Deficiencies of Bohm Trajectories in View of Basic Quantum Principles

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Received 3 April 2016, Accepted 11 April 2016, Published 25 May 2016

Abstract: Quantum mechanics has been one of the most successful theories in physics, yet its foundation has remained a subject of discussion ever since it was incepted in the 1920s. While the Copenhagen interpretation represents the main-stream view, recent years have witnessed revived interest in the alternative *deterministic, or pilot-wave, interpretation*, pioneered by Madelung, de Broglie, and Bohm. It has been argued that these two interpretations are basically equivalent. In this article we show that this is not true. We exhibit the approximate nature of particle trajectories in Bohm's quantum mechanics. They follow the streamlines of a superfluid in Madelung's reformulation of the Schrödinger wave function, around which the proper particle trajectories perform their quantum mechanical fluctuations that ensure Heisenberg's uncertainty relation between position and momentum. These fluctuations explain the apparent discrepancy in the double-slit interference intensities between Bohmian mechanics and observations. They are also the reason for the non-existence of a possible radiation that would be emitted by an electron if its physical trajectory were deflected by the Bohmian quantum potential.

Keywords: Quantum Mechanics; Bohmian Quantum Mechanics; Bohm Trajectories; Pilot-wave, Interpretation; Madelung; de Broglie; Bohm
PACS (2010): 03.65.-w; 03.65.Ta; 03.75.Nt

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1 Introduction

In modern work on quantum mechanics (QM), one often reads, as justification of the effort, remark made by Richard Feynman in a 1964 lecture [1] that he thinks "it is safe to say that no one understands quantum mechanics". Similarly, Murray Gell-Mann in

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his lecture at the 1976 Nobel Conference regrets that "Niels Bohr brainwashed the whole generation of theorists into thinking that the job (of finding an adequate presentation of quantum mechanics) was done 50 years ago" [2]. Thus there is no wonder that even now reputable scientists are trying to get our deterministic thinking in line with quantum theory [3].

A theory of this type has been proposed a long time ago. It is based on an observation made as early as 1926, during the inceptive days of QM, by Madelung [4, 5]. He demonstrated that the Schrödinger equation can be transcribed into a hydrodynamic form, in which the Schrödinger field becomes the probability amplitude of the fluid. This was later referred to as the "Madelung quantum hydrodynamic" interpretation. Around the same time, de Broglie presented a deterministic interpretation of QM at the 1927 Solvay Conference, which was further developed by Bohm in 1952 to its present form [6].

It has been long believed by the experts on foundations of quantum mechanics that Bohmian mechanics and standard quantum mechanics are observationally equivalent [6, 7, 8, 10, 9]. They are merely different ontological interpretations on what exactly happens to a quantum particle, say electron, in a physical process. There are two salient features in Bohmian mechanics. One, quantum processes are inherently nonlocal, manifested by the *quantum potential* that permeates the entire space-time. Two, guided by this quantum potential, a test particle will execute deterministic motion (which Bohm called causal). That is, the particle's position and momentum are simultaneously specified throughout space-time. This is in drastic contrast to the basic notion of Heisenberg's uncertainty principle. It therefore appears meaningful to investigate the equivalence of these two formulations of QM in some details.

In this note we want to demonstrate that Bohmian way of doing QM is *not* equivalent, but a certain semiclassical approximation to proper Schrödinger QM. To do this, we invoke the second quantization reformulation of the Schrödinger equation as an N-body system. Following Madelung's original philosophy, we identify this N-body system as a superfluid and characterize its physical properties by its particle current density $\mathbf{J}_k$ and its superfluid velocity $\mathbf{V}^s_k$. This superfluid embodies de Broglie's *pilot wave* that guides the motion of a single particle. Under this setting, single-particle movements will be found from fluctuating paths around the superfluid streamlines in a semi-classical approximation. The fluctuations explain the discrepancy in the double-slit interference intensities between that derived from Bohmian mechanics and quantum mechanics. They are also the reason for the non-existence of a radiation that would have to be emitted by an electron if its physical trajectory is deflected by the Bohmian quantum potential.

2 Pilot Wave

In order to elucidate our point, it is useful to invoke the second-quantized reformulation of Schrödinger QM [12] as a functional integral over a Schrödinger field $\psi(\mathbf{x}, t)$ via the quantum-mechanical partition function [13]

$$Z = \int \mathcal{D}\psi\mathcal{D}\psi^\dagger\mathcal{D}\lambda e^{i[\mathcal{A}+\int dt\lambda(t)(N-N_0)]/\hbar}, \tag{1}$$

in which

$$\mathcal{A} = \int dtd^3x\, \psi^\dagger(i\hbar\partial_t - H)\psi \tag{2}$$

is the action and

$$H = \frac{\hat{\mathbf{p}}^2}{2m} + \hat{V}(\mathbf{x}) \tag{3}$$

the Hamiltonian of the system. The integral over the Lagrangian multiplyer λ guarantees that the particle number

$$N = \int d^3x\, \psi^\dagger\psi \equiv \int d^3x\rho(\mathbf{x},t) \tag{4}$$

is fixed to render the specific value N_0.

In the operator language of QM, the second-quantized theory is formulated in terms of *field operators* $\hat{\psi}(\mathbf{x},t)$ which are defined from the particle annihilation operators as $\hat{\psi}(\mathbf{x},t) \equiv e^{iHt/\hbar}\hat{a}_{\mathbf{x}}e^{-iHt/\hbar}$. The N-body wave functions arise from this by forming matrix elements of the states $|\psi(t)\rangle$ in a Fock space $\langle\hat{a}_{\mathbf{x}_1},\ldots,\hat{a}_{\mathbf{x}_N}|$:

$$\Psi_N(\mathbf{x}_1,\ldots,\mathbf{x}_N;t) = \langle\mathbf{x}_1\ldots,\mathbf{x}_N|\psi(t)\rangle \tag{5}$$

We shall interpret this N-body wave function as de Broglie's *pilot wave* of the particles. Taking the operator version of the action (2) in the N-particle Fock space it reads

$$\mathcal{A}_N = \int dt \int d\mathbf{X}\, \Psi_N^*(\mathbf{X},t)(i\hbar\partial_t - \hat{H}_N)\Psi_N(\mathbf{X},t) \tag{6}$$

where $\mathbf{X}$ collects the N-particle positions $(\mathbf{x}_1,\ldots,\mathbf{x}_N)$, and

$$\hat{H}_N = -\sum_\nu \left[\frac{\hbar^2}{2m}\partial^2_{\mathbf{x}_\nu} + V(\mathbf{x}_\nu)\right]. \tag{7}$$

The N-body wave function (5) satisfies the Schrödinger equation

$$\hat{H}_N\Psi_N(\mathbf{x}_1,\ldots,\mathbf{x}_N;t) = i\hbar\partial_t\Psi_N(\mathbf{x}_1,\ldots,\mathbf{x}_N;t). \tag{8}$$

At this point Madelung [4, 5] factorized in 1926 the wave function as a product of a real wave function R and a phase $e^{iS/\hbar}$,

$$\Psi_N \equiv Re^{iS/\hbar}, \tag{9}$$

with $R = \sqrt{\rho}$, and derived from the the Schrödinger equation the classical Hamilton-Jacobi equation for S, apart from an extra *quantum potential* [2]

$$V_\mathrm{q} = -\sum_{k=1}^{N} \frac{\hbar^2}{2m} \frac{\Delta_k R}{R}. \tag{10}$$

The full equation reads

$$\begin{aligned} i\partial_t R - \frac{1}{\hbar} R\partial_t S = \frac{\hbar}{2m} \sum_{k=1}^{N} \Bigg[& R \left(\frac{1}{\hbar} \boldsymbol{\nabla}_k S \right)^2 \\ & -2i\boldsymbol{\nabla}_k R \cdot \frac{1}{\hbar} \boldsymbol{\nabla}_k S - iR\frac{1}{\hbar}\Delta_k S \Bigg] + \frac{1}{\hbar}(V_\mathrm{c} + V_\mathrm{q})R, \end{aligned} \tag{11}$$

where $\Delta_k \equiv \nabla_k^2$ is the Laplace operator and $V_\mathrm{c} \equiv \sum_{n=1}^{N} V(\mathbf{x}_n)$. This is the way that led Madelung to the interpretation of the Schrödinger field as a probability amplitude of a quantum fluid. In light of present-day experiments on low-temperature Bose-Einstein condensates (BEC), we shall prefer to identify this liquid as a superfluid. From the N-particle formulation we identify the current density of each individual particle as Schrödinger current density

$$\mathbf{J}_k \equiv -i\frac{\hbar}{2m} \Psi_N^*(\mathbf{Q}, t) \overset{\leftrightarrow}{\boldsymbol{\nabla}}_k \Psi_N(\mathbf{Q}, t), \tag{12}$$

where $\boldsymbol{\nabla}_k = (\partial_1, \ldots, \partial_D)$, and the particle number density

$$\rho_N \equiv \Psi_N^* \Psi_N. \tag{13}$$

Here we may identify the *superfluid velocity* $\mathbf{V}_k^s$ by the relation

$$\rho_N \mathbf{V}_k^s \equiv \mathbf{J}_k. \tag{14}$$

By integrating the superfluid velocity over time one obtains the trajectory of the fluid density element, i.e., the streamline of the superfluid in configuration space.

3 Bohmian Quantum Mechanics

Collecting the imaginary parts in (11) yields the continuity equation

$$\partial_t R^2 = -\sum_{k=1}^{N} \boldsymbol{\nabla}_k (\mathbf{v}_k R^2), \tag{15}$$

where $m\mathbf{v}_k = \mathbf{p}_k = \boldsymbol{\nabla}_k S$, whereas the real parts give

$$\partial_t S + \frac{1}{2m} \sum_{k=1}^{N} \left[(\boldsymbol{\nabla}_k S)^2 \right] + V_\mathrm{c} + V_\mathrm{q} = 0. \tag{16}$$

[2] This term was derived in Bohm's 1952 paper [6], but was already stated in Madelung's 1926 paper [4] as a consequence of quantum physics.

This is the place where we can make the link between QM and Bohm's theory. We observe that one can replace the gradient kinetic term in the field action (2) by setting [14]

$$\psi^\dagger \frac{\hat{\mathbf{p}}^2}{2m}\psi \to m\,\frac{\mathbf{j}^2}{2\rho}. \tag{17}$$

where ρ is the fluctuating particle density defined in (4) and

$$\mathbf{j} \equiv \frac{1}{2m}\psi^\dagger \overset{\leftrightarrow}{\boldsymbol{\nabla}}\psi \tag{18}$$

is the current density of the fluctuating field. Classically, this may be interpreted as describing a cloud of particles streaming with a velocity

$$\mathbf{v} = \frac{\mathbf{j}}{\rho}. \tag{19}$$

This velocity field can be introduced into the quantum mechanical partition function (1) as a dummy auxiliary velocity variable by rewriting it as

$$Z = \int \mathcal{D}\psi\mathcal{D}\psi^\dagger\mathcal{D}\mathbf{v}\mathcal{D}\lambda e^{i[\mathcal{A}'+\int dt\lambda(t)(N-N_0)]/\hbar}, \tag{20}$$

where

$$\mathcal{A}' = \int dtd^3x\,\psi^\dagger(i\hbar\partial_t - H)\psi + m\int dtd^3x\,\frac{\rho}{2}\left(\mathbf{v} - \frac{\mathbf{j}}{\rho}\right)^2. \tag{21}$$

If the auxiliary field $\mathbf{v}$ is fully integrated out of the partition function, we recover the correct Schrödinger quantum mechanics.

By integrating $\mathbf{v}$ over time along the streamlines, we obtain $\mathbf{x}(t) = \int_0^t dt\,\mathbf{v}$ and interprete this as the deterministic position of the *quantum particle*. On the basis of the superfluid picture introduced in the previous section, the Bohmian deterministic QM is based on the *assumption* that the streamlines of superfluid velocity may be interpreted as the possible actual trajectories of the single particle under consideration.

4 Path Integral Representation of Bohm's QM

The path integral approach to Bohmian mechanics is not new. Philippidis et al. [15] invoked it to calculate the interference pattern behind a double-slit. The reader familiar with the standard path integral representation of QM [16, 17] will recognize that the partition function (1) is simply the second-quantized version [18] of the canonical path integral:

$$(\mathbf{x}_b t_b|\mathbf{x}_a t_a) = \int_{\mathbf{x}(t_a)=\mathbf{x}_a}^{\mathbf{x}(t_b)=\mathbf{x}_b} \mathcal{D}'\mathbf{x} \int \frac{\mathcal{D}\mathbf{p}}{2\pi\hbar} e^{i\mathcal{A}[\mathbf{p},\mathbf{x}]/\hbar}. \tag{22}$$

with the canonical action

$$\mathcal{A}[\mathbf{p},\mathbf{x}] = \int_{t_a}^{t_b} dt\left[\mathbf{p}(t)\dot{\mathbf{x}}(t) - \frac{\mathbf{p}^2(t)}{2m} - V(\mathbf{x}(t))\right]. \tag{23}$$

We note that the first term in this action guarantees the validity of Heisenberg's uncertainty relation between $\mathbf{p}$ and $\mathbf{x}$. If we integrate out the fluctuating momentum paths, the amplitude takes the form

$$(\mathbf{x}_b t_b|\mathbf{x}_a t_a) = \int_{\mathbf{x}(t_a)=\mathbf{x}_a}^{\mathbf{x}(t_b)=\mathbf{x}_b} \mathcal{D}'\mathbf{x}\, e^{i\mathcal{A}_F[\mathbf{x}]/\hbar} \tag{24}$$

with the action

$$\mathcal{A}_F[\mathbf{x}] = \frac{m}{2}\int_{t_a}^{t_b} dt\left[\dot{\mathbf{x}}^2(t) - V(\mathbf{x})\right], \tag{25}$$

which was used by Feynman [16, 17] to calculate quantum mechanical amplitudes via path integrals by summing over all histories of $\mathbf{x}(t)$ in $\mathbf{x}$-space.

The Bohmian QM is obtained by approximating the path integrals over the fluctuating momenta in two steps. First, one rewrites the initial path integral (22) with the help of a dummy velocity path $\mathbf{v}(t)$ as

$$(\mathbf{x}_b t_b|\mathbf{x}_a t_a) = \int_{\mathbf{x}(t_a)=\mathbf{x}_a}^{\mathbf{x}(t_b)=\mathbf{x}_b} \mathcal{D}'\mathbf{x} \int \mathcal{D}\mathbf{v} \int \frac{\mathcal{D}\mathbf{p}}{2\pi\hbar} e^{i\mathcal{A}'[\mathbf{p},\mathbf{v},\mathbf{x}]/\hbar}, \tag{26}$$

in which the action $\mathcal{A}[\mathbf{p},\mathbf{x}]$ of (22) has been replaced by

$$\mathcal{A}'[\mathbf{p},\mathbf{v},\mathbf{x}] = \frac{m}{2}\int_{t_a}^{t_b} dt\left[\mathbf{v}(t) - \frac{\mathbf{p}(t)}{m}\right]^2 + \mathcal{A}[\mathbf{p},\mathbf{x}]. \tag{27}$$

The Gaussian path integral over all $\mathbf{v}(t)$'s ensures that (26) is the same as the amplitude (22). Second, one approximates the path integral over $\mathbf{v}(t)$ in a certain semiclassical way by selecting only the extremum of the first term in (27), i.e., by assuming the velocity $\mathbf{v}(t)$ to be *equal* to $\mathbf{V}(t) \equiv \mathbf{p}(t)/m$ at each instant of time, rather than performing its proper harmonic quantum fluctuations dancing around $\mathbf{V}(t)$ [19] to satisfy $\mathbf{v}(t) = \mathbf{V}(t)$ only on the average. We note that this approximation destroys the validity of Heisenberg's uncertainty relation. By integrating $\mathbf{V}(t)$ over time one obtains functions $\mathbf{X}(t)$ which in Bohm's theory are considered to be the trajectories of the quantum particle guided by the pilot wave. It is therefore evident that Bohmian mechanics is not equivalent to proper QM.

5 Double-Slit Experiment

A commonly invoked *gedanken experiment* in QM is the scattering of a stream of particles on a double-slit which gives evidence on the wave-particle duality. It shows that interference patterns produced by massive particles are analogous to those produced by light waves in Young's experiment. Due to the smallness of the de Broglie wavelength for electrons, the double-slit experiment remained gedanken until 1961 when Claus Jönsson performed it successfully [20] and found an interference pattern in good agreement with Young's formula in the Fraunhofer zone:

$$I(\theta) = I_0(\sin\alpha/\alpha)^2\cos^2\beta, \tag{28}$$

where $\alpha = (a/\lambda)\pi \sin\theta, \beta = (d/\lambda)\pi \sin\theta$, a is the width of the slit, d the separation of the two slits, and θ the deflection angle relative to the symmetry line between the two slits. As is well-known, the interference pattern governed by the factor $\cos^2\beta$ is modulated by the square of the cardinal sine function $(\sin\alpha/\alpha)^2$ that governs the single-slit diffraction. Subsequent double-slit experiments were performed with photons, electrons, neutrons, atoms, and even molecules, again in good agreement with Young's formula [21, 22, 23, 24, 25].

In the context of Bohmian mechanics, the pattern of electrons behind a double-slit was first calculated explicitly by Philippidis et al. [15] using the physical parameters of Jönsson's experimental setup as their numerical example. The associated quantum potential is plotted in Fig. 1, while the Bohmian particle trajectories are shown in Fig. 2. These calculations have often been cited and reproduced in the literature (see for example [5, 27]) as a strong argument for Bohmian mechanics. However, the Philippidis et al. interference deviates significantly from the experimental result, which agrees perfectly with quantum mechanics [28]. Specifically, the ratio of intensities between the central primary peak and the second peak according to Bohmian mechanics, which can be deduced from Fig. 2 by counting the number of Bohmian trajectories crossing the screen in each corresponding *constructive interference zone*, is roughly $26 : 8 \sim 3.25$. Whereas Young's formula using the same set of parameters gives the ratio $1 : 0.97 \sim 1.03$: The latter agrees with QM and experiment.

What could be the origin of this significant discrepancy? In his book, Holland stressed the fundamental difference between the Feynman paths and the Bohmian trajectories, which he called quantum paths [5]. "For example, in the two-slit experiment a path starting in one of the slits and crossing the axis of symmetry of the apparatus is possible for Feynman but forbidden for us (Bohmian mechanics)." (p. 267 of [5]). That is, the trajectories for electrons passing, for example, the left slit will only reach the left side of the center divide on the screen. As we have recognized, a Bohmian trajectory is a semiclassical approximation of the fluctuating QM paths that *dance* around it. These fluctuating QM paths are not confined to one side of the symmetry line. This explains why Bohmian trajectories have an excessive probability at the center and do not produce correct double-slit interference patterns predicted by QM.

6 A Spurious Radiation

One other consequence of treating the semiclassical averaged velocity $\mathbf{V}(t)$ as the actual particle velocity $\mathbf{v}(t)$ is that such a deterministic motion may induce a spurious radiation that would be emitted by an electron if its physical trajectory is deflected by the Bohmian quantum potential. Invoking the double-slit experiment as an example, it was shown in [29] that the drastic variations of the quantum potential (see Fig. 1) in the direction transverse to electron's motion from the slits to the screen would inevitably induce radiation if the particle does execute Bohmian deterministic classical trajectory, with the emission angle following the direction of the *canyon* where the particle crosses.

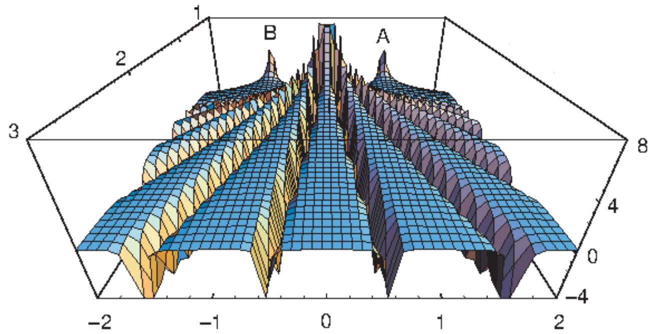

Fig. 1 Plot of the quantum potential $V_q(\mathbf{x})$ looking back from the screen to double-slit A & B, taken from [29]. This quantum potential was first calculated in [15].

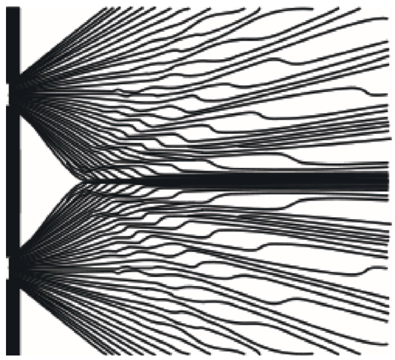

Fig. 2 Bohmian trajectories in a double-slit experiment [15]. The key parameters used in [15] follow that in Jönsson's experiment [28]: slit half-width $a/2 = 0.1 \times 10^{-4}$cm, slit separation $d = 1 \times 10^{-4}$cm, distance between the slits and the screen $x = 35$cm, and the electron de Broglie wavelength $\lambda = 5.5 \times 10^{-10}$cm.

This would result in a discrete bright-and-dark stripe pattern on the screen due to such radiation, which exactly complements the interference pattern of electrons that eventually all travel on the *plateau*.

With the realization that the Bohmian trajectories are semiclassical approximation to the actual fluctuating QM trajectories, we believe that this spurious radiation effect should not occur. It would be interesting, nevertheless, to investigate this spurious radiation experimentally.

7 Multivalued Nature of S

As experimentalists are in the process of investigating detailed properties of Bohmian quantum mechanics [30], they should be aware that an important aspect of that theory is still absent in (11) and (16). That is, the function S is really a *multivalued function* of configuration space and time [14]. Its derivatives $\boldsymbol{\nabla}_k S(\mathbf{Q}, t)$ are defined only modulo integer multiples of $2\pi\hbar$ times a delta function in some area A to be denoted by $\delta_k(\mathbf{Q}, A; t)$. It is defined by the integral

$$\boldsymbol{\delta}_k(\mathbf{Q}, A; t) \equiv \int_{A(t)} d^{3N-3}\bar{Q} \int d\mathbf{A}_k \; \delta(\mathbf{Q} - \bar{\mathbf{Q}}), \tag{29}$$

where $\int d^{3N-3}\bar{Q}$ runs only over the configuration space of all $\bar{\mathbf{q}}_i$ except $\bar{\mathbf{q}}_k$, and the vector $\bar{\mathbf{q}}_k$ is integrated over the area A [14]. Therefore the Bohm equation (11) for the pilot wave is correct only if the gradients of S in that equation are replaced by

$$\boldsymbol{\nabla}_k S(\mathbf{Q},t) \to \boldsymbol{\nabla}_k S(\mathbf{Q},t) - 2\pi m\hbar \boldsymbol{\delta}_k(\mathbf{Q},A;t), \tag{30}$$

where A denotes possible surfaces across which the phase jumps by an amount $2\pi m\hbar$, with some integer m. In analogy, a charged particle circulating around an infinitely thin magnetic flux line along the z-axis has a wave function $e^{im\phi}$, where ϕ is the azimuthal angle in cylindrical coordinates. The replacement of (30) in (16) accounts for this effect in general. By analogy with the theory of plasticity, we shall denote the extra term as $\mathbf{S}_k^P = 2\pi m\boldsymbol{\delta}_k(\mathbf{Q},A;t)$ and call it the *plastic deformation* of the eikonal S,

Similarly we have to replace the time derivative in the first terms of (11) and (16) as

$$\begin{aligned}\partial_t S(\mathbf{Q},t) &\to \partial_t S(\mathbf{Q},t) - 2\pi n\hbar\delta(t-t(\mathbf{Q})) \\ &= \partial_t S(\mathbf{Q},t) - S_t^P(\mathbf{Q},t).\end{aligned} \tag{31}$$

After these replacements the Bohm equation (16) gives a complete description of the motion of a gas of Bose particles in a zero-temperature condensate if the gas is sufficiently dilute that there are practically no interactions among the particles. In the presence of electromagnetism, the plastic deformations of the eikonal are modified by the usual minimal replacement rules.

Note that (11) is also the hydrodynamic description of a field $\Psi(\mathbf{Q},t)$ emerging from a standard Ginzburg-Landau action [33], the only difference is that here the field depends on all $3N$ configuration coordinates in $\mathbf{Q}$, rather than only a single coordinate $\mathbf{x}$, as in the original Ginzburg-Landau action, which is a mean-field approximation to a second-quantized many-body action [34].

8 Summary

In this note we have demonstrated that Bohmian mechanics and proper QM are not equivalent. However this aspect, that is, determinism vs. indeterminism, is independent of the issue of nonlocality [35]. As is demonstrated by the famous Bell inequality [36], QM is inherently nonlocal. This aspect is particularly transparent in the Madelung-de Broglie-Bohm formulation manifested through the superfluid BEC in the multidimensional configuration space.

Acknowledgments

We thank Gerhard Birkl, Jürgen Dietel, Basil Hiley and Stefan Teufel for interesting discussions.

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