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# Electromagnetic Media in pp-wave Spacetime

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**Abstract:** The physical configuration of a dielectric transformation media is investigated, whilst it is subjected in a spacetime formed by gravitational waves, namely the pp-wave. Furthermore, such media is also manipulated to resemble the spacetime itself, and we deal with peculiar cases of transformations, exploited by the media.


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## 1 Introduction

The connection between theory of transformation optics (TO) and engineered metamaterials [1, 2, 3] stems from the capability of these materials to be designed flexibly, to have variety of applications which are proposed by the theory. These may include cloaking devices [4], invisibility devices [5, 6] and perfect lenses [7]. It may also become important to develop a joint between general theory of relativity and man-made metamaterials. The importance of general relativistic modifications has been noted in engineering [8], for example in Global Positioning System (GPS). However, when fabrication of an optical device is desired which has to undergo general relativistic effects (for example, an orbiting telescope containing a superlens [9], or a satellite antenna based on TO [10]), these modifications has to be applied in interpreting the behavior of electromagnetic fields in materials, when for instance it is considered in curved spacetime [11]. Here it should be noted that despite the fact that general relativity and TO basically share a same mathematical language, it is a crucial task to fully understand an optical device in the context of general relativity, since TO is discussed on a fixed background, whereas general relativity is a dynamic theory of spacetime.

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This is the original idea of transformation optics. The underlying physics of TO, firstly proposed by Eddington [12], is that the trajectory of light in an arbitrary vacuum curved spacetime, could be regenerated in an appropriate dielectric media, residing in Minkowski spacetime. Later, Gordon [13] discussed the way of finding the analog curved vacuum spacetime, out of a definite dielectric, using an effective optical metric. However it was Plebanski [14] who found out that the constitutive equations for electromagnetic fields in an arbitrary vacuum curved spacetime, are equivalent to those in an appropriate dielectric media in Minkowski spacetime. Further, De Felice [15] and afterwards Reznik [16], used this equivalence in studying gravitational systems and De Felice generalized it to Friedmann-Robertson-walker spacetime in a universe with no spatial curvature. Note that, TO is based on transformation media performing coordinate transformations, which were initially considered to be purely spatial transformations [6, 17, 18, 19]. However since De Felice's approach is linked to differential geometry, one can perform both space and time transformations [9].

As stated above, the most important feature of the Plebanski's equivalence, is providing the possibility of studying an analog system in Minkowski spacetime that mimics some aspects of gravitational systems, like trajectories of light. Because of complexity of such systems and also lack of a unified theory of gravitation, some remarkably distinguishable phenomena are not still completely understood, even if they have been observationally approved. For example, there are some experimental evidences for stimulated emissions from an analog system [see for example [20]], that may provide some kind of Hawking-like radiation [21]. However since this appears to be a quantum phenomena, we have yet not been able to properly explain it within available gravitational theories. Therefore it seems that if we could resemble spacetime properties using a definite dielectric with certain configuration, it might be possible to facilitate the investigation of gravitational systems, experiencing that spacetime.

In this paper we consider such simplification to study the some features of a spacetime constructed by gravitational waves, by identifying a pp-wave geometry and a dielectric in Minkowski spacetime. However the method we exploit is not the Plebanski- De Felice approach, because of its limitations, specially lack of covariance which prompts us to take only a certain class of transformations.

Note that, in accordance with what Plebanski-De Felice TO relies on, the appearance of magnetoelectric coupling can be regarded as the velocity of an isotropic media at low speeds. However instead, we will apply a covariant theory of TO, introduced in [22] , developed in [23] and studied in the context of vacuum solutions of general relativity in [24] and [11]. This method, because of its covariance, provides a greater freedom in considering diversified types of motions and transformations (even for time-varying dielectric [25]), while designing materials.

The paper is organized as follows: in section 2, a general survey on classical electrodynamics is made. In section 3, we provide a constructive introduction to the mentioned covariant approach of TO, and this method will be used for a moving dielectric in a vacuum pp-wave spacetime, in order to specifying some characteristics of such media. In section 4, we construct a dielectric analog of pp-wave spacetime and assuming a peculiar transformation, we state that some sort of coordinate singularity can be obtained, which may provide a electromagnetically cloaked region. The concluding remarks are given in section 5, and also some relations which are in use in this paper, are brought in the appendix.

## 2 Classical Electrodynamics

The covariant formulation of classical electrodynamics is based on its tensorial representation on a Riemannian manifold. The reader may find detailed discussions in [26, 27, 28]. For a manifold with spacetime metric $\boldsymbol{g}$, the field strength 2-form $\boldsymbol{F}$, in terms of the potential co-vector $\boldsymbol{A}$, is defined by the exterior derivative

$$\boldsymbol{F} = \mathrm{d}\boldsymbol{A}. \tag{1}$$

The co-vector $\boldsymbol{A} = A_\mu$ itself consists of the electric field $\overrightarrow{E}$ and the magnetic flux $\overrightarrow{B}$. Consequently, the tensorial form of the field strength tensor in Minkowski spacetime and local Cartesian frame reads as

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{2}$$

Moreover, in terms of the electric flux $\overrightarrow{D}$ and the magnetic field $\overrightarrow{H}$, the excitation 2-form $\boldsymbol{G}$ may be written in the following tensorial form:

$$G_{\mu\nu} = \begin{pmatrix} 0 & H_x & H_y & H_z \\ -H_x & 0 & D_z & -D_y \\ -H_y & -D_z & 0 & D_x \\ -H_z & D_y & -D_x & 0 \end{pmatrix}. \tag{3}$$

Together with (2), this provides the Maxwell's equations.

$$\mathrm{d}\boldsymbol{F} = 0,$$

$$\mathrm{d}\boldsymbol{G} = \boldsymbol{J}, \tag{4}$$

where $\boldsymbol{J}$ is the current 3-form [27]. On the other hand, linear dielectric materials admit the following relation between $\boldsymbol{F}$ and $\boldsymbol{G}$

$$\boldsymbol{G} = \boldsymbol{\chi}(\star\boldsymbol{F}), \tag{5}$$

or in component form

$$G_{\mu\nu} = \chi_{\mu\nu}{}^{\alpha\beta}(\star \boldsymbol{F})_{\alpha\beta}. \tag{6}$$

Here, the Hodge dual $\star$ is a map on a manifold $(M, \boldsymbol{g})$, which for 2-forms is defined as

$$\star_{\alpha\beta}{}^{\mu\nu} = \frac{1}{2}\sqrt{|g|}\ \epsilon_{\alpha\beta\sigma\rho} g^{\sigma\mu} g^{\rho\nu}, \tag{7}$$

with $\epsilon_{\alpha\beta\sigma\rho}$ as the Levi-Civita tensor and $g$ the determinant of metric. In equation (5), $\star\boldsymbol{F}$ maps $\boldsymbol{F}$ to another 2-form on the same manifold. Furthermore, the susceptibility $\boldsymbol{\chi}$ contains the material characteristics. Being also a linear material, the vacuum configured by [22]

$$(\chi_{\text{vac}})_{\mu\nu}{}^{\alpha\beta} =$$

$$\frac{1}{2} \times \begin{pmatrix}
\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&1&0&0\\ -1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&1&0\\ 0&0&0&0\\ -1&0&0&0\\ 0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ -1&0&0&0 \end{pmatrix} \\
\begin{pmatrix} 0&-1&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&-1&0&0\\ 0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&-1&0&0 \end{pmatrix} \\
\begin{pmatrix} 0&0&-1&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&0&0\\ 0&0&-1&0\\ 0&1&0&0\\ 0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&-1&0 \end{pmatrix} \\
\begin{pmatrix} 0&0&0&-1\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&0&0\\ 0&0&0&-1\\ 0&0&0&0\\ 0&1&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&-1\\ 0&1&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}
\end{pmatrix}, \tag{8}$$

may perform $\boldsymbol{\chi}_{\text{vac}}(\star\boldsymbol{F}) = \star\boldsymbol{F}$. Maintaining above definitions, the classical theory of electrodynamics becomes structured by means of the constitutive equation (5), which results in the following collected form of six independent relations in Minkowski spacetime:

$$\overrightarrow{H} = \breve{\mu}^{-1}\,\overrightarrow{B} + \breve{\gamma}_1{}^*\,\overrightarrow{E}, \qquad \overrightarrow{D} = \breve{\varepsilon}^*\,\overrightarrow{E} + \breve{\gamma}_2{}^*\,\overrightarrow{B}. \tag{9}$$

Equation (9) may be rearranged to

$$\overrightarrow{B} = \breve{\mu}\,\overrightarrow{H} + \breve{\gamma}_1\,\overrightarrow{E}, \qquad \overrightarrow{D} = \breve{\varepsilon}\,\overrightarrow{E} + \breve{\gamma}_2\,\overrightarrow{H}. \tag{10}$$

The 3-dimensional matrices $\breve{\varepsilon}$, the permittivity, $\breve{\mu}$, the permeability and $\breve{\gamma}$, the magneto-electric coupling are viable to provide an equivalence between (9) and (10) by relating

$$\breve{\mu} = \left(\breve{\mu}^{-1}\right)^{-1}, \qquad \breve{\varepsilon} = \breve{\varepsilon}^* - \breve{\gamma}_2{}^*\breve{\mu}\,\breve{\gamma}_1{}^*,$$

$$\breve{\gamma}_1 = -\breve{\mu}\breve{\gamma}_1{}^*, \qquad \breve{\gamma}_2 = \breve{\gamma}_2{}^*\breve{\mu}. \tag{11}$$

The way in which these $3 \times 3$ matrices constitute the components of $\boldsymbol{\chi}$, has been given in the appendix.

## 3 Covariant Formulation of Transformation Optics

A brief survey on a covariant formulation of TO is given here, which has been demonstrated in [22] and developed in [23]. Consider an initial manifold $(M, \boldsymbol{g}, \star)$ with field configuration $(\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{J})$ and material distribution $\boldsymbol{\chi}$ on which the Maxwell's equations $\mathrm{d}\boldsymbol{F} = 0$ and $\mathrm{d}\boldsymbol{G} = \boldsymbol{J}$ and constitutive equations $\boldsymbol{G} = \boldsymbol{\chi}(\star\boldsymbol{F})$ are valid, is mapped to its sub-manifold $M \subseteq \widetilde{M}$ by $T : M \longrightarrow \widetilde{M}$ (see Figure 1). The Maxwell's and constitutive equations also hold on $\widetilde{M}$; $\mathrm{d}\widetilde{\boldsymbol{F}} = 0$ and $\mathrm{d}\widetilde{\boldsymbol{G}} = \widetilde{\boldsymbol{J}}$. Parallel to geometric transformations however, the physical fields are transformed by $\mathcal{T}$, the inverse of $T$, and therefore $\boldsymbol{F}$ and $\boldsymbol{G}$ are mapped to $\widetilde{\boldsymbol{F}}$ and $\widetilde{\boldsymbol{G}}$ by $\mathcal{T}^*$, the pullback of $\mathcal{T}$, i.e. [22]

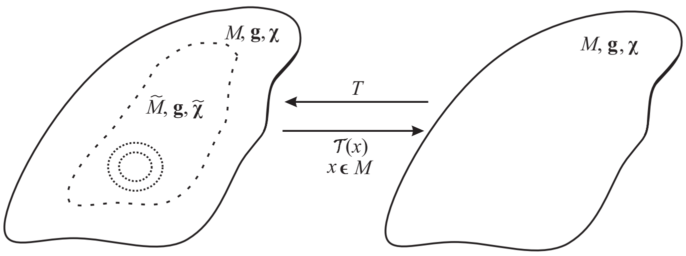


**Figure 1** Manifold $M$ and its sub-manifold $\widetilde{M}$, which is obtained using a map like $T$ from $M$ to $\widetilde{M}$. $\widetilde{M}$ contains a material $\widetilde{\boldsymbol{\chi}}$. The map could be defined, in a way to cerate a hole in $\widetilde{M}$ (cloaked region).

$$\widetilde{\boldsymbol{G}} = \mathcal{T}^*(\boldsymbol{G}) = \mathcal{T}^*\left(\boldsymbol{\chi}(\star\boldsymbol{F})\right) = \widetilde{\boldsymbol{\chi}}\left(\star\mathcal{T}^*(\boldsymbol{F})\right). \tag{12}$$

For every point $x \in \widetilde{M}$, we can relate the material distributions in $\widetilde{M}$ and $M$, as follows:

$$\widetilde{\chi}_{\lambda\kappa}{}^{\tau\eta}(x) = -\Lambda^{\alpha}{}_{\lambda}\Lambda^{\beta}{}_{\kappa}\chi_{\alpha\beta}{}^{\mu\nu}|_{\mathcal{T}(x)}\ \star_{\mu\nu}{}^{\sigma\rho}|_{\mathcal{T}(x)}\left(\Lambda^{-1}\right)^{\pi}{}_{\sigma}\left(\Lambda^{-1}\right)^{\theta}{}_{\rho}\star_{\pi\theta}{}^{\tau\eta}|_{x}, \tag{13}$$

where $\boldsymbol{\Lambda}$ is the Jacobian of the transformation $\mathcal{T}(x)$. In (13) the first $\star$ is calculated on $M$ and the other one on $\widetilde{M}$. Note that, if the initial media is supposed to be a vacuum, then the first $\boldsymbol{\chi}$ has to be the one defined in (8), however non-vacuum initial media in this covariant formulation has also been considered in [29]. Now using (8) and the fact that $\boldsymbol{\chi}_{vac}\star = \star$, the material distribution in $\widetilde{M}$ is characterized by

$$\widetilde{\chi}_{\lambda\kappa}{}^{\tau\eta}(x) = -\Lambda^{\alpha}{}_{\lambda}\Lambda^{\beta}{}_{\kappa}\star_{\alpha\beta}{}^{\sigma\rho}|_{\mathcal{T}(x)}\left(\Lambda^{-1}\right)^{\pi}{}_{\sigma}\left(\Lambda^{-1}\right)^{\theta}{}_{\rho}\star_{\pi\theta}{}^{\tau\eta}|_{x}, \tag{14}$$

by means of which, one can relate a vacuum initial manifold with an arbitrary metric, to its sub-manifold containing a material $\widetilde{\boldsymbol{\chi}}$.

### 3.1 Moving Dielectric in pp-Wave Spacetime

In this subsection, it is assumed that a transformation media is moving in a region, constructed by gravitational waves and determined by the pp-wave spacetime metric. We discuss that how will be the dielectric's configuration, when special coordinate transformations are performed. Also some examples are given.

The pp-wave line element in $(u, v, x, y)$ coordinate system reads as [30, 31]

$$ds^2 = -c^2 d\tau^2 = \Phi(u, x, y) du^2 + dudv + dx^2 + dy^2, \tag{15}$$

implying

$$g_{\mu\nu} = \begin{pmatrix} \Phi(u, x, y) & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{16}$$

Keeping $c = 1$, the 4-velocity vectors are derived from the following equation

$$g_{\mu\nu} U^\mu U^\nu = -1, \tag{17}$$

for time-like geodesics, with $U^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$. According to (15), this results in

$$-1 = \Phi \dot{u}^2 + \dot{u}\dot{v} + \dot{x}^2 + \dot{y}^2, \tag{18}$$

with dot standing for $\frac{\mathrm{d}}{\mathrm{d}\tau}$. This is equivalent to

$$e_0^\mu e_{0\mu} = \eta_{00}, \tag{19}$$

where $e_A^\mu = \frac{\partial x^\mu}{\partial x^A}$ is the set of basis vectors, transforming a 1-form in Minkowskian manifold (parameterized by $x^A$) to a 1-form in pp-wave manifold (parameterized by $x^\mu$). The coordinates in pp-wave formalism, consist of two light-cone elements, namely $u$ and $v$;

$$\begin{aligned} u &= t - z, \\ v &= t + z. \end{aligned} \tag{20}$$

Also, the only non-zero components of Christoffel symbols in this geometry, are

$$\begin{aligned} \Gamma^v_{uu} &= \frac{\partial \Phi}{\partial u}, \\ \Gamma^v_{iu} &= \frac{\partial \Phi}{\partial x^i}, \\ \Gamma^i_{uu} &= \frac{1}{2}\frac{\partial \Phi}{\partial x^i}, \end{aligned} \tag{21}$$

where $i = 2, 3$. Therefore, the geodesic equations imply that

$$\ddot{u} = 0 \qquad \rightarrow \qquad \dot{u} = \text{const.} = \alpha. \tag{22}$$

Therefore $\dot{u}$ could be considered as a constant of motion.

## 3.2 Transformation Optics in pp-Wave Spacetime

Now we are at a stage of exploiting the covariant formulation of TO introduced in section 3. Let us consider a transformation media moving in pp-wave spacetime. The covariant relation (14) for a definite coordinate transformation $\mathcal{T}(x)$, results in the configuration of the considered dielectric in same spacetime. However, what a local observer examines is the same configuration obtained in Minkowski spacetime. Therefore we are in need of a transformation operator, to transform $\widetilde{\boldsymbol{\chi}}$ in pp-wave spacetime, to its Minkowskian correspondent, $\hat{\boldsymbol{\chi}}$. To obtain the proper transformation matrix we pursue the following procedure. The matrix consists of the basis vectors $e^{\mu}_{A}$, exhibiting the following traits:

- Each vector, respectively must give the time-like and space-like results; i.e.

$$g_{\mu\nu}e^{\mu}_{A}e^{\nu}_{B} = \eta_{AB}. \tag{23}$$

- Orthogonality condition:

$$g_{\mu\nu}e^{\mu}_{A}e^{\nu}_{B} = 0, \tag{24}$$

for $A \neq B$.

We consider the dielectric to move along the $z$ direction with coordinate velocity $\frac{\mathrm{d}z}{\mathrm{d}t} = v_z$. Therefore the $x$ and $y$ coordinates are left unchanged. The zero components of the set of basis vectors, namely $e^{\mu}_{0}$, is equivalent to the velocity 4-vector. We have

$$e^{\mu}_{0} = (\dot{u}, \dot{v}, 0, 0). \tag{25}$$

According to the geodesic result (22), and the orthogonality condition, we obtain

$$e^{\mu}_{0} = (\alpha, -\frac{1+\alpha^2\Phi}{\alpha}, 0, 0). \tag{26}$$

For the 1st and 2nd basis vectors, we could use the pure coordinate differentiation, since they are exactly revealing the Minkowskian representation. Therefore

$$e^{\mu}_{1} = \frac{\partial x^{\mu}}{\partial x} = (0, 0, 1, 0),$$
$$e^{\mu}_{2} = \frac{\partial x^{\mu}}{\partial y} = (0, 0, 0, 1). \tag{27}$$

The 3rd vector however, is dependent to $e^{\mu}_{0}$ which due to the orthogonality condition (23), becomes

$$e^{\mu}_{3} = (\alpha, \frac{1-\alpha^2\Phi}{\alpha}, 0, 0). \tag{28}$$

All these vectors, satisfy both mentioned conditions. Using them, one can conclude the transformation matrix as

$$S_{A}{}^{\mu} = \begin{pmatrix} \alpha & -\frac{1+\alpha^2\Phi(u,x,y)}{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha & \frac{1-\alpha^2\Phi(u,x,y)}{\alpha} & 0 & 0 \end{pmatrix}. \tag{29}$$

It is straightforward to see that

$$S_A{}^{\mu} S_B{}^{\nu} g_{\mu\nu} = \eta_{AB}. \tag{30}$$

$S_A{}^{\mu}$ in (29) transforms a 1-form $\theta_\mu$ in pp-wave spacetime, to $\theta_A$ in local frame. To transform vectors, we must use the matrix $S^A{}_{\mu}$; the transpose of the inverse of $S_A{}^{\mu}$. Finally, one can relate [32]

$$\hat{\chi}_{AB}{}^{CD} = S_A{}^{\lambda} S_B{}^{\kappa} S^C{}_{\tau} S^D{}_{\eta}\, \widetilde{\chi}_{\lambda\kappa}{}^{\tau\eta}, \tag{31}$$

to obtain the locally examined $\hat{\boldsymbol{\chi}}$ form its pp-wave correspondent $\widetilde{\boldsymbol{\chi}}$.

## 3.3 Examples

Having the transformation matrix obtained, we still need to consider a coordinate transformation to be performed by the transformation media, otherwise, the media becomes unnoticeable. Let

$$\mathcal{T}(u, v, x, y) = (u', v', x', y'),$$

to transform the components from the media (unprimed) to the vacuum space (primed).

### 3.3.1 Transformation on the null coordinates

As a first step, let us take

$$(u', v', x', y') = \mathcal{T}(u, v, x, y) = (f(u), v, x, y). \tag{32}$$

The corresponding Jacobian reads as

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \partial_u f(u) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{33}$$

Using (16), (7), (33) and (14) together with (31), and comparing with (A.1), one obtains the dielectric $3 \times 3$ vector, defined in the constitutive equations (9) and (10) in Minkowski spacetime.

$$\hat{\varepsilon} = \hat{\mu} = \begin{pmatrix} \frac{1}{-\Phi(u,x,y)\alpha^2 + \Phi(f(u),x,y) f'(u)\alpha^2 + 1} & 0 & 0 \\ 0 & \frac{1}{-\Phi(u,x,y)\alpha^2 + \Phi(f(u),x,y) f'(u)\alpha^2 + 1} & 0 \\ 0 & 0 & \frac{1}{f'(u)} \end{pmatrix},$$

$$\hat{\gamma}_1 = \hat{\gamma}_2^T = \frac{\alpha^2\left(\Phi(u,x,y) - f'(u)\Phi(f(u),x,y)\right)}{\alpha^2 f'(u)\Phi(f(u),x,y) + \alpha^2(-\Phi(u,x,y)) + 1}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{34}$$

Note that $f' \equiv \partial_u f(u)$. These results show that, the potential $\Phi$ appears explicitly in the dielectric characteristics. In the localized limit $f(u) \sim u$, we get the Newtonian result

$$\hat{\varepsilon} = \hat{\mu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{35}$$

which is the identity matrix. Here, $\Phi \equiv \Phi(u,v,x,y)$. This result also corresponds to an isotropic dielectric. Moreover, the limit, when applied for the magnetoelectric coupling term $\hat{\gamma}$, yields

$$\hat{\gamma}_1 = (\alpha^2 - 1)\Phi \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{36}$$

So one can see that the magnetoelectric terms do not vanish in the Newtonian limit. Traditionally, non-vanishing coupling terms are in accord with moving materials [33], which expose direct impacts on the permittivity values, i.e. $\hat{\varepsilon}$. In this case, the non-vanishing $\hat{\gamma}$ implies the initial motion of the dielectric, in the pp-wave spacetime. However, for $\alpha^2 = 1$, i.e. for

$$(\dot{u})^2 = (\frac{du}{d\tau})^2 = 1,$$

these terms will vanish. In other words we have

$$\frac{du}{d\tau} \equiv \frac{d(t-z)}{d\tau} = \dot{t} - \dot{z} = \gamma(1 - v_z) = 1.$$

Since for low speeds, $\gamma \approx 1$, then the above relation results in $\dot{z} = 0$, or $v_z = 0$; no motion along $z$ direction, or in our case, no motion at all.

#### 3.3.2 $y$-dependent transformation of $x$

In this case, the transformation is supposed to be preformed on the $x$ coordinate, orthogonal to the dielectric's direction of motion, namely the $\hat{z}$ direction. Let us investigate, what the contribution of media's velocity, would be in the transformation media characteristics.

The coordinate transformation is supposed to be

$$\mathcal{T} : \mathcal{T}(u,v,x,y) \longrightarrow (u',v',x',y') = (u,v,f(x,y),y), \tag{37}$$

where the function $f(x, y)$ acts on the $x$ coordinate, with a minor coupling to the $y$ coordinate. The corresponding Jacobian reads as

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & f_{,x} & f_{,y} \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{38}$$

Same procedure as in the previous example, if pursued for the transformation (38), yields

$$\hat{\varepsilon} = \hat{\mu} = \frac{1}{-1 + \alpha^2\Phi - \alpha^2\Phi_f} \begin{pmatrix} -\frac{1+f_{,y}^2}{f_{,x}} & f_{,y} & 0 \\ f_{,y} & -f_{,x} & 0 \\ 0 & 0 & f_{,x} \end{pmatrix} ,$$

$$\hat{\gamma}_1 = \hat{\gamma}_2^T = \frac{\alpha^2}{-1 + \alpha^2\Phi - \alpha^2\Phi_f} [\Phi - \Phi_f] \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{39}$$

with $\Phi_f \equiv \Phi(u, f(x, y), y)$ and $\alpha = \dot{u}$. In this case, the vanishing magnetoelectric coupling terms, are followed by the following items:

- $\alpha = 0$: therefore we have

$$\dot{u} \equiv \frac{du}{d\tau} = 0 \quad \Rightarrow \quad u = \delta = \text{const.}$$

  Or one can write

$$\frac{d(t - z)}{d\tau} = \dot{t} - \dot{z} = \gamma - v_z\gamma = 0 \quad \Rightarrow \quad v_z = 1.$$

- $\Phi = \Phi_f$: this implies that $f(x, y) \equiv f(x) = x$. Therefore $\hat{\gamma}_1 = \hat{\gamma}_2 = 0$, and

$$\hat{\varepsilon} = \hat{\mu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$

None of the above items refers to a stationary situation, for which $v_z = 0$. Therefore, one may infer that for perpendicular transformations, the magnetoelectric coupling terms do not explicitly retain the motion of the transformation media. Likewise, the vanishing magnetoelectric coupling terms here, do not lead to vanishing medium velocity.

## 4 Dielectric Analog of pp-wave Spacetime

Having introduced the covariant formulation of TO, here we mention that it is also possible to establish the concept of dielectric analogs. This important implementation of a dielectric analog is exploited here to investigate the characteristics of pp-wave spacetime, in a locally flat Minkowskian frame. The covariant method which is applied here, has been presented in [24], however since the initial vacuum manifold is supposed to be the pp-wave spacetime, therefore the light waves (rays) in this case are those which are supposed to have interactions with the gravitational waves (to obtain insights into this concept, please see Ref. [31]). We are concerning with the propagation of light waves in a dielectric, analogous to the pp-wave spacetim (see Figure 2). We begin with pp-wave

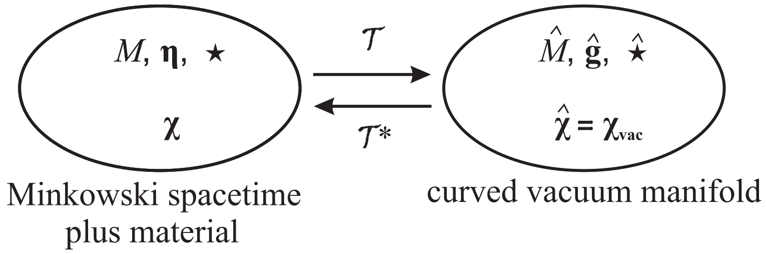


**Figure 2** The map $\mathcal{T}$ transforms the Minkowskian manifold $M$ , containing a material $\boldsymbol{\chi}$ , to the vacuum curved manifold $\hat{M}$. Pullback of this map, $T^{*}$, performs an associate map, taking differential forms in the cotangent bundle of $\hat{M}$, into the cotangent bundle of $M$ and here, appears to be the reverse map.

metric, written in Cartesian coordinates

$$\mathrm{d}s^2 = [1+\Phi(u,x,y)]\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + [\Phi(u,x,y)-1]\mathrm{d}z^2 - 2\Phi(u,x,y)\,\mathrm{d}t\mathrm{d}z. \tag{40}$$

One can note that, in contrast with the previous section, we wrote the metric in Cartesian coordinates. This is because of the fact that, the resultant dielectric analog is also characterized in the same coordinate system and as it is seen in Figure 2, the coordinate transformation $\mathcal{T}(t,x,y,z) = (t',x',y',z')$, maps Minkowski spacetime to vacuum pp-wave. However, as we will seen below, this map is not always in need of a peculiar expression other than the identity map. In contrast with the previous discussion where the dielectric had to perform a coordinate transformation ( otherwise it became unnoticeable ), the dielectric analog, because of different geometric background, could be valid even when the identity transformation is applied.

Turning back to our discussion, the susceptibility in a local frame is obtained by the following tensor representation [24]:

$$\chi_{\lambda\kappa}{}^{\tau\eta}(x) = -\Lambda^{\alpha}{}_{\lambda}\Lambda^{\beta}{}_{\kappa}\,\hat{\star}_{\alpha\beta}{}^{\sigma\rho}|_{\mathcal{T}(x)}\left(\Lambda^{-1}\right)^{\pi}{}_{\sigma}\left(\Lambda^{-1}\right)^{\theta}{}_{\rho}\,\star_{\pi\theta}{}^{\tau\eta}|_{x}, \tag{41}$$

where $\chi_{\lambda k}{}^{\tau\eta}(x)$ is obtained in Minkowski spacetime. Putting $\mathcal{T} = \mathcal{T}_0$, the identity map, results in $\Lambda^{\mu}{}_{\nu} = \mathbf{1}_4$ and equation (41) becomes

$$\chi_{\lambda\kappa}{}^{\tau\eta}(x) = -\hat{\star}_{\lambda\kappa}{}^{\sigma\rho}|_{x}\;\star_{\sigma\rho}{}^{\tau\eta}|_{x}, \tag{42}$$

in which the first Hodge dual is calculated in pp-wave spacetime, without any coordinate transformation. According to this identity transformation, using (40) and (42) and comparing with (A.1), our impedance matched dielectric analog in Minkowski spacetime, is characterized by the following vectors:

$$\breve{\mu} = \breve{\varepsilon} = \begin{pmatrix} -\frac{1}{g_{00}(x)} & 0 & 0 \\ 0 & -\frac{1}{g_{00}(x)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{43}$$

$$\breve{\gamma}_1 = (\breve{\gamma}_2)^T = \begin{pmatrix} 0 & 1 - \frac{1}{g_{00}(x)} & 0 \\ \frac{1}{g_{00}(x)} - 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{44}$$

where $g_{00}(x) = 1 + \Phi(u, x, y)$, according to the metric in (40). Note that the above magnetoelectric couplings are caused by the off-diagonal terms in the metric, which could be regarded as the time-dependency of the dielectric analog.

## 4.1 Square Cloak: electromagnetic invisibility

In order to create a cloaked region, one should consider a transformation from the vacuum spacetime (pp-wave), to a square shell (with $z$ =const.) in Minkowski spacetime. Such transformation has been derived in [34] as below:

$$T(t', x', y', z') = (t, x, y, z) = \left(t',\, x'\left(\frac{s_2 - s_1}{s_2}\right) + s_1,\, y'\left(\frac{s_2 - s_1}{s_2} + \frac{s_1}{x'}\right),\, z'\right) \tag{45}$$

for $0 < x' \leq s_2$, $-s_2 < y' \leq s_2$, $|y'| < |x'|$ and $|z'| < \infty$. This transformation, maps the points within $0 \leq x' \leq s_2$ to $s_1 \leq x \leq s_2$, which creates the empty region $0 \leq x \leq s_1$. Note that, this transformation only covers the shaded region in Figure 3; to obtain the whole region $s_1 \leq x \leq s_2$, one should use a rotation matrix for angles $\frac{\pi}{2}$, $\pi$ and $\frac{3\pi}{2}$. As before, we should consider the pullback of $T$, namely $\mathcal{T}$ in order to transform the fields. We have

$$\mathcal{T}(t, x, y, z) = (t', x', y', z') = (t,\, f(x),\, g(x, y),\, z) \tag{46}$$

where $f(x) = \frac{s_2 - s_1}{s_2}(x - s_1)$ and $g(x, y) = \frac{s_2(x - s_1)}{(s_2 - s_1)}\frac{y}{x}$. In (46), the map is defined for $s_1 \leq x \leq s_2$, $-s_2 < y \leq s_2$ with $|y| < |x|$ and $|z| < \infty$. Note that, the map is not defined for $x < s_1$. According to (46), now the corresponding Jacobian matrix reads as

$$\Lambda^{\mu}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{s_2}{s_2 - s_1} & 0 & 0 \\ 0 & \frac{y s_1 s_2}{x^2(s_2 - s_1)} & \frac{s_2(x - s_1)}{x(s_2 - s_1)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{47}$$

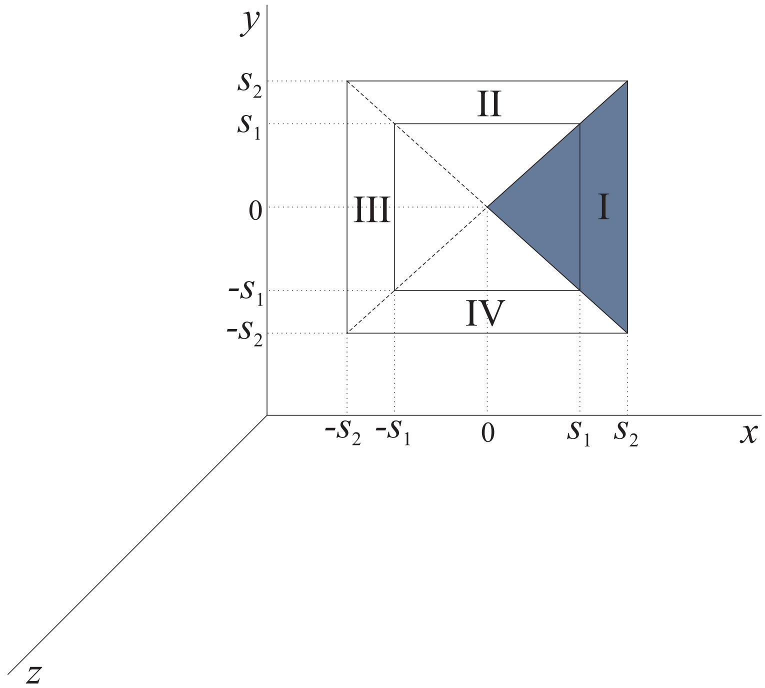


**Figure 3** A square cloak. The transformation (45), maps to the shaded region I. The whole region is obtained by using a rotation matrix.

Substitution of (46) and (47) in (41) results in the following vectors, corresponding to same parameters in constitutive equations (9) and (10), in region I of Figure 3:

$$\check{\mu}_{\mathrm{I}} = \check{\varepsilon}_{\mathrm{I}} = g_{00}(\mathcal{T}(x))^{-1} \begin{pmatrix} \frac{s_1-x}{x} & \frac{s_1 y}{x^2} & 0 \\ s_1 y & \frac{x^4+{s_1}^2y^2}{(s_1-x)x^3} & 0 \\ 0 & 0 & -\frac{{s_2}^2(s_1-x)}{(s_1-s_2)^2x}\, g_{00}(\mathcal{T}(x)) \end{pmatrix}, \tag{48}$$

$$(\check{\gamma}_1)_{\mathrm{I}} = {(\check{\gamma}_2)_{\mathrm{I}}}^T = g_{00}(\mathcal{T}(x))^{-1} \begin{pmatrix} 0 & g_{00}(\mathcal{T}(x))-1 & 0 \\ 1-g_{00}(\mathcal{T}(x)) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{49}$$

where $g_{00}(\mathcal{T}(x)) = 1+\Phi(u, f(x), g(x,y))$ according to metric (15), maintaining the coordinate transformation (46). One can note that the form of the magnetoelectric couplings, in (44) and (49) are the same since the off-diagonal term of the pp-wave metric remains the same, consequently the dielectric analog does not experience any further time-dependent alternations. Also as it was noted above, relations (48) and (49) only cover the region I in Figure 3, therefore we shall apply the following rotations, to obtain the whole region:

$$\check{\mu}_{\mathrm{II}} = \check{\varepsilon}_{\mathrm{II}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \check{\mu}_{\mathrm{I}}, \quad (\check{\gamma}_1)_{\mathrm{II}} = {(\check{\gamma}_2)_{\mathrm{II}}}^T = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot (\check{\gamma}_1)_{\mathrm{I}}, \tag{50}$$

$$\check{\mu}_{\mathrm{III}} = \check{\varepsilon}_{\mathrm{III}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \check{\mu}_{\mathrm{I}}, \ \ (\check{\gamma}_1)_{\mathrm{III}} = (\check{\gamma}_2)_{\mathrm{III}}{}^T = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot (\check{\gamma}_1)_{\mathrm{I}}, \tag{51}$$

$$\check{\mu}_{\mathrm{IV}} = \check{\varepsilon}_{\mathrm{IV}} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \check{\mu}_{\mathrm{I}}, \ \ (\check{\gamma}_1)_{\mathrm{IV}} = (\check{\gamma}_2)_{\mathrm{IV}}{}^T = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot (\check{\gamma}_1)_{\mathrm{I}}. \tag{52}$$

The vectors in (48)-(52), characterize a region, in which no electromagnetic wave can enter. Therefore we created a cloaked region, through which no optical data can be exchanged and consequently has been electromagnetically isolated. In fact, dielectric configurations are not interacting gravitational waves, but just electromagnetic waves. However, we should bear in mind that the analog dielectric obtained above, is indeed replaced by the pp-wave spacetime and if any real-world opto-gravitational interactions do exist, the aforementioned cloaked region will be capable of exhibiting them. We should note also that, such cloaked region is indeed caused by a coordinate transformation. Therefore such a singularity is a coordinate singularity, note a real one. However, because of its optical characteristics, this region may be regarded as a trapped surface, which has several correspondences in regular solutions to general relativity.

## 5 Conclusion

The configuration of a transformation media, when it is moving in a region constructed by gravitational waves, was discussed as the main aim. To fulfill this task, a covariant formulation of transformation optics was applied. It is essential for this transformation media to be applied to light waves. In this work however, one may consider some possible opto-gravitational interactions in the region, and performing some definite coordinate transformations, may demonstrate specific behaviors. Moreover, we also constructed a dielectric analog of the mentioned region, and showed that this analog has to be magnetoelectric. According to a peculiar coordinate transformation, we also dealt with a 2-dimensional electromagnetically isolated region; no light rays may pierce nor emanate from the region. This is traditionally related to electromagnetic [35] or acoustic [36] cloaks.

# A Susceptibility Reference Frame

Susceptibility reference matrix in Cartesian coordinates [23]:

$$\chi_{\mu\nu}{}^{\alpha\beta} = \frac{1}{2}\begin{pmatrix} \mathbf{0} & * & * & * \\ \mathcal{M} & \mathbf{0} & * & * \\ \mathcal{N} & \mathcal{O} & \mathbf{0} & * \\ \mathcal{P} & \mathcal{Q} & \mathcal{R} & \mathbf{0} \end{pmatrix}, \tag{A.1}$$

where

$$\mathcal{M} = \begin{pmatrix} 0 & -\mu^{-1}_{xx} & -\mu^{-1}_{xy} & -\mu^{-1}_{xz} \\ \mu^{-1}_{xx} & 0 & -\gamma^*_{1xz} & \gamma^*_{1xy} \\ \mu^{-1}_{xy} & \gamma^*_{1xz} & 0 & -\gamma^*_{1xx} \\ \mu^{-1}_{xz} & -\gamma^*_{1xy} & \gamma^*_{1xx} & 0 \end{pmatrix},$$

$$\mathcal{N} = \begin{pmatrix} 0 & -\mu^{-1}_{yx} & -\mu^{-1}_{yy} & -\mu^{-1}_{yz} \\ \mu^{-1}_{yx} & 0 & -\gamma^*_{1yz} & \gamma^*_{1yy} \\ \mu^{-1}_{yy} & \gamma^*_{1yz} & 0 & -\gamma^*_{1yx} \\ \mu^{-1}_{yz} & -\gamma^*_{1yy} & \gamma^*_{1yx} & 0 \end{pmatrix},$$

$$\mathcal{O} = \begin{pmatrix} 0 & -\gamma^*_{2zx} & -\gamma^*_{2zy} & -\gamma^*_{2zz} \\ \gamma^*_{2zx} & 0 & -\varepsilon^*_{zz} & \varepsilon^*_{zy} \\ \gamma^*_{2zy} & \varepsilon^*_{zz} & 0 & -\varepsilon^*_{zx} \\ \gamma^*_{2zz} & -\varepsilon^*_{zy} & \varepsilon^*_{zx} & 0 \end{pmatrix},$$

$$\mathcal{P} = \begin{pmatrix} 0 & -\mu^{-1}_{zx} & -\mu^{-1}_{zy} & -\mu^{-1}_{zz} \\ \mu^{-1}_{zx} & 0 & -\gamma^*_{1zz} & \gamma^*_{1zy} \\ \mu^{-1}_{zy} & \gamma^*_{1zz} & 0 & -\gamma^*_{1zx} \\ \mu^{-1}_{zz} & -\gamma^*_{1zy} & \gamma^*_{1zx} & 0 \end{pmatrix},$$

$$\mathcal{O} = \begin{pmatrix} 0 & \gamma^*_{2yx} & \gamma^*_{2yy} & \gamma^*_{2yz} \\ -\gamma^*_{2yx} & 0 & \varepsilon^*_{yz} & -\varepsilon^*_{yy} \\ -\gamma^*_{2yy} & -\varepsilon^*_{yz} & 0 & \varepsilon^*_{yx} \\ -\gamma^*_{2yz} & \varepsilon^*_{yy} & -\varepsilon^*_{yx} & 0 \end{pmatrix},$$

$$\mathcal{R} = \begin{pmatrix} 0 & -\gamma^*_{2xx} & -\gamma^*_{2xy} & -\gamma^*_{2xz} \\ \gamma^*_{2xx} & 0 & -\varepsilon^*_{xz} & \varepsilon^*_{xy} \\ \gamma^*_{2xy} & \varepsilon^*_{xz} & 0 & -\varepsilon^*_{xx} \\ \gamma^*_{2xz} & -\varepsilon^*_{xy} & \varepsilon^*_{xx} & 0 \end{pmatrix} .$$

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