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Topology of States on a Black Hole Event Horizon

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Abstract: In this letter the relationship between entanglement and spacetime is examined as a phase change in the vacuum at the stretched horizon. This phase change occurs with respect to a homotopy type defined on the horizon according to the ensemble of Planck unit areas.

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1 Introduction

The holographic principle stems from the observation by ‘t Hooft [1] that a quantum bit (qubit) of information can be isolated on a Planck area of the event horizon. The Planck length is found through the identification of the deBroglie wavelength of a particles with a standing wave determined by its Schwarzschild radius. This leads to $\ell_p = \sqrt{G\hbar/c^3}$. A unit of area on an event horizon $\mathcal{A}_p = G\hbar/c^3$ which is determined by the Newtonian gravitational constant uniquely for $\hbar = c = 1$. The area of an event horizon is then proportional to the amount of information a black hole contains. This is digital qubit form of the Bekenstein relationship between the area of a black hole and entropy $S = k\mathcal{A}/4\ell_p$ [2].

An observer may choose to observe the black hole at a distance, or the IR limit of red shift energy, or to probe the horizon on a fine grained scale by observing it close up. The IR limit is difficult, particularly if the black hole is large and cold, and information available from Hawking radiation. Up close the observer may elect to hover over the event horizon and experience extreme Hawking-Unruh radiation. The UV limit is equivalent to scatting quantum fields on the horizon, or stretched horizon, at very high energy. This leads to an uncertainty in being able to isolate a qubit. As that energy approaches the Planck energy $E = \sqrt{\hbar c^5/G}$ the uncertainty becomes nearly infinite.

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A wave function spread out on the stretched horizon is analogous to a wave entering a multi-slit screen. An observer can only ascertain which Planck unit of area the particle is at with the application of a powerful "microscope." However, as with the Heisenberg microscope this wildly perturbs the particle. The meaning of geometry emerges only for a horizon with sufficient area so there are units of area far enough apart so an angular resolution exists.

The large N qubit, or equivalently large number of Planck units of area, limit has Bott periodicity and describes a large horizon with a partition of states. Bott periodicity is a topological property of Lie groups, where periods of 2, 4 and 8 added to the rank of a Lie group recovers the same topology. This leads to physics of the horizon that involves a phase change. This is a critical phase change which occurs at the Hagedorn temperature. This physics is associated with probing the horizon on a small scale, or equivalently hovering above the horizon with an extreme acceleration [3].

2 Vacuum Phase on the Horizon

The two slit experiment wave function is a superposition of states through the two slits. An ensemble of experiments produces a wave pattern on the detecting plate. Now consider the resolution of a screen and the Planck length. Suppose a wave emerges from a spherical screen of radius R with pixel size Δx. The wave reaches the center in $T ~=~ R/c$. The total number of pixels on this screen is $4\pi R^2/\Delta x^2$, which per time $2R/c$ is equal to the Planck rate of transmission $1/T_p$, and so $\Delta x ~=~ \sqrt{2\pi RcT_p}$, which is very small but much larger than the Planck scale. For the two slit experiment the resolution of the two slits is set by a limit

$$\Delta x ~=~ \sqrt{DcT_p} \tag{1}$$

for $D ~\simeq~ 2\pi R$. For smaller two slits separation the double slit experiment is indistinguishable from a single slit experiment.

The angular uncertainty in the wave is $\Delta\theta ~=~ \Delta x/D \simeq~ cT_p/D$, and angular uncertainty reduces at larger distance. Direction has a clearer physical meaning at larger scales. The means square uncertainty in distance becomes $\langle x\rangle^2 ~=~ DcT_p/\sqrt{2\pi}$, which is considerably larger than the Planck area. Distance emerges at large scales with increased number of Planck units of information on a screen. If physics could be measured on the Planck scale, the angular uncertainty would be huge. This is the Heisenberg microscope [7], which illustrates how spatial directions on the Planck scale are not well defined. What is important is the topology induced by the slits a 2D holographic system. The screen and slits define "topological logic gate" information about spacetime. The quantum logic gate is a reversible unitary processor that utilizes a Hadamard matrix, such as a C-NOT gate [4]. Spacetime is potentially emergent from entanglement [5], and the above example is a double slit version of how entanglement forms the building blocks of geometry. The structure of spacetime, such as the $SO(3,~1)$, or the spacetime symmetry $\mathfrak{cl}(1,~3)$ in a Clifford basis is a method for working quantum information [6].

These nontrivial topological properties are similar to category theory of types [8], and motives [9]. In this approach distances or even paths are not relevant, but rather instead topologically distinct setting of a logic gate. The fundamentally important information is not the position in space or time of a particle, but rather the ”processing” of qubits according to logic gates with topological features. The wave function spreads across the horizon so it impossible to determine which gate does the processing, but each gate is entangled with a qubit for the particle escaping to infinity. Every particle interacting with the black hole has an amplitude for escape to infinity, which can include the emission of Hawking radiation. The larger the horizon area means the holographic screen has a greater resolution or angular certainty $\Delta\theta \simeq cT_p/D$ decreases. The black hole becomes more classical and the geometry of gravitation assumes more meaning.

A path integral is a distribution of corresponding paths, and these paths can wind through a space in topologically distinct ways. These topologically distinct paths wind in space with homology, which corresponds to a topological invariant such as a Morse index [10]. Consider a path integral $Z = \int D[A]e^{-iS(A)}$. The gauge potential transforms by the group g as $A' = gAg^{-1} + gdg^{-1}$ with the gauge fields $F = dA + A \wedge A$ transforming as $F' = gFg^{-1}$ and,

$$e^{-iS} \rightarrow e^{-igSg^{-1}} = ge^{-iS}g^{-1} \tag{2}$$

The Lagrangian for this action determined by gauge potentials A_μ is an $N \times N$ matrix given by the indices $a,\ b$ in A^{ab}_μ

$$\mathcal{L} = \frac{N}{\lambda}Tr\left(F_{\mu\nu}F^{\mu\nu}\right), \tag{3}$$

for $\lambda = g^2N$ the $'$t Hooft coupling. This is the standard Yang-Mills (YM) theory, now taken in a large N limit in the gauge symmetry. The gauge group is $SU(N)$ or $SO(N)$ and we have for $N = 2m$ that $SU(2m)$ is homeomorphic to the Grassmannian $G_m(\mathbb{C}^{2m})$ due to the universal bundle theorem [11] and,

$$\pi_i G_{(m,2m)} \simeq \pi_{i+1}SU(2m). \tag{4}$$

The AdS_5 has the isometry group $SO(4,\ 2) \simeq SU(2,\ 2)$ for superconformal theories $SU(4) \simeq SO(6)$ [11]. The case $m = 2$ is the CFT corresponding to $SU(2,\ 2)$ for twistor space.

The large N limit in the AdS/CFT correspondence [12] $lim_{N\rightarrow\infty}su(N)$, and we have the Bott periodicity theorem at work with the gauge group. There is a cyclic homotopy to unitary groups so that $\pi_n(U(N)) = \pi_{n+2}(U(N))$ [13] which applies for $N \rightarrow \infty$ [13]. Further, the loop space structure has an eightfold periodicity for stable complex bundles. For BU the classifying space of stable bundles the two-fold loop space Ω^2BU is constructed from the loops space functor Ω such that homotopy equivalent to a set of a countable number of copies of BU, $\Omega^2BU \simeq \mathbb{Z} \times BU$, which connects with the two-fold period $\Omega^2U \simeq U$. This extends to orthogonal groups and the entire 8- fold periodic structure [14]. The fundamental field theoretic content is a set of field configurations $\phi = [X,\ BSU(N)]$ which exist in the homotopy of the configuration space. The configuration space is governed by the number of possible Planck units of horizon area $= N$, and the partition of field amplitudes.

The partition of field amplitudes is determined by the tensor components of states in Clebsch-Gordon decompositions. States $|\psi\rangle = \psi^{i_1,\dots,i_n}_{j_1,\dots,j_m}|0^{j_1,\dots,j_m}_{i_1,\dots,i_n}\rangle$ are built from tensors with a complete set of Youngs Tableaux descriptions for the set of irreducible representations. For a succession of $n \times n$ matrices with $n \rightarrow \infty$ there are $p(n)$ possible Young Tableaux for each n and irreducible representations the density of states is

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty}\left(\frac{1}{1 - \omega^n}\right). \tag{5}$$

The partition function $p(n)$ is the integer partition, where the circle approximation for this is the Hardy-Ramanujan formula. It is also easy to show the integer partition function gives the density of states of a superstring [15].

The Bott periodicity demonstrates a cyclic topological nature to these quantum fields. The topological nature of these fields is invariant with respect to the size N, and it is possible to take the limit $N \rightarrow \infty$, for N even. The topological nature of these fields remains the same as the partition of these functions changes. The Symmetry Protected Topological (SPT) state is an energy gapped phase with the Hawking temperature. The SPT states with a symmetry given by a group $\mathcal{G}$ have topological orders given by the cohomology $H^2(\mathcal{G},\ U(1))$ [16]. For the case $d = 2$ the 2-space plus time model with the Euclideanized $\mathcal{G} = SO(3)$ gives the $2 + 1$ spin Hall effect and the time reverse symmetry group $\mathbb{Z}$. The projective representation of this group is found by the quotient with the normalizer of that group $P\mathcal{G} = \mathcal{G}/N$ with the map $\pi : \mathcal{G} = P\mathcal{G}$ that defines a bundle of lift elements. These elements λ obey for $g \in \mathcal{G}$ the rule $\lambda(g,\ g') = \sigma(g,\ g')\lambda(g)\lambda(g)$, where $\sigma(g,\ g')$ is a Schur multiplier. The projective representation of the group mods out the action of this normalizer. This is cohomology $H^2(\mathcal{G},\ N)$ with the normalizer $N = U(1)$ [16].

By Bott periodicity the topological properties of all $SU(2n)$ groups are equivalent. $SU(2) \simeq SO(3)$, means the topological properties of these fields on this surface is that of the $2 + 1$ spin Hall effect. The stretched horizon is a tangle of loops and handles, which determine this large N expansion. However, the primary topological field properties are those of the smallest group $SU(2)$. The SPT states apply for large N, and the stretched horizon exists in a different phase. This topological phase is additional hair or fuzz on the stretched horizon. This hair is proposed by Mathur as a resolution to the firewall problem [17]. The periodicity and the extension for large N in $SU(2N)$ is a way the entanglement of states on the horizon acquire high resolution and become more classical. The occurrence of a sharp horizon is equivalent to the entanglement of states.

There has been a remarkably similar result with working on information as a fundamental unit in quantum physics [18]. The space for Bell states is $\mathcal{H}^4 \sim \mathbb{C}^2$ in four dimensions. The group for general qubits is $SU(2^n)$ and the Bloch vector representation is $SO(2^n - 1)$, where for $n = 2$ this gives the Bloch sphere. The $SO(3)$ is then the standard unit for $d = 3 = 2^2 - 1$, which due to Bott periodicity on even $SU(2N)$

$= SU(2^n)$ gives the same $\hat{\mathcal{G}}_{qubit} = SO(3)$. This returns us then to the perspective of emergent spacetime and Clifford algebra, such as illustrated in [6]

3 Discussion

A considerable difficulty has appeared in physics that seeks to make gravitation commensurate with quantum mechanics. It is known that Hawking radiation is entangled with the black hole whence it escapes. The heuristic of a particle anti-particle pair in the vacuum splitting so one enters the black hole and the other escapes gives a qualitative sense of this. This means that a black hole emits radiation and its entanglement entropy increase. This is a standard observation where a cavity with hot atoms will become entangled with the radiation it emits. However, eventually the cavity will release quantum states entangled with prior emitted quantum states and its entanglement entropy decreases [19]. This does not happen with entanglement entropy with black holes, and entanglement entropy grows continually until it violates the Bekenstein bound. This is the basis of the AMPS paper [20] argument that unitary conservation of information and the equivalence principle are not compatible.

A phase change in quantum states near the horizon permits states to spread into a set of SPT states which comprise fuzzball states. The quantum critical, or analogous thermal induced, phase occur from two perspectives. An accelerated or asymptotic observer witnesses quantum states on the horizon shifted to the UV and at the stretched horizon there is a phase change. This observer sees the rapid passage of events and a rapid flux of radiation in and out of the black hole. The hot Hawking radiation would exhibit a thermal induced phase approaching the Hagedorn temperature, where a phase change occurs. The quantum critical and thermal induced phases are complements of each other. The occurrence of states in this SPT phase provides the fuzzball set of states. This provides an avenue for the resolution of the AMPS firewall problem [17].

The density of states of the black hole is given by the integer partition function. The density of states for strings has a generating function that in an asymptotic limit or approximation is given by the Hardy-Ramanujan formula. The mass of a black hole is its energy and the number of Planck areas on the horizon means that $E_n = c\sqrt{n}$ and the partition function for a Schwarzschild black hole is

$$Z(\beta) = \sum_E g(E) e^{-\beta E} \tag{6}$$

for a counting of states leads to a form of the Hardy-Ramanujan formula, similar to the analysis in [21]. The large N expansion of a YM field on the horizon leads to the integer partition.

The 8 fold periodicity of leads us into the nature of $SO(8)$, which has real postive spinor reprentations for $SO(8n)$ as $n = 1, 2, \dots$ and exhibits Bott periodicity. $SO(8)$ is also embedded in $SO(9,1)$ and is the vector representation of $SO(8)$ it is also the dual spinor reprentations $\sigma \rightarrow M\sigma$ and the Hermitian dual $\sigma^\dagger \rightarrow \sigma^\dagger M^\dagger$. These two spinor representations form a 2×2 repreentation. These are the transverse degrees of freedom of $SO(9,1)$, where the real vector representations are $SO(9)$ and the two spinor representations are transverse in $SO(9,1)$. The $SO(9)$ is part of the F_4 sequence $SO(9) \rightarrow F_{52/36} \rightarrow OP^2$ that diagonalizes the Freudenthal triple system with real eigenvalues. The additional two spinors plus the vector is a triality of $SO(8)$. This means the 3×3 matrix representation diagonalizes the full 3×3 Fruedental triple system [22]. This means that quantum information is a gauge-like system with exceptional algebraic structure.

The black hole is an enigmatic system. From a classical perspective it only has three possible observable parameters, mass, charge and angular momentum. Yet recent research has found black holes as quantum states of a remarkable complexity [23]. This is in keeping with the holographic principle which permits black hole event horizons to act as processors and memory storage systems of qubits. Clearly this is a different phase of a black hole that is radically different from the standard classical general relativistic black hole. In [3] it is argued that the horizon with a high level of quantum complexity, such as storage of a large number of quantum bits, exists in a different phase of quantum states, similar to quantum criticality. This phase occurs at extremely high temperatures as observed very near the horizon. Further, the cyclicity of quantum states due to Bott periodicity indicates that much of the quantum information that exists in the universe is in fact redundant. There is a massive amount of redundancy in the number of degrees of freedom in the universe.

Gravitation is a theory on how quantum states on local inertial frames observe a constant vacuum. Similarly an observer on an accelerated frame is in a constant vacuum. However a distant observer witnesses an inertial frame evolve through different vacua. This is in particular with the region within string lengths of the horizon where the vacuum exists in another quantum critical phase. On a path integral there is only one possible path that is inertial in a classical sense. The other paths are accelerated by quantum fluctuations. This is an equivalence principle of the vacuum, for the quantum gravitation path integral over the moduli space gives a smooth transformation between inertial and accelerated paths. The two perspectives on the nature of the horizon are ultimately equivalent. The distinction between them apparent in current theories is ultimately an illusion.

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