilibrium when $a > 3$. This is a **supercritical** Neimark-Sacker bifurcation (see Figure 2 below).



$a = 2.999$ and initial value $(0.001, 0.002)$. $a = 3$ and initial value $(0.001, 0.002)$. $a = 3.001$ and initial value $(0.001, 0.002)$.

Figure 2 Phase portraits of system T_ξ

Next, for $a_0 = -3 \in (-\infty, -1) \cup (1, +\infty) \setminus \{\frac{5}{4}, 2\}$, we find $b_0 = \frac{-2a_0^2+4a_0}{2a_0-1} = 30/7$. So, if we fix $b = 30/7$ then $a = -3$ is a bifurcation value. Figure 5 shows that the equilibrium is asymptotically stable if $a < -3$ and unstable for $a \geq -3$ (weakly at $a = -3$), while a unique and unstable closed invariant curve exists for $a < -3$. This is a **subcritical** Neimark-Sacker bifurcation (see Figure 3 below).



$a = -3.001$ and initial value $(0.5, 0.5)$. $a = -3$ and initial value $(0.5, 0.5)$. $a = -2.999$ and initial value $(0.5, 0.5)$.

Figure 3 Phase portraits of system T_ξ

2.4 The existence and the stability of cycles of the system T_ξ

2.4.1 The case of cycles of order 2

A cycle of order 2 of T_ξ is a point $M(x, y)$ of $\mathbb{R}^2$, satisfying $T_\xi \circ T_\xi(M) = M$ and $T_\xi(M) \neq M$. The calculation of $T_\xi \circ T_\xi$ gives $T_\xi \circ T_\xi(x, y) = (x'', y'')$ such that:

$$\begin{cases} x'' = a(x - x^3) + b(y - y^3) \\ y'' = a(y - y^3) + b\left[a(x - x^3) + b(y - y^3) - \left(a(x - x^3) + b(y - y^3)\right)^3\right] \end{cases}.$$

So, a cycle $M(x, y)$ of order 2 of T_ξ must satisfy the system

$$\begin{cases} x = a(x - x^3) + b(y - y^3) \\ y = a(y - y^3) + b(x - x^3) \end{cases},$$

with the condition $x \neq y$.

We observe that if we impose to M the additional condition $y = -x$ (with $x \neq 0$), a simple calculation gives (if $a - b \in (-\infty, 0) \cup (1, +\infty)$) the two solutions:

$$(x, y) = \left(\sqrt{1 - \frac{1}{a - b}}, -\sqrt{1 - \frac{1}{a - b}}\right) \quad \text{and} \quad \left(-\sqrt{1 - \frac{1}{a - b}}, \sqrt{1 - \frac{1}{a - b}}\right).$$

So, we obtain the following

Proposition 2.11. *If $a - b \in (-\infty, 0) \cup (1, +\infty)$ then the map T_ξ has at least the cycle of order 2: $C = \{M_1, M_2\}$, where*

$$M_1\left(\sqrt{1 - \frac{1}{a - b}}, -\sqrt{1 - \frac{1}{a - b}}\right) \quad \textit{and} \quad M_2\left(-\sqrt{1 - \frac{1}{a - b}}, \sqrt{1 - \frac{1}{a - b}}\right).$$

The following proposition establishes a necessary and sufficient condition for this cycle $C = \{M_1, M_2\}$ of T_ξ to be non-hyperbolic.

Proposition 2.12. *Let* $(a,b) \in \mathbb{R}^2$ *such that* $a-b \in (-\infty,0) \cup (1,+\infty)$. *Then the preceding cycle* $C=\{M_1,M_2\}$ *of* T_ξ *is non-hyperbolic if and only if one of the following conditions holds:*

- $a^2-b^2-a-2b=0$.
- $a \in (-\infty,-1) \cup (1,+\infty)$ *and* $b=\frac{2a^2-4a}{2a-1}$.

Proof 2.13. The jacobian matrix of T_ξ at M_i $(i=1,2)$ is given by:

$$JT_\xi(M_i) = \begin{pmatrix} 0 & 1 \\ a\left(-2+\frac{3}{a-b}\right) & b\left(-2+\frac{3}{a-b}\right) \end{pmatrix},$$

which has the characteristic polynomial

$$\chi(\lambda) = \lambda^2 + b\left(2-\frac{3}{a-b}\right)\lambda + a\left(2-\frac{3}{a-b}\right).$$

Now, we distinguish the three following cases according to the sign of the discriminant Δ of χ.

1^st^ case: (If $\Delta \geq 0$). In this case, the cycle C is non-hyperbolic if and only if one of the roots of χ is equal to 1 or -1; that is $\chi(-1)=0$ or $\chi(1)=0$.

Since $a-b \in (-\infty,0) \cup (1,+\infty)$, it is easy to see that $\chi(-1) \neq 0$. So C is non-hyperbolic if and only if $\chi(1)=0$; which gives the condition

$$a^2-b^2-a-2b=0.$$

2^nd^ case: (If $\Delta < 0$). In this case, χ has two complex conjugate roots λ_1 and λ_2 $(\lambda_2=\overline{\lambda_1})$. So, the cycle C is non-hyperbolic if and only if $|\lambda_1|=1$. But since

$$|\lambda_1|=1 \Leftrightarrow |\lambda_1|^2=1 \Leftrightarrow \lambda_1\overline{\lambda_1}=1 \Leftrightarrow \lambda_1\lambda_2=1$$

and $\lambda_1\lambda_2 = a\left(2-\frac{3}{a-b}\right)$, it follows that C is non-hyperbolic if and only if

$$a\left(2-\frac{3}{a-b}\right)=1 \tag{5}$$

which gives

$$b=\frac{2a^2-4a}{2a-1} \tag{6}$$

On the other hand, the condition $\Delta<0$ is equivalent to:

$$b^2\left(2-\frac{3}{a-b}\right)^2 - 4a\left(2-\frac{3}{a-b}\right) < 0.$$

By substituting in this last equation $2-\frac{3}{a-b}$ by $\frac{1}{a}$ (according to (5)), we get (after simplifying):

$$b^2-4a^2<0,$$

that is

$$(b+2a)(b-2a)<0.$$

Next, by substituting in this last inequality b by $\frac{2a^2-4a}{2a-1}$ (according to (6)), we get

$$\left(\frac{6a^2-6a}{2a-1}\right)\left(\frac{-2a^2-2a}{2a-1}\right)<0,$$

that is

$$(a+1)(a-1)>0,$$

which is realized if and only if $a \in (-\infty,-1)\cup(1,+\infty)$.
In conclusion, for this case, C is non-hyperbolic if and only if $a \in (-\infty,-1)\cup(1,+\infty)$ and $b=\frac{2a^2-4a}{2a-1}$.
This completes the proof of the proposition.

2.4.2 The general case

Figure 4 below shows the existence of attractive cycles of orders 1 to 14. The black areas correspond to the values of the couple (a,b) for which there is no cycle of order ≤ 14. These black areas may correspond to the existence of chaotic attractors and the white areas correspond to the non-existence of attractor in the phase plane.

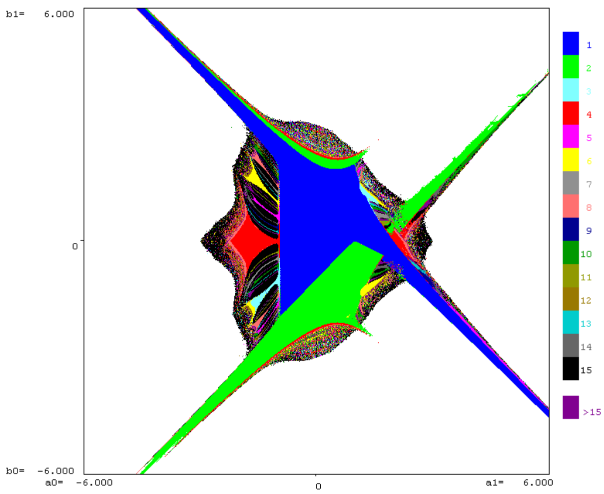


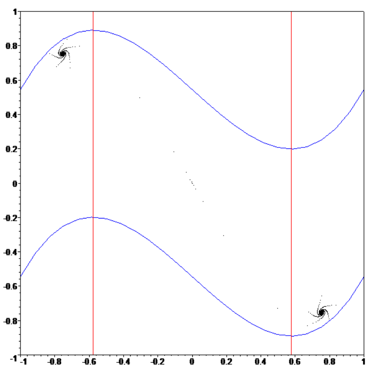
Figure 4 Stability and existence of attractive cycles of T_ξ

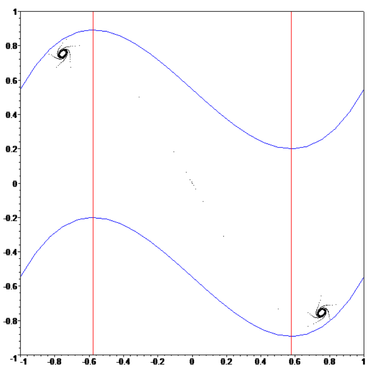
2.5 Numerical simulations concerning some other types of bifurcations

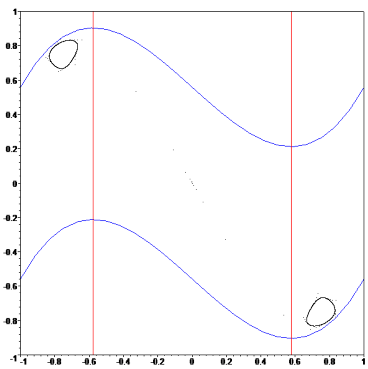
Now, we present some numerical simulations including Neimark-Sacker bifurcations of cycles of order 2 and bifurcations of closed invariant curves of T_ξ. Furthermore, several chaotic attractors are observed for some particular values of a and b.

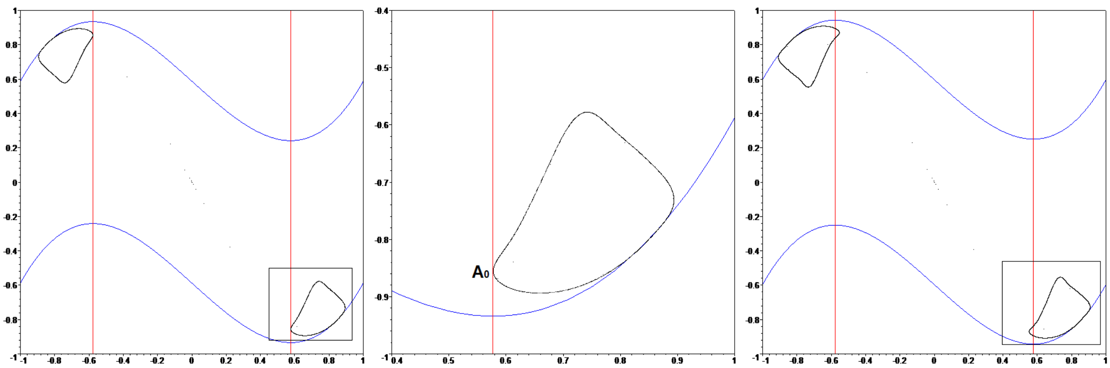
Let $a_0 = 1.417 \in (-\infty,-1)\cup(1,+\infty)$ and $b_0 = \frac{2a_0^2-4a_0}{2a_0-1} = -0.9$. By fixing b equal to b_0 and varying the parameter a from the value $1.415 < a_0$ to the value $2.18 > a_0$, we have observed the following situations which we have illustrated in the figures below, inserted chronologically and in which the lines coloured red and blue respectively correspond to the critical curves LC_{-1} and LC. Note that all these figures are obtained using Maple software, starting from the initial value $(x, y) = (0.001, 0.002)$ and using 10 000 iterations. Note also that the figures labelled by indexed letters (like $(\boldsymbol{d_1})$) are obtained by zooming a part of the figures which are labelled by those letters.

(1) For $a = 1.415$, we have a stable cycle of order 2 of focus type: $C = \{M_1, M_2\}$, where $M_1(-0.7536\ldots, 0.7536\ldots)$ and $M_2(0.7536\ldots, -0.7536\ldots)$ (see Figure $(\boldsymbol{a})$).
(2) For $a_0 := 1.417$, we observe the appearance of a Neimark-Sacker bifurcation from the preceding cycle C (see Figure $(\boldsymbol{b})$).
(3) For $a_0 < a < a_1 := 1.527$, we observe an invariant cycle of closed curves $\{\Gamma^1, \Gamma^2\}$ born from the destabilization of the cycle C (see Figure $(\boldsymbol{c})$).
(4) For $a = a_1 := 1.527$, the closed curve Γ^1 comes into contact with the branch L^1_{-1}; hence a_1 is a bifurcation value of an invariant cycle of closed curves (see Figures $(\boldsymbol{d})$ and $(\boldsymbol{d_1})$). The same phenomenon holds for the second closed curve Γ^2.
(5) For $a = a_2 := 1.55$, the closed curve Γ^1 comes into contact with L^1_{-1} in A_0 and B_0; hence a_2 is a bifurcation value of an invariant cycle of closed curves (see Figures $(\boldsymbol{e})$, $(\boldsymbol{e_1})$ and $(\boldsymbol{e_2})$).
(6) For $a_2 < a \leq 1.65$, the curve Γ^1 cuts L^1_{-1} at several points and the preceding bifurcation creates oscillations of Γ^1 along the L^2_n's. For example, for $a_3 := 1.6$ and $a_4 := 1.65$, the successive iterations of rank n of the couple $(A_0, B_0) \in \Gamma^1 \cap L^1_{-1}$ by T_ξ are the tangential points of contact between Γ^1 and L^2_n; this fact is responsible for changing the shape of Γ^1 (see Figures $(\boldsymbol{f})$, $(\boldsymbol{f_1})$, $(\boldsymbol{g})$ and $(\boldsymbol{g_1})$).
(7) For $a_5 := 1.999 \leq a \leq 2.18$, we observe the appearance of chaotic attractors (see Figures $(\boldsymbol{h})$, $(\boldsymbol{i})$ and $(\boldsymbol{j})$).
(8) For $(a, b) \in \{(-2.14, -0.90), (-3.41, 4.70), (3.16, -1.20)\}$, other chaotic attractors appear in the phase plane (see Figures $(\boldsymbol{k})$, $(\boldsymbol{l})$ and $(\boldsymbol{m})$).

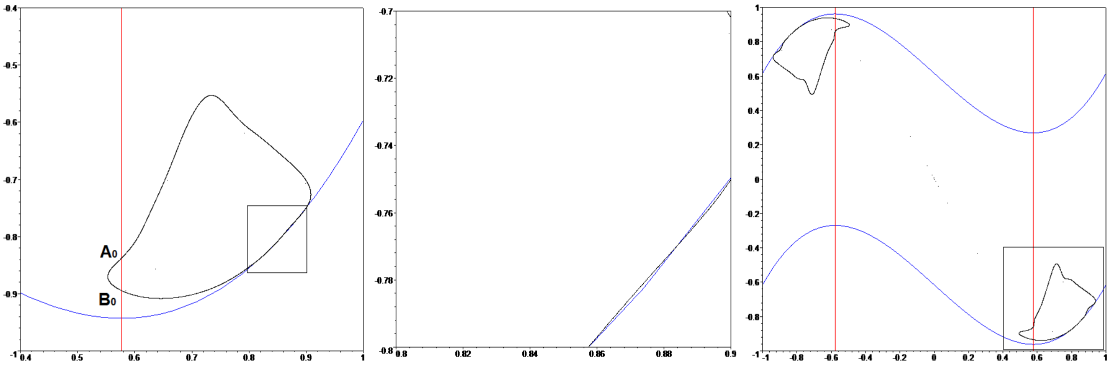

$(\boldsymbol{a})$ **:** $a = 1.415$


$(\boldsymbol{b})$ **:** $a = 1.417$

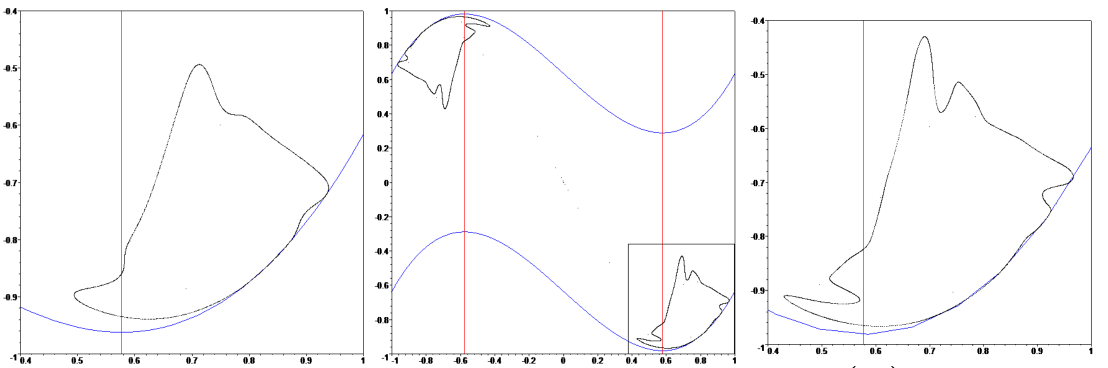

$(\boldsymbol{c})$ **:** $a = 1.45$



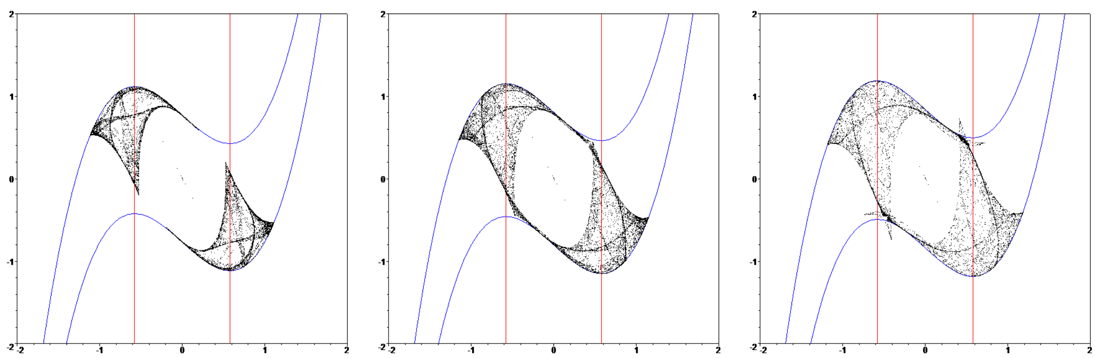
(d) : $a = 1.527$ (d_1) : $a = 1.527$ (e) : $a = 1.55$



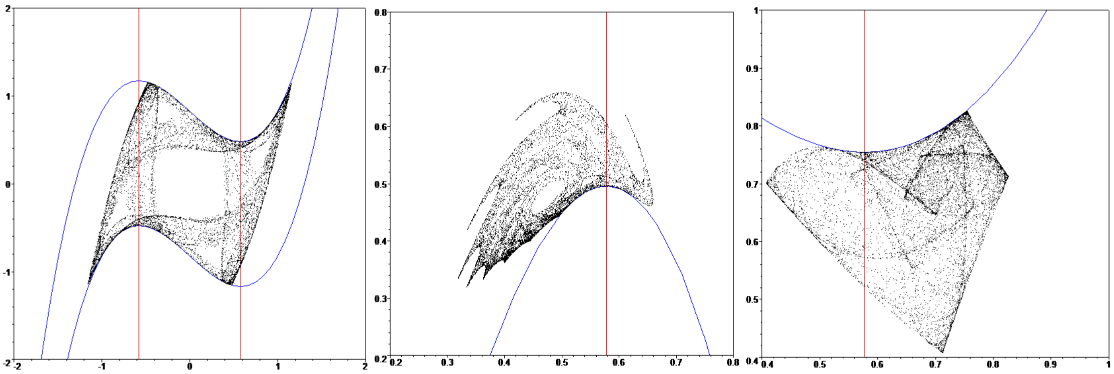
(e_1) : $a = 1.55$ (e_2) : $a = 1.55$ (f) : $a = 1.6$



(f_1) : $a = 1.6$ (g) : $a = 1.65$ (g_1) : $a = 1.65$



(h) : $a = 1.999$ (i) : $a = 2.09$ (j) : $a = 2.18$



(k) **:** $a = -2.14$ and $b = -0.9$ ***(l)*** **:** $a = -3.412$ and $b = 4.70$ ***(m)*** **:** $a = 3.16$ and $b = -1.2$

Figure 5 Phase portraits of system (T)

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