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The Wave Function and Quantum Reality

Shan Gao*

Institute for the History of Natural Sciences, Chinese Academy of Sciences, Beijing 100190, P. R. China

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Abstract: This article analyzes the ontological meaning of the wave function. According to protective measurement, the mass and charge of a charged quantum system are distributed in space, and the mass and charge density in each position is proportional to the modulus squared of the wave function of the system there. It is shown that the mass and charge density is formed by the ergodic motion of a localized particle with the total mass and charge of the system. Moreover, it is argued that the ergodic motion is not continuous but discontinuous and random. Based on this result, we suggest that the wave function in quantum mechanics describes the state of random discontinuous motion of particles, and the modulus squared of the wave function gives the probability density of the particles *being* in certain locations in space.

Keywords: Quantum Mechanics; Protective Measurement; Wave Function; Quantum Reality; Quantum Interpretation
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1 Introduction

The physical meaning of the wave function is an important interpretative problem of quantum mechanics. Notwithstanding more than eighty years' developments of the theory, however, this is still a debated issue. It has been widely argued that the probability interpretation is not wholly satisfactory because of resorting to the vague concept of measurement - though it is still the standard interpretation in textbooks nowadays (Bell 1990). On the other hand, the ontological meaning of the wave function is also in dispute in the realistic alternatives to quantum mechanics such as the de Broglie-Bohm theory and the many-worlds interpretation (de Broglie 1928; Bohm 1952; Everett 1957; De Witt and Graham 1973). In view of this unsatisfactory situation, it seems that we need a new starting point to solve the fundamental interpretive problem of quantum mechanics.

* Email: gaoshan@ihns.ac.cn

The meaning of the wave function is often analyzed in the context of conventional impulse measurements, for which the coupling interaction between the measured system and the measuring device is of short duration and strong. Even though the wave function of a quantum system is in general extended over space, an ideal position measurement will collapse the wave function and can only detect the system in a random position in space. Thus it seems natural to assume that the wave function is only related to the probabilities of these random measurement results as in the standard probability interpretation. However, it has been known that there exists another kind of measurement that can avoid the collapse of the wave function, namely the protective measurement (Aharonov and Vaidman 1993; Aharonov, Anandan and Vaidman 1993; Aharonov, Anandan and Vaidman 1996). Protective measurement also uses a standard measuring procedure, but with a weak and long duration coupling interaction and an appropriate procedure to protect the measured wave function from collapsing. These differences permit protective measurement to be able to gain more information about the measured quantum system and its wave function, and thus it may help to unveil more physical content of the wave function. In this paper, we will analyze the possible implications of protective measurement for the ontological meaning of the wave function.

The plan of this paper is as follows. In Section 2, we first introduce the principle of protective measurement. It is stressed that protective measurement can measure the expectation values of observables for a single quantum system, and these expectation values are physical properties of the system, not properties of an ensemble of identical systems. Section 3 then analyzes a typical example of such properties, the mass and charge density. According to protective measurement, the mass and charge of a charged quantum system are distributed throughout space, and the mass and charge density in each position is proportional to the modulus squared of the wave function of the system there. In Section 4, the physical origin of the mass and charge density is investigated. It is shown that the mass and charge density of a quantum system is formed by the time average of the ergodic motion of a localized particle with the total mass and charge of the system. Moreover, it is argued that classical ergodic models, which assume continuous motion of particles, are inconsistent with quantum mechanics, and the ergodic motion is discontinuous and random. Based on this result, we suggest in Section 5 that the wave function in quantum mechanics describes the state of random discontinuous motion of particles, and at a deeper level, it represents the property of the particles that determines their random discontinuous motion. On this interpretation, the modulus squared of the wave function not only gives the probability density of the particles *being found* in certain locations in space as the probability interpretation holds, but also gives the objective probability density of the particles *being* there. In Section 6, we briefly analyze the possible implications of the new interpretation of the wave function for the solutions to the measurement problem. Conclusions are given in the last section.

2 Protective Measurements

For a conventional measurement such as an impulsive measurement, if the measured system, prior to the measurement of a variable A, is not in an eigenstate of A, then its state will be invariably entangled with the state of the device due to the interaction. A protective measurement differs from the conventional measurement in that the measured state is protected from entangling and changing appreciably when the measurement is being made[2]. A universal protection scheme is via the quantum Zeno effect. Let's see how this can be done.

Let $|\psi\rangle$ be the measured state of a single system at a given instant $t = 0$. To protect this state from being changed, we make projective measurements of an observable $P(t)$, for which $|\psi\rangle$ is a nondegenerate eigenstate, a large number of times which are dense in the measurement interval $[0, T]$ (Aharonov, Anandan and Vaidman 1993). For example, $P(t)$ is measured in $[0, T]$ at times $t_n = (n/N)T, n = 1, 2, ..., N$, where N is an arbitrarily large number. At the same time, an observable A is measured in the interval $[0, T]$ by an independent measurement described by the following Hamiltonian:

$$H(t) = H_S + H_D + g(t)PA, \tag{1}$$

where H_S and H_D are the free Hamiltonians of the measured system and the measuring device, respectively, and P is the momentum conjugate to the pointer variable X of the device. The time-dependent coupling strength $g(t)$ is a smooth function normalized to $\int dtg(t) = 1$ in $[0, T]$, and $g(0) = g(T) = 0$. The initial state of the pointer at $t = 0$, $|\phi(0)\rangle$, is supposed to be a Gaussian wave packet of eigenstates of X with width w_0, centered around the eigenvalue x_0.

Then the branch of the state of the combined system after T, in which each projective measurement of $P(t_n)$ results in the state of the measured system being in $|\psi\rangle$, is given by

$$\begin{aligned} |t = T\rangle &= |\psi\rangle \langle\psi|e^{-\frac{i}{\hbar}\frac{T}{N}H(t_N)}... |\psi\rangle \langle\psi|e^{-\frac{i}{\hbar}\frac{T}{N}H(t_2)} |\psi\rangle \langle\psi| \\ &\quad \times e^{-\frac{i}{\hbar}\frac{T}{N}H(t_1)} |\psi\rangle |\phi(0)\rangle \\ &= |\psi\rangle \langle\psi|e^{-\frac{i}{\hbar}\frac{T}{N}g(t_N)PA}... |\psi_1\rangle \langle\psi|e^{-\frac{i}{\hbar}\frac{T}{N}g(t_2)PA} |\psi_1\rangle \langle\psi| \\ &\quad \times e^{-\frac{i}{\hbar}\frac{T}{N}g(t_1)PA} |\psi_1\rangle |\phi(T)\rangle , \end{aligned} \tag{2}$$

where $|\psi_1\rangle$ is the state of the system after it evolves from the state $|\psi\rangle$ under the Hamiltonian H_S for a time interval T/N, and $|\phi(T)\rangle$ is the state of the device when it evolves under the Hamiltonian H_D after T. Here it is assumed that $[H_D, PA]$ is zero or its effect is negligible in the interval $[0, T]$ for simplicity. Thus in the limit of $N \to \infty$, we have

[2] It might be worth noting that there appeared numerous objections to the validity of protective measurements (see, e.g. Unruh 1994; Rovelli 1994; Ghose and Home 1995; Uffink 1999), and these objections have been answered (Aharonov, Anandan and Vaidman 1996; Dass and Qureshi 1999; Vaidman 2009; Gao 2013a). For a more detailed introduction to protective measurement see Gao (2014).

$$|t=T\rangle = |\psi\rangle\, e^{-\frac{i}{\hbar}\int_0^T g(t)\langle\psi|A|\psi\rangle Pdt}\,|\phi(T)\rangle = |\psi\rangle\, e^{-\frac{i}{\hbar}<A>P}\,|\phi(T)\rangle\,, \tag{3}$$

where $< A >\equiv \langle\psi|A|\psi\rangle$ is the expectation value of the measured observable A in the measured state ψ. Since the total probability of other branches is proportional to T^2/N to the first order of N, the above state will be the state of the combined system after T when $N \to \infty$[3]. It can be seen that the exponential operator in Eq. (3) shifts the center of the pointer by an amount $\langle A\rangle$, namely that the state of the pointer after T is $\langle x + \langle A\rangle\,|\phi(T)\rangle$. This indicates that the result of the protective measurement is the expectation value of the measured observable in the measured state, and moreover, the measured state is not changed by the protective measurement.

It is worth stressing that under the above protection procedure the measurement of an observable is not necessarily weak (when compared with the system's Hamiltonian), and the measurement time T is not necessarily long enough so that the measurement interaction is adiabatic either (cf. Vaidman 2008)[4]. A stronger measurement with a shorter measurement time is better if only the projective measurements can be made frequently during the measurement (i.e. the condition $N \to \infty$ can be approximately valid). Certainly, the measurement must be weaker than the projective measurements which protect the measured state from being changed.

To sum up, we have demonstrated that for an arbitrary but known state of a quantum system at a given instant, we can protect the state from being changed via the quantum Zeno effect by frequent projective measurements, and an independent measurement of an observable A, which is made at the same time, yields the expectation value of the observable in the measured state[5].

By a conventional measurement on a single quantum system, one obtains one of the eigenvalues of the measured observable, and the expectation value of the observable can only be obtained as the statistical average of eigenvalues for an ensemble of identically prepared systems. Thus it seems surprising that a protective measurement can yield the expectation value of the measured observable only from a single quantum system. In fact, the appearance of expectation values as measurement results is quite natural when the measured state is not changed and the entanglement during the conventional measurement does not take place as for protective measurements[6]. In this case, the

[3] Note that this result, like the quantum Zeno effect, does not depend on a particular formulation of quantum mechanics, and especially, it is independent of whether wavefunction collapse is real or apparent.

[4] According to Vaidman (2008), "Apart from protection, the procedure consists of a standard von Neumann measurement with weak coupling which is switched on and, after a long time, switched off, adiabatically."

[5] As we will argue later, in order to analyze the physical meaning of the wave function, we need not to measure the time evolution of the wave function, but only to measure the wave function at a given instant.

[6] The measured state being unchanged permits the state as well as the expectation values of observables in the state to be measurable. In this sense, protective measurement is not special; it is just the *very* way to measure the actual state of a quantum system at a given instant. By comparison, a non-protective

evolution of the combining state is

$$|\psi(0)\rangle\,|\phi(0)\rangle \to |\psi(t)\rangle\,|\phi(t)\rangle\,, t > 0 \tag{4}$$

where $|\psi(t)\rangle$ is the same as $|\psi(0)\rangle$ up to a phase factor during the measurement time interval $[0, T]$. Then by Ehrenfest's theorem we have

$$\frac{d}{dt}\langle\psi(t)\phi(t)|X\,|\psi(t)\phi(t)\rangle = -g(t)\langle\psi(0)|A\,|\psi(0)\rangle\,, \tag{5}$$

which further leads to

$$\langle\psi(T)\phi(T)|X\,|\psi(T)\phi(T)\rangle - \langle\psi(0)\phi(0)|X\,|\psi(0)\phi(0)\rangle = \langle\psi(0)|A\,|\psi(0)\rangle\,. \tag{6}$$

This means that the shift of the center of the pointer of the device gives the expectation value of the measured observable in the measured state.

Note that in some special cases the universal protection procedure via the quantum Zeno effect is not necessary, and the system's Hamiltonian can help protect its state from changing when the measurement interaction is weak and adiabatic. For example, for a quantum system in a discrete nondegenerate energy eigenstate, the system itself supplies the protection of the state due to energy conservation. By the adiabatic theorem, the adiabatic interaction during the measurement ensures that the measured system cannot make a transition from one discrete energy eigenstate to another. Moreover, according to the first order perturbation theory, for any given value of P, the energy of the measured energy eigenstate shifts by an infinitesimal amount: $\delta E = \langle H_I\rangle = g(t)P\langle A\rangle$, and the corresponding time evolution $e^{-iP\langle A\rangle/\hbar}$ then shifts the pointer by the expectation value $\langle A\rangle$. For degenerate energy eigenstates, we may not use the universal protection procedure either. The simplest way is to add a protective potential to change the energies of the other states and lift the degeneracy. Then the measured state remains unchanged, but is now protected by energy conservation like nondegenerate energy eigenstates.

Since the wave function can be reconstructed from the expectation values of a sufficient number of observables, the wave function of a quantum system can be measured by a series of protective measurements. Let the explicit form of the measured state at a given instant t be $\psi(x)$, and the measured observable A be (normalized) projection operators on small spatial regions V_n having volume v_n:

$$A = \begin{cases} \frac{1}{v_n}, & \text{if } x \in V_n, \\ 0, & \text{if } x \notin V_n. \end{cases} \tag{7}$$

measurement such as a conventional impulsive measurement will change the measured state, and the resulting measurement result (i.e. one of the eigenvalues of the measured observable) does not reflect the actual state of the measured system. Moreover, when a quantum system interacts with another quantum system under general non-protective conditions, its state also evolves in time, and thus the expectation values of observables do not manifest themselves explicitly in the interaction either. For example, the interaction between two charged quantum systems are not directly dependent on the expectation values of their charges, but described by the potential terms in the Schrödinger equation.

A protective measurement of A then yields

$$\langle A\rangle = \frac{1}{v_n}\int_{V_n} |\psi(x)|^2 dv, \tag{8}$$

which is the average of the density $\rho(x) = |\psi(x)|^2$ over the small region V_n. Similarly, we can measure another observable $B = \frac{\hbar}{2mi}(A\nabla + \nabla A)$. The measurement yields

$$\langle B\rangle = \frac{1}{v_n}\int_{V_n} \frac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*)dv = \frac{1}{v_n}\int_{V_n} j(x)dv. \tag{9}$$

This is the average value of the flux density $j(x)$ in the region V_n. Then when $v_n \to 0$ and after performing measurements in sufficiently many regions V_n we can measure $\rho(x)$ and $j(x)$ everywhere in space. Since the wave function $\psi(x,t)$ can be uniquely expressed by $\rho(x,t)$ and $j(x,t)$ (except for an overall phase factor), the whole wave function of the measured system at a given instant can be measured by protective measurements.

We have been discussing the protective measurement of a single quantum system. The scheme of protective measurement can also be extended to a many-particle system (Anandan 1993). If the system is in a product state, then one can easily measuring each state of the individual systems protectively. If the system is in an entangled state, one need to add an appropriate protection procedure to the whole system, and then the entangled state of the system can be protectively measured. However, it is worth noting that the realization of such protective measurements relies on the availability of corresponding protective interactions, which is limited by existing physical interactions[7].

Lastly, we stress that the validity of the scheme of protective measurement does not rely on the standard von Neumann formulation of measurement, in which it might be debatable to represent a macroscopic device with a single wave function. In the above formulation of protective measurement, the measuring system can be a microscopic system such as an electron, and the shift of the center of the wave packet of the measuring system is only determined by the Schrödinger equation. Since the state of the measured system is not changed during the protective measurement, a large number of identical measuring systems can be used to protectively measure the original measured system, and the centers of their wave packets have the same shift. Then the shift can be read out by conventional measurements of the ensemble of these identical measuring systems, for which the probability distribution of the results satisfies the Born rule. In a word, the scheme of protective measurement is only based on the Schrödinger equation (for microscopic systems) and the Born rule, and especially, it is independent of whether wavefunction collapse is real or not.

[7] For example, it seems that the entangled spatial wave function of a many-body system, which lives on configuration space, cannot be protectively measured. The reason is that a protective measurement cannot be performed in different positions in space at the same time.

3 On the Mass and Charge Density of a Quantum System

According to a recent analysis of protective measurement, the expectation values of observables are properties of a single quantum system (Gao 2015). Typical examples of such properties are the mass and charge density of a quantum system. In this section, we will present a detailed analysis of this property, as it may have important implications for the physical meaning of the wave function.

3.1 A Heuristic Argument

The mass and charge of a classical system always localize in a definite position in space at each moment. For a charged quantum system described by the wave function $\psi(x,t)$, how do its mass and charge distribute in space then? We can measure the total mass and charge of the quantum system by the gravitational and electromagnetic interactions and find them in certain region of space. Thus it seems that the mass and charge of a quantum system must also exist in space with a certain distribution. Before we discuss the answer given by protective measurement, we will first give a heuristic argument.

The Schrödinger equation of a charged quantum system under an external electromagnetic potential may provide a clue to the answer. The equation is

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = [-\frac{\hbar^2}{2m}(\nabla - \frac{iQ}{\hbar}A)^2 + Q\varphi]\psi(x,t), \tag{10}$$

where m and Q are the mass and charge of the system, respectively, φ and A are the electromagnetic potential. The electrostatic interaction term $Q\varphi\psi(x,t)$ in the equation indicates that the interaction exists in all regions where the wave function of the system, $\psi(x,t)$, is nonzero, and thus it seems to suggest that the charge of the system also distributes throughout these regions. If the charge does not distribute in some regions where the wave function is nonzero, then there will not exist an electrostatic interaction there. Note that φ is a classical potential and it does distribute throughout space. Furthermore, since the integral $\int_{-\infty}^{\infty} Q|\psi(x,t)|^2 d^3x$ is the total charge of the system, the charge density in space, if indeed exists, will be $Q|\psi(x,t)|^2$. Similarly, the mass density can be obtained from the Schrödinger equation of a quantum system under an external gravitational potential:

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = [-\frac{\hbar^2}{2m}\nabla^2 + mV_G]\psi(x,t). \tag{11}$$

The gravitational interaction term $mV_G\psi(x,t)$ in the equation also suggests that the (passive gravitational) mass of the quantum system distributes throughout the whole region where its wave function $\psi(x,t)$ is nonzero, and the mass density in space is $m|\psi(x,t)|^2$.

3.2 The Answer of Protective Measurement

In the following, we will show that protective measurement provides a more convincing argument for the existence of mass and charge density. The mass and charge density of a single quantum system, as well as its wave function, can be measured by protective measurement as expectation values of certain observables (Aharonov and Vaidman 1993). For example, a protective measurement of the flux of the electric field of a charged quantum system out of a certain region will yield the expectation value of its charge inside this region, namely the integral of its charge density over this region. Similarly, we can also measure the mass density of a quantum system by a protective measurement of the flux of its gravitational field in principle (Anandan 1993)[8].

Consider a quantum system in a discrete nondegenerate energy eigenstate $\psi(x)$. We take the measured observable A_n to be (normalized) projection operators on small spatial regions V_n having volume v_n:

$$A_n = \begin{cases} \frac{1}{v_n}, & \text{if } x \in V_n, \\ 0, & \text{if } x \notin V_n. \end{cases} \tag{12}$$

The protective measurement of A_n then yields

$$\langle A_n \rangle = \frac{1}{v_n} \int_{V_n} |\psi(x)|^2 dv = |\psi_n|^2, \tag{13}$$

where $|\psi_n|^2$ is the average of the density $\rho(x) = |\psi(x)|^2$ over the small region V_n. Then when $v_n \to 0$ and after performing measurements in sufficiently many regions V_n we can measure $\rho(x)$ everywhere in space.

When the observable A_n and the corresponding interaction Hamiltonian are physically realized by the electromagnetic or gravitational interaction between the measured system and the measuring device, what the above protective measurement measures is in fact the charge or mass density of the quantum system[9], and its result indicates that the mass and charge density of the system in each position x is proportional to the modulus squared of its wave function there, namely the density $\rho(x)$. In the following, we will give a concrete example to illustrate this important result (see also Aharonov, Anandan and Vaidman 1993).

3.3 A Specific Example

Consider the spatial wave function of a single quantum system with negative charge Q (e.g. $Q = -e$)

$$\psi(x,t) = a\psi_1(x,t) + b\psi_2(x,t), \tag{14}$$

[8] Since the principle of protective measurements is based on the established parts of quantum mechanics, their results as predicted by quantum mechanics are reliable even though the measurements have not been performed in practice.

[9] This important point was also stressed by Aharonov, Anandan and Vaidman (1993).

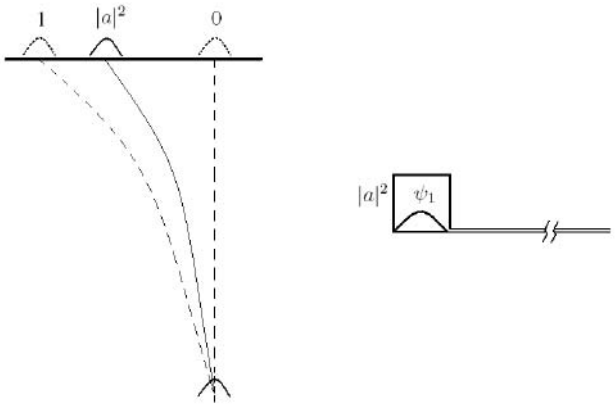

Fig. 1 Scheme of a protective measurement of the charge density of a quantum system

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where $\psi_1(x,t)$ and $\psi_2(x,t)$ are two normalized wave functions respectively localized in their ground states in two small identical boxes 1 and 2, and $|a|^2+|b|^2=1$. An electron, which initial state is a Gaussian wave packet narrow in both position and momentum, is shot along a straight line near box 1 and perpendicular to the line of separation between the boxes. The electron is detected on a screen after passing by box 1. Suppose the separation between the boxes is large enough so that a charge Q in box 2 has no observable influence on the electron. Then if the system were in box 2, namely $|a|^2=0$, the trajectory of the electron wave packet would be a straight line as indicated by position "0" in Fig.1. By contrast, if the system were in box 1, namely $|a|^2=1$, the trajectory of the electron wave packet would be deviated by the electric field of the system by a maximum amount as indicated by position "1" in Fig.1. We first suppose that $\psi(x,t)$ is unprotected, then the wave function of the combined system after interaction will be

$$\psi(x,x',t)=a\varphi_1(x',t)\psi_1(x,t)+b\varphi_2(x',t)\psi_2(x,t), \tag{15}$$

where $\varphi_1(x',t)$ and $\varphi_2(x',t)$ are the wave functions of the electron influenced by the electric fields of the system in box 1 and box 2, respectively, the trajectory of $\varphi_1(x',t)$ is deviated by a maximum amount, and the trajectory of $\varphi_2(x',t)$ is not deviated and still a straight line. When the electron is detected on the screen, the above wave function will collapse to $\varphi_1(x',t)\psi_1(x,t)$ or $\varphi_2(x',t)\psi_2(x,t)$. As a result, the detected position of the electron will be either "1" or "0" in Fig.1, indicating that the system is in box 1 or 2 *after* the detection. This is a conventional impulse measurement of the projection operator on the spatial region of box 1, denoted by A_1. A_1 has two eigenstates corresponding to the system being in box 1 and 2, respectively, and the corresponding eigenvalues are 1 and 0, respectively. Since the measurement is accomplished through the electrostatic interaction between two charges, the measured observable A_1, when multiplied by the charge Q, is actually the observable for the charge of the system in box 1, and its eigenvalues are Q and 0, corresponding to the charge Q being in boxes 1 and 2, respectively. Such a measurement cannot tell us the charge distribution of the system in each box *before* the

measurement.

Now let's make a protective measurement of A_1. Since $\psi(x,t)$ is degenerate with its orthogonal state $\psi^{'}(x,t) = b^*\psi_1(x,t) - a^*\psi_2(x,t)$, we need an artificial protection procedure to remove the degeneracy, e.g. joining the two boxes with a long tube whose diameter is small compared to the size of the box[10]. By this protection $\psi(x,t)$ will be a nondegenerate energy eigenstate. The adiabaticity condition and the weakly interacting condition, which are required for a protective measurement, can be further satisfied when assuming that (1) the measuring time of the electron is long compared to $\hbar/\Delta E$, where ΔE is the smallest of the energy differences between $\psi(x,t)$ and the other energy eigenstates, and (2) at all times the potential energy of interaction between the electron and the system is small compared to ΔE. Then the measurement of A_1 by means of the electron trajectory is a protective measurement, and the trajectory of the electron is only influenced by the expectation value of the charge of the system in box 1. In particular, when the size of box 1 can be omitted compared with the separation between it and the electron wave packet, the wave function of the electron will obey the following Schrödinger equation:

$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t} = -\frac{\hbar^2}{2m_e}\nabla^2\psi(\vec{r},t) - k\frac{e\cdot|a|^2Q}{|\vec{r}-\vec{r_1}|}\psi(\vec{r},t), \tag{16}$$

where m_e is the mass of electron, k is the Coulomb constant, $\vec{r_1}$ is the position of the center of box 1, and $|a|^2Q$ is the expectation value of the charge Q in box 1. Correspondingly, the trajectory of the center of the electron wave packet, $\vec{r_c}(t)$, will satisfy the following equation by Ehrenfest's theorem:

$$m_e\frac{d^2\vec{r_c}}{dt^2} = -k\frac{e\cdot|a|^2Q}{|\vec{r_c}-\vec{r_1}|(\vec{r_c}-\vec{r_1})}. \tag{17}$$

Then the electron wave packet will reach the position "$|a|^2$" between "0" and "1" on the screen as denoted in Fig.1. This shows that the result of the protective measurement is the expectation value of the projection operator A_1, namely the integral of the density $|\psi(x)|^2$ in the region of box 1. When multiplied by Q, it is the expectation value of the charge Q in the state $\psi_1(x,t)$ in box 1, namely the integral of the charge density $Q|\psi(x)|^2$ in the region of box 1. In fact, as Eq. (16) and Eq. (17) clearly indicate, this is what the protective measurement really measures.

As we have argued in the last section, the result of a protective measurement reflects the objective property or physical state of the measured system. Thus the result of the above protective measurement, namely the expectation value of the charge Q in the state $\psi_1(x,t)$, $|a|^2Q$, will reflect the actual charge distribution of the system in box 1. In other words, the result indicates that there exists a charge $|a|^2Q$ in box 1. In the following, we will give another two arguments for this conclusion.

First of all, let's analyze the result of the protective measurement. Suppose we can

[10] It is worth stressing that the added protection procedure depends on the measured state, and different states need different protection procedures in general.

continuously change the measured state from $|a|^2 = 0$ to $|a|^2 = 1$ (and adjust the protective interaction correspondingly). When $|a|^2 = 0$, the single electron will reach the position "0" of the screen one by one, and it is incontrovertible that no charge is in box 1. When $|a|^2 = 1$, the single electron will reach the position "1" of the screen one by one, and it is also incontrovertible that there is a charge Q in box 1. Then when $|a|^2$ assumes a numerical value between 0 and 1 and the single electron reaches the position "$|a|^2$" between "0" and "1" on the screen one by one, the results should similarly indicate that there is a charge $|a|^2Q$ in the box by continuity. The point is that the definite deviation of the trajectory of the electron will reflect that there exists a definite amount of charge in box 1.[11] Next, let's analyze the equation that determines the result of the protective measurement, namely Eq. (17). It gives a more direct support for the existence of a charge $|a|^2Q$ in box 1. The r.h.s of Eq. (17) is the formula of the electric force between two charges located in different spatial regions. It is incontrovertible that e is the charge of the electron, and it exists in the position $\vec{r}$. Then $|a|^2Q$ should be the other charge that exists in the position $\vec{r_1}$. In other words, there exists a charge $|a|^2Q$ in box 1.

In conclusion, protective measurement shows that a quantum system with mass m and charge Q, which is described by the wave function $\psi(x,t)$, has mass density $m|\psi(x,t)|^2$ and charge density $Q|\psi(x,t)|^2$ in space, respectively[12]. This conclusion is mainly based on the linear Schrödinger evolution and the Born rule. In the above example, the linear Schrödinger evolution determines the deviation of the electron wave packet, and the Born rule is needed to obtain the information about the center of the electron wave packet detected on the screen.

4 The Physical Origin of Mass and Charge Density

We have argued that the mass and charge of a quantum system are distributed throughout space, and the mass and charge density in each position is proportional to the modulus squared of the wave function of the system there. In this section, we will further investigate the physical origin of the mass and charge density. As we will see, the answer may provide an important clue to the physical meaning of the wave function.

Historically, the charge density interpretation for electrons was originally suggested by Schrödinger when he introduced the wave function and founded wave mechanics (Schrödinger 1926). Although the existence of the charge density of an electron can

[11] Any physical measurement is necessarily based on some interaction between the measured system and the measuring system. One basic form of interaction is the electrostatic interaction between two electric charges as in our example, and the existence of this interaction during a measurement, which is indicated by the deviation of the trajectory of the charged measuring system such as an electron, means that the measured system also has the charge responsible for the interaction. If one denies this point, then it seems that one cannot obtain any information about the measured system by the measurement. Note that the arguments against the naive realism about operators and the eigenvalue realism in the quantum context are irrelevant here (Daumer et al 1997; Valentini 2010).

[12] Strictly speaking, the mass density is $m|\psi(x)|^2 + \psi^* H\psi/c^2$ in the non-relativistic domain, but the second term is very small compared with the first term and can be omitted.

provide a classical explanation for some phenomena of radiation, its explanatory power is very limited. In fact, Schrödinger clearly realized that the charge density cannot be classical because his equation does not include the usual classical interaction between the densities. Presumably since people thought that the charge density could not be measured and also lacked a consistent physical picture, this initial interpretation of the wave function was soon rejected and replaced by Born's probability interpretation (Born 1926). Now protective measurement re-endows the charge distribution of an electron with reality by a more convincing argument. The question is then how to find a consistent physical explanation for it[13]. Our following analysis can be regarded as a further development of Schrödinger's idea to some extent. The twist is: that the charge distribution is not classical does not imply its non-existence; rather, its existence points to a non-classical picture of quantum reality hiding behind the mathematical wave function.

4.1 The Mass and Charge Density is Effective

As noted earlier, the expectation values of observables are the properties of a quantum system either at a precise instant or during an infinitesimal time interval. Correspondingly, the mass and charge distribution of a quantum system, which can be protectively measured as the expectation values of certain observables, has two possible existent forms: it is either real or effective. The distribution is real means that it exists throughout space at the same time. The distribution is effective means that there is only a localized particle with the total mass and charge of the system at every instant, and the time average of its motion during an infinitesimal time interval forms the effective distribution. Moreover, since the integral of the formed mass and charge density in any region is required to be equal to the expectation value of the total mass and charge in the region, the motion of the particle must be ergodic. In the following, we will determine the existent form of the mass and charge distribution of a quantum system.

If the mass and charge distribution is effective, then there will exist no gravitational and electrostatic self-interactions of the effective distribution, as there is only a localized particle at every instant. This is consistent with the superposition principle of quantum mechanics and the derivation of the mass and charge distribution based on protective measurement. By contrast, if the mass and charge distribution is real, then there will exist gravitational and electrostatic self-interactions of the real distribution, as the distribution exists throughout space at the same time[14]. The existence of the gravitational and

[13] Note that the proponents of protective measurement did not give an analysis of the origin of the charge distribution. According to them, this type of measurement implies that the wave function of a single quantum system is ontological, i.e., that it is a real physical wave (Aharonov, Anandan and Vaidman 1993).

[14] That a real charge distribution has electrostatic self-interaction has been confirmed not only in the classical domain but also in the quantum domain for many-body systems. For example, two charged quantum systems such as two electrons have electrostatic interaction, and thus a real charge distribution containing these two charges has electrostatic self-interaction. Thus it is reasonable to expect that this assumption also holds true for individual quantum systems.

electrostatic self-interactions is inconsistent with the superposition principle of quantum mechanics (at least for microscopic systems such as electrons). Moreover, as we will show below, the existence of the electrostatic self-interaction for the charge distribution of an electron also contradicts experimental observations.

Interestingly, the Schrödinger-Newton equation, which was proposed by Diosi (1984) and Penrose (1998), just describes the gravitational self-interaction of the mass density. The equation for a single quantum system can be written as

$$i\hbar\frac{\partial\psi(\mathbf{x},t)}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{x},t)-Gm^2\int\frac{|\psi(\mathbf{x}',t)|^2}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}'\psi(\mathbf{x},t)+V\psi(\mathbf{x},t), \tag{18}$$

where m is the mass of the quantum system, V is an external potential, G is Newton's gravitational constant[15]. If there is also an electrostatic self-interaction, then the equation for a free quantum system with mass m and charge Q will be

$$i\hbar\frac{\partial\psi(\mathbf{x},t)}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{x},t)+(kQ^2-Gm^2)\int\frac{|\psi(\mathbf{x}',t)|^2}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}'\psi(\mathbf{x},t). \tag{19}$$

Note that the gravitational self-interaction is attractive, while the electrostatic self-interaction is repulsive. It has been shown that the measure of the potential strength of the gravitational self-interaction is $\varepsilon^2=(\frac{4Gm^2}{\hbar c})^2$ for a free system with mass m (Salzman 2005). This quantity represents the strength of the influence of the self-interaction on the normal evolution of the wave function; when $\varepsilon^2\approx 1$ the influence is significant. Similarly, for a free charged system with charge Q, the measure of the potential strength of the electrostatic self-interaction is $\varepsilon^2=(\frac{4kQ^2}{\hbar c})^2$. As a typical example, for a free electron the potential strength of the electrostatic self-interaction will be $\varepsilon^2=(\frac{4ke^2}{\hbar c})^2\approx 1\times 10^{-3}$. This indicates that the electrostatic self-interaction will have a remarkable influence on the evolution of the wave function of a free electron[16]. If such an interaction indeed exists, it should have been detected by precise interference experiments on electrons. As another example, consider the electron in the hydrogen atom. Since the potential of the electrostatic self-interaction is of the same order as the Coulomb potential produced by the nucleus, the energy levels of hydrogen atoms will be remarkably different from those predicted by quantum mechanics and confirmed by experiments. Therefore, the electrostatic self-interaction cannot exist for a charged quantum system such as an electron.

To sum up, the superposition principle of quantum mechanics requires that the mass and charge distribution of a quantum system such as an electron is not real but effective; at every instant there is only a localized particle with the total mass and charge of the system, while during an infinitesimal time interval the time average of the ergodic motion

[15] It has been argued that the existence of a gravitational self-interaction term in the Schrödinger-Newton equation does not have a consistent Born rule interpretation (Adler 2007). The reason is that the probability of simultaneously finding a particle in different positions is zero.

[16] By contrast, the potential strength of the gravitational self-interaction for a free electron is $\varepsilon^2=(\frac{4Gm_e^2}{\hbar c})^2\approx 4\times 10^{-89}$. This kind of gravitational self-interaction is too weak to be detected presently (Salzman and Carlip 2006).

of the particle forms the effective mass and charge distribution[17], and the mass and charge density in each position is proportional to the modulus squared of the wave function of the system there.

4.2 The Ergodic Motion of a Particle is Discontinuous

Which sort of ergodic motion then? If the ergodic motion of a particle is continuous, then it can only form the effective mass and charge density during a finite time interval. But the effective mass and charge density is required to be formed by the ergodic motion of the particle during an infinitesimal time interval (*not* during a finite time interval) at a given instant. Thus it seems that the ergodic motion of the particle cannot be continuous. This is at least what the existing quantum mechanics says. However, there may exist a possible loophole here. Although the classical ergodic models that assume continuous motion are inconsistent with quantum mechanics due to the existence of a finite ergodic time, they may be not completely precluded by experiments if only the ergodic time is extremely short. After all quantum mechanics is only an approximation of a more fundamental theory of quantum gravity, in which there may exist a minimum time scale such as the Planck time. Therefore, we need to investigate the classical ergodic models more thoroughly.

Consider an electron in a one-dimensional box in the first excited state $\psi(x)$ (Aharonov and Vaidman 1993). Its wave function has a node at the center of the box, where its charge density is zero. Assume the electron performs a very fast continuous motion in the box, and during a very short time interval its motion generates an effective charge distribution. Let's see whether this distribution can assume the same form as $e|\psi(x)|^2$, which is required by protective measurement[18]. Since the effective charge density is proportional to the amount of time the electron spends in a given position, the electron must be in the left half of the box half of the time and in the right half of the box half of the time. But it can spend no time at the center of the box where the effective charge density is zero; in other words, it must move at infinite velocity at the center. Certainly, the appearance of velocities faster than light or even infinite velocities may be not a fatal problem, as our discussion is entirely in the context of non-relativistic quantum mechanics, and especially the infinite potential in the example is also an ideal situation. However, it seems difficult to explain why the electron speeds up at the node and where the infinite energy required for the acceleration comes from. Moreover, the sudden acceleration of the electron near the node may also result in large radiation (Aharonov, Anandan and

[17] At a particular time the charge density is either zero (if the electron is not there) or singular (if the electron is inside the infinitesimally small region including the space point in question).

[18] Note that in Nelson's stochastic mechanics, the electron, which is assumed to undergo a Brownian motion, moves only within a region bounded by the nodes (Nelson 1966). This ensures that the theory can be equivalent to quantum mechanics in a limited sense. Obviously this sort of motion is not ergodic and cannot generate the required charge distribution. This conclusion also holds true for the motion of particles in some variants of stochastic mechanics (Bell 1986; Vink 1993), as well as in the de Broglie-Bohm theory (de Broglie 1928; Bohm 1952).

Vaidman 1993), which is inconsistent with the predictions of quantum mechanics. Again, it seems very difficult to explain why the accelerating electron does not radiate here.

Let's further consider an electron in a superposition of two energy eigenstates in two boxes $\psi_1(x) + \psi_2(x)$. In this example, even if one assumes that the electron can move with infinite velocity (e.g. at the nodes), it cannot *continuously* move from one box to another due to the restriction of box walls. Therefore, any sort of continuous motion cannot generate the effective charge distribution $e|\psi_1(x) + \psi_2(x)|^2$. One may still object that this is merely an artifact of the idealization of infinite potential. However, even in this ideal situation, the model should also be able to generate the effective charge distribution by means of some sort of ergodic motion of the electron; otherwise it will be inconsistent with quantum mechanics. On the other hand, it is very common in quantum optics experiments that a single-photon wave packet is split into two branches moving along two well separated paths in space. The wave function of the photon disappears outside the two paths for all practical purposes. Moreover, the experimental results are not influenced by the environment and experimental setup between the two paths of the photon. Thus it is very difficult to imagine that the photon performs a continuous ergodic motion back and forth in the space between its two paths.

In view of these serious drawbacks of the classical ergodic models and their inconsistency with quantum mechanics, we conclude that the ergodic motion of particles cannot be continuous. If the motion of a particle is discontinuous, then the particle can readily move throughout all regions where the wave function is nonzero during an arbitrarily short time interval at a given instant. Furthermore, if the probability density of the particle appearing in each position is proportional to the modulus squared of its wave function there at every instant, the discontinuous motion can also generate the right mass and charge distribution. This will solve the above problems of the classical ergodic models. The discontinuous ergodic motion requires no existence of a finite ergodic time. Moreover, a particle undergoing discontinuous motion can also move from one region to another spatially separated region, no matter whether there is an infinite potential wall between them, and such discontinuous motion is not influenced by the environment and experimental setup between these regions either. Besides, discontinuous motion can also solve the problems of infinite velocity and accelerating radiation. The reason is that no classical velocity and acceleration can be defined for discontinuous motion, and energy and momentum will require new definitions and understandings as in quantum mechanics.

In summary, we have argued that the mass and charge distribution of a quantum system, which can be measured by protective measurement, is not real but effective. Moreover, the effective mass and charge distribution is formed by the discontinuous motion of a localized particle, and the probability density of the particle appearing in each position is proportional to the modulus squared of its wave function there.

4.3 An Argument for Random Discontinuous Motion

Although the above analysis demonstrates that the ergodic motion of a particle is discontinuous, it doesn't say that the discontinuous motion must be random. In particular, the randomness of the result of a quantum measurement may be only apparent. In order to know whether the motion of particles is random or not, we need to analyze the cause of motion. For example, if motion has no *deterministic* cause, then it will be random, only determined by a probabilistic cause. This may also be the right way to find how particles move. Since motion involves change in position, if we can find the cause or instantaneous condition determining the change[19], we will be able to find how particles move in reality.

Let's consider the simplest states of motion of a free particle, for which the instantaneous condition determining the change of its position is a constant during the motion. In logic the instantaneous condition can only be deterministic or indeterministic. That the instantaneous condition is deterministic means that it leads to a deterministic change of the position of the particle at a given instant. That the instantaneous condition is indeterministic means that it only determines the probability of the particle appearing in each position in space at a given instant. If the instantaneous condition is deterministic, then the simplest states of motion of the free particle will have two possible forms. The first one is continuous motion with constant velocity, and the equation of motion of the particle is $x(t + dt) = x(t) + vdt$, where the deterministic instantaneous condition v is a constant[20]. The second one is discontinuous motion with infinite average velocity; the particle performs a finite jump along a fixed direction at every instant, where the jump distance is a constant, determined by the constant instantaneous condition[21]. On the other hand, if the instantaneous condition is indeterministic, then the simplest states of motion of the free particle will be random discontinuous motion with even position probability distribution. At each instant the probability density of the particle appearing in every position is the same.

In order to know whether the instantaneous condition is deterministic or not, we need to determine which sort of simplest states of motion are the solutions of the equation of free motion in quantum mechanics (i.e. the free Schrödinger equation). According to the analysis in the last subsection, the momentum eigenstates of a free particle, which are the solutions of the free Schrödinger equation, describe the ergodic motion of the particle with even position probability distribution in space. Therefore, the simplest states of motion with a constant probabilistic instantaneous condition are the solutions of the equation of free motion, while the simplest states of motion with a constant deterministic instantaneous condition are not.

When assuming that (1) the simplest states of motion of a free particle are the solu-

[19] The word "cause" used here only denotes a certain instantaneous condition determining the change of position, which may appear in the laws of motion. Our analysis is independent of whether the condition has causal power or not.

[20] This deterministic instantaneous condition has been often called intrinsic velocity (Tooley 1988).

[21] In discrete space and time, the motion will be a discrete jump across space along a fixed direction at each time unit, and thus it will become continuous motion with constant velocity in the continuous limit.

tions of the equation of free motion; and (2) the instantaneous condition determining the position change of a particle is always deterministic or indeterministic for any state of motion, the above result then implies that motion, no matter whether it is free or forced, has no deterministic cause, and thus it is random and discontinuous, only determined by a probabilistic cause. The argument may be improved by further analyzing these two seemingly reasonable assumptions, but we will leave this for future work.

5 The Wave Function as a Description of Random Discontinuous Motion of Particles

In classical mechanics, we have a clear physical picture of motion. It is well understood that the trajectory function $x(t)$ in classical mechanics describes the continuous motion of a particle. In quantum mechanics, the trajectory function $x(t)$ is replaced by a wave function $\psi(x,t)$. If the particle ontology is still viable in the quantum domain, then it seems natural that the wave function should describe some sort of more fundamental motion of particles, of which continuous motion is only an approximation in the classical domain, as quantum mechanics is a more fundamental theory of the physical world, of which classical mechanics is an approximation. The analysis in the last section provides a strong support for this conjecture. It shows that a quantum system such as an electron is a localized particle that undergoes random discontinuous motion, and the probability density of the particle appearing in each position is proportional to the modulus squared of its wave function there. As a result, the wave function in quantum mechanics can be regarded as a description of the more fundamental motion of particles, which is essentially discontinuous and random. In this section, we will give a more detailed analysis of random discontinuous motion and the meaning of the wave function.

5.1 An Analysis of Random Discontinuous Motion of Particles

Let's first make clearer what we mean when we say a quantum system such as an electron is a particle. The picture of particles appears from our analysis of the mass and charge density of a quantum system. As we have shown in the last section, the mass and charge density of an electron, which is measurable by protective measurement and proportional to the modulus squared of its wave function, is not real but effective; it is formed by the ergodic motion of a localized particle with the total mass and charge of the electron. If the mass and charge density is real, i.e., if the mass and charge distributions at different locations exist at the same time, then there will exist gravitational and electrostatic interactions between the distributions, the existence of which not only contradicts experiments but also violates the superposition principle of quantum mechanics. It is this analysis that reveals the basic existent form of a quantum system such as an electron in space and time. An electron is a particle[22]. Here the concept of a particle is used

[22] However, the analysis cannot tell us the precise size and possible structure of an electron.

in its usual sense. A particle is a small localized object with mass and charge, and it is only in one position in space at an instant. However, as we have argued above, the motion of an electron described by its wave function is not continuous but discontinuous and random in nature. We may say that an electron is a quantum particle in the sense that its motion is not continuous motion described by classical mechanics, but random discontinuous motion described by quantum mechanics.

Next, let's analyze the random discontinuous motion of particles. From a logical point of view, for the random discontinuous motion of a particle, the particle must have an instantaneous property (as a probabilistic instantaneous condition) that determines the probability density of it appearing in every position in space; otherwise the particle would not "know" how frequently it should appear in each position in space. This property is usually called indeterministic disposition or propensity in the literature[23], and it can be represented by $\varrho(x,t)$, which satisfies the nonnegative condition $\varrho(x,t) \geqslant 0$ and the normalization relation $\int_{-\infty}^{+\infty} \varrho(x,t)dx = 1$. As a result, the position of the particle at every instant is random, and its trajectory formed by the random position series is also discontinuous at every instant[24].

Unlike the deterministic continuous motion, the trajectory function $x(t)$ no longer provides a useful description for random discontinuous motion. In the following, we will give a strict description of random discontinuous motion of particles based on measure theory. For simplicity but without losing generality, we will mainly analyze the one-dimensional motion that corresponds to the point set in two-dimensional space and time. The results can be readily extended to the three-dimensional situation.

We first analyze the random discontinuous motion of a single particle. Consider the state of motion of the particle in finite intervals Δt and Δx near a space-time point (t_i,x_j) as shown in Fig. 2. The positions of the particle form a random, discontinuous trajectory in this squared region[25]. We study the projection of this trajectory in the t-axis, which is a dense instant set in the time interval Δt. Let W be the discontinuous trajectory of the particle and Q be the squared region $[x_j, x_j + \Delta x] \times [t_i, t_i + \Delta t]$. The dense instant set can be denoted by $\pi_t(W \cap Q) \in \Re$, where π_t is the projection on the t-axis. According

[23] Note that the propensity here denotes single case propensity. For a helpful analysis of the single-case propensity interpretation of probability in GRW theory see Frigg and Hoefer (2007). In addition, it is worth stressing that the propensities possessed by particles relate to their objective motion, not to the measurements on them. By contrast, according to the existing propensity interpretations of quantum mechanics, the propensities a quantum system has relate only to measurements; a quantum system possesses the propensity to exhibit a particular value of an observable if the observable is measured on the system (see Suárez 2004 for a comprehensive analysis).

[24] However, there is an exception. When the probability density function is a special δ-function such as $\delta(x - x(t))$, where $x(t)$ is a continuous function of t, the motion of the particle is deterministic and continuous. In addition, even for a general probability density function it is still possible that the random position series forms a continuous trajectory, though the happening probability is zero.

[25] Recall that a trajectory function $x(t)$ is essentially discontinuous if it is not continuous at every instant t. A trajectory function $x(t)$ is continuous if and only if for every t and every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that whenever a point t_0 has distance less than δ to t, the point $x(t_0)$ has distance less than ε to $x(t)$.

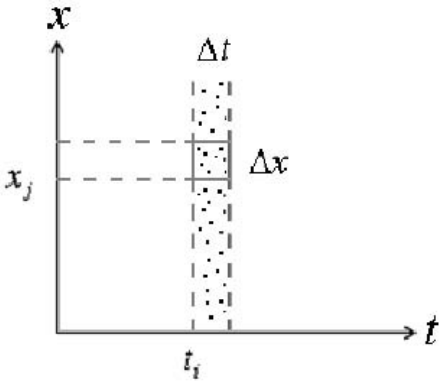

Fig. 2 The description of random discontinuous motion of a single particle

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to the measure theory, we can define the Lebesgue measure:

$$M_{\Delta x,\Delta t}(x_j,t_i)=\int_{\pi_t(W\cap Q)\in\Re}dt. \tag{20}$$

Since the sum of the measures of all such dense instant sets in the time interval Δt is equal to the length of the continuous time interval Δt, we have:

$$\sum_j M_{\Delta x,\Delta t}(x_j,t_i)=\Delta t. \tag{21}$$

Then we can define the measure density as follows:

$$\rho(x,t)=\lim_{\Delta x,\Delta t\to 0}M_{\Delta x,\Delta t}(x,t)/(\Delta x\cdot\Delta t). \tag{22}$$

This quantity provides a strict description of the position distribution of the particle or the relative frequency of the particle appearing in an infinitesimal space interval dx near position x during an infinitesimal interval dt near instant t, and it satisfies the normalization relation $\int_{-\infty}^{+\infty}\rho(x,t)dx=1$ by Eq. (21). Note that the existence of the limit relies on the continuity of the evolution of $\varrho(x,t)$, the property of the particle that determines the probability density of it appearing in every position in space. In fact, $\rho(x,t)$ is determined by $\varrho(x,t)$, and there exists the relation $\rho(x,t)=\varrho(x,t)$. We call $\rho(x,t)$ position measure density or position density in brief.

Since the position density $\rho(x,t)$ changes with time in general, we may further define the position flux density $j(x,t)$ through the relation $j(x,t)=\rho(x,t)v(x,t)$, where $v(x,t)$ is the velocity of the local position density. It describes the change rate of the position density. Due to the conservation of measure, $\rho(x,t)$ and $j(x,t)$ satisfy the continuity equation:

$$\frac{\partial\rho(x,t)}{\partial t}+\frac{\partial j(x,t)}{\partial x}=0. \tag{23}$$

The position density $\rho(x,t)$ and position flux density $j(x,t)$ provide a complete description of the state of random discontinuous motion of a single particle[26].

The description of the motion of a single particle can be extended to the motion of many particles. For the random discontinuous motion of N particles, we can define joint position density $\rho(x_1,x_2,...x_N,t)$ and joint position flux density $j(x_1,x_2,...x_N,t) = \rho(x_1,x_2,...x_N,t)v(x_1,x_2,...x_N,t)$. They also satisfy the continuity equation:

$$\frac{\partial\rho(x_1,x_2,...x_N,t)}{\partial t}+\sum_{i=1}^{N}\frac{\partial j(x_1,x_2,...x_N,t)}{\partial x_i}=0. \tag{24}$$

When these N particles are independent, the joint position density can be reduced to the direct product of the position density for each particle, namely $\rho(x_1,x_2,...x_N,t) = \prod_{i=1}^{N}\rho(x_i,t)$. Note that the joint position density $\rho(x_1,x_2,...x_N,t)$ and joint position flux density $j(x_1,x_2,...x_N,t)$ are not defined in the real three-dimensional space, but defined in the 3N-dimensional configuration space.

5.2 Interpreting the Wave Function

Although the motion of particles is essentially discontinuous and random, the discontinuity and randomness of motion are absorbed into the state of motion, which is defined during an infinitesimal time interval and represented by the position density $\rho(x,t)$ and position flux density $j(x,t)$. Therefore, the evolution of the state of random discontinuous motion of particles may obey a deterministic continuous equation. By assuming that the nonrelativistic equation of random discontinuous motion is the Schrödinger equation in quantum mechanics, both $\rho(x,t)$ and $j(x,t)$ can be expressed by the wave function in a unique way[27]:

$$\rho(x,t)=|\psi(x,t)|^2, \tag{25}$$

$$j(x,t)=\frac{\hbar}{2mi}[\psi^*(x,t)\frac{\partial\psi(x,t)}{\partial x}-\psi(x,t)\frac{\partial\psi^*(x,t)}{\partial x}]. \tag{26}$$

Correspondingly, the wave function $\psi(x,t)$ can be uniquely expressed by $\rho(x,t)$ and $j(x,t)$ (except for a constant phase factor):

$$\psi(x,t)=\sqrt{\rho(x,t)}e^{im\int_{-\infty}^{x}\frac{j(x',t)}{\rho(x',t)}dx'/\hbar}. \tag{27}$$

[26] It is also possible that the position density $\rho(x,t)$ alone provides a complete description of the state of random discontinuous motion of a particle. Which one is right depends on the laws of motion. As we will see later, quantum mechanics requires that a complete description of the state of random discontinuous motion of particles includes both the position density and position flux density.

[27] Note that the relation between $j(x,t)$ and $\psi(x,t)$ depends on the concrete evolution under an external potential such as electromagnetic vector potential. By contrast, the relation $\rho(x,t)=|\psi(x,t)|^2$ holds true universally, independent of the concrete evolution.

In this way, the wave function $\psi(x,t)$ also provides a complete description of the state of random discontinuous motion of particles. For the motion of many particles, the joint position density and joint position flux density are defined in the 3N-dimensional configuration space, and thus the many-particle wave function, which is composed of these two quantities, is also defined in the 3N-dimensional configuration space.

Interestingly, we can reverse the above logic in some sense, namely by assuming the wave function is a complete objective description for the motion of particles, we can also reach the random discontinuous motion of particles, independent of our previous analysis. If the wave function $\psi(x,t)$ is a complete description of the state of motion for a single particle, then the quantity $|\psi(x,t)|^2dx$ will not only give the probability of the particle being found in an infinitesimal space interval dx near position x at instant t (as required by quantum mechanics), but also give the objective probability of the particle being there at the instant. This accords with the common-sense assumption that the probability distribution of the measurement results of a property is the same as the objective distribution of the values of the property in the measured state. Then at instant t the particle will be in a random position where the probability density $|\psi(x,t)|^2$ is nonzero, and during an infinitesimal time interval near instant t it will move throughout the whole region where the wave function $\psi(x,t)$ spreads. Moreover, its position density in each position is equal to the probability density there. Obviously this kind of motion is random and discontinuous.

One important point needs to be pointed out here. Since the wave function in quantum mechanics is defined at an instant, not during an infinitesimal time interval, it should be regarded not simply as a description of the state of random discontinuous motion of particles, but more suitably as a description of the property of the particles that determines their random discontinuous motion at a deeper level[28]. In particular, the modulus squared of the wave function represents the property that determines the probability density of the particles appearing in certain positions in space at a given instant (this means $\varrho(x,t) \equiv |\psi(x,t)|^2$). By contrast, the position density and position flux density, which are defined during an infinitesimal time interval at a given instant, are only a description of the state of the resulting random discontinuous motion of particles, and they are determined by the wave function. In this sense, we may say that the motion of particles is "guided" by their wave function in a probabilistic way.

5.3 Further Discussions

We have been analyzing random discontinuous motion of particles in position space. Does the picture of random discontinuous motion exist for other observables such as momentum and energy? Since there are also momentum wave functions etc in quantum mechanics, it seems tempting to assume that the above interpretation of the wave function in position space also applies to the wave functions in momentum space etc. This means that when a particle is in a superposition of the eigenstates of an observable, it also undergoes random

[28] For a many-particle system in an entangled state, this property is possessed by the whole system.

discontinuous motion among the eigenvalues of this observable. For example, a particle in a superposition of momentum eigenstates also undergoes random discontinuous motion among all momentum eigenvalues. At each instant the momentum of the particle is definite, randomly assuming one of the momentum eigenvalues with probability given by the modulus squared of the wave function at this momentum eigenvalue, and during an infinitesimal time interval around the instant the momentum of the particle spreads throughout all momentum eigenvalues.

However, there is also another possibility, namely that the picture of random discontinuous motion exists only for position, while momentum and energy etc are not instantaneous properties of a particle and they do not undergo random discontinuous change either. There are several reasons supporting this possibility. The first is that our previous arguments for random discontinuous motion of particles apply only to position, not to other observables such as momentum and energy etc. For example, since the interaction Hamiltonian for a many-particle system relates to the positions of these particles, not to their momenta and energies, the previous analysis of electrostatic self-interaction applies only to position. Next, the Kochen-Specker theorem requires that under certain reasonable assumptions only a certain number of observables can be assigned definite values at all times (Kochen and Specker 1967). This strongly suggests that the picture of random discontinuous motion exist only for a certain number of observables. Moreover, since there are infinitely many observables besides position and these observables arguably have the same status, this may further imply that the picture of random discontinuous motion does not exist for any observable other than position. Lastly, the meaning of observables as Hermitian operators acting on the wave function lies in the corresponding ways to decompose (and also to measure) the same wave function. For example, position and momentum reflect two ways to decompose the same spatial wave function. In this sense, the existence of random discontinuous motion for momentum will be redundant.

Therefore, it seems more reasonable to assume that the picture of random discontinuous motion exists only for position. On this view, the position of a particle is the only instantaneous property of the particle defined at instants (besides its wave function), while momentum and energy are properties relating to the state of motion of the particle (e.g. momentum and energy eigenstates), which is formed by the motion of the particle during an infinitesimal time interval around a given instant[29]. Certainly, when a particle is in a momentum or energy eigenstate, we may still say that the particle has definite momentum or energy, whose value is the corresponding eigenvalue. Moreover, when a particle is in a momentum or energy superposition state and the momentum or energy branches are well separated in space, we may also say that the particle has definite momentum or energy in each separated region.

Finally, we note that spin is a more distinct property. Since the spin of a free particle is always definite along one direction, the spin of the particle does not undergo random

[29] Note that the particle position here is different from the position property represented by the position observable in quantum mechanics, and the latter is also a property relating only to the state of motion of the particle such as position eigenstates.

discontinuous motion, though a spin eigenstate along one direction can always be decomposed into two different spin eigenstates along another direction. But if the spin state of a particle is entangled with its spatial state due to interaction and the branches of the entangled state are well separated in space, the particle in different branches will have different spin, and it will also undergo random discontinuous motion between these different spin states. This is the situation that usually happens during a spin measurement.

6 Possible Implications for Solving the Measurement Problem

In this section, we will briefly discuss possible implications of the new interpretation of the wave function for the solutions to the measurement problem. That is, we will briefly analyze how to solve the measurement problem when assuming the interpretation of the wave function in terms of random discontinuous motion of particles is true.

It can be seen that random discontinuous motion of particles, unlike the continuous motion of particles in the de Broglie-Bohm theory or Bohmian mechanics, does not provide a solution to the measurement problem. This is not against expectation, since it only provides an ontological interpretation of the wave function, and what the precise laws of motion are still needs to be determined. As we will argued below, however, this ontological interpretation of the wave function, if it is true, may also have implications for the solutions to the measurement problem.

An important aspect of the measurement problem is to explain the origin of the Born probabilities or the probabilities of measurement results. According to the interpretation of the wave function in terms of random discontinuous motion of particles, the ontological meaning of the modulus squared of the wave function of an electron in a given position is that it represents the probability density that the electron as a particle appears in this position, while according to the Born rule, the modulus squared of the wave function of the electron in the position also gives the probability density that the electron is found there. It is hardly conceivable that these two probabilities have no connection. On the contrary, it seems natural to assume that the origin of the Born probabilities is the random discontinuous motion of particles. If this assumption turns out to be true, then it will have significant implications for the solution to the measurement problem, because the existing solutions have not accommodated this assumption. In Bohmian mechanics (Goldstein 2013), the Born probabilities are epistemic. In the latest formulation of the many-worlds interpretation (Wallace 2012), the Born probabilities are subjective. In dynamical collapse theories, although the Born probabilities are objective, it is usually assumed that the randomness originates from a classical noise field independent of the wave function of the studied system (Ghirardi 2011). In short, none of these main solutions to the measurement problem assumes that the Born probabilities originate from the wave function itself.

Therefore, if the Born probabilities originate from the objective probabilities inherent in the random discontinuous motion of particles described by the wave function, then all these realistic alternatives to quantum mechanics need to be reformulated. The reformu-

lation may be easier for some alternatives, but more difficult for others. For example, it is relatively easy to find a dynamical collapse model where the chooser or the noise source that collapses the wave function is the underlying random discontinuous motion of particles (Gao 2013b). However, it seems difficult to find a new formulation of Bohmian mechanics in which the probabilities of measurement results are objective and come from the wave function. Moreover, it seems that the many-worlds interpretation and the many-minds interpretation cannot be reformulated in terms of the objective probabilities inherent in the random discontinuous motion of particles either. A detailed analysis of the relationship between the suggested interpretation of the wave function and stochastic hidden variables theories or Bell's Everett(?) theory will be given in another separate paper.

Certainly, if the interpretation of the wave function in terms of random discontinuous motion of particles is not true, then the above implications will be totally irrelevant. However, these analyses at least indicate that understanding the origin of the Born probabilities may be a key to solving the measurement problem. Moreover, if this interpretation of the wave function is not true, then either the above arguments supporting the interpretation are problematic or the basic realistic assumption used in these arguments is not true. This realistic assumption is that the wave function of a quantum system at each instant describes the state of a physical entity or many physical entities either at the instant or during an infinitesimal interval around the instant. A formulation of Bohmian mechanics does reject this assumption, and it assumes a nomological interpretation of the wave function (Goldstein 2013). On the other hand, if one accepts this realistic assumption, then one needs to find loopholes in our arguments in order to avoid the above implications for the solutions to the measurement problem.

7 Conclusions

In this paper, we argue that the ontological meaning of the wave function may be derived with the help of protective measurement. There are three key steps in the derivation. First, protective measurement, which is based on the established parts of quantum mechanics, shows that the mass and charge of a charged quantum system are distributed throughout space, and the mass and charge density in each position is proportional to the modulus squared of the wave function of the system there. Next, the superposition principle of quantum mechanics requires that the mass and charge distribution is effective, that is, it is formed by the ergodic motion of a localized particle with the total mass and charge of the system. Lastly, the consistency of the formed distribution with that predicted by quantum mechanics requires that the ergodic motion of the particle is discontinuous, and the probability density of the particle appearing in every position is equal to the modulus squared of its wave function there. Based on this analysis, we suggest that the wave function in quantum mechanics describes the state of random discontinuous motion of particles, and at a deeper level, it may represent the property of the particles that determines their random discontinuous motion. In particular, the modulus

squared of the wave function determines the probability density of the particles appearing in certain positions in space.

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