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The Eternal Rainbow Universe

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Abstract: There are many strong theoretical motivations on why there can be a Lorentz invariance violation due to quantum gravity effects at length scales of the order of Planck length. In this present study, we consider some theoretical arguments which either favors the idea of Lorentz invariance violation or modifies the action of the Lorentz group acting on momentum space. All these ideas predicts a modification in the dispersion relation that depends on some functions known as Rainbow functions. With simple calculations and arguments we show that there are some unique choices of the Rainbow functions which predict an infinitely existing universe near the Plank scale and hence resolve the big bang singularity.

Keywords: Lorentz Invariance Violation; Rainbow Functions; Eternal Rainbow Universe; Big Bang Theory; Quantum Gravity
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Einstein's general theory of relativity is by far the most successful theory in describing the universe on a macroscopic scale. The classical theory is well tested in the weak field regime but essentially breaks down in the regime where it predicts singularities, beyond which the classical notion of spacetime cannot be extended [1]. For example, the big bang singularity appears in almost all the cosmological models which are exact solutions of Einstein's equations subject to some symmetries. At the singularity the classical theory breaks down as curvatures diverge and we believe that some profound quantum theory of gravity should replace the classical theory at these scales. There are several approaches or proposals for a complete quantum theory of gravity and the common feature among them is the existence of a minimum measurable length scale which is the Planck length. This minimal length is expected to play the role of a natural cut-off and resolve the curvature singularities that exists in the classical theory. Any quantum theory of gravity is expected to induce Lorentz invariance violation at Planck scale. This is due to the fact

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that the dynamical degrees of freedom for gravity are geometric which upon quantization give rise to the granularity of space at the Planck scale and this is consistent with the idea of minimum spatial length. This violation is expected to occur despite the fact that the classical theories are locally Lorentz invariant. As studied by several authors, Lorentz invariance violation at Planck scale can be an artifact of spacetime discreteness [2], spontaneous breaking of Lorentz symmetry in string field theory [3], spin network of LQG [4] or consideration of non-commutative approaches to geometry [5]. In a very optimistic sense, consideration of Lorentz invariance violation may help us understand different approaches of QG in a better way, at least theoretically. At low energies the effect of Lorentz invariance violation is expected to get suppressed by (m/m_P) or powers of this. The most common way to study this is by considering the modified dispersion relations (MDRs) and it has been argued that these MDRs can satisfactorily explain threshold anomalies of ultra high energetic cosmic rays and TeV photons [6–8]. A detailed review can be found in [9].

Camelia separately and Magueijo and Smolin developed a theoretical framework known as the doubly special relativity (or DSR) that can naturally accommodate MDRs [10, 11]. DSR can be treated as an extension of special relativity with an extra principle that all observers agree that there is an invariant energy scale (E_P). DSR is one among other possibilities of constructing a non-linear Lorentz transformation in momentum space which implies a deformed Lorentz symmetry which in turn modifies the dispersion relation of special relativity with corrections of high energy scales. Some recent developments include [12]. We motivate ourselves in considering Lorentz invariance deformation scenarios through DSR and Rainbow Gravity but it must be noted that *Lorentz invariance violation and Lorentz invariance deformation are conceptually different scenarios.*

In DSR the non-linear transformations in momentum space increase the difficulty in defining the dual position space. Magueijo and Smolin [13] proposed a solution known as the doubly general relativity (DGR) which assumes that a test particle can also feel the spacetime background and this depends on the parametrization E/E_P. Here E can be considered as the energy scale at which spacetime geometry is probed. However in this deformed Einstein's gravity we have only one particle and that is the graviton. Therefore the particle probing the spacetime will be the graviton produced by the fluctuations of the spacetime itself. So we will not have a single metric for the spacetime, but a one parameter family of metrics which depends on the energy (momentum) of these test particles, forming a *Rainbow* geometry. This approach is called the *Rainbow Gravity* and if taken into consideration, spacetime is endowed with two arbitrary functions $f(E/E_P)$ and $g(E/E_P)$, commonly known as Rainbow functions, having the following properties $\lim_{E\to 0} f, g\ (E/E_P) = 1$. $f(E/E_P)$ and $g(E/E_P)$ appear in the solutions of the modified Einsteins field equations

$$G_{\mu\nu}(E/E_P) = 8\pi G(E/E_P)\, T_{\mu\nu}(E/E_P) + g_{\mu\nu}\Lambda(E/E_P) \quad , \tag{1}$$

where $G(E/E_P)$ is an energy dependent Newton's constant, defined such that $G(0)$ is the low-energy Newton's constant and $\Lambda(E/E_P)$ is an energy dependent cosmological constant. Some recent studies on Rainbow formalism can be found in [14, 15].

In general any departure from Lorentz invariance will lead to the modification of the dispersion relation in the form

$$E^2\ f^2(E/E_P) - p^2\ g^2(E/E_P) = m^2 \quad . \tag{2}$$

As as effective model let us study the FRW cosmology in the framework of Rainbow gravity [16]. The line element for the Rainbow modified spatially flat FRW universe can be written as

$$ds^2 = -\frac{1}{f^2(E/E_P)}dt^2 + \frac{a^2}{g^2(E/E_P)}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \quad . \tag{3}$$

The corresponding modified Friedmann equations and the conservation equation become

$$\left[H - \frac{\dot{g}(E/E_P)}{g(E/E_P)}\right]^2 = \frac{8\pi G}{3f^2(E/E_P)}\rho \quad , \tag{4}$$

$$\dot{H} + \left(\frac{\dot{g}(E/E_P)}{g(E/E_P)}\right)^2 - \frac{\ddot{g}(E/E_P)}{g(E/E_P)} = -4\pi G\frac{(\rho+P)}{f^2(E/E_P)} - \left(H - \frac{\dot{g}(E/E_P)}{g(E/E_P)}\right)\frac{\dot{f}(E/E_P)}{f(E/E_P)} \tag{5}$$

and

$$\dot{\rho} = -3\left(H - \frac{\dot{g}(E/E_P)}{g(E/E_P)}\right)(\rho+P) \quad . \tag{6}$$

Let us now consider an ensemble of ultra high energetic relativistic particles, which is relevant for very early universe, in thermal equilibrium having average energy $\langle E\rangle \sim T$. Under general thermodynamical considerations, along with the equation of state $P = (\gamma-1)\rho$ and the conservation equation we can write the average energy in the form [17]

$$\langle E\rangle = c\ \gamma\ \rho^{\frac{\gamma-1}{\gamma}} \quad , \tag{7}$$

where c is a constant and not much relevant for our purpose. The relation depends only on the equation of state parameter and not on the form of MDR chosen. Note that $\gamma = 4/3$ represents radiation.

We now discuss briefly the general conditions on the Rainbow functions that leads to non-singular Rainbow cosmology. If we use the first modified Friedmann equation in the conservation equation we get

$$\dot{\rho} = -\sqrt{24\pi G}\frac{\gamma\ \rho^{3/2}}{f(\rho)} = \kappa(\rho) \quad , \tag{8}$$

where $f(E/E_P)$ can be written as $f(\rho)$ with the help of equation (7) and lets denote the r.h.s as $\kappa(\rho)$. This first order dynamical system is well studied in cosmological contexts in [18] ([19] is useful for more general applications). Knowing the fixed points of the function $\kappa(\rho)$, i.e., its zeros (denoted as ρ_i's) and its asymptotic behavior enables one to qualitatively describe the behavior of the general solution without actually solving the system. For resolving finite-time singularities we would like to show the existence of an upper bound for the density ρ, having a fixed point ρ_1, which is reached at an infinite time, or to show the existence of a point at which the density is unbounded (a potential singularity) but reached in an infinite time, i.e., not a physical singularity. One can show that finite-time singularities (including big bang singularity) are absent if one of the following is true [18]:

- If f grows asymptotically as $\sqrt{\rho}$, or faster. For example, if $f \sim \rho^s$, where $s \geq 1/2$, one can calculate the time to reach a potential singularity by integrating equation (8) starting from some initial finite density ρ^* to an infinite one. The integral diverges and this means that the time to reach this potential singularity is infinite, therefore, it is not a finite-time singularity and hence, not physical.
- If f^{-1} is differentiable and has a zero at $\rho = \rho_1$ other than $\rho = 0$, then the cosmological solution is nonsingular and interpolates monotonically between ρ_1 and 0.

With a particular choice of $f(E/E_P)$ and $g(E/E_P)$ a possible resolution of the big bang singularity was proposed in [20] in the context of quantum cosmology with a perfect fluid. Here we would like to study how the choice of the Rainbow functions $f(E/E_P)$ and $g(E/E_P)$ as predicted by MDR and DSR plays an important role in possible resolution of the big bang singularity in these modified cosmological models. One of the most interesting forms of MDR, has been suggested by Amelino-Camelia, *et el.* in [6, 7] for massless particles which has the following form

$$E^2 \left(\frac{e^{E/E_P} - 1}{E/E_P} \right)^2 - p^2 = 0 \tag{9}$$

where E_P is the Planck energy scale near which the dispersion relation is modified. This modified dispersion relation has been introduced by Amelino-Camelia, *et al.* to explain the astrophysical observations of the hard spectra coming from gamma-ray bursts at cosmological distances. With (7) we can evaluate the form of $f(\rho)$ and the corresponding equation of (8) in this case can be written as

$$\dot{\rho} = -\sqrt{24\pi}\ \frac{\gamma^2 \rho^{3/2} \rho^{\frac{\gamma-1}{\gamma}}}{\left(\mathrm{Exp}[\gamma \rho^{\frac{\gamma-1}{\gamma}}] - 1 \right)} = \kappa_1(\rho) \quad . \tag{10}$$

From now we will work in units where $c = E_P = G = 1$. In Fig. (1) we have plotted $\dot{\rho}$ versus ρ for radiation and we can see that the density is not bounded which might be a sign of singularity. We can calculate the time taken by the solution to evolve from a finite density $\rho*$ (lets say) to an infinite one which is a potential singularity by integrating equation (10). So for radiation

$$t = \int_{\rho*}^{\infty} \frac{d\rho}{\kappa_1(\rho)} \to \infty \quad . \tag{11}$$

We remember that this integral is *finite* and non-zero in the standard case. Here we see that the time required for the solution to reach infinite density from a finite one is infinite

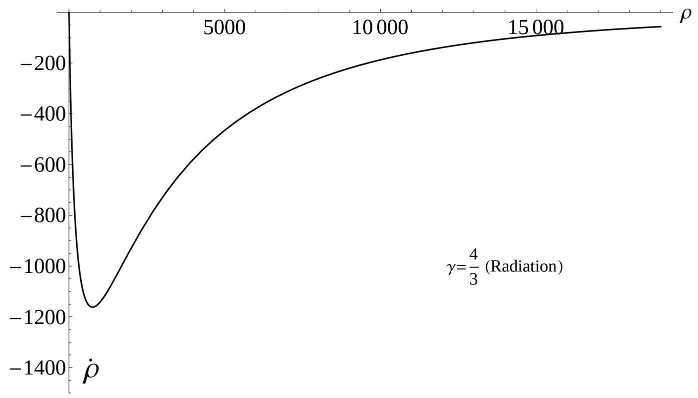

Fig. 1 $\dot{\rho}$ versus ρ for equation (10)

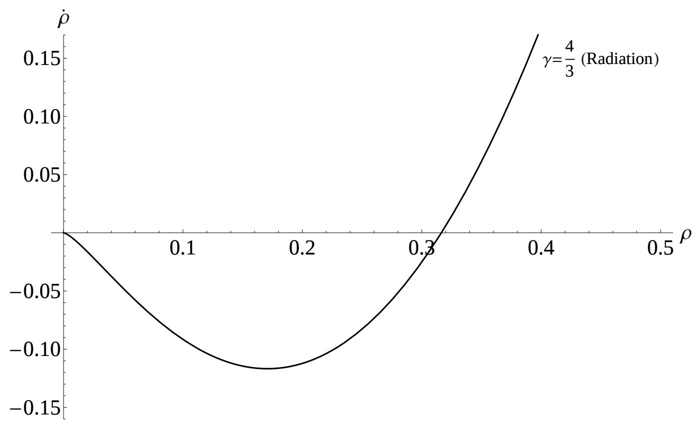

Fig. 2 $\dot{\rho}$ versus ρ for equation (13). Here we have considered $\rho_P = 1$. In this scale the fixed points are at $\rho = 0$ and $\rho = 81/256$.

and hence there is no finite time singularity in this cosmological model. In addition to this we can also find that the solution takes infinite time to reach the fixed point $\rho = 0$ from a finite ρ which we believe to be a sensible result.

Now we study another choice of Rainbow functions

$$f(E/E_P) = g(E/E_P) = \frac{1}{1 - E/E_{Pl}} \quad , \tag{12}$$

which was suggested by Magueijo and Smolin in [11,13]. This particular choice is capable of giving a theory with constant velocity of light and also solves the horizon problem. With this form of f and g we can write the modified conservation equation as

$$\dot{\rho} = -\sqrt{24\pi}\ \gamma\ \rho^{3/2} \left(1 - \gamma\ \rho^{\frac{\gamma-1}{\gamma}}\right) = \kappa_2(\rho) \quad . \tag{13}$$

Here one can observe that the above solution has two fixed points, $\rho = 0$ and $\rho = \gamma^{\frac{\gamma}{1-\gamma}}$, which is showing that the solution is nonsingular and interpolate between $\rho = 0$ and $\rho \sim \rho_P$. In Fig. (2) we have plotted $\dot{\rho}$ vs ρ for radiation ($\gamma = 4/3$). One can show the absence of finite-time singularities by calculating the time necessary to reach any of the two fixed

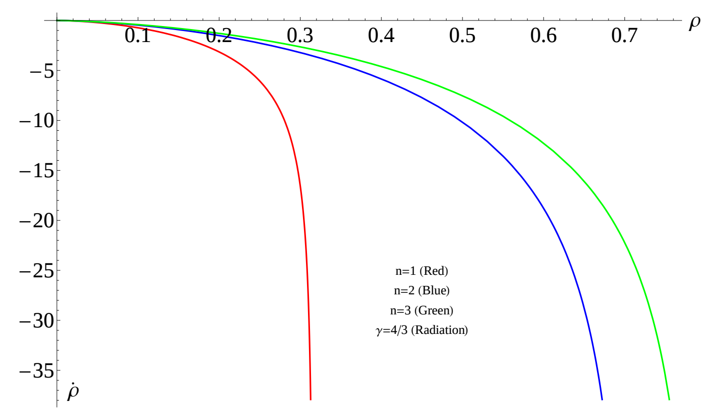

Fig. 3 $\dot{\rho}$ versus ρ for equation (16). Here we have considered $E_P = 1$
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points $\rho_f = 0$ or $\rho_f = 81/256\ \rho_P$ starting from a finite density $0 < \rho^* < 81/256\rho_P$. The time is

$$t = \int_{\rho*}^{\rho_f} \frac{d\rho}{\kappa_2(\rho)} \to \infty \quad , \tag{14}$$

which means that the time necessary to reach a fixed point is infinite. This introduce a possible resolution for the big bang singularity in this model. This gives a solution which is non-singular and has two fixed points.

Another interesting MDR was considered in [21] and which can be expressed as

$$E^2 - p^2 - m^2 \simeq \pm \frac{2}{n+1} E^2 \left(\frac{E}{E_P} \right)^n \quad , \tag{15}$$

where the constant n may get determined experimentally. For this we get the modified conservation equation as

$$\dot{\rho} = -\sqrt{24\pi}\ \gamma\ \rho^{3/2} \left[1 - \frac{2}{n+1}\ \gamma^n\ \rho^{\frac{n(\gamma-1)}{\gamma}} \right]^{-1/2} = \kappa_3(\rho) \quad . \tag{16}$$

In Fig. (3) we have plotted $\dot{\rho}$ versus ρ for $\gamma = 4/3$ and we see there is only one fixed point in this case, which is at $\rho = 0$. But for our analysis a fixed point at a non-zero value of ρ is very crucial. So we cannot claim that there is a resolution of the big-bang singularity with this choice of the Rainbow functions.

Another interesting choice of the Rainbow functions was used by the authors in [15]

$$f(E/E_P) = \left(1 + \beta \frac{E}{E_P} \right) e^{-\alpha E^2/E_P^2} \qquad \text{and} \qquad f(E/E_P) = 1 \quad . \tag{17}$$

Here $\alpha = 1/4$ and $\beta = -(4/3)\sqrt{\alpha/\pi}$. This choice is motivated from Noncommutative Geometry models and here we can write the modified conservation equation as

$$\dot{\rho} = -\sqrt{24\pi}\ \frac{\gamma\ \rho^{3/2}\ e^{\alpha\gamma^2 \rho^{\frac{2(\gamma-1)}{\gamma}}}}{\left(1 + \beta\ \gamma\ \rho^{\frac{\gamma-1}{\gamma}} \right)} = \kappa_4(\rho) \quad . \tag{18}$$

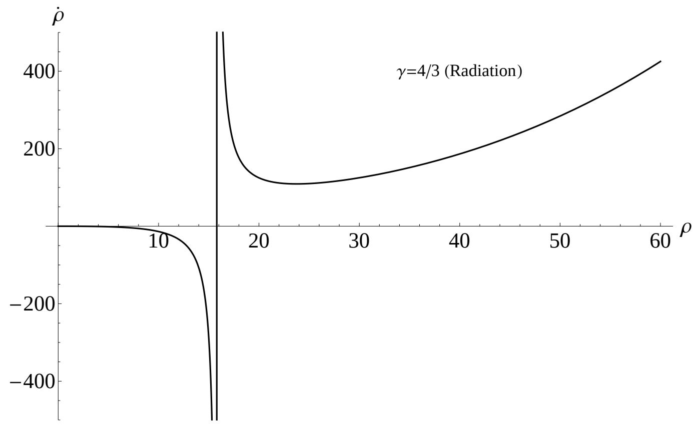

Fig. 4 $\dot{\rho}$ versus ρ for equation (18). Here we have considered $E_P = 1$

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In Fig. (4) we have plotted $\dot{\rho}$ versus ρ for $\gamma = 4/3$. Here we can see that there is a fixed point at $\rho = 0$ but the solution is not continuous at some finite value of ρ and hence our analysis will not hold.

So in the present work we have studied the effect of Rainbow gravity in the cosmological scenario of the very early universe. We have briefly stated the general conditions of the fixed point analysis for having a nonsingular universe in the Rainbow framework. We have restricted our study to four Rainbow functions that are well motivated and available in recent literature. A substantial part of this study is already reported in [22]. In the first two cases the general cosmological solutions reveal the absences of big bang singularities and both solutions are nonsingular. These nonsingular solutions can also be expressed in exact forms. In general the Friedmann equations gets modified in the Rainbow gravity formalism by the so called Rainbow functions. The modified conservation equation allow us to evaluate a relation between the test particle energy and the energy density whose solutions played a major role in studying the singularity. In the first case, we notice that the universe takes infinite amount of time to reach $\rho \to \infty$ and $\rho = 0$ from a finite value of ρ. In the second case, we find the system exhibits two fixed points and the solution takes infinite time to reach the fixed points which represents a non-singular solution. This hints towards an existence of a Rainbow universe which existed forever. In the third choice there is only one fixed point at $\rho = 0$ and the solution is not bounded, so we cannot comment anything on the potential singularity. Similar situation arises with the fourth choice also, where the solution is not continuous and hence our analysis cannot be done. So it is not the case where with any arbitrary choice of the Rainbow functions we can get an ageless universe. The first two choices are in some sense unique for the infinite age of the universe with no big bang. They may be considered with extra care for the study of Greisen-Zatsepin-Kuzmin (GZK) limit of ultra high energy cosmic rays.

There are many strong motivations, at least theoretically, for Lorentz invariance violation at Planck scale. This includes the protection of Higgs mass upto the Planck scale against large radiative corrections in Grand Unified theories. Even if we assume that there is a order one Lorentz invariance violation at the Planck scale and study its consequences with modified dispersion relations or doubly special relativity then we can see that there are unique choices of the Rainbow functions which predicts an infinitely existed universe with no singularities.

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