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Thermodynamics of hot quantum scalar field in a $(D+1)$ dimensional curved spacetime

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Abstract: We use the brick wall model to calculate the free energy of quantum scalar field in a curved spacetime (D +1) dimensions. We find the thermodynamics properties of quantum scalar field in several scenarios: Minkowski spacetime, Schwarzschild spacetime and BTZ spacetime. For the cases analyzed, the thermodynamical properties of quantum scalar field is exactly with the reported. It was found that the entropy of the gas is proportional to the horizon area in a gravity field strong, which is consistent with the holographic principle.

Keywords: Brick wall, black holes, holographic principle
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1 Introduction

In the early 70's, a connection between gravity and thermodynamics is established. Where the horizon for a black hole has an associated temperature

$$T_H = \frac{\hbar \kappa_0}{2\pi} \tag{1}$$

and entropy

$$S_{BH} = \frac{1}{4l_{pl}^2} A. \tag{2}$$

This entropy is proportional to black hole surface. The Bekenstein-Hawking entropy is considered a true thermodynamic entropy of the black holes. In a typical thermodynamic system, the thermal properties must reflect the microscopic physics. The temperature is a measure of average energy of particles and entropy counts the number of microstates of the system.

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The first models emerged to explain the origin of entropy arising from Gibbons/Hawking studies and model t'Hooft brick wall. For the former model (Euclidean approach) provides no insight into the dynamical origin of S_{BH}.

The latter (brick wall model) studies behavior of scalar fields near to black hole horizon [2]. Under the last model, the S_{BH} is related with the vacuum fluctuations in strong gravitational fields [1]. Because, an observer it at rest with black hole horizon sees a thermal bath of particles on the horizon [2].

In this point, we must clarify the concept of brick wall. Following to Israel [3]:

The brick wall is treated here as a real physical barrier. To prevent misunderstanding this conception must be distinguished from a quite different one, according to which the wall is fictitious, merely a mathematical ***cutoff*** *used to regularize the* $(\Delta r)^{-1}$ *and* $log\Delta r$ *terms in the expression for the thermal entropy .*

Our main purpose in this paper is study of hot quantum scalar field in a $(D+1)$ dimensional curved spacetime. Under the brick wall model, we calculate the free energy F_D of hot quantum scalar field in $(D+1)$ dimensions.

For this study D are n dimensional space, where $n = 2, 3...$ This method allows find the black hole entropy S_D and this is proportional to area.

In the section two, we calculate the free energy for hot quantum scalar field in a $(D+1)$ dimensional curved spacetime. Under standard prescription we derive the entropy for this scenery.

In the section three, we revised the thermodynamics of a hot quantum field in $(3+1)$ dimensional flat spacetime. We study the particular case of $D = 3$ and we find that the thermodynamics properties are same reported for [1] and [4].

In the section four, we study the BTZ black hole, we find that entropy under method explained en the section two is proportional al area.

In the end, we present discussions, we conclude that this model corresponds to a generalization of Brick wall model in $(D+1)$ dimensions.

2 Thermodynamics of ideal gas in $(D+1)$ dimensions

We consider a spacetime with metric

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2 \tag{3}$$

this includes the spacetimes Schwarzschild, Reissner-Nordstronm and (anti-) de Sitter in spacetime (3+1) dimensions [2].

For this spacetime Fursaev showed that free energy F_3 for a bosonic field in three spatial dimensions near to black hole horizon is [1]

$$F_3 = -\frac{1}{\pi^2}\zeta(4)\int T^4\sqrt{-g}d^3x, \tag{4}$$

where $\zeta(4)$ is Riemann function zeta, T is temperature of the bosonic gas, g is the determinant of metric and d^3x is differential element of volume spatial.

Free energy in $(2+1)$ for a gas of bosons is

$$F_2 = -\frac{1}{\pi}\zeta(3)\int T^3\sqrt{-g}d^2x. \tag{5}$$

If we consider a $(D+1)$-dimensional spacetime with a metric [5]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^{D-1}d\Omega^{D-1}. \tag{6}$$

and metric tensor $g_{\mu\nu}$ given by

$$g_{\mu\nu} = \begin{pmatrix} -f(r) & \dots & 0 \\ \vdots & \frac{1}{f(r)} & \dots \\ 0 & \dots & r^{D-1}\Omega \end{pmatrix}, \tag{7}$$

Then, the free energy is

$$F_D = -\frac{1}{\pi^{D-1}}\zeta(D+1)\int T^{D+1}\sqrt{-g}d^Dx, \tag{8}$$

where T is temperature of ideal gas and obey the Tolman's law

$$T(r) = \frac{T_\infty}{\sqrt{f(r)}} \tag{9}$$

and g is the determinant of metric tensor

$$g = r^{D-1}\Omega. \tag{10}$$

Their square root is

$$\sqrt{-g} = \sqrt{r^{D-1}\Omega} = r^{\frac{D-1}{2}}\Omega^{1/2}, \tag{11}$$

where r is radial part and Ω is solid angle in D dimensions. The differential element of volume spatial d^Dx, is the product of a radial part (dr) and an angular part $(d^{D-1}\Omega)$

$$d^Dx = dr d^{D-1}\Omega. \tag{12}$$

Thus, the free energy is reduced to

$$F_D = -\frac{1}{\pi^{D-1}}\zeta(D+1)\int \frac{T_\infty^{D+1}}{f(r)^{\frac{D+1}{2}}} r^{\frac{D-1}{2}} dr\Omega^{1/2}d^{D-1}\Omega. \tag{13}$$

Separating radial part and angular part in (13), we can write free energy as

$$F_D = -\frac{1}{\pi^{D-1}}\zeta(D+1)T_\infty^{D+1}\int \frac{r^{\frac{D-1}{2}}}{f(r)^{\frac{D+1}{2}}}dr\int \Omega^{1/2}d^{D-1}\Omega. \tag{14}$$

The first integral in (14) is associated to radial part and second one to angular part. Following S. Kolelar and T. Padmanabhan [5], and can be written as

$$F_D = -\frac{1}{\pi^{D-1}}\zeta(D+1)T_\infty^{D+1}\Omega\int\frac{r^{\frac{D-1}{2}}}{f(r)^{\frac{D+1}{2}}}dr. \tag{15}$$

Where Ω is solid angle in D dimensions.

Consider the following conditions [5]:

$$f(r_H) = 0,$$

which defines the horizon

$$2\kappa = f'(r_H)$$

It is a finite temperature in the vicinity of the black hole. Near to horizon, we write $f(r)$ as [2, 5, 10]

$$f(r) = f'(r_H)(r - r_H). \tag{16}$$

According to the above, the integral of (15) reduces to

$$\int\frac{r^{\frac{D-1}{2}}}{f(r)^{\frac{D+1}{2}}}dr = \int\frac{r^{\frac{D-1}{2}}}{(f'(r_H)(r-r_H))^{\frac{D+1}{2}}}dr = \frac{1}{f'(r_H)^{\frac{D+1}{2}}}\int\frac{r^{\frac{D-1}{2}}}{(r-r_H)^{\frac{D+1}{2}}}dr. \tag{17}$$

Defining $u = r - r_H$

$$\frac{1}{f'(r_H)^{\frac{D+1}{2}}}\int\frac{r^{\frac{D-1}{2}}}{(r-r_H)^{\frac{D+1}{2}}}dr = \frac{1}{f'(r_H)^{\frac{D+1}{2}}}\int\frac{(u+r_H)^{\frac{D-1}{2}}}{u^{\frac{D+1}{2}}}du. \tag{18}$$

And using a binomial expansion

$$(r_H+u)^{\frac{D-1}{2}} = r_H^{\frac{D-1}{2}} + \frac{1}{1!}\left(\frac{D-1}{2}\right)r_H^{\frac{D-3}{2}}u + \frac{1}{2!}\left(\frac{D-1}{2}\right)\left(\frac{D-3}{2}\right)r_H^{\frac{D-3}{2}}u^2 + \ldots + u^{\frac{D-1}{2}}. \tag{19}$$

So, (18) rewrites

$$\frac{1}{f'(r_H)^{\frac{D+1}{2}}}\int\frac{(u+r_H)^{\frac{D-1}{2}}}{u^{\frac{D+1}{2}}}du =$$
$$\frac{1}{f'(r_H)^{\frac{D+1}{2}}}\int\frac{1}{u^{\frac{D+1}{2}}}\left[r_H^{\frac{D-1}{2}} + \frac{1}{1!}\left(\frac{D-1}{2}\right)r_H^{\frac{D-3}{2}}u + \frac{1}{2!}\left(\frac{D-1}{2}\right)\left(\frac{D-3}{2}\right)r_H^{\frac{D-3}{2}}u^2 + \ldots + u^{\frac{D-1}{2}}\right]du. \tag{20}$$

Then the integral in (18) is rewritten as (20) [5]. S. Kolelar and T. Padmanabhan says: the main contribution to this integral comes from the lower limit of the integral $\frac{r_H^{\frac{D-1}{2}}}{u^{\frac{D+1}{2}}}$. Because near the horizon $u = h$ and $\frac{h}{r_H} \ll l_P$ [5].

$$\frac{1}{f'(r_H)^{\frac{D+1}{2}}}\int\frac{r_H^{\frac{D-1}{2}}}{u^{\frac{D+1}{2}}}du = -\frac{2r_H^{\frac{D-1}{2}}}{f'(r)^{\frac{D+1}{2}}(D-1)h^{\frac{D-1}{2}}} \tag{21}$$

where l_p is the Planck length. Near the horizon, we can replace h to l_p as a distance own

$$l_p = \int_{r_H}^{H} \frac{dr}{\sqrt{f(r)}} \approx 2\sqrt{\frac{h}{f'(r_h)}}, \tag{22}$$

solvig h and replace in (21)

$$\int \frac{r^{\frac{D-1}{1}}}{f(r)^{\frac{D+1}{2}}} dr = -\frac{2^D r_H^{\frac{D-1}{2}}}{(D-1)\left[f'(r_H)\right]^D l_p^{D-1}} \tag{23}$$

Finally we write the free energy for ideal gas in $(D+1)$ dimensions in the following form

$$F_D = -\frac{\zeta(D+1)}{(D-1)\pi^{D-1}} \left[\frac{A_{\frac{D-1}{2}}}{l_p^{D-1}}\right] \frac{T_\infty^{D+1}}{\kappa^D} \tag{24}$$

where the product $A_{\frac{D-1}{2}} = \Omega r_H^{\frac{D-1}{2}}$ is the horizon area in $(D+1)$ dimensions. The free energy is an *off-shell* prescription and expresses in four independent variables:

- The temperature T_∞,
- geometrical characteristics $(D+1)$ spacetime dimensions, horizon's area A and surface gravity κ.

The entropy is recoverable from the free energy by the standard prescription

$$S_D = -\left(\frac{\partial F}{\partial T}\right) \tag{25}$$

$$S_D = \frac{\zeta(D+1)}{(D-1)\pi^{D-1}} \left[\frac{A_{\frac{D-1}{2}}}{l_P^{D-1}}\right] \frac{(D+1)T_\infty^D}{\kappa^D}. \tag{26}$$

At this point, we observe that this is an *off-shell* prescription [2, 6]. Because, geometrical quantities (A y κ) are kept fixed when the temperature is varied [2].

3 Particular case $(3+1)$ dimensional flat spacetime

If we take the equation (8) and we do $g_{\mu\nu} = \eta_{\mu\nu}$ where $\eta_{\mu\nu}$ is tensor of Minkowski for a flat spacetime. So the equation (8) become in

$$F_D = -\frac{1}{\pi^{D-1}} \zeta(D+1) \int T^{D+1} \sqrt{-\eta} d^D x, \tag{27}$$

where $\sqrt{-\eta} = 1$, $T = T_\infty$ and $V_D = \int d^D x$. Then, we have

$$F_D = -\frac{1}{\pi^{D-1}} \zeta(D+1) T_\infty^{D+1} V_D. \tag{28}$$

We find the entropy of the hot quantum scalar field in the Minkowski spacetime like

$$S_D = -\frac{\partial F}{\partial T_\infty}$$

$$S_D = \frac{(D+1)}{\pi^{D-1}}\zeta(D+1)T_\infty^D V_D. \tag{29}$$

In particular the case when $D = 3$ for a flat spacetime. The free energy is

$$F_3 = -\frac{\pi^2}{90}T_\infty^4 V_3 \tag{30}$$

and the entropy is

$$S_3 = \frac{2}{45}\pi^2 T_\infty^3 V_3 \tag{31}$$

other properties of hot quantum scalar field in the minkowski spacetime are well known [19].

4 Particular case $(3+1)$ dimensional curved spacetime

If the temperature of quantum field is $T_\infty = T_H = \frac{\kappa}{2\pi}$, so the entropy is $S_3|_{T_\infty = T_H}$ using (26)

$$S_3|_{T_\infty = T_H} = \frac{1}{360\pi l_p^2}A \tag{32}$$

it is agree with Fursaev's result!! The quantum field is supposed to be in thermal equilibrium with the horizon [6].

Another hand, if $D = 3$, (24) and (26) reduced to

$$F_3 = -\frac{\pi^2}{180}\frac{T_\infty^4 A}{l_p^2 \kappa^3} \tag{33}$$

$$S_3 = \frac{\pi^2}{45}\frac{T_\infty^3 A}{l_p^2 \kappa^3} \tag{34}$$

such that (26) is entropy of bosonic field in $(3+1)$. For a photon gas in Schwarzschild spacetime, their entropy is [4]

$$S_3 = \frac{1}{45}\frac{\pi^2 k_B^4 c^3}{\hbar^3}\left[\frac{A}{l_P^2 \kappa^3}\right]T_\infty^3. \tag{35}$$

Also we calculate other thermodynamics properties with standard recipes:

- The internal energy of quantum field, $dE = dF + T_\infty dS$, is

$$E_D = \frac{\zeta(D+1)}{(D-1)\pi^{D-1}}\left[\frac{A_{\frac{D-1}{2}}}{l_p^{D-1}}\right]\frac{DT_\infty^{D+1}}{\kappa^D}. \tag{36}$$

  If $D = 3$ the internal energy

$$E_3 = \frac{\pi^2}{60}\frac{AT_\infty^4}{l_p^2 \kappa^3}, \tag{37}$$

  that is the internal energy of photon gas in Schwarzschild spacetime [4].

- The specific heat is

$$C_V = \left(\frac{\partial E}{\partial T_\infty}\right)_V$$

$$C_V = \frac{\zeta(D+1)}{(D-1)\pi^{D-1}} \left[\frac{A_{\frac{D-1}{2}}}{l_p^{D-1}} \right] \frac{D(D+1)T_\infty^D}{\kappa^D}, \tag{38}$$

with $D = 3$

$$C_{V-3} = \frac{\pi^2}{15} \frac{AT_\infty^3}{l_p^2 \kappa^3}. \tag{39}$$

- And the pressure of quantum field near to horizon is

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T_\infty} = \frac{1}{l_P}\left(\frac{\partial F}{\partial A}\right)_{T_\infty}$$

$$P_D = \frac{\zeta(D+1)}{(D-1)\pi^{D-1}} \left[\frac{1}{l_p^D} \right] \frac{T_\infty^{D+1}}{\kappa^D} \tag{40}$$

with $D = 3$

$$P_3 = \frac{\pi^2}{180} \frac{T_\infty^4}{l_p^3 \kappa^3}. \tag{41}$$

Is interesting to note that the thermodynamic properties described by (33), (34), (37), (39) and (41) have already been studied by the authors in the case of a photon gas in the Schwarzschild spacetime [4] under the approximation of brick wall model by Mukohyama and Israel for a hot quantum scalar field near the horizon in the Boulware state [2].

5 BTZ spacetime

This spacetime is interesting because the BTZ black holes are asymptotically anti de Sitter. In gravity (2+1) the curvature is constant ($R = -6/l^2$). This black holes don't have points and regions in which the curvature is divergent. However, the BTZ black holes have a horizon, an ergosphere and thermodynamic properties to the classical solutions of General Relativity [9].

Their metric is

$$ds^2 = -\left(\frac{r^2 - r_+^2}{l^2}\right) dt^2 + \frac{1}{\left(\frac{r^2 - r_+^2}{l^2}\right)} dr^2 + r^2 d\phi, \tag{42}$$

with a negative cosmological constant $\lambda = -\frac{1}{l^2}$. The area of horizon of BTZ black hole is the length $2\pi r_+$ [1, 8].

We calculate the black hole entropy for this spacetime with method of the section II. The first step is to use (8) in which $D = 2$ spatial dimensions, then we have

$$F_2 = -\frac{1}{\pi}\zeta(3) \int T^3 \sqrt{-g} d^2x, \tag{43}$$

in agree with (9); (11), $\sqrt{-g} = r^{1/2}\Omega^{1/2}$ and (12) $d^2x = dr d\Omega$. The free energy in the BTZ spacetime is rewritten as

$$F_2 = -\frac{1}{\pi}\zeta(3) \int \left[\frac{T_\infty}{f(r)^{1/2}}\right]^3 r^{1/2}\Omega^{1/2} dr d\Omega = -\frac{1}{\pi}\zeta(3) T_\infty^3 \int \Omega^{1/2} d\Omega \int \frac{r^{1/2}}{f(r)^{3/2}} dr, \tag{44}$$

where the solid angle is $\Omega = \int \Omega^{1/2} d\Omega = 2\pi$ and $f(r) = \left(\frac{r^2 - r_+^2}{l^2}\right)$ for the BTZ spacetime. We consider the last integral is agree with the condition (16), the integral reduces to

$$\int \frac{r^{1/2}}{f(r)^{3/2}} dr = \frac{1}{[f'(r_+)]^{3/2}} \int \frac{r^{1/2}}{(r - r_+)^{3/2}} dr, \tag{45}$$

this integral is similar to (17) y it can solve the same form. With $u = r - r_+$, thus (45) become in

$$\int \frac{r^{1/2}}{f(r)^{3/2}} dr = \frac{1}{[f'(r_+)]^{3/2}} \int \frac{(r_+ + u)^{1/2}}{u^{3/2}} du. \tag{46}$$

And using a binomial expansion

$$(r_+ + u)^{1/2} = r_+^{1/2} + \frac{1}{2} r_+^{-1/2} u + \ldots + u^{1/2}. \tag{47}$$

Then the integral (46) is rewritten as

$$\frac{1}{[f'(r_+)]^{3/2}} \int \frac{(r_+ + u)^{1/2}}{u^{3/2}} du =$$

$$\frac{1}{[f'(r_+)]^{3/2}} \int \frac{\left[r_+^{1/2} + \frac{1}{2} r_+^{-1/2} u + \ldots + u^{1/2}\right]}{u^{3/2}} du \approx \frac{1}{[f'(r_+)^{3/2}]} \int \frac{r_+^{1/2}}{u^{3/2}} du = -\frac{2 r_+^{1/2}}{f'(r_+)^{3/2}} . \frac{1}{\sqrt{u}}. \tag{48}$$

Under the condition of S. Kolekar and T. Padmanabhan [5]. Again, near to horizon $u = h$ and we replace h to l_p as distance own in agree with (22)

$$\int \frac{r^{1/2}}{f(r)^{3/2}} = -\frac{2^2 r_+^{1/2}}{[f'(r)]^2 \, l_p}. \tag{49}$$

We write the free energy for hot quantum scalar field in BTZ spacetime as

$$F_2 = -\frac{1}{\pi} \zeta(3) T_\infty^3 \left[\Omega r_+^{1/2}\right] \frac{2^2}{2^2 \kappa^2 l_p} = -\frac{1}{\pi} \zeta(3) \frac{T_\infty^3}{\kappa^2} \frac{\left[A_{\frac{2-1}{2}}\right]}{l_p} \tag{50}$$

where the area of horizon is $A_{\frac{2-1}{2}} = \Omega r_+^{1/2} = 2\pi r_+$. We can obtain the entropy as

$$S_2 = \frac{3}{\pi} \zeta(3) \frac{T_\infty^2}{\kappa^2} \left[\frac{2\pi r_+}{l_p}\right]. \tag{51}$$

Again, if the temperature of quantum field is $T_\infty = T_H = \frac{\kappa}{2\pi}$ and the entropy to the temperatura Hawking is

$$S_2|_{T_\infty = T_H} = \frac{3}{\pi} \zeta(3) \frac{1}{\kappa^2} \frac{\kappa^2}{4\pi^2} \left[\frac{2\pi r_+}{l_p}\right]$$

$$S_2|_{T_\infty = T_H} = \frac{2\pi r_+ a}{l_p}. \tag{52}$$

The mass M of the BTZ black hole is defined as

$$M = \frac{r_+^2}{8l^2 G_3}, \tag{53}$$

where G_3 is the 3D gravitational coupling and has the dimension of length [1].

Thus, $S_2|_{T_\infty = T_H}$ is

$$S_2|_{T_\infty = T_H} = \frac{2\pi a}{l_p}\sqrt{8l^2 M G_3} \tag{54}$$

6 Summary and Discussion

The last result shows that entropy of ideal gas is proportional to area in $(D+1)$ dimensions, when this is in equilibrium state with the horizon at temperature T_∞.

Under the brick wall model for $(D+1)$ dimensional curved spacetime, the entropy can be determined by response of the free energy of the system to change of temperature given by (25) [1, 2, 4, 6]. Also the distinction between *thermodynamical* and *statistical* entropies disappears in this model, because the geometrical and thermal variables are kept independent. Observe that this is an off-shell prescription [2, 6].

The free energy was made from the brick wall model, which studies quantum fields close to the horizon [2]. Following this model, thus affirms Fursaev and Mukohyama: entropy defined in (26) and corresponds to the Bekenstein-Hawking entropy S_{BH} for quantum fields near the horizon in spacetime $(D{+}1)$. And also depends on the geometrical and thermal quantities as expressed Mukohyama [1, 2]

$$S_D \propto \left[\frac{A_{\frac{D-1}{2}}}{(D-1)l_P^{D-1}}\right]\frac{(D+1)T_\infty^D}{\kappa^D} \tag{55}$$

if $T = T_H$, then $S_D = S_{BH}$, that are defined only for geometrical quantities in $(D+1)$ dimensions [2].

We find that the entropy to the Hawking temperature in BTZ black hole is proportional to

$$S_2|_{T_\infty = T_H} \propto 2\pi\sqrt{l^2 M}, \tag{56}$$

This result is consistent with [1, 8].

From the above we can consider that the present model is a generalization of the study of hot quantum scalar field in a $(D+1)$ dimensional curved spacetime of brick wall model. In the case when $D = 3$ the usual space, the entropy is reduced to the results reported by [1] and [4].

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