The DKP Equation in the Woods-Saxon Potential Well: Bound States

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Abstract: We solve the Duffin-Kemmer-Pétiau equation in the presence of a spatially one-dimensional symmetric potential well. We compute the scattering state solutions and we derive conditions for transmission resonances. The bound solutions are derived by a graphic study and the appearance of the antiparticle bound state is discussed.

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1 Introduction

The presence of strong fields introduces quantum phenomena, such as supercriticallity and spontaneous pair production which is one of the most interesting non-perturbative phenomena associated with the charged quantum vacuum.

During the last decades, a great effort has been made in understanding quantum processes in strong fields. The discussion of overcritical behavior of bosons requires a full understanding of the single particle spectrum, and consequently of the exact solutions to the Klein-Gordon (KG) equation. Recently, transmission resonances for the KG [1] and

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Duffin-Kemmer-Pétiau (DKP) equation [2, 3, 4] in the presence of a Woods-Saxon (WS) potential barrier have been computed. The transmission coefficient as a function of the energy and the potential amplitude shows a behavior that resembles the one obtained for the Dirac equation in [5].

The KG equation in the WS potential well [6] was solved and it was shown that analogous to the square well potential, there is a critical point V_{cr} where the bound antiparticle mode appears to coalesce with the bound particle.

In the present article, we solve the DKP equation in the WS potential well and we make a graphical study for the resonance transmissions. Among the advantages of working with the WS potential we have to mention that, in the one-dimensional case, the DKP equation as well as the KG and Dirac equations are solvable in terms of special functions and therefore the study of bound states and scattering processes becomes more tractable. We show that the antiparticle bound states arise for the WS potential well, which is a smoothed out form of the square well. The interest in computing bound states and spontaneous pair creation processes in such potentials lies in the fact that they possess properties that could permit us to determine how the shape of the potential affects the pair creation mechanism.

The article is structured as follows: Section 2 is devoted to solving the DKP equation in the presence of the one dimensional WS potential well. In Section 3 We derive the equation governing the eigenvalues corresponding to the bound states and compute the bound states. Finally, in Section4, we briefly summarize our results.

2 The DKP Equation in the WS Ptential Well

The DKP equation [7, 8, 9] is a natural manner to extend the covariant Dirac formalism to the case of scalar (spin 0) and vectorial (spin 1) particles when interacting with an electromagnetic field. It will be written as ($\hbar = c = 1$):

$$[i\beta^{\mu}(\partial_{\mu} + ieA_{\mu}) - m]\,\psi(\mathbf{r}, t) = 0 \tag{1}$$

where the matrices β^{μ} verify the DKP algebra:

$$\beta^{\mu}\beta^{\nu}\beta^{\lambda} + \beta^{\lambda}\beta^{\nu}\beta^{\mu} = g^{\mu\nu}\beta^{\lambda} + g^{\nu\lambda}\beta^{\mu} \tag{2}$$

where the convention for the metric tensor is here $g^{\mu\nu} = diag(1, -1, -1, -1)$. The algebra (2) has three irreductible representations whose degrees are $1, 5,$ and 10. The first one is trivial, having no physical content, the second and the third ones correspond respectively to the scalar and vectorial representations. For the spin 0, the β^{μ} are given by:

$$\beta^{0} = \begin{pmatrix} \theta & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}; \; \beta^{i} = \begin{pmatrix} \mathbf{0} & \rho^{i} \\ -\rho^{i}_{T} & \mathbf{0} \end{pmatrix}; i = 1, 2, 3 \tag{3}$$

with

$$\rho^1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \rho^2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\rho^3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \ \theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4}$$

the ρ_T denoting the transposed matrix of ρ, and $\mathbf{0}$ the zero matrix. For the spin 1, the β^μ are given by:

$$\beta^0 = \begin{pmatrix} 0 & \overline{0} & \overline{0} & \overline{0} \\ \overline{0}^T & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \overline{0}^T & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \overline{0}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} ; \ \beta^i = \begin{pmatrix} 0 & \overline{0} & e_i & \overline{0} \\ \overline{0}^T & \mathbf{0} & \mathbf{0} & -is_i \\ -e_i^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \overline{0}^T & -is_i & \mathbf{0} & \mathbf{0} \end{pmatrix} ; \ i = 1, 2, 3 \tag{5}$$

with

$$e_1 = (1, 0, 0) \, ; e_2 = (0, 1, 0) \, ; e_3 = (0, 0, 1) \, ; \overline{0} = (0, 0, 0) \tag{6}$$

$\mathbf{0}$ and $\mathbf{1}$ denoting respectively the zero matrix and the unity matrix, and the s_i being the standard nonrelativistic (3×3) spin 1 matrices:

$$s_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, s_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, s_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{7}$$

The DKP particles we consider are in interaction with the WS potential defined by:

$$V_r(z) = \frac{-V_0}{1 + \exp\left(\frac{|z| - a}{r}\right)} \tag{8}$$

where V_0 is real and positive, $a > 0$ and $r > 0$ are real, positive and adjustable.

The form of the WS potential is shown in the Fig.1, from which one readily notices that for a given value of the width parameter a, as the shape parameter r decreases $(r \longrightarrow 0^+)$, the WS potential reduces to a square well with smooth walls:

$$V(z) = -V_0 \theta (a - |z|)$$
$$= \begin{cases} -V_0 & \text{for } |z| \leqslant a \\ 0 & \text{for } |z| > a \end{cases} \tag{9}$$

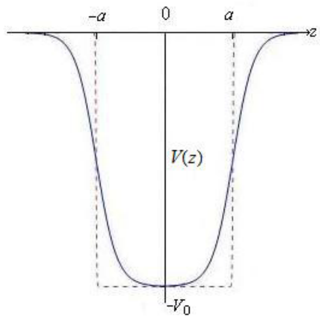

Fig. 1 The WS potential for $a = 2$, with $r = \frac{1}{3}$ (solide line) and $r = \frac{1}{100}$ (dotted line)

The interaction being scalar and independent of time, one can choose for $\psi(z,t)$ the form $e^{-iEt}\kappa\left(z\right)$, so one gets the following eigenvalue equation:

$$\left[\beta^{0}\left(E-eV\right)+i\beta^{3}\frac{d}{dz}-m\right]\kappa\left(z\right)=0 \tag{10}$$

with $\kappa\left(z\right)^{T}=\left(\varphi,\mathbf{A},\mathbf{B},\mathbf{C}\right)$, $\mathbf{A},\mathbf{B}$ and $\mathbf{C}$ being respectively vectors of components A_i, B_i and C_i; $i = 1, 2, 3$. According to the equations they satisfy, one gathers the components of $\kappa\left(z\right)$ this way

$$\Psi^{T}=\left(A_{1},A_{2},B_{3}\right),\Phi^{T}=\left(B_{1},B_{2},A_{3}\right),\Theta^{T}=\left(C_{2},-C_{1},\varphi\right);\text{ and }C_{3}=0 \tag{11}$$

with

$$\mathbf{O}_{KG}\Psi=0 \tag{12}$$

$$\begin{pmatrix}\Phi\\ \Theta\end{pmatrix}=\begin{pmatrix}\frac{E-eV}{m}\\ \frac{i}{m}\frac{d}{dz}\end{pmatrix}\otimes\Psi \tag{13}$$

then one will designate by $\phi\left(z\right)^{T}=\left(\Psi,\Phi,\Theta\right)$ the solution of (10) [2],
$O_{KG}=\frac{d^{2}}{dz^{2}}+\left[\left(E-eV\right)^{2}-m^{2}\right]$ being the Klein-Gordon "KG" operator.

By the following, one will follow the same steps as for the barrier potential [2], where one will replace V_0 by $-V_0$, then one gets the asymptotic behavior of the wave function at $|z| \longrightarrow \infty$:

$$\begin{pmatrix}\Psi\\ \Omega\\ \Theta\end{pmatrix}\underset{z\longrightarrow-\infty}{\longrightarrow}Ae^{-ik(z+a)}\begin{pmatrix}1\\ \frac{E}{m}\\ -\frac{i\mu}{rm}\end{pmatrix}\otimes V+Be^{ik(z+a)}\begin{pmatrix}1\\ \frac{E}{m}\\ \frac{i\mu}{rm}\end{pmatrix}\otimes V \tag{14}$$

$$\begin{pmatrix}\Psi\\ \Omega\\ \Theta\end{pmatrix}\underset{z\longrightarrow+\infty}{\longrightarrow}Ce^{ik(z-a)}\begin{pmatrix}1\\ \frac{E}{m}\\ \frac{i\mu}{rm}\end{pmatrix}\otimes V+De^{-ik(z-a)}\begin{pmatrix}1\\ \frac{E}{m}\\ -\frac{i\mu}{rm}\end{pmatrix}\otimes V \tag{15}$$

with the following definitions of the coefficients:

B and D are respectively the coefficients of the incoming waves from $-\infty \longrightarrow 0$ and from $+\infty \longrightarrow 0$.

A and C are respectively the coefficients of the reflected and transmitted wave.

V is a constant vector of dimension (3×1) :

$$\mathbf{V} = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} \tag{16}$$

The coefficients of reflection $\mathbf{R}$ and transmission $\mathbf{T}$ will be given by [2] :

$$\mathbf{R} = \frac{1}{4}\left|\lambda^{2\mu}\right|^2 \left|\frac{F_6}{F_5} + \frac{F_2}{F_1}\right|^2 \tag{17}$$

and

$$\mathbf{T} = \frac{1}{4}\left|\lambda^{2\mu}\right|^2 \left|\frac{F_6}{F_5} - \frac{F_2}{F_1}\right|^2 \tag{18}$$

with

$$\begin{aligned}
&\lambda = \tfrac{1}{1+\exp(-\frac{a}{r})} \\
&F_1 = {}_2F_1\left(\alpha_1, \beta_1, \gamma_1, \lambda\right) \\
&F_2 = {}_2F_1\left(\alpha_2, \beta_2, \gamma_2, \lambda\right) \\
&F_3 = {}_2F_1\left(\alpha_1+1, \beta_1+1, \gamma_1+1, \lambda\right) \\
&F_4 = {}_2F_1\left(\alpha_2+1, \beta_2+1, \gamma_2+1, \lambda\right) \\
&F_5 = \left[-\mu + \lambda\left(\mu - \nu\right)\right] F_1 + \lambda\left(1-\lambda\right) \tfrac{\alpha_1\beta_1}{\gamma_1} F_3 \\
&F_6 = \left[\mu - \lambda\left(\mu + \nu\right)\right] F_2 + \lambda\left(1-\lambda\right) \tfrac{\alpha_2\beta_2}{\gamma_2} F_4
\end{aligned} \tag{19}$$

and

$$\begin{cases}
\alpha = \left(\mu + \nu + \frac{1}{2}\right) - \dfrac{\nu_0}{2} \\
\beta = \left(\mu + \nu + \frac{1}{2}\right) + \dfrac{\nu_0}{2} \\
\gamma = 1 + 2\mu \\
\mu^2 = r^2\left(m^2 - E^2\right), \mu = irk \\
\nu^2 = r^2\left[m^2 - \left(E + eV_0\right)^2\right],\ \nu = irp \ \text{ with } p \text{ real} \\
\nu_0 = \sqrt{\left(1 - 2reV_0\right)\left(1 + 2reV_0\right)}
\end{cases} \tag{20}$$

Remark that one will distinguish two cases: $|E| > m$, i.e. k is real, which solutions are called scattering states [2, 3, 4], and $|E| < m$, i.e. k is imaginary and which solutions are bound states.

$$\begin{cases} \alpha_1 = \left(-\mu+\nu+\frac{1}{2}\right) - \frac{\nu_0}{2} \\ \beta_1 = \left(-\mu+\nu+\frac{1}{2}\right) + \frac{\nu_0}{2} \\ \gamma_1 = 1-2\mu \end{cases}, \begin{cases} \alpha_2 = \left(\mu+\nu+\frac{1}{2}\right) - \frac{\nu_0}{2} \\ \beta_2 = \left(\mu+\nu+\frac{1}{2}\right) + \frac{\nu_0}{2} \\ \gamma_2 = 1+2\mu \end{cases} \tag{21}$$

3 Bound States

To get the transmission coefficient for bound states in terms of E and of $(-eV_0)$, we proceed to solve numerically the equation (18). When varying the depth $(-eV_0)$ of the square well, we get:

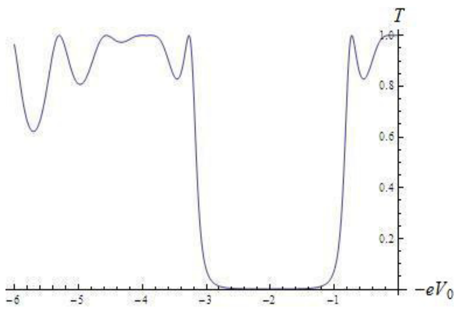

Fig. 2 The **T** coefficient in terms of the depth $(-eV_0)$ of the potential, for $a = 2$, $E = -2m$ and $m = 1$

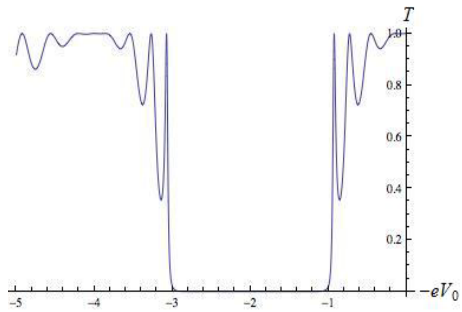

Fig. 3 The **T** coefficient in terms of the depth $(-eV_0)$ of the potential, for $a = 4$, $E = -2m$ and $m = 1$

and when varying the energy E of the particle, we get:

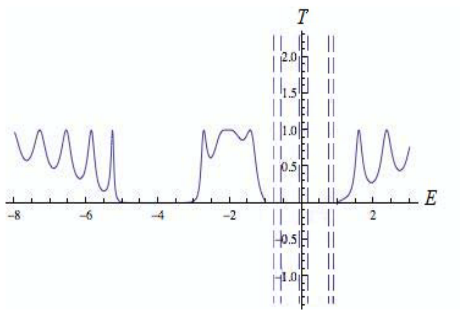

Fig. 4 The **T** coefficient in terms of the energy E for $m = 1, eV_0 = 4, a = 2$. The energies of the bound states are depicted by dashed lines.

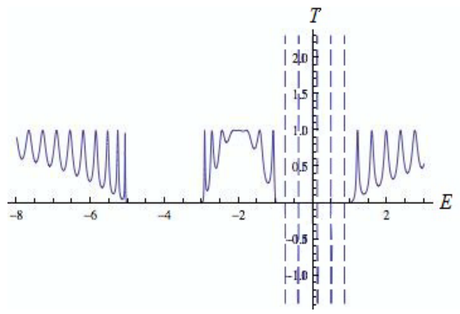

Fig. 5 The **T** coefficient in terms of the energy E for $m = 1, eV_0 = 4, a = 4$. The energies of the bound states are depicted by dashed lines.

Fig. 2 and Fig. 3 show that as in the Dirac [10] and the KG cases [6], the transmission coefficient vanishes for values of the potential strength $E - m < -eV_0 < E + m$, and transmission resonances appear for $-eV_0 > E + m$. They also show that the width of the transmission resonances decreases as the parameter a decreases.

Fig. 4 and Fig. 5, show that the occurrence of the transmission resonances increases with the width a of the square well. As in the case of the Dirac particle [10], we have significant structures of resonance in the particle continuum for $E > m$, and in the antiparticle continuum for $E < -eV_0 - m$. Antiparticles with lower energy ($-eV_0 - m < E < -eV_0 + m$) only penetrate the well with a probability which decreases with the width a of the well. Hence **T** is about zero in our case. However, in the domain $-eV_0 + m < E < -m\,(\Longrightarrow eV_0 > 2m)$, there is the possibility that the incoming wave meets a bound state, and thus penetrates the potential domain more or less unhindered. At the point where by extrapolation of the spectrum of the bound states (see Fig. 6), one would expect the quasi bound state, **T** is equal to 1. The dived bound state in this way becomes perceptible as a resonance in the scattering spectrum below $E = -m$.

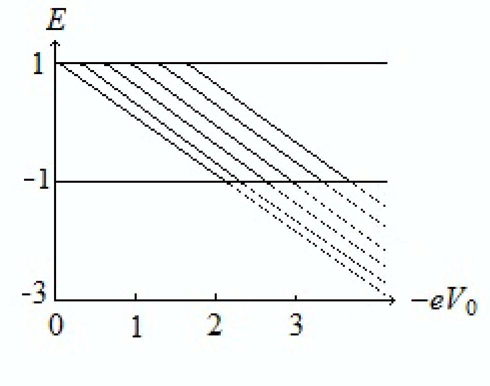

Fig. 6 Eigenvalue spectrum for $m = 1, a = 4$.

In Fig. 6, the energies of the dived states corresponding to resonances are depicted by dashed lines. They are extracted from the maxima of the transmission coefficients of Fig. 5

By the following, one wants to get the dependence of the spectrum of bound states (i.e. $|E| < m$) on the potential strength V_0. One uses for this aim, the unitary condition that coefficients **R** and **T** verify, i.e. $\mathbf{R} + \mathbf{T} = 1$,
which leads to:

$$1 = \frac{1}{2}\left|\lambda^{2\mu}\right|^2 \left[\left|\frac{F_6}{F_5}\right|^2 + \left|\frac{F_2}{F_1}\right|^2\right] \tag{22}$$

So

$$\frac{2}{|\lambda^{2\mu}|^2} = \left|\frac{[\mu - \lambda(\mu+\nu)]\,{}_2F_1(\alpha_2,\beta_2,\gamma_2,\lambda) + \lambda(1-\lambda)\frac{\alpha_2\beta_2}{\gamma_2}\,{}_2F_1(\alpha_2+1,\beta_2+1,\gamma_2+1,\lambda)}{[-\mu + \lambda(\mu-\nu)]\,{}_2F_1(\alpha_1,\beta_1,\gamma_1,\lambda) + \lambda(1-\lambda)\frac{\alpha_1\beta_1}{\gamma_1}\,{}_2F_1(\alpha_1+1,\beta_1+1,\gamma_1+1,\lambda)}\right|^2 + \left|\frac{{}_2F_1(\alpha_2,\beta_2,\gamma_2,\lambda)}{{}_2F_1(\alpha_1,\beta_1,\gamma_1,\lambda)}\right|^2 \tag{23}$$

One proceeds to solve numerically the equation (23) and thus one determines the energy spectrum of the bound solutions for several sets of parameters r and a, using the Gauss hypergeometric function:

$$F(\alpha,\beta,\gamma,z) = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-z)^{-\alpha} \times \frac{\Gamma(\alpha-\beta+1)\Gamma(\gamma-\alpha-\beta)}{\Gamma(1-\beta)\Gamma(\gamma-\beta)}$$

$$+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(-z)^{-\beta}\times\frac{\Gamma(\beta-\alpha+1)\Gamma(\gamma-\beta-\alpha)}{\Gamma(1-\alpha)\Gamma(\gamma-\alpha)}\qquad|\arg(-z)|<\pi \tag{24}$$

For that, one puts $\lambda_\epsilon=\lambda-i\varepsilon$, then:

$$F(\alpha,\beta,\gamma,\lambda)=\lim_{\epsilon\to 0}F(\alpha,\beta,\gamma,\lambda_\epsilon)$$

so one obtains:

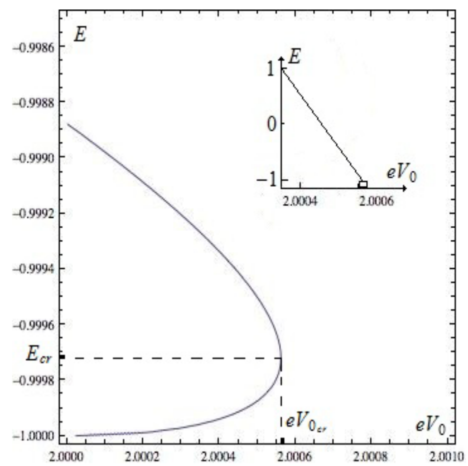

Fig. 7 Bound state spectrum for $m=1, a=1, r=0.00015$

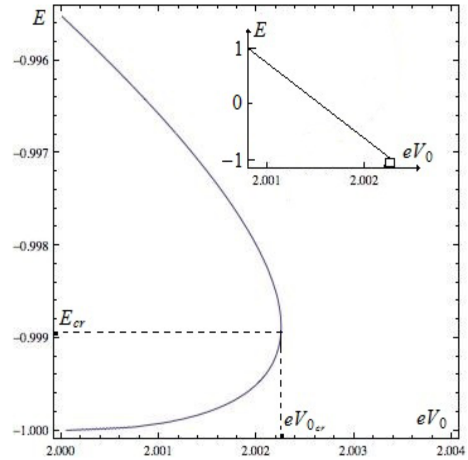

Fig. 8 Bound state spectrum for $m=1, a=4, r=0.0003$

In Fig. 7 (resp. Fig. 8), the bound state for the antiparticle appears for the domain $2.0004<eV_0<2.0006$ (resp. $2.002<eV_0<2.003$). For $eV_0\simeq 2.00055m$ (resp. $eV_0\simeq 2.0022m$), the well becomes supercritical. The lowest bound state enters the lower continuum and can there be realized as a resonance in the transmission coefficient. This critical value is depicted by $(eV_0)_{cr}$, and its corresponding energy by E_{cr}. The appearance of these bound antiparticle states is corresponding with the short range of the potential.

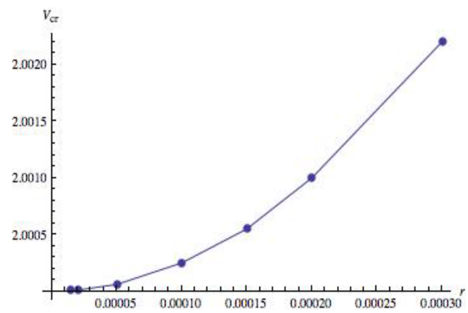

Fig. 9 Critical potential $eV_{0c}r$ versus the shape parameter r for $m=1, a=4$

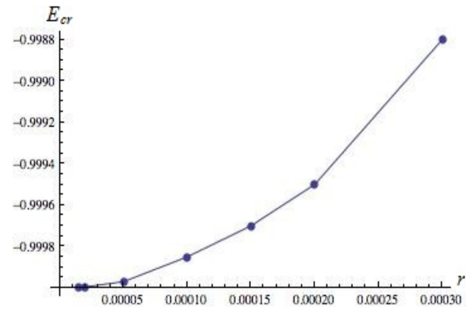

Fig. 10 Critical energy E_cr versus the shape parameter r for $m=1, a=4$

Fig. 9 shows that when the shape parameter r increases, the critical potential value $eV_{0_{cr}}$ where the bound antiparticle mode appears to coalesce with the bound particle increases.

Fig. 10 shows that as in the KG case [6] , when the shape parameter r increases, the critical energy value E_{cr} for which antiparticle state appears increases.

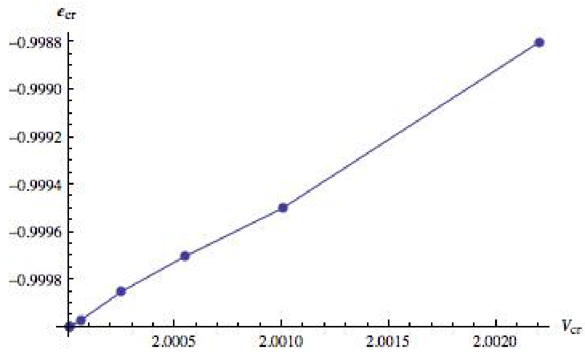

Fig. 11 Critical energy E_{cr} versus critical potential eV_{0cr} for $m = 1, a = 4$

For $a = 4$, we have moved the shape parameter r from 0.000015 to 0.0003. Fig. 11 shows the behavior of the critical energy value E_{cr} versus the critical potential $eV_{0_{cr}}$. We notice that as the value of $eV_{0_{cr}}$ increases, the energy value for which the antiparticle state appears increases.

4 Conclusion

We have showed a similarity in behavior between DKP, KG and Dirac particles, when interacting with a one-dimensional potential well. The resonances being interpreted as the signature for spontaneous pair creation, we have demonstrated that the WS potential well is able to bind particles. These resonances do not exist for subcritical potentials.

Transmission resonances for the one dimensional DKP equation possesses the same rich structure that we observe for the Dirac and the KG equations. For the DKP and KG particles, this can be interpreted as a demonstration of the equivalence between DKP and KG theories. For DKP and Dirac particles in a one-dimensional potential well, the bound state always exists, independent of the depth and the width of the potential. This being opposite with the corresponding three-dimensional problem where not every potential well has a bound state.

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