EJTP **13, No. 36** (2016) 99–106 *Electronic Journal of Theoretical Physics*

Validity of Korteweg-de-Vries Equation for Arterial Pulse Waves

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Received 24 December 2015, Accepted 24 April 2016, Published 10 November 2016

Abstract: Korteweg-de-Vries equation (KdV) is very popular tool to describe nonlinear waveform like cnoidal waves or solitons. The long waves within environments with weak dispersion are the object of it. Many authors recognize pulse waves (PW) in blood vessels as varieties of such nonlinear waves. Therefore, the solution of the problem of the dispersion for pulse waves is either solid scientific support or vice versa of this widespread viewpoint. This paper first presents the solution of above problem with methods of dimension analysis. This technique is utterly independent of assumptions that are backbone of KdV theory. The results of analysis show that the dispersion of PW depends just of two parameters. Both of them are proportional to wavenumber. The ratio of them is equal to large Reynolds number. Thus, we can keep only one of them. It is dimensionless wave number: $u = \dfrac{2\,\pi\,D}{\lambda} << 1$, where D is inner diameter of vessel, and λ is the wavelength. For our purposes important is the result about weak dispersion of pulse waves. The dependence of frequency vs. the wavenumber is almost linear with minor cubic adding. Pulse wave is quite long wave as it follows from dimensional analysis. Both these outcomes confirm the validity of KdV equation for pulse waves into blood vessels.

Keywords: Korteweg-de-Vries Equation, Validity, Dimensional Analysis, Dispersion, Pulse Waves, Dimensionless Wave Number
PACS (2010): 02.60.Cb, 87.19.U-, 47.63.Cb
MSC: 92C99

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1 Introduction

The pulse wave (PW) is prominent complex phenomenon within blood circulation. There are several methods of pulse wave measurements based on different principles. Such measurements are standard for different medical diagnostic purposes [1 , 2].

In 1895 the Danish scientists Diderik Johannes Korteveg and Gustav de Vries studied equations, which obey a wave on the water, in the absence of water currents and with constant viscosity of fluid. The equation of Korteveg-de-Vries (KdV) first described the case of long waves on shallow water [3 , 4].

The KdV equation is in intensive using of hemodynamics for mathematical modeling of blood pulse waves [5 , 6 , 7]. Indeed, the pulse wave (PW) resembles solitary waves in many respects [7 , 8]. It comes about cnoidal waves as well as solitons. Both are exact solutions of the KdV equation [9 , 10].

There exist the rich traditions of dimensional analysis within fluid mechanics [11 , 12]. Yet, this topic tends be overlooked into hemodynamics sources. We can to refer on the single paper [13]. The dimension analysis is especially useful because its conclusions do not depend on the basic assumptions of KdV-theory. Consequently, they could confirm or refute its legitimacy.

The aim of this paper is the determination of the general form of dispersion law for pulse waves into blood vessels by dimensional analysis method. Weak or strong dispersion should be independent arguments ”pro et contra” respectively as for validity of KdV equation regarding to PW.

2 Dimensional Analysis

We are going here to apply the methods [12 , 13] i.e. the classic way of dimensional analysis for above-mentioned problem. To do this, let us determine the set of so-called controlling variables and constants that significantly affect the spread of pulse waves. The short list of these: the cyclic frequency of the PW (ω), wave number (k), kinematic viscosity of blood (μ), D - effective inner diameter of the artery, blood density (ρ) and effective modulus of elasticity of the arterial walls (C) [1 , 2 , 5].

Some equation of state unites the selected variables:

$$F(\omega, \kappa, \mu, D, \rho, C) = 0. \tag{1}$$

Let us to define the dimensions of above mentioned control factors respectively:

$$\left[\left(\frac{1}{time}\right), \left(\frac{1}{length}\right), \left(\frac{length^2}{time}\right), length, \left(\frac{mass}{length^3}\right), \left(\frac{mass}{length\, time^2}\right)\right]. \tag{2}$$

Therefore, we need only three fundamental units (length, time and mass: L, T, M) to describe the dimensions of all variables. It is obvious from the above list of dimensions. Hence, the equation of state (1) may be rewritten as function only of three dimensionless factors. These three factors are combinations of six primary variables in accordance with the famous π-theorem [12].

The dimension of each of these three factors ($\pi_i, i = 1, 2, 3$) is obviously equal to one. This fact looks as follows in the monomial form:

$$[\pi_{1,2,3}] = [\omega]^{r_1}[\kappa]^{r_2}[\mu]^{r_3}[D]^{r_4}[\rho]^{r_5}[C]^{r_6} = \\ = L^{-r_2+2r_3+r_4-3r_5-r_6}T^{-r_1-r_3-2r_6}M^{r_5+r_6} = 1, \tag{3}$$

here $r_j, (j = 1..6)$ - are degree indicators, the square brackets denote dimensions of variables according to (2).

Reader can easy write a system of linear equations for these indicators using the expression (3):

$$\left.\begin{aligned} -r_2 + 2r_3 + r_4 - 3r_5 - r_6 = 0 \\ r_1 + r_3 + 2r_6 = 0 \\ r_5 + r_6 = 0 \end{aligned}\right\}. \tag{4}$$

Get this system in relation to the three indicators that presenting via the other three:

$$\left.\begin{aligned} r_2 &= r_3 + r_4 - r_1 \\ r_5 &= \frac{r_1 + r_3}{2} \\ r_6 &= \frac{-r_1 - r_3}{2} \end{aligned}\right\}. \tag{5}$$

Let us account the solutions (5) into monomial expression (3):

$$[\omega]^{r_1}[\kappa]^{r_3+r_4-r_1}[\mu]^{r_3}[D]^{r_4}[\rho]^{\left(\frac{r_1+r_2}{2}\right)}[C]^{\left(\frac{-r_1-r_2}{2}\right)} = 1. \tag{6}$$

Further we are going to apply the method of extremes [14] for unambiguous identification of the rest unknown indicators: r_1, r_3, r_4. Taking the logarithm of the both sides of the expression (6), we write down it as a function of those indicators:

$$r_1 \ln\omega + (r_3 + r_4 - r_1)\ln\kappa + r_3\ln\mu + r_4\ln D + \\ + \left(\frac{r_1 + r_3}{2}\right)\ln\rho - \left(\frac{r_1 + r_3}{2}\right)\ln C = 0 \tag{7}$$

Then find the partial derivatives of the function (7) on its variables r_1, r_3, r_4, looking for extreme:

$$\ln\omega - \ln\kappa + \frac{\ln\rho}{2} - \frac{\ln C}{2} = 0 \tag{8}$$

$$\ln\kappa + \ln\mu + \frac{\ln\rho}{2} - \frac{\ln C}{2} = 0 \tag{9}$$

$$\ln\kappa + \ln D = 0 \tag{10}$$

The solutions of (8-10) have such forms:

$$\omega = \kappa\sqrt{\frac{C}{\rho}}; \qquad \mu = \frac{1}{\kappa}\sqrt{\frac{C}{\rho}}; \qquad D = \frac{1}{\kappa}, \tag{11}$$

here $\sqrt{\frac{C}{\rho}}$ is well-known Moens-Korteweg velocity of course.

We can now find the above-mentioned three dimensionless factors with solutions (11):

$$\pi_1 = \frac{\omega}{\kappa\sqrt{\frac{C}{\rho}}}; \qquad \pi_2 = \frac{\mu\,\kappa}{\sqrt{\frac{C}{\rho}}}; \qquad \pi_3 = \kappa D. \tag{12}$$

Currently we can rewrite the equation of state (1) in such a form:

$$\pi_1 = f(\pi_2, \pi_3). \tag{13}$$

Because parameters $(\pi_2, \pi_3\ )$ both are proportional to wave number κ we can to write down:

$$\omega(\kappa) = \left(\sqrt{\frac{C}{\rho}}\right)\kappa\, f(\kappa), \tag{14}$$

here the function of wavenumber depends on two dimensionless factors:

$$f(\kappa) = f(\pi_2, \pi_3). \tag{15}$$

3 Dispersion: Analysis and Estimates

Equation (14) is actually the general law of dispersion for the frequency of the pulse waves. It is the connection between the frequency and wavenumber. The dispersion takes place if only the function $f(\kappa)$ is not a constant. Just the dependence of the function $f(\kappa)$ (15) on wave number determines the non-linearity of the dispersion law (14). Opposite, the dispersion law becomes linear under condition $f(\kappa) = const$. The dispersion is absent in such a case.

Important is that the ratio of two variables of function $f(\kappa)$ (13) is exactly equal to Reynolds's number:

$$\frac{\pi_3}{\pi_2} = \frac{D}{\mu}\sqrt{\frac{C}{\rho}} = \mathrm{Re}. \tag{16}$$

This number describes the contribution of the viscous forces in the current mode of the stream.

Let us to put for evaluations [1]: $D \approx 2.4{\cdot}10^{-2}\ m$, $\mu \approx 3.3{\cdot}10^{-6}\ m^2/s$, $C \approx 5{\cdot}10^6\ Pa$, $\rho \approx 1.06 \cdot 10^3\ kg/m^3$.

Then we can obtain about 7 m/s for Moens-Korteweg speed and big Reynolds number: Re$\approx 5 \cdot 10^4 >> 1$. Hence, the Moens-Korteweg speed significantly exceeds the typical value for stationary blood flow (about $0.2 - 0.5\ m/s$ [1]). The Reynolds number for the pulse waves testifies the negligible contribution of the viscous force. This force is presented by dimensionless factor π_2.

Thus, the extremely small factor $\pi_2 << \pi_3$ may be neglected in (14), (15) according to the recommendations of the theory [12]. From the viewpoint of Physics, it means choosing the model of ideal fluids for blood flow, neglecting the effects of viscous friction on the spread of PW. We can to rewrite the equation of state (14) in more compact and simpler form based on such considerations:

$$\omega = \left(\sqrt{\frac{C}{\rho}}\right) \kappa\, f(\kappa D). \tag{17}$$

Estimations of the pulse wavelength is provided in [1 , 5] though the direct measurements are unknown for us. These evaluations belongs to rather wide range: $\lambda \approx (2.5 - 5.5)\ m$. Nonetheless, this allows us to evaluate the dimensionless factor $\pi_3 = \kappa D$:

$$\kappa\, D = \frac{2\pi D}{\lambda} \in [2.7 - 6.0] \cdot 10^{-2} << 1. \tag{18}$$

Hence, the dimensionless wavenumber (18) is clearly small and PW shall be taking as the long waves. It is the first argument for KdV-equation.

Let us decompose the function of right side (17) close to zero point ($\kappa D \to 0$). We hope that the function has a finite limit and proper derivatives in this point [12]:

$$f(\kappa\, D) \approx A + B\kappa D + \frac{1}{2}\, G(\kappa D)^2. \tag{19}$$

Here A, B, G are some digital dimensionless constants. If function (17) has not only the finite limit but and an extreme at zero point then the constant B is equal to zero. Under such condition, the dispersion law converts itself to the form:

$$\omega = \left(\sqrt{\frac{C}{\rho}}\right) \kappa \left(A + \frac{1}{2}\, G\kappa^2 D^2\right). \tag{20}$$

The dispersion law (20) would be one-to-one with that reported in [5] if additionally $A = 1$, $G = \dfrac{1}{12}$:

$$\omega = \left(\sqrt{\frac{C}{\rho}}\right) \kappa \left(1 + \frac{\kappa^2 D^2}{24}\right). \tag{21}$$

The concretization of dimensionless constants in (21) is not the sole and main distinction between expressions (20) and (21). More important is that (21) has been obtained in framework of KdV theory whereas (20) beyond it.

Both dispersion laws (20) and (21) are practically linear because each next factor within brackets is much smaller of first. We shall to recognize the dispersion as very weak for the reason defined by inequality (18) for dimensionless wavenumber.

The phase propagation velocity and the same for energy spread are determined by (21) and practically coinciding:

$$\nu_{ph} = \left(\sqrt{\frac{C}{\rho}}\right) \left(1 + \frac{1}{24}\,(\kappa D)^2\right) \tag{22}$$

$$\nu_{gr} = \left(\sqrt{\frac{C}{\rho}}\right) \left(1 + \frac{3}{24}\,(\kappa D)^2\right) \tag{23}$$

Such an almost coincidence is typical for environments with weak dispersion.

4 Conclusions

The pulse waves are long waves in comparison with the channel sizes. The small values of dimensionless wavenumbers confirm this statement. It is an argument for validity of KdV theory.

The dispersion of pulse waves is extremely weak because minor dimensionless wavenumber defines it. That is the second argument for legitimacy of KdV equation for pulse waves.

Dimensional analysis and estimates are the way that allows achieving both above arguments for validity of KdV equation as for blood pulse waves.

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