Spin and Zitterbewegung in a Field Theory of the Electron.[(‡)]

Erasmo Recami [1] [*] [†] and Giovanni Salesi [2]

[1]*Facoltà di Ingegneria, Università Statale di Bergamo, 24044–Dalmine (BG), Italy; INFN–Sezione di Milano, Milan, Italy; and*
[1]*DECOM, Faculty of Electrical Engineering (FEEC), State University at Campinas (UNICAMP), Campinas, Brazil.*
[2]*Facoltà di Ingegneria, Università Statale di Bergamo, 24044–Dalmine (BG), Italy; INFN–Sezione di Milano, Milan, Italy.*

Abstract: In previous papers we investigated the classical theory of the electron spin, and its "quantum" version, by using Clifford algebras; and we ended with a new non-linear Dirac-like equation (NDE). We re-address in this review the whole subject, and extend it, by adopting however the ordinary tensorial language, within the first quantization formalism. In particular, we re-derive the NDE for the electron field, showing it to be associated with a new conserved probability current. Incidentally, the Dirac equation results from the former by averaging over a Zbw cycle.

Afterward, we derive an equation of motion for the 4-velocity field, which allows regarding the electron as an extended-like object with a classically intelligible internal structure.

We carefully study the solutions of the NDE; with special attention to those implying (at the classical limit) light-like helical motions, since they appear to be the most adequate solutions for the electron description.

At last we introduce a natural generalization of our approach, for the case in which an external electromagnetic potential is present.

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1   Premise

In previous work, we investigated the classical Barut–Zanghi (BZ) theory[1] for the electron spin, which involves internal Zitterbewegung (zbw) motions along cylindrical helices, by using the powerful language of Clifford algebras[2]. The "quantum" version of such approach lead us[2], in particular, to a new, non-linear, Dirac-like equation (NDE).

In this review article we implement a first quantization of the BZ theory for the free electron, by adopting however the *ordinary* tensorial language. In so doing, we re-derive the NDE for the electron quantum field. Such equation involves *a new probability current J^μ*, which is shown to be a conserved quantity, endowed (except in the "trivial" case corresponding to the standard Dirac theory) with a zbw motion having the typical frequency $\omega = 2m$, where m is the electron mass.

Then, we derive a new equation of motion for the field velocity v^μ quite different from that of Dirac's theory. Indeed, it allows us referring to realistic internal motions endowed with an intuitive kinematical meaning. The consequence is that our electron results to be an extended-like particle, with a classically intelligible internal structure. Afterwards, we write down the general solution of the NDE, which appears to be a superposition of plane waves with positive and negative frequencies; in the present theory such superposition results to entail always a positive field energy, both for particles and for antiparticles.

We analyse the kinematical (zbw) structure of v^μ in the center-of-mass frame as a function of the initial spinor $\psi(0)$. After having shown that, for extended-like particles like our electron, quantity $v^2 \equiv v_\mu v^\mu$ in general is *not* constant in time (quite differently from the scalar particle case, in which $v^2 = 1$ is always constant), we look for the particular NDE solutions that on the contrary imply a constant v^2; and we find this case to correspond to a circular, uniform zbw motion with $\omega = 2m$ and orbital radius $R = |\boldsymbol{v}|/2m$ (where $|\boldsymbol{v}|$, and therefore R, depend on the particular NDE solution considered). Even more, the simple requirement of a uniform (classical) motion seems to play the role of the ordinary quantization conditions for the z-component of the field spin s; namely, $v^2 =$ constant does imply $s_z = \pm\frac{1}{2}$, that is to say, the electron polarization. We also examine, then, the oscillating linear motions associated with unpolarized electrons.

Special attention is devoted to the light-like ($v^2 = 0$) helical motions, either clockwise or anti-clockwise, for which the orbital radius results to be equal to half a Compton wavelength. In fact, as already mentioned, such motions appear to be the most adequate for the electron description, both kinematically and physically, and correspond to the electromagnetic properties of the electron, such as Coulomb field and intrinsic magnetic moment.

2   Introduction

Attempts to put forth classical theories for the electron are known since almost a century.[3−6] For instance, Schrödinger's suggestion[4] that the electron spin was related to the Zitterbewegung (internal) motion did originate a large amount of subsequent work, including Pauli's. In ref.[7] one can find, for instance, even the proposal of models with clockwise and anti-clockwise "inner motions" as classical analogues of quantum relativistic spinning particles and antiparticles, respectively. Most of the cited works did not pay attention, however, to the transition from the classical theory to its quantum version, i.e., to finding out a wave equation whose classical limit be consistent with the assumed model.

Among the approaches in which on the contrary a passage to the quantum version was performed, let us quote the papers by Barut and Pavsic.[1,8] Namely, those authors first considered a classical lagrangian[1] $\mathcal{L}$ describing (internal) helical motions; and second —after having passed to the hamiltonian formalism— they constructed a quantum version of their model and thus derived (in their opinion) the Dirac equation. Such results are really quite interesting and stimulating. We show below, however, that from the BZ theory one does naturally derive a non-linear Dirac-like equation,[2] rather than the Dirac (eigenvalue) equation itself, whose solutions are only a subset of the former's.

Many further results have been met,[2,9], as we were saying, by using the powerful Clifford algebra language.[10] Due to their general interest and importance, we reformulate here the results appeared in refs.[2] by the *ordinary* tensorial language: This will allow us to show more plainly and clearly, and more easily outline, their geometrical and kinematical significance, as well as their field theoretical implications. In so doing, also we shall further develop such a theory; for instance: (i) by showing the strict correlation existing between electron polarization and zbw kinematics (in the case of both time-like and –with particular attention, as we know— light-like internal helical motions); and (ii) by forwarding a probabilistic interpretation of the NDE current (after having shown it to be always conserved).

3   The classical Barut–Zanghi (BZ) theory

The existence of an "internal" motion (inside the electron) is denounced, besides by the presence of spin, by the remarkable fact that, according to the standard Dirac theory, the electron four-impulse p^μ is *not* parallel to the four-velocity: $v^\mu \neq p^\mu/m$; moreover, while $[p^\mu, H] = 0$ so that p^μ is a conserved quantity, on the contrary v^μ is *not* a constant of the motion: $[v^\mu, H] \neq 0$. Let us recall that indeed, if ψ is a solution of Dirac equation, only the "quantum average" (or probability current) $<v^\mu> \equiv \psi^\dagger v^\mu \psi \equiv \psi^\dagger \gamma^0 \gamma^\mu \psi \equiv \overline{\psi} \gamma^\mu \psi$ is constant in time (and space), since it equals p^μ/m.

This suggests, incidentally, that to describe a spinning particle at least four inde-

pendent canonically conjugate variables are necessary; for example:

$$x^\mu, \ p^\mu; \ v^\mu, \ \mathcal{P}^\mu,$$

or alternatively (as we are going to see):

$$x^\mu, \ p^\mu; \ z, \ \overline{z}$$

where z and $\overline{z} \equiv z^\dagger \gamma^0$ are ordinary $\mathbb{C}^4$–bispinors.

In the BZ theory,[1] the classical electron was actually characterized, besides by the usual pair of conjugate variables (x^μ, p^μ), by a second pair of conjugate classical *spinorial* variables $(z, \overline{z})$, representing internal degrees of freedom, which are functions of the proper time τ of the electron global center-of-mass (CM) system. The CM frame is the one in which at every instant of time it is $\boldsymbol{p} = 0$ (but in which, for spinning particles, $\boldsymbol{v} \equiv \dot{\boldsymbol{x}}$ is *not* zero, in general!). Barut and Zanghi, then, introduced a classical lagrangian that in the free case $(A_\mu = 0)$ writes $[c = 1]$

$$\mathcal{L} = \frac{1}{2} i\lambda(\dot{\overline{z}}z - \overline{z}\dot{z}) + p_\mu(\dot{x}^\mu - \overline{z}\gamma^\mu z) \tag{1}$$

where λ has the dimension of an action. [The extension of this lagrangian to the case with external fields, $A^\mu \neq 0$, has been treated elsewhere]. One of the consequent (four) motion equations yields directly the particle four-velocity:

$$\dot{x}^\mu \equiv v^\mu = \overline{z}\gamma^\mu z . \tag{1'}$$

We are not writing down explicitly the spinorial indices of $\overline{z}$ and z.

Let us explicitly notice that, in the classical theories for elementary spinning particles, it is convenient[11] to split the motion variables as follows

$$x^\mu \equiv \xi^\mu + X^\mu; \quad v^\mu = w^\mu + V^\mu, \tag{2}$$

where ξ^μ and $w^\mu \equiv \dot{\xi}^\mu$ describe the *translational, external* or *drift* motion, i.e. the motion of the CM, whilst X^μ and $V^\mu \equiv \dot{X}^\mu$ describe the *internal* or *spin* motion.

From eq.(1) one can see[1] that also

$$H \equiv p_\mu v^\mu = p_\mu \overline{z}\gamma^\mu z \tag{3}$$

is a constant of the motion (and precisely it is the CMF energy); being H, as one may easily check, the BZ hamiltonian in the CMF, we can suitably set $H = m$, quantity m being the particle rest-mass. In this way, incidentally, we obtain, even for spinning particles, the ordinary relativistic constraint (usually met for scalar particles):

$$p_\mu v^\mu = m . \tag{3'}$$

The general solution of the equations of motion corresponding to lagrangian (1), with $\lambda = -1$ [and $\hbar = 1$], is:

$$z(\tau) = [\cos(m\tau) - i\frac{p_\mu \gamma^\mu}{m} \sin(m\tau)]z(0) , \tag{4a}$$

$$\overline{z}(\tau) = \overline{z}(0)[\cos(m\tau) + i\frac{p_\mu \gamma^\mu}{m}\sin(m\tau)] \ , \tag{4b}$$

with $p^\mu = \text{constant};\ \ p^2 = m^2;\ $ and finally:

$$\dot{x}^\mu \equiv v^\mu = \frac{p^\mu}{m} + [\dot{x}^\mu(0) - \frac{p^\mu}{m}]\cos(2m\tau) + \frac{\ddot{x}^\mu}{2m}(0)\sin(2m\tau) \ . \tag{4c}$$

This general solution exhibits the classical analogue of the phenomenon known as Zitterbewegung: in fact, the velocity v^μ contains the (expected) term p^μ/m plus a term describing an oscillating motion with the characteristic zbw frequency $\omega = 2m$. The velocity of the CM will is given of course by $w^\mu = p^\mu/m$.

Notice that, instead of adopting the variables z and $\overline{z}$, we can work in terms of the spin variables, i.e., in terms of the set of dynamical variables

$$x^\mu, \ p^\mu; \ v^\mu, S^{\mu\nu}$$

where

$$S^{\mu\nu} \equiv \frac{i}{4}\overline{z}[\gamma^\mu, \gamma^\nu]z \ ; \tag{5a}$$

then, we would get the following motion equations:

$$\dot{p}^\mu = 0 \ ; \ \ v^\mu = \dot{x}^\mu \ ; \ \ \dot{v}^\mu = 4S^{\mu\nu}p_\nu \ ; \ \ \dot{S}^{\mu\nu} = v^\nu p^\mu - v^\mu p^\nu \ . \tag{5b}$$

By varying the action corresponding to $\mathcal{L}$, one finds as generator of space-time rotations the conserved quantity $J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$, where $L^{\mu\nu} \equiv x^\mu p^\nu - x^\nu p^\mu$ is the orbital angular momentum tensor, and $S^{\mu\nu}$ is just the particle spin tensor: so that $\dot{J}^{\mu\nu} = 0$ implies $\dot{L}^{\mu\nu} = -\dot{S}^{\mu\nu}$.

Let us explicitly observe that the general solution (4c) represents a helical motion in the ordinary 3-space: a result that has been met also by means of other,[#1] alternative approaches.[12,13]

4   Field theory of the extended–like electron

The natural way of "quantizing" lagrangian (1) is that of reinterpreting the classical spinors z and $\overline{z}$ as Dirac field spinors, say ψ and $\overline{\psi}$ [in the following the Dirac spinors will be merely called "spinors", instead of bi-spinors, for simplicity's sake]:

$$z \to \psi \ ; \ \ \overline{z} \to \overline{\psi} \ ;$$

which will lead us below to the conserved probability current $J^\mu = m\,\overline{\psi}\gamma^\mu\psi/p^0$. Recall that here the operators $(x^\mu, p^\mu; \psi, \overline{\psi})$ are field variables; for instance,[2] $\psi = \psi(x^\mu);\ \overline{\psi} =$

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[1] [#1] Alternative approaches to the kinematical description of the electron spin have been proposed, e.g., by Pavsic and Barut in refs.[12,13]. In connection with Pavsic's approach,[12] we would like here to mention that the classical angular momentum was defined therein as $s \equiv 2\beta v \wedge a/\sqrt{1 - v^2}$, whilst in the BZ theory it is $s \equiv r \wedge mw$, where $a \equiv \dot{v}$. Both quantities s result to be parallel to p.

$\overline{\psi}(x^\mu)$.   Thus, the quantum version of eq.(1) is the field lagrangian

$$\mathcal{L} = \frac{i}{2}\lambda(\dot{\overline{\psi}}\psi - \overline{\psi}\dot{\psi}) + p_\mu(\dot{x}^\mu - \overline{\psi}\gamma^\mu\psi) \qquad (6)$$

that refers (so as in the classical case) to free electrons with *fixed* impulse p^μ.   The four Euler–Lagrange equations, with $-\lambda = \hbar = 1$, yield the following motion equations:

$$\begin{cases} \dot{\psi} + ip_\mu\gamma^\mu\psi = 0 & (7a) \\[2mm] \dot{x}^\mu = \overline{\psi}\gamma^\mu\psi & (7b) \\[2mm] \dot{p} = 0 \, , & (7c) \end{cases}$$

besides the hermitian adjoint of eq.(7a), holding for $\overline{\psi} = \psi^+\gamma^0$.   In eqs.(7), the invariant time τ is still the CMF proper time, and it is often put $p_\mu\gamma^\mu \equiv \not{p}$.

We can pass to the hamiltonian formalism by introducing the field hamiltonian corresponding to the energy in the CMF:

$$H = p_\mu\overline{\psi}\gamma^\mu\psi \, , \qquad (8)$$

which easily yields *the noticeable equation*[#2]

$$p_\mu\overline{\psi}\gamma^\mu\psi = m \qquad (9)$$

This *non-linear* equation, satisfied by all the solutions of eqs.(7), is very probably the simplest non-linear *Dirac–like* equation.[14] *Another* non-linear Dirac–like equation, quite equivalent to eq.(7a) but employing the generic coordinates x^μ and no longer the CMF proper time, was anticipated in the first one of refs.[2]; it is obtained by inserting the identity

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\partial}{\partial x^\mu} \equiv \dot{x}^\mu\partial_\mu \qquad (10)$$

into eq.(7a). In fact, one gets $i\dot{x}^\mu\partial_\mu\psi = p_\mu\gamma^\mu\psi$,   and, since $\dot{x}^\mu = \overline{\psi}\gamma^\mu\psi$ because of eq.(7b), one arrives at the important equation:

$$i\overline{\psi}\gamma^\mu\psi\partial_\mu\psi = p_\mu\gamma^\mu\psi \, . \qquad (11)$$

A more general equation, in which $p_\mu\gamma^\mu$ is replaced by m,

$$i\overline{\psi}\gamma^\mu\psi\partial_\mu\psi = m\psi \, ,$$

can be easily obtained (cf. the last one of refs.[2]) by releasing the fixed–p^μ condition.

Let us notice that, differently from eqs.(6)–(7), equation (11) can be valid a priori even for massless spin $\frac{1}{2}$ particles, since the CMF proper time does not enter it any longer.

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[1] #2 In refs.[12] it was moreover assumed that $p_\mu\dot{x}^\mu = m$, which actually does imply the very general relation $p_\mu\overline{\psi}\gamma^\mu\psi = m$, but *not* the Dirac equation $p_\mu\gamma^\mu\psi = m\psi$, as claimed therein; these two equations in general are not equivalent.

The remarkable non-linear equation (11) corresponds to the whole system of eqs.(7): quantizing the BZ theory, therefore, does not lead to the Dirac equation, but rather to the non-linear, Dirac-like equation (11), that we call NDE.

The eq.(11) might be even adopted in substitution for the free Dirac (eigenvalue) equation, since it apparently admits a sensible classical limit [which describes an internal periodic motion with frequency $2m$, implying the existence of an intrinsic angular momentum tensor: the "spin tensor" $S^{\mu\nu}$ of eq.(5a)]. Moreover, eq.(11) admit assigning a natural, simple physical meaning to the negative frequency waves.[15] In other words, eqs.(6)–(11) seem to allow us overcoming the wellknown problems related with physical meaning and time evolution of the position operator x^{μ} and of the velocity operator $\dot{x}^{\mu}$: we shall come back to this point. [We always indicate a variable and the corresponding operator by the same simbol]. In terms of field quantities (and no longer of operators), eq.(11) corresponds[2] to the four motion equations

$$\dot{p}^{\mu} = 0; \quad v^{\mu} = \dot{x}^{\mu}; \quad \dot{v}^{\mu} = 4S^{\mu\nu}p_{\nu}; \quad \dot{S}^{\mu\nu} = v^{\nu}p^{\mu} - v^{\mu}p^{\nu} , \qquad (12a)$$

in which now v^{μ} and the spin tensor $S^{\mu\nu}$ are the field quantities $v^{\mu} \equiv \overline{\psi}(x)\gamma^{\mu}\psi(x)$ and

$$S^{\mu\nu} = \frac{i}{4}\overline{\psi}(x)[\gamma^{\mu}, \gamma^{\nu}]\psi(x) .$$

By deriving the third one of eqs.(12a), and using the first one of them, we obtain

$$\ddot{v}^{\mu} = 4\dot{S}^{\mu\nu}p_{\nu} ; \qquad (12b)$$

by substituting now the fourth one of eqs.(12a) into eq.(12b), and imposing the previous constraints $p_{\mu}p^{\mu} = m^2, \quad p_{\mu}v^{\mu} = m$, we get the time evolution of the field four-velocity:

$$v^{\mu} = \frac{p^{\mu}}{m} - \frac{\ddot{v}^{\mu}}{4m^2} . \qquad (13)$$

Let us recall, for comparison, that the corresponding equation *for the standard Dirac case*[1−7] was devoid of a classical, realistic meaning because of the known appearance of an imaginary unit i in front of the acceleration:[16]

$$v^{\mu} = \frac{p^{\mu}}{m} - \frac{i}{2m}\dot{v}^{\mu} . \qquad (13')$$

One can observe, incidentally, that by differentiating the relation $p_{\mu}v^{\mu} = m = $ constant, one immediately gets that the (internal) acceleration $\dot{v}^{\mu} \equiv \ddot{x}^{\mu}$ is orthogonal to the electron impulse p^{μ} since $p_{\mu}\dot{v}^{\mu} = 0$ at any instant. To conclude, let us recall that the Dirac electron has no classically meaningful internal structure; on the contrary, our electron, an *extended–like* particle, does possess an internal structure, and internal motions which are all kinematically, geometrically acceptable and meaningful: As we are going to see.

5   General solution of the new non-linear, Dirac–like equation (NDE), and conservation of the probability current

In a generic frame, the general solution of eq.(11) can be easily shown to be the following $[\not p \equiv p_\mu \gamma^\mu]$:

$$\psi(x) = [\frac{m - \not p}{2m} \, e^{ip_\mu x^\mu} + \frac{m + \not p}{2m} \, e^{-ip_\mu x^\mu}] \, \psi(0) \; ; \tag{14a}$$

which, in the CMF, reduces to

$$\psi(\tau) = [\frac{1 - \gamma^0}{2} \, e^{im\tau} + \frac{1 + \gamma^0}{2} \, e^{-im\tau}] \, \psi(0) \; , \tag{14b}$$

or, in a simpler form, to:

$$\psi(\tau) = [\cos(m\tau) - i\gamma^0 \sin(m\tau)] \, \psi(0) \; , \tag{14c}$$

quantity τ being the particle proper time, as above. Let us explicitly observe that, by introducing eq.(14a), or eq.(14b), into eq.(9), one obtains that every solution of eq.(11) does correspond to the CMF field hamiltonian $H = m > 0$, even if it appears (as expected in any theories with zbw) to be a suitable superposition of plane waves endowed with positive and negative frequencies.[15] Notice that superposition (14a) is a solution of eq.(11), due to the non-linearity of such an equation, *only* for suitable pairs of plane waves, with weights

$$\frac{m \pm \not p}{2m} = \Lambda_\pm \; ,$$

respectively, which are nothing but the usual projectors Λ_+ (Λ_-) over the positive (negative) energy states of the standard Dirac equation. In other words, the plane wave solution (for a fixed value of p) of the Dirac eigenvalue equation $\not p \psi = m\psi$ is a particular case of the general solution of eq.(11): namely, for either

$$\Lambda_+ \psi(0) = 0 \qquad \text{or} \qquad \Lambda_- \psi(0) = 0 \; . \tag{15}$$

Therefore, the solutions of the Dirac eigenvalue equation are a *subset* of the set of solutions of our NDE. It is worthwhile to repeat that, for each fixed p, the wave function $\psi(x)$ describes both particles and antiparticles: all corresponding however to positive *energies*, in agreement with the reinterpretation forwarded in refs.[15] (as well as with the already mentioned fact that we can always choose $H = m > 0$).

We want now to study, eventually, the probability current J^μ corresponding to the wave functions (14a,b,c). Let us define it as follows:

$$J^\mu \equiv \frac{m}{p^0} \, \overline{\psi}\gamma^\mu\psi \tag{16}$$

where the normalization factor m/p^0 (the 3-volume V being assumed to be equal to 1, as usual; so that $p^0 V \equiv p^0$) is required to be such that the classical limit of J^μ, that is $(m/p^0)\, v^\mu$, equals $(1; \boldsymbol{v})$, like for the ordinary probability currents. Notice also that sometimes in the literature p^0 is replaced by $\mathcal{E}$; and that $J^0 \equiv 1$, which means that we have one particle inside the unitary 3-volume $V = 1$. This normalization allows us to recover, in particular, the Dirac current $J_{\rm D}^\mu = p^\mu/p^0$ when considering the (trivial) solutions, without zbw, corresponding to relations (15). Actually, if we substitute quantity $\psi(x)$ given by eq.(14a) into eq.(16), we get

$$J^\mu = \frac{p^\mu}{p^0} + E^\mu \cos(2p_\mu x^\mu) + H^\mu \sin(2p_\mu x^\mu) \ , \tag{16$'$}$$

where

$$E^\mu \equiv J^\mu(0) - p^\mu/p^0 \ ; \quad H^\mu \equiv \dot{J}(0)/2m \ . \tag{16$''$}$$

If we now impose conditions (15), we have $E^\mu = H^\mu = 0$ and get therefore the Dirac current $J^\mu = J_{\rm D}^\mu = \text{constant} = p^\mu/p^0$. Let us notice too that the normalization factor $\sqrt{m/p^0}$ cannot be inserted into ψ and $\overline{\psi}$, as it would seem convenient, because of the non-linearity of eq.(11) and/or of constraint (9).

Since $p_\mu E^\mu \equiv p_\mu J^\mu(0) - p_\mu p^\mu/p^0 = m^2/p^0 - m^2/p^0 = 0$ (where we used eq.(9) for $x = 0$) and since $p_\mu H^\mu \equiv p_\mu \dot{J}^\mu(0)/2m = 0$ obtained deriving both members of eq.(9) —note incidentally that both E^μ and H^μ are orthogonal to p^μ— it follows that

$$\partial_\mu J^\mu = 2p_\mu H^\mu \cos(2px) - 2p_\mu E^\mu \sin(2px) = 0 \ . \tag{16$'''$}$$

We may conclude, with reference to equation (11), that our current J^μ is *conserved*: We are therefore allowed to adopt the usual probabilistic interpretation of fields $\psi, \overline{\psi}$. Equation (16') does clearly show that the conserved current J^μ, as well as its classical limit $(m/p^0)v^\mu$ [see eq.(4c)], are endowed with a Zitterbewegung–type motion: precisely, with an oscillating motion having the CMF frequency $\Omega = 2m \simeq 10^{21}\mathrm{s}^{-1}$ and period $T = \pi/m \simeq 10^{-20}$s (we may call Ω and T the zbw frequency and period, respectively).

From eq.(16') one can immediately verify that in general

$$J^\mu \neq p^\mu/p^0 \ , \quad J^\mu \equiv J^\mu(x) \ ;$$

whilst the Dirac current $J_{\rm D}^\mu$ for the free electron with fixed p, as already mentioned, is constant:

$$J_{\rm D}^\mu = p^\mu/p^0 = \ \text{constant} \ ,$$

corresponding to *no* zbw. In other words, our current behaves differently from Dirac's, even if both of them obey[#3] the constraint [cf. eq.(9)]

$$p_\mu J^\mu = p_\mu J_{\rm D}^\mu = m^2/p^0 \ .$$

[#3] In the Dirac case, this is obtained by getting from the ordinary Dirac equation, $p_\mu \gamma^\mu \psi_{\rm D} = m\psi_{\rm D}$, the non-linear constraint $p_\mu \overline{\psi}_{\rm D} \gamma^\mu \psi_{\rm D} = m\overline{\psi}_{\rm D}\psi_{\rm D}$, and therefore by replacing $\overline{\psi}_{\rm D}\psi_{\rm D}$ by m/p^0, consistently with the ordinary normalization $\psi_{\rm D} = e^{-ipx}u_p/\sqrt{2p^0}$, with $\overline{u}_p u_p = 2m$.

It's noticeable, moreover, that our current J^μ goes into the Dirac one, not only in the no-zbw case of eq.(15), but also when averaging it (in time) over a zbw period:

$$< J^\mu >_{\text{zbw}} = \frac{p^\mu}{p^0} \equiv J_{\text{D}}^\mu \ . \tag{17}$$

In the next section we study the kinematical zbw structure of J^μ, relative to some given solutions of the NDE: Let us stress that this structure is the same for both J^μ and its classical limit mv^μ/p^0 (and that in the CMF it is $mv^\mu/p^0 \equiv v^\mu$). That is to say, the probability current stream-lines correspond just to the classical world-lines of a pointlike particle, in agreement with the Correspondence principle. Consequently, we can study the kinematical features of our J^μ by means of the analysis of the time evolution of the four-velocity v^μ.

6   Uniform motion solutions of the NDE

Before examining the solutions, let us stress that we do use the term *electron*[#4] to indicate the whole spinning system associated with the geometrical center of the helical trajectories (which corresponds to the center of the electron Coulomb field[17]). let us repeat that this geometrical center is always at rest in the CMF, i.e., in the frame where $\boldsymbol{p}$ and $\boldsymbol{w}$ (but *not* $\boldsymbol{v} \equiv \boldsymbol{V}$) vanish [cf. eqs.(2)]. On the contrary, we shall call $\mathcal{Q}$ the pointlike object moving along the helix; we shall refer to it as to the electron "constituent", and to its (internal) movement as to a "sub-microscopic" motion.

We need first of all to make explicit the kinematical definition of v^μ, rather different from the common (scalar particle) one. In fact, from the very definition of v^μ, one obtains

$$v^\mu \equiv \mathrm{dx}^\mu/\mathrm{d}\tau \equiv (\mathrm{dt}/\mathrm{d}\tau; \mathrm{d}\boldsymbol{x}/\mathrm{d}\tau) \equiv (\frac{\mathrm{dt}}{\mathrm{d}\tau}; \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{dt}} \frac{\mathrm{dt}}{\mathrm{d}\tau})$$

$$= (1/\sqrt{1 - \boldsymbol{w}^2}; \ \boldsymbol{u}/\sqrt{1 - \boldsymbol{w}^2}) \ , \qquad [\boldsymbol{u} \equiv \mathrm{d}\boldsymbol{x}/\mathrm{dt}] \tag{18}$$

[where, let us recall, $\boldsymbol{w} = \boldsymbol{p}/m$ is the velocity of the CM in the chosen reference frame (that is, in the frame in which the quantities x^μ are measured)]. Below, it will be convenient to choose as reference frame the CMF (even if quantities as $v^2 \equiv v_\mu v^\mu$ are frame invariant); so that [cf. definition (2)]:

$$v_{\text{CM}}^\mu = V^\mu \equiv (1; \boldsymbol{V}) \ , \tag{19}$$

wherefrom one deduces for the speed $|\boldsymbol{V}|$ of the internal motion (i.e., for the zbw speed) the new conditions:

$$0 < V^2(\tau) < 1 \quad \Leftrightarrow \quad 0 < \boldsymbol{V}^2(\tau) < 1 \qquad \text{(time-like)}$$

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1 #4 Usually we speak about electrons, but this theory could be applied to all leptons.

$$V^2(\tau) = 0 \quad \Leftrightarrow \quad \boldsymbol{V}^2(\tau) = 1 \qquad \text{(light-like)} \qquad (19')$$

$$V^2(\tau) < 0 \quad \Leftrightarrow \quad \boldsymbol{V}^2(\tau) > 1 \qquad \text{(space-like) ,}$$

where $V^2 = v^2$. Notice that, in general, the value of $\boldsymbol{V}^2$ does vary with τ; except in special cases (e.g., the case of polarized particles: as we shall see).

Our NDE in the CMF (where, let us remember, $J^\mu \equiv v^\mu$) can be written as

$$(\overline{\psi}\gamma^\mu\psi)i\partial_\mu\psi = m\gamma^0\psi \qquad (20)$$

whose general solution is eq.(14b) or, equivalently, eq.(14c). In correspondence to it, we also have [due to eq.(16')] that

$$J^\mu = p^\mu/m + E^\mu\cos(2m\tau) + H^\mu\sin(2m\tau) \equiv V^\mu \qquad (21a)$$

$$V^2 \equiv J^2 = 1 + E^2\cos^2(2m\tau) + H^2\sin^2(2m\tau) + 2E_\mu H^\mu\sin(2m\tau)\cos(2m\tau) . \qquad (21b)$$

Let us select the solutions ψ of eq.(20) corresponding to constant V^2 and A^2, where $A^\mu \equiv \mathrm{d}V^\mu/\mathrm{d}\tau \equiv (0; \boldsymbol{A})$, quantity $V^\mu \equiv (1; \boldsymbol{V})$ being the zbw velocity. Therefore, we shall suppose in the present frame that quantities

$$V^2 = 1 - \boldsymbol{V}^2 \; ; \quad A^2 = -\boldsymbol{A}^2$$

are constant in time:

$$V^2 = \text{constant} ; \quad A^2 = \text{constant} , \qquad (22)$$

so that $\boldsymbol{V}^2$ and $\boldsymbol{A}^2$ are constant in time too. (Notice, incidentally, that we are dealing exclusively with the internal motion, in the CMF; thus, our results are quite independent of the global 3-impulse $\boldsymbol{p}$ and hold both in the relativistic and in the non- relativistic case). The requirements (22), inserted into eq.(21b), yield the following constraint

$$\begin{cases} E^2 = H^2 & (23a) \\[2mm] E_\mu H^\mu = 0 . & (23b) \end{cases}$$

Constraints (23) are necessary and sufficient (initial) conditions to get a circular uniform motion (the only uniform and finite motion conceivable in the CMF). Since both E and H do not depend on the time τ, also eqs.(23) hold for any time instant. In the euclidean 3-dimensional space, and at any instant of time, constraints (23) read:

$$\begin{cases} \boldsymbol{A}^2 = 4m^2\boldsymbol{V}^2 & (24a) \\[2mm] \boldsymbol{V} \cdot \boldsymbol{A} = 0 & (24b) \end{cases}$$

which correspond to a uniform circular motion with radius

$$R = |\boldsymbol{V}|/2m . \qquad (24c)$$

Quantity R is the "Zitterbewegung radius"; the zbw frequency was already found to be always $\Omega = 2m$. Quantities E, H (with $p^0 = m$) given in eqs.(16″) are functions of the initial spinors $\psi(0), \overline{\psi}(0)$. Bearing in mind that in the CMF $v^0 = 1$ [cf. eq.(19)] and therefore $\overline{\psi}\gamma^0\psi = 1$ (which, incidentally, implies the CMF normalization $\psi^\dagger\psi = 1$), one gets

$$E^\mu \equiv -\frac{p^\mu}{m} + \overline{\psi}(0)\gamma^\mu\psi(0) = (0;\; \overline{\psi}(0)\vec{\gamma}\psi(0)) \;. \tag{24'a}$$

By substituting into the second one of eqs.(16″) the expression (14c) for the general solution of the NDE, we finally have

$$H^\mu = i\overline{\psi}(0)[\gamma^0\gamma^\mu - g^{0\mu}]\psi(0) = (0;\; i\overline{\psi}(0)\vec{\alpha}\psi(0)) \;, \tag{24'b}$$

where $\vec{\gamma} \equiv (\gamma^1, \gamma^2, \gamma^3)$, while $\vec{\alpha} \equiv \gamma^0\vec{\gamma}$ and $g^{\mu\nu}$ is the metric. Therefore, conditions (23) or (24) can be written in spinorial form, for any time τ, as follows

$$\begin{cases} (\overline{\psi}\vec{\gamma}\psi)^2 = -(\overline{\psi}\vec{\alpha}\psi)^2 & (25a) \\[2mm] (\overline{\psi}\vec{\gamma}\psi) \cdot (\overline{\psi}\vec{\alpha}\psi) = 0 \;. & (25b) \end{cases}$$

At this point, let us show that this classical uniform circular motion, occurring around the z-axis (which in the CMF can be chosen arbitrarily, while in a generic frame is parallel to the global three-impulse $\boldsymbol{p}$, as we shall see), does just correspond to the case of *polarized* particles with $s_z = \pm\frac{1}{2}$. It may be interesting to notice, once more, that in this case the classical requirements (23) or (24) —namely, the uniform motion conditions— play the role of the ordinary quantization conditions $s_z = \pm\frac{1}{2}$. Now, let us first observe [cf. eq.(5a)] that in the CMF the z-component of the spin vector

$$s_z = S^{12} \equiv \frac{i}{4}\overline{\psi}(\tau)[\gamma^1, \gamma^2]\psi(\tau)$$

can actually be shown to be a constant of the motion. In fact, by easy calculations on eq.(14c), one finds s_z to be independent of τ:

$$s_z(\tau) = \frac{i}{2}\overline{\psi}(0)\gamma^1\gamma^2\psi(0) = \frac{1}{2}\overline{\psi}(0)\Sigma_z\psi(0) \;; \tag{26}$$

where $\vec{\Sigma}$ is the spin operator, that in the standard representation reads

$$\vec{\Sigma} \equiv \begin{pmatrix} \vec{\sigma} & 0 \\ & \\ & \\ 0 & \vec{\sigma} \end{pmatrix}$$

quantities $\sigma_x, \sigma_y, \sigma_z$ being the well-known Pauli 2×2 matrices.

Then, it is straightforward to realize that the most general spinors $\psi(0)$ satisfying the conditions

$$s_x = s_y = 0 \tag{27a}$$

$$s_z = \frac{1}{2}\overline{\psi}(0)\Sigma_z\psi(0) = \pm\frac{1}{2} \tag{27b}$$

are (in the standard representation) of the form

$$\psi_{(+)}^{\mathrm{T}}(0) = (\ a\ 0\ |\ 0\ d\ ) \tag{28a}$$

$$\psi_{(-)}^{\mathrm{T}}(0) = (\ 0\ b\ |\ c\ 0\ )\ , \tag{28b}$$

respectively; and obey in the CMF the normalization constraint $\psi^{\dagger}\psi = 1$. [It can be easily shown that, for generic initial conditions, it is $-\frac{1}{2} \leq s_z \leq \frac{1}{2}$]. In eqs.(28) we separated the first two from the second two components, bearing in mind that in the standard Dirac theory (and in the CMF) they correspond to the positive and negative frequencies, respectively. With regard to this, let us observe that the "negative-frequency" components c and d do *not* vanish at the non-relativistic limit [since, let us repeat, it is $\boldsymbol{p} = 0$ in the CMF]; but the field hamiltonian H is nevertheless positive and equal to m, as already stressed.

Now, from relation (28a) we are able to deduce that (with $* \equiv$ complex conjugation)

$$< \vec{\gamma} > \equiv \overline{\psi}\vec{\gamma}\psi = 2(\mathrm{Re}[a^*d], +\mathrm{Im}[a^*d], 0)$$
$$< \vec{\alpha} > \equiv \overline{\psi}\vec{\alpha}\psi = 2i(\mathrm{Im}[a^*d], -\mathrm{Re}[a^*d], 0)$$

and analogously, from eq.(28b), that

$$< \vec{\gamma} > \equiv \overline{\psi}\vec{\gamma}\psi = 2(\mathrm{Re}[b^*c], -\mathrm{Im}[b^*c], 0)$$
$$< \vec{\alpha} > \equiv \overline{\psi}\vec{\alpha}\psi = 2i(\mathrm{Im}[b^*c], +\mathrm{Re}[b^*c], 0)\ ,$$

which just imply relations (25):

$$\begin{cases} < \vec{\gamma} >^2 = - < \vec{\alpha} >^2 \\ \\ < \vec{\gamma} > \cdot < \vec{\alpha} > = 0\ . \end{cases}$$

In conclusion, the (circular) polarization conditions eqs.(27) imply the internal zbw motion to be uniform and circular ($V^2 = $ constant; $A^2 = $ constant); eqs.(27), in other words, do imply that s_z is conserved and quantized, at the same time.

Notice that, when passing from the CMF to a generic frame, eq.(27) transforms into

$$\lambda \equiv \frac{1}{2}\overline{\psi}\frac{\vec{\Sigma} \cdot \boldsymbol{p}}{|\boldsymbol{p}|}\psi \ = \ \pm\frac{1}{2} \ = \ \text{constant}\ . \tag{29}$$

Therefore, to get a uniform motion around the $\boldsymbol{p}$-direction [cf. equations (4c) or (16')], we have to request that the helicity λ be constant (over space and time), and quantized in the ordinary way, i.e., $\lambda = \frac{1}{2}$. We shall come back to the question of the double sign $\pm\frac{1}{2}$ in the case of the light-like helical trajectories; here, for simplicity, let us confine to the + sign.

It may be interesting, now, to calculate $|\boldsymbol{V}|$ as a function of the spinor components a and d. With reference to eq.(28a), since $\psi^\dagger \psi \equiv |a|^2 + |d|^2 = 1$, we obtain (for the $s_z = +\frac{1}{2}$ case):

$$\boldsymbol{V}^2 \equiv < \vec{\gamma} >^2 = 4|a^*d|^2 = 4|a|^2 \, (1-|a|^2) \tag{30a}$$

$$\boldsymbol{A}^2 \equiv (2im < \vec{\alpha} >)^2 = 4m^2\boldsymbol{V}^2 = 16m^2|a|^2 \, (1-|a|^2) \, , \tag{30b}$$

and therefore the normalization (valid now in any frame, at any time)

$$\overline{\psi}\psi = \sqrt{1-\boldsymbol{V}^2} \, , \tag{30c}$$

which show that to the same speed and acceleration there correspond two spinors $\psi(0)$, related by an exchange of a with d. From eq.(30a) we derive also that, as $0 \le |a| \le 1$, it is:

$$0 \le \boldsymbol{V}^2 \le 1 \, ; \quad 0 \le \overline{\psi}\psi \le 1 \, . \tag{30d}$$

Correspondingly, from eq.(24c) we obtain for the zbw radius $0 \le R \le \frac{1}{2}m$.

The second one of eqs.(30d) is a *new*, rather interesting (normalization) boundary condition. From eq.(30c) one can easily see that: (i) for $\boldsymbol{V}^2 = 0$ (no zbw) we have $\overline{\psi}\psi = 1$ and ψ is a "Dirac spinor"; (ii) for $\boldsymbol{V}^2 = 1$ (light-like zbw) we have $\overline{\psi}\psi = 0$ and ψ is a "Majorana spinor"; (iii) for $0 < \boldsymbol{V}^2 < 1$ we meet, instead, spinors with properties "intermediate" between the Dirac and the Majorana ones.

As an example, let us write down the "Caldirola[18] solution" $\psi(0)$, corresponding to the zbw speed $\sqrt{3}/2$ and to the zbw radius $\sqrt{3}/4m$, and yielding correct values for the zero and first order contributions to the electron magnetic moment (for simplicity we chose a and d real):

$$\psi^{\mathrm{T}}(0) = (\frac{1}{2} \quad 0 \mid 0 \quad \frac{\sqrt{3}}{2}) \, ;$$

as well as that got by interchanging $1/2$ and $\sqrt{3}/2$.

The "Dirac" case,[2,9] corresponding to $\boldsymbol{V}^2 = \boldsymbol{A}^2 = 0$, that is to say, corresponding to no zbw internal motion, is merely represented (apart from phase factors) by the spinors

$$\psi^{\mathrm{T}}(0) \equiv (1 \quad 0 \mid 0 \quad 0) \tag{31}$$

and (interchanging a and d)

$$\psi^{\mathrm{T}}(0) \equiv (0 \quad 0 \mid 0 \quad 1) \, . \tag{31'}$$

The spinorial quantities (31), (31'), together with the analogous ones for $s_z = -\frac{1}{2}$, satisfy eq.(15): i.e., they are also solutions (in the CMF) of the Dirac eigenvalue equation. This is the unique case in which the zbw disappears, while the field spin is still present! In fact, even in terms of eqs.(31)–(31'), one still gets that $\frac{1}{2}\overline{\psi}\Sigma_z\psi = +\frac{1}{2}$.

Since we have been discussing the classical limit (v^μ) of a quantum quantity (J^μ), let us add that even the well-known change in sign of the fermion wave function, under

360º-rotations around the z-axis, gets in our theory a natural classical interpretation. In fact, a 360º-rotation of the coordinate frame around the z-axis (passive point of view) is indeed equivalent to a 360º-rotation of the constituent $\mathcal{Q}$ around the z-axis (active point of view). On the other hand, as a consequence of the latter transformation, the zbw angle $2m\tau$ does vary of 360º, the proper time τ does increase of a zbw period $T = \pi/m$, and the pointlike constituent does describe a complete circular orbit around z-axis. At this point it is straightforward to notice that, since the period $T = 2\pi/m$ of the $\psi(\tau)$ in eqs.(14b)–(14c) is twice as large as the zbw orbital period, the wave function of eqs.(14b)–(14c) does suffer a phase–variation of 180º only, and then does change its sign: as it occurs in the standard theory.

To conclude this Section, let us shortly consider the interesting case obtained when *releasing* the conditions (22)–(25) (and therefore abandoning the assumption of circular uniform motion), requiring instead that:

$$|a| = |c| \quad \text{and} \quad |b| = |d| \ , \tag{32}$$

so to obtain an internal oscillating motion along a constant straight line; where we understood $\psi(0)$ to be written, as usual,

$$\psi^{\mathrm{T}}(0) \equiv (a \ \ b \mid c \ \ d) \ .$$

For instance, one may choose either

$$\psi^{\mathrm{T}}(0) \equiv \frac{1}{\sqrt{2}}(1 \ \ 0 \mid 1 \ \ 0) \ , \tag{32'}$$

or $\psi^{\mathrm{T}}(0) \equiv \frac{1}{\sqrt{2}}(1 \ \ 0 \mid i \ \ 0)$, or $\psi^{\mathrm{T}}(0) \equiv \frac{1}{2}(1 \ \ -1 \mid -1 \ \ 1)$, or $\psi^{\mathrm{T}}(0) \equiv \frac{1}{\sqrt{2}}(0 \ \ 1 \mid 0 \ \ 1)$, and so on.

In case (32'), for example, one actually gets

$$< \vec{\gamma} > = (0, 0, 1) \ ; \ < \vec{\alpha} > = (0, 0, 0)$$

which, inserted into eqs.(24'a), (24'b), yield

$$E^{\mu} = (0; 0, 0, 1) \ ; \qquad H^{\mu} = (0; 0, 0, 0) \ .$$

Therefore, because of eq.(21a), we have now a linear, oscillating motion [for which equations (22), (23), (24) and (25) do *not* hold: here $V^2(\tau)$ does vary from 0 to 1!] along the z-axis:

$$V_x(\tau) = 0; \quad V_y(\tau) = 0; \quad V_z(\tau) = \cos(2m\tau) \ .$$

This new case could describe an unpolarized, mixed state, since it is

$$\boldsymbol{s} \equiv \frac{1}{2}\overline{\psi}\vec{\Sigma}\psi = (0, 0, 0) \ ,$$

in agreement with the existence of a linear oscillating motion.

7   The light-like helical case.

Let us go back to the case of circular uniform motions (in the CMF), for which conditions (22)-(25) hold, with $\psi(0)$ given by eqs.(28). Let us fix our attention, however, on the special case of light-like motion[2]. The spinor fields $\psi(0)$, corresponding to $V^2 = 0; \boldsymbol{V}^2 = 1$, are given by eqs.(28) with $|a| = |d|$ for the $s_z = +\frac{1}{2}$ case, or $|b| = |c|$ for the $s_z = -\frac{1}{2}$ case; as it follows from eqs.(30) for $s_z = +\frac{1}{2}$ and from the analogous equations

$$\boldsymbol{V}^2 = 4|b^*c| = 4|b|^2(1 - |b|^2) \tag{30'a}$$

$$\boldsymbol{A}^2 = 4m^2\boldsymbol{V}^2 = 16m^2|b|^2(1 - |b|^2) \,, \tag{30'b}$$

holding for the case $s_z = -\frac{1}{2}$. It can be easily shown that a difference in the phase factors of a and d (or of b and c, respectively) does not change the kinematics, nor the rotation direction, of the motion; but it does merely shift the zbw phase angle at $\tau = 0$. Thus, we may choose *purely real* spinor components (as we did above). As a consequence, the *simplest* spinors may be written as follows

$$\psi_{(+)}^{\mathrm{T}} \equiv \frac{1}{\sqrt{2}}(1 \;\; 0 \mid 0 \;\; 1) \tag{33a}$$

$$\psi_{(-)}^{\mathrm{T}} \equiv \frac{1}{\sqrt{2}}(0 \;\; 1 \mid 1 \;\; 0) \;; \tag{33b}$$

and then

$<\vec{\gamma}>_{(+)} = (1,0,0) \;; \; <\vec{\alpha}>_{(+)} = (0,-i,0)$

$<\vec{\gamma}>_{(-)} = (1,0,0) \;; \; <\vec{\alpha}>_{(-)} = (0,i,0)$

which, inserted into eqs.(24'), yield

$E^{\mu}{}_{(+)} = (0;1,0,0) \;; \qquad H^{\mu}{}_{(+)} = (0;0,1,0) \,.$

$E^{\mu}{}_{(-)} = (0;1,0,0) \;; \qquad H^{\mu}{}_{(-)} = (0;0,-1,0) \,.$

With regard to eqs.(33), let us observe that eq.(21a) implies for $s_z = +\frac{1}{2}$ an *anti-clockwise*[6,2] internal motion, with respect to the chosen z-axis:

$$V_x = \cos(2m\tau); \quad V_y = \sin(2m\tau); \quad V_z = 0 \,, \tag{34}$$

that is to say

$$\begin{cases} X = (2m)^{-1}\sin(2m\tau) + X_0 \\[2mm] Y = -(2m)^{-1}\cos(2m\tau) + Y_0 \\[2mm] Z = Z_0 \;; \end{cases} \tag{34'}$$

and a *clockwise* internal motion for $s_z = -\frac{1}{2}$:

$$V_x = \cos(2m\tau); \quad V_y = -\sin(2m\tau); \quad V_z = 0 , \tag{35}$$

that is to say

$$\begin{cases} X = (2m)^{-1}\sin(2m\tau) + X_0 \\[2mm] Y = (2m)^{-1}\cos(2m\tau) + Y_0 \\[2mm] Z = Z_0 . \end{cases} \tag{35'}$$

Let us explicitly observe that spinor (33a), related to $s_z = +\frac{1}{2}$ (i.e., to an anti-clockwise internal rotation), gets equal weight contributions from the positive–frequency spin-up component and from the negative–frequency spin-down component, in full agreement with our "reinterpretation" in terms of particles and antiparticles, given in refs.[15]. Analogously, spinor (33b), related to $s_z = -\frac{1}{2}$ (i.e., to a clockwise internal rotation), gets equal weight contributions from the positive-frequency spin-down component and the negative-frequency spin-up component.

As we have seen above [cf. eq.(29)], in a *generic* reference frame the polarized states are characterized by a helical uniform motion around the $\boldsymbol{p}$-direction; thus, the $\lambda = +\frac{1}{2}$ [$\lambda = -\frac{1}{2}$] spinor will correspond to an anti-clockwise [a clockwise] helical motion with respect to the $\boldsymbol{p}$-direction.

Going back to the CMF, we have to remark that in this case eq.(24c) yields for the zbw radius R the traditional result:

$$R = \frac{|\boldsymbol{V}|}{2m} \equiv \frac{1}{2m} \equiv \frac{\lambda}{2} , \tag{36}$$

where λ is the Compton wave-length. Of course, $R = 1/2m$ represents the maximum (CMF) size of the electron, among all the uniform motion solutions (A^2 =const; V^2 =const); the minimum, $R = 0$, corresponding to the Dirac case with no zbw ($V = A = 0$), represented by eqs.(31), (31'): that is to say, the Dirac free electron is a pointlike, extensionless object.

Finally, let us to underline that the present light-like solutions, among all the uniform motion, polarized state solutions, are likely the most suitable solutions for a complete and "realistic" (i.e., classically meaningful) picture of the free electron. Really, the uniform circular zbw motion with speed c seems to be the sole that allows us —if we think the electric charge of the whole electron to be carried around by the internal constituent $\mathcal{Q}$— to obtain the electron Coulomb field and magnetic dipole (with the correct strength $\mu = e/2m$), *simply by averaging* over a zbw period[17,19] the electromagnetic field generated by the zbw circular motion of $\mathcal{Q}$ (and therefore oscillating with the zbw frequency). Moreover, only in the light-like case the electron spin can be actually regarded as totally arising from the internal zbw motion, since the *intrinsic* term $\Delta^{\mu\nu}$ entering the BZ theory[1] does *vanish* when $|\boldsymbol{V}|$ tends to c.

8   Generalization of the NDE for the non-free cases.

Let us eventually pass to consider the presence of *external* electromagnetic fields: $A^\mu \neq 0$. For the non-free case, Barut and Zanghi[1] proposed the following lagrangian

$$\mathcal{L} = \frac{1}{2}i(\dot{\overline{\psi}}\psi - \overline{\psi}\dot{\psi}) + p_\mu(\dot{x}^\mu - \overline{\psi}\gamma^\mu\psi) + eA_\mu\overline{\psi}\gamma^\mu\psi \qquad (37)$$

which in our opinion should be better rewritten in the following form, obtained directly from the free lagrangian *via* the minimal prescription procedure:

$$\mathcal{L} = \frac{1}{2}i(\dot{\overline{\psi}}\psi - \overline{\psi}\dot{\psi}) + (p_\mu - eA_\mu)(\dot{x}^\mu - \overline{\psi}\gamma^\mu\psi), \qquad (38)$$

all quantities being expressed as functions of the (CMF) proper time τ, and the generalized impulse being now $p^\mu - eA^\mu$. We shall call as usual $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$.

Lagrangian (38) does yield, in this case, the following system of differential equations:

$$\begin{cases} \dot{\psi} + i(p_\mu - eA_\mu)\gamma^\mu\psi = 0 & (39a) \\[2mm] \dot{x}^\mu = \overline{\psi}\gamma^\mu\psi & (39b) \\[2mm] \dot{p}^\mu - e\dot{A}^\mu = eF^{\mu\nu}\dot{x}^\mu\ . & (39c) \end{cases}$$

As performed in Sect.**4**, we can insert the identity (10) into eqs.(39a), (39b), and exploit the definition of the velocity field, eq.(39c). We easily get the following *five* first–order *differential equations* (one scalar plus one vector equation) in the five independent variables ψ and p^μ:

$$\begin{cases} i(\overline{\psi}\gamma^\mu\psi)\partial_\mu\psi = (p_\mu - eA_\mu)\gamma^\mu\psi & (40a) \\[2mm] (\overline{\psi}\gamma^\nu\psi)\partial_\nu(p^\mu - eA^\mu) = eF^{\mu\nu}\overline{\psi}\gamma_\nu\psi\ , & (40b) \end{cases}$$

which are now *field* equations (quantities $\psi, \overline{\psi}, p$ and A being all functions of x^μ).

The solutions $\psi(x)$ of system (40) may be now regarded as the classical spinorial fields for relativistic spin-$\frac{1}{2}$ fermions, in presence of an electromagnetic potential $A^\mu \neq 0$. We can obtain from eqs.(40) well-defined time evolutions, both for the CMF velocity p^μ/m and for the particle velocity v^μ. A priori, by imposing the condition of finite motions, i.e., $v(\tau)$ and $p(\tau)$ periodic in time (and ψ vanishing at spatial infinity), one will be able to find a discrete spectrum, out from the continuum set of solutions of eqs.(40). Therefore, without solving any eigenvalue equation, within our field theory we can individuate discrete spectra of energy levels for the stationary states, in analogy with what we already found in the free case (in which the *uniform* motion condition implied the z-componens s_z of spin s to be discrete). The case of a uniform, external magnetic field has been treated in ref.[16], where —incidentally— also the formal resolvent of system (39) is given. We shall expand on this point elsewhere: having in mind, especially, the applications to classical problems, so as the hydrogen atom, the Stern and Gerlach

experiment, the Zeeman effect, and tunnelling through a barrier.

9   Conclusions.

In this paper we have analyzed and studied the electron internal motions (predicted by the BZ theory), as functions of the initial conditions, for both time-like and space-like speeds. In so doing, we revealed the noticeable *new* kinematical properties of the velocity field for the spinning electron. For example, we found that quantity $v^2 \equiv v_\mu v^\mu$ may assume any real value and be variable in time, differently from the ordinary (scalar particle) case in which it is always either $v^2 = 1$ or $v^2 = 0$. By requiring v^2 to be constant in time (uniform motions), we found however that [since the CMF does *not* coincide with the rest frame!] quantity v^2 can assume all values in the interval 0 to 1, depending on the value of $\psi(0)$.

Moreover, this paper shows clearly the correlation existing between electron polarization and Zitterbewegung. In particular, in Sect.**6** we show that the requirement $\mathbf{s} = (0, 0, \pm\frac{1}{2})$ [i.e., that the classical spin magnitude, corresponding to the average quantum spin magnitude, be $\frac{1}{2}$] corresponds to the requirement of uniform motions for $\mathcal{Q}$, and vice-versa. We found the zbw oscillation to be a uniform circular motion, with frequency $\omega = 2m$, and with an orbital radius $R = |\boldsymbol{v}|/2m$ that in the case of light-like orbits equals half the Compton wave-length; the clockwise (anti-clockwise) helical motions corresponding to the spin-up (spin-down) case.

We introduced also an equation for the 4-velocity field (different from the corresponding equation of Dirac theory), which allows an intelligible description of the electron internal motions: thus overcoming the well-known problems about the physical meaning of the Dirac *position operator* and its time evolution.[20] Indeed, the equation for the standard Dirac case was devoid of any intuitive kinematical meaning, because of the appearance of the imaginary unit i in front of the acceleration (which is related to the non-hermiticity of the velocity operator in that theory).

Finally, we have considered a natural generalization of the NDE for the non-free case, which allows a classical description of the interaction between a relativistic fermion and an external electromagnetic field (a description we shall deepen elsewhere).

Acknowledgements

structure of spinning particles, to magnetic poles, and to the use of Clifford algebras in space-time physics.

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