Minkowski Momentum of an MHD Wave

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Abstract: Momentum of an MHD wave has been examined from the view point of the electromagnetic momentum expression proposed by Minkowski. Basic calculations confirm that the Minkowski momentum is the sum of electromagnetic momentum and the momentum of the medium, as proposed in some of the past literature. This result has been explicitly confirmed by an example of an MHD wave, whose dynamics can be easily and precisely calculated from basic equations.

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1 Introduction

The Abraham-Minkowski controversy has been discussed by a number of authors over a hundred years. Minkowski [1] proposed the electromagnetic momentum density in a dielectric medium must be $\mathbf{D} \times \mathbf{B}$, and Abraham [2,3] proposed $\mathbf{E} \times \mathbf{H}$ for that (in the present letter symbols have conventional meanings, e.g., $\mathbf{E}$ = electric field, unless otherwise stated). There have been published numerous papers on this problem both theoretically and experimentally, but the final conclusion is still yet to come (see, e.g., [4,5] for reviews).

Several authors [6,7,8] pointed that the electromagnetic field inevitably affect the dynamics of the medium to change its energy-momentum, and therefore, the energy-momentum of an electromagnetic wave must include the contribution of the medium. The present letter claims that the Minkowski momentum is the sum of electromagnetic momentum and the momentum of the medium. Feigel [6] obtained a similar

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result based on the Noether's theorem using Lagrangian formulation. Compared to this elegant approach, the calculation in the present letter is rather down-to-earth type, which is more closer to the Minkowski's original derivation. This approach is less elegant, however, easier to understand its meaning intuitively. Note that the author does not claim this calculation is a novel result; it is to give readers a concise knowledge on this problem. The consideration stated above is confirmed by an example of an MHD wave in a collisionless magnetized plasma. The author also would like to remark the derivation of MHD wave momentum flux here is not novel. The purpose of the present letter is to suggest that the momentum of plasma waves is favorable evidence for the Minkowski momentum.

Usually the electromagnetic behavior of an ordinary medium is complicated and need to calculate microscopic states of molecules, which is difficult to solve exactly. A collisionless plasma is, in contrast, easy to calculate its response to the electromagnetic field from the classical basic equations (Maxwell equations and Newtonian equation of motion). Here in this letter we use the MHD approximation, which is the approximation often used to investigate large scale phenomena in plasmas. If one wishes it is possible to derive an exact solution without the MHD approximation to confirm the same result.

The word "plasma" refers to a gas contains charged particles and interacts with electromagnetic field. Usually the negative and positive charges of a plasma particles are equal in total and the fluid is neutral as a whole; local charge accumulations may take place but they are canceled to vanish when integrated over a large enough volume.

In the present letter we treat a simplest case, where the plasma contains no neutral particles and consists of equal number protons and electrons only. Further, we assume the plasma density is low enough to neglect the collisions between particles. This situation is not unrealistic at all. On the contrary, such plasmas are commonly found in most of space environments except the peripheral region of celestial bodies such as the sun or earth.

When we are interested in large scale phenomena in collisionless plasmas, the MHD approximation can give a good description; the words "large scale" mean much larger than the temporal and spatial scales of Larmor motion. In such scales, a plasma can be treated with two simple equations, namely, Ohm's law and fluid equation of motion. When the collisions are negligible, the conductivity becomes infinitely large and the Ohm's law is reduced to the condition $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$. This is called "frozen-in condition" since fluid elements of a plasma behave as if "frozen" into the magnetic field under this condition.

From the frozen-in condition we can determine the plasma bulk motion from the electromagnetic field; the momentum of plasma fluid is calculated from this motion. In addition the polarization is calculated from the equation of motion, and therefore, the electric flux density can be obtained to estimate the Minkowski momentum. Comparing these two results we find the Minkowski momentum is the sum of the electromagnetic momentum and the momentum carried by the plasma fluid.

The derivation from the MHD equation stated above is intuitive and easy to understand, however, it is not the derivation from the microphysics of charged particles. To understand the microscopic aspect of the plasma dynamics, it is desirable to derive the same result without the knowledge of the MHD theory. Therefore, a derivation based on the very basic Newtonian equation of motion is also provided.

2 Basics

We briefly review the meaning of polarization/magnetization vector in this section. Microscopic Ampere's law in a medium is

$$-\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0^{-1} \nabla \times \mathbf{B} = \mathbf{J} \tag{1}$$

Suppose there is no external current, and $\mathbf{J}$ consists of the polarization current $\mathbf{J}_P$ and magnetization current $\mathbf{J}_M$, which are generated in response to the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ respectively. We introduce the polarization vector $\mathbf{P}$ and magnetization vector $\mathbf{M}$ such that

$$\frac{\partial}{\partial t}\mathbf{P} = \mathbf{J}_P\,, \quad \nabla \times \mathbf{M} = \mathbf{J}_M \tag{2}$$

In this context, $\mathbf{P}$ and $\mathbf{M}$ should be understood as convenient mathematical expressions to represent the response of the medium to the electromagnetic field, rather than real physical entities. This is somewhat similar to introducing electric potential ϕ to handle electric fields $\mathbf{E}$. What we actually measure is $\mathbf{E}$, but the calculation becomes easier when we introduce a mathematical tool called potential ϕ.

Let us introduce macroscopic polarization and magnetization vector $\bar{\mathbf{P}}$ and $\bar{\mathbf{M}}$ (bar means average) by averaging over a microscopically large but macroscopically small volume. These macroscopic vectors are created in response to the electromagnetic field, thus, can be expressed as a functional of $\mathbf{E}$ and $\mathbf{B}$ in general. For example, $\bar{\mathbf{P}}$ may be written as something like

$$\bar{\mathbf{P}}(\mathbf{x}, t) = \int_{-\infty}^{0} dt' \int d\mathbf{x}'\, f(\mathbf{E}(t - t', \mathbf{x} - \mathbf{x}'), \mathbf{B}(t - \tau, \mathbf{x} - \mathbf{x}'))\,.$$

If the above complex relations can be approximated with simple linear relations, the calculation will be considerably simplified. Actually there are materials, namely, dielectric and magnetic materials, for which the above expression can be well approximated by simple linear and local relations as

$$\bar{\mathbf{P}} = \chi_P \bar{\mathbf{E}}\,, \quad \bar{\mathbf{M}} = \chi_M \bar{\mathbf{B}}\,. \tag{3}$$

The fields $\bar{\mathbf{D}}$ and $\bar{\mathbf{H}}$ are then defined as macroscopic quantities as

$$\bar{\mathbf{D}} = \varepsilon_0 \bar{\mathbf{E}} + \bar{\mathbf{P}} = \varepsilon \bar{\mathbf{E}}\,, \quad \bar{\mathbf{H}} = \mu_0^{-1} \bar{\mathbf{B}} + \bar{\mathbf{M}} = \mu^{-1} \bar{\mathbf{B}} \tag{4}$$

The linear coefficients χ_P and χ_M are matrices in general because the medium may not be isotropic (as in our example of magnetized plasmas).

Note that these are not exact expressions; just phenomenological approximation that is valid only within certain ranges of field strength and time/spatial scales. Also, there are materials for which the above expressions are not appropriate. For example, collisionless plasmas we will examine in this letter do not have such a linear relation for magnetic field and flux.

The macroscopic polarization and magnetization vector $\bar{\mathbf{P}}$ and $\bar{\mathbf{M}}$ are often introduced as the sum of polarization/magnetization of each molecule in the material. This is not the only microscopic mechanism to cause the polarization or magnetization. The derivation of macroscopic Maxwell equations is valid as long as the linear and local relations (3) are appropriate. Actually, the polarization vector $\bar{\mathbf{P}}$ is not due to the molecular polarization, but derived from the Larmor motion of charged particles, in our example of an MHD plasma in this letter.

3 Minkowski Momentum

This section is for the review on the argument favorable to the Minkowski momentum; the calculations here are close to the original Minkowski's. The momentum of microscopic electromagnetic field is $\varepsilon_0 \mathbf{E} \times \mathbf{B}$ and its conservation law is

$$\frac{\partial}{\partial t}(\varepsilon_0 \mathbf{E} \times \mathbf{B}) + \nabla \cdot T + (\mathbf{J}_P + \mathbf{J}_M) \times \mathbf{B} + (Q_P + Q_M)\mathbf{E} = 0 \,, \tag{5}$$

where T is the Maxwell stress tensor and we denote $\partial T_{ij}/\partial x_i = (\nabla \cdot T)_j$ in short. The polarization/magnetization charge Q_P and Q_M are the result of polarization/magnetization current ($\partial Q_{P,M}/\partial t = \nabla \mathbf{J}_{P,M}$). The charge due to magnetization current vanishes when averaged, $\bar{Q}_M = 0$ since $\bar{\mathbf{J}}_M$ satisfies (2).

The third and fourth term of (5) are the Lorentz and Coulomb force acting on the medium, and therefore, it can expressed by the momentum change of the medium.

$$(\mathbf{J}_p + \mathbf{J}_M) \times \mathbf{B} + (Q_P + Q_M)\mathbf{E} = \frac{\partial}{\partial t}\mathbf{g} + \nabla \cdot T_M \,, \tag{6}$$

where $\mathbf{g}$ and T_M are the momentum density and stress tensor of the medium. It should be noted that the right hand side of the above expression has mathematical ambiguity. If we define new values of momentum/stress by $\mathbf{g}' = \mathbf{g} + \mathbf{a}$ and $T' = T + G$ with arbitrary vector $\mathbf{a}$ and tensor G that satisfy $\partial \mathbf{a}/\partial t = \nabla G = 0$, they also satisfy the above equation. Therefore, $\mathbf{g}$ and T do not necessarily have to be the total momentum/stress of the medium. For example, the medium may contain a part that does not interact with the electromagnetic field, and such part causes this ambiguity.

From (4) we obtain

$$\bar{\mathbf{J}}_P \times \bar{\mathbf{B}} = \frac{\partial}{\partial t}(\bar{\mathbf{P}} \times \bar{\mathbf{B}}) + \chi_P \left[(\bar{\mathbf{E}} \cdot \nabla)\bar{\mathbf{E}} + \frac{1}{2}\nabla \bar{\mathbf{E}}^2 + \bar{\mathbf{E}}(\nabla \bar{\mathbf{E}}) \right] , \tag{7}$$
$$\bar{\mathbf{J}}_M \times \bar{\mathbf{B}} = \chi_M \left[(\bar{\mathbf{B}} \triangledown)\bar{\mathbf{B}} + \frac{1}{2}\nabla \bar{\mathbf{B}}^2 \right] .$$

Here we neglected cross terms of fluctuation in averaging as usually done in this kind of calculation, e.g., $\overline{\mathbf{P} \times \mathbf{B}} = \bar{\mathbf{P}} \times \bar{\mathbf{B}}$. Combining (5), (6) and (7) we obtain

$$\frac{\partial}{\partial t}(\varepsilon_0 \bar{\mathbf{E}} \times \bar{\mathbf{B}} + \bar{\mathbf{g}}) + \nabla \cdot (\bar{T} + \bar{T}_M) = \frac{\partial}{\partial t}(\bar{\mathbf{D}} \times \bar{\mathbf{B}}) + \nabla \cdot \bar{T}' = 0 \, , \tag{8}$$

where $\bar{T}'$ is the stress tensor in a dielectric medium defined as

$$\bar{T}'_{ij} = \bar{E}_i \bar{D}_j + \mu_0^{-1} \bar{B}_i \bar{H}_j - \frac{1}{2}\delta_{ij}(\bar{\mathbf{E}} \cdot \bar{\mathbf{D}} + \bar{\mathbf{B}} \cdot \bar{\mathbf{H}}) \, , \tag{9}$$

which is the sum of the fluxes of electromagnetic momentum and momentum carried by the medium.

Now that we understand the Minkowski momentum includes the part of the medium then so should be for the energy. The energy is equivalent to mass in relativity, thus the energy flux of the medium must include mass, and then the flux may take the form of $\bar{\mathbf{D}} \times \bar{\mathbf{B}}$ to make the energy momentum symmetric.

4 Drift and Current in MHD Plasma

In this section we give a brief derivation of the drift and current invoked by an electric field in a MHD plasma, which we will need to analyze the momentum associated with an electromagnetic wave (an MHD wave) in the next section. Both E cross B drift and polarization current are very basic ingredients in the MHD theory, and readers familiar with plasma physics can skip this section.

Firstly, we give derivation using two MHD equations, namely, equation of motion and Ohm's law. The MHD theory regards a plasma as a conductive fluid, and the current is related to the electric field with the conductivity σ as

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \sigma^{-1}\mathbf{J} \, , \tag{10}$$

where $\mathbf{v}$ is the bulk velocity of the plasma fluid, i.e., the velocity of the gravicenter of plasma particles. If the density of the plasma is low enough to be able to ignore collisions between plasma particles, the conductivity is supposed to infinitely large; the right hand side of (10) vanishes in this case. Taking the outer product of $\mathbf{B}$ and (10) with vanishing right hand side, we arrive at

$$\mathbf{v}_\perp = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \, , \tag{11}$$

where $\mathbf{v}_\perp$ is the velocity component perpendicular to the magnetic field $\mathbf{B}$. The above expression means that the perpendicular motion of the MHD plasma is determined by the outer product of $\mathbf{E}$ and $\mathbf{B}$; this is called E cross B drift in plasma physics.

Now that we have the velocity of an MHD plasma, we move on to the current associated with plasma motion. To this end, we start from the equation of motion, which is

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{1}{\rho}(\nabla P + \mathbf{J}\times\mathbf{B})\,, \tag{12}$$

where ρ and P are the mass density and plasma pressure respectively. Usually a charge separation in an MHD plasma is negligibly small so the Coulomb force term does not appear in the equation of motion. Here in this letter we treat the simplest case, where nonlinearity and the plasma pressure are negligible; the same conclusion can be obtained when these effects are included. The current perpendicular to the magnetic field is obtained by taking the outer product of (12) and $\mathbf{B}$, which is

$$\mathbf{J}_\perp = \frac{\rho}{B^2}\left(\mathbf{B}\times\frac{\partial \mathbf{v}}{\partial t}\right)\,. \tag{13}$$

This current is called inertial current or polarization current, and corresponds to $\bar{\mathbf{J}}_P$ in Section II.

Hereafter we assume a specific case where the plasma is immersed in a stationary background magnetic field from an external source; plasmas in the earth's magnetosphere or nuclear fusion devices (tokamak etc.) are examples of such a situation. We further simplify the problem to have the magnetic field uniform; we take the z coordinate in its direction as $B_z = B_0$.

Suppose a linearly polarized plane wave is propagating in the z direction. We take the x direction along its electric field (E_x), and consequently the wave magnetic field is in the y direction (B_y). From (11) we obtain

$$v_y = \frac{E_x}{B_0}\,. \tag{14}$$

and substituting this into (13) we have

$$J_x = \frac{\rho}{B_0^2}\frac{\partial E_x}{\partial t} \tag{15}$$

The time derivative of B_y vanishes by taking the outer product with B_0 in the above expression. The polarization vector $\mathbf{P}$ can be calculated from the definition (2) as

$$P_x = \frac{\rho E_x}{B_0^2}\,. \tag{16}$$

The above expression has the form form of (3) and thus we obtain

$$D_x = \left(\varepsilon_0 + \frac{\rho}{B_0^2}\right)E_x\,, \tag{17}$$

where $V_A = B_0/\sqrt{\mu_0 \rho}$. From this expression we can obtain an electromagnetic wave propagating in the magnetic field direction with wave velocity $\sqrt{c^2 + V_A^2}$; this wave is called "Alfven wave".

Combining the above expression with (14) we arrive at

$$(\mathbf{D} \times \mathbf{B})_y = D_x B_0 = \varepsilon_0 E_x B_0 + \rho v_y \,. \tag{18}$$

The above expression means the electromagnetic momentum proposed by Minkowski is the sum of the electromagnetic momentum $\varepsilon_0 E_x B_0$ and momentum of the MHD plasma ρv_y.

5 Maicroscopic Derivation

The essential properties of an MHD plasma to obtain our result (18) is the E cross B drift (11) and the inertial current (13). We have derived these two from the MHD equations in the above. This derivation is standard and intuitive, however, it is not from the basic equation and underlying microphysics is not transparent. In the following we derive the same result from the very basic equation of motion of a particle in electromagnetic fields.

Suppose a charged particle with charge e and mass m is in a uniform background magnetic filed in the z direction $B_z = B_0$. What we would like to calculate is the particle motion in response to the time varying electric field exerted perpendicularly to the magnetic field. We take the direction of the electric field along the x axis.

Then the equation of motion becomes

$$\begin{aligned} \frac{\partial u_x}{\partial t} &= \pm \Omega u_y \pm \frac{e}{m} E(t) \,, \\ \frac{\partial u_y}{\partial t} &= \mp \Omega u_x \,, \end{aligned} \tag{19}$$

where u_x and u_y are the particle velocity and $\Omega = eB_0/m$; double sign corresponds to positive and negative particles. The motion in the z direction is not important here and we assume $u_z = 0$ for simplicity. The particle motion is a superposition of so called Lamor motion, which is the motion with $E(t) = 0$, and the effect of $E(t)$. Thus we denote

$$\begin{aligned} u_x &= u_0 \cos(\Omega t + \varphi) + u_x' \,, \\ u_y &= \pm u_0 \sin(\Omega t + \varphi) + u_y' \,, \end{aligned} \tag{20}$$

where the first term of each expression represents the Lamor motion; φ and u_0 is determined from the initial condition. The second terms are called drift motion, or drifts, invoked in response to $E(t)$. Here we assume the frequency of the electric field is much smaller than the frequency of Larmor motion. The drift motion is linear to the electric field, therefore, $|\partial u_x/\partial t| \ll \Omega |u_x|$. This assumption is valid for MHD waves; in other words, the MHD theory is limited to such low frequency phenomena.

Substituting (20) into (19) we obtain

$$\frac{\partial^2 u'_x}{\partial t^2} = \Omega^2 u'_x \pm \frac{e}{m}\frac{\partial}{\partial t}E(t)$$

Neglecting the left hand side because of $|\partial u_x/\partial t| \ll \Omega |u_x|$ we obtain

$$u'_x = \mp \frac{e}{m\Omega^2}\frac{\partial}{\partial t}E(t)\,. \tag{21}$$

Since u_y must vanish when $E(t) = 0$ we have

$$u'_y = \frac{E(t)}{B_0}\,. \tag{22}$$

Now that the motion of each particle is obtained, we can calculate the bulk motion of the plasma and the current in it. Suppose the plasma consists of equal number of electrons and protons, whose masses are m_e and m_p respectively; the number density of each species is the same n.

The velocity of a plasma $\mathbf{v}$ is defined as the velocity of gravicenter of particles. The contribution of the Larmor motion is canceled out when averaged, and thus drift velocity determines the plasma bulk velocity. The y component of the drift velocity u'_y is the same for all particles (22), thus we have (14). The velocity in the x direction (21) may be rewritten as

$$u'_x = \mp \frac{1}{B_0\Omega}\frac{\partial}{\partial t}E(t)\,, \tag{23}$$

therefore negligibly small compared to u'_y when the frequency of the MHD wave is much smaller than Ω. Therefore, we can write (11) for the plasma bulk velocity $\mathbf{v}_\perp$.

The drift in the x direction is negligible for the bulk motion, however, it is not so for the current. Since the drift in the y direction has the same sign for electrons and protons, there is no net current in the y direction. In contrast, the current exists in the x direction because (21) has opposite sign for electrons and protons. By summing up the current carried by each particle, we arrive at

$$J_x = \frac{\rho}{B_0^2}\frac{\partial}{\partial t}E(t)\,, \tag{24}$$

where ρ is the mass density $\rho = n(m_i + m_e)$. Now we see the current is expressed as in (13).

6 Concluding Remarks

Momentum carried by an MHD wave is examined from the view point of Abraham-Minkoski controversy. It has been confirmed that the total momentum (electromagnetic momentum plus momentum of the medium) is expressed in the form of $\mathbf{D} \times \mathbf{B}$

as proposed by Minkowski. The derivation of an MHD wave is not new at all, and based on the very simple and basic two properties of an MHD plasma, the frozen-in condition (14) and polarization current (13), namely. It would not be exaggeration when one says the whole kingdom of MHD plasma physics would fall if these two basic properties were wrong.

Here in the present letter we examined a simplest case of an parallel (to the **B** field) propagating MHD wave, but similar calculations can be done for more complicated plasma waves to confirm the result here. Collisionless plasmas contain a wide variety of wave phenomena, and the properties of waves can be precisely calculated at least in the linear limit. Calculation of the Minkowski momentum for various plasma waves would be a good exercise to understand the Abraham-Minkowski controversy.

What we have shown in the present letter is that the Minkowski momentum can be self consistent description of the total momentum of an electromagnetic wave in a polarizable medium. This does not necessarily mean the Abraham momentum is wrong and inconsistent; it might be possible to give Abraham momentum another appropriate meaning to make it consistent. For example, Barnett [9] recently argued both Abraham and Minkowski momentum can be consistent when we interpret the former as kinetic momentum and latter as canonical momentum. It is out of our scope here to examine this argument, however, it should be noted the legitimacy of the Minkowski momentum does not automatically exclude the validity of the Abraham momentum.

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