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# Bell Length as Non-Local Correlation Distance

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**Abstract:** A quantum correlation between two qubits in a Bohmian framework is here investigated. From the analysis of the quantum correctors in entropy it is possible to derive a characteristic length of entanglement which measures the range of a quantum informational gradient, or the maximum degree of de-localization of a system. We have entitled this measure of non-local effects to J. S. Bell.


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## 1 Introduction

On the occasion of de Broglies ninetieth birthday, John Bell wrote in 1982: “But in 1952 I saw the impossible done. It was in papers by David Bohm. Bohm showed explicitly how parameters could indeed be introduced, into non-relativistic wave mechanics, with the help of which the indeterministic description could be transformed into a deterministic one. More importantly, in my opinion, the subjectivity of the orthodox version, the necessary reference to the “observer”, could be eliminated” [1].

With these words, Bell gave start to that “experimental metaphysics” [2] based on the systematic exploration of entanglement processes as well as the new territories of quantum information. All that also implied a greater and greater theoretical and experimental consideration of Bohms version [3, 4, 5], in particular of the central role of the quantum

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potential, which allows an elegant understanding of non-locality.

Since their first consideration, Bell inequalities have become a complex matter (and a PACS number!) because entanglement processes crucially depend on the characteristics of the correlated systems and on the influence of the environment [6]. According to current research, there exist different versions of the quantum entropy and consequently different entropies associated with the quantum potential (see, for a review [3]). In particular, Bohms potential as shown in [7, 8] can be considered, following Hiley, a source of active information whose geometric counterpart indicates a departure from the classical case. Although a more robust formulation of non-locality in terms of quantum field theory has to be developed yet, this entropic reading of the quantum potential allows us to compare the geometries of the Bohm trajectories and the Feynman paths and to connect them to the microstates of the quantum entropy [9]. In this work we take into consideration the entanglement between two quantum rotors (for example, two qbits). Our analysis is based on the role of entropy linked to the quantum potential modifications. We propose a new quantity, the Bell Length, as an indicator of the strength of a non local quantum correlation. In chapter 2 we adopt a definition of quantum entropy and we analyze the underlying informational geometry. In chapter 3 we define a new quantum correlation entropic length ( Bell Length) measuring the degree of the non-local correlations in a quantum system. After analyzing in chapter 4 the well-positive nature of Bell Length and its dependence on coordinates in the case of quantum harmonic oscillator, in chapter 5 we apply the Bell Length to the problem of quantum rotors. In chapter 6 we propose some perspectives introduced by the Bell length in the analysis of spin-spin correlations in an entangled qubit pair. Finally, in chapter 7 we study an appropriate entropic generalization of Bell-CHSH inequalities.

## 2 A Quantum Counterpart of a Boltzmann-type Entropy

In a series of recent works [10, 11, 3] it has been highlighted the utility of a quantum counterpart of a Boltzmann-type entropy:

$$S_Q = -\frac{1}{2}\ln\rho, \tag{1}$$

where $\rho = |\psi(\vec{x},t)|^2$ is the probability density associated with the wave function $\psi(\vec{x},t)$ of the physical system. The quantum entropy (1) can be interpreted as the physical entity that describes the correlation degree of the Bohmian trajectories in the density distribution $\rho$. Starting from the quantum entropy (1), the quantum potential can be expressed as

$$Q = -\frac{\hbar^2}{2m}(\nabla S_Q)^2 + \frac{\hbar^2}{2m}\left(\nabla^2 S_Q\right), \tag{2}$$

i.e. it emerges as an information channel given by the sum of two quantum correctors linked to the quantum entropy. In fact, the two Bohmian equations of motion deriving from the decomposition of the Schrdinger equation describing the motion of the corpuscle associated with the wave function and the continuity equation, respectively become

$$\frac{|\nabla S|^2}{2m} - \frac{\hbar^2}{2m}(\nabla S_Q)^2 + V + \frac{\hbar^2}{2m}\left(\nabla^2 S_Q\right) = -\frac{\partial S}{\partial t} \tag{3}$$

and

$$\frac{\partial S_Q}{\partial t} = -\left(\vec{v} \cdot \nabla S_Q\right) + \frac{1}{2}\nabla \cdot \vec{v} \tag{4}$$

Equation (3) provides an energy conservation law in which the term $-\frac{\hbar^2}{2m}(\nabla S_Q)^2$ can be interpreted as the quantum corrector of the kinetic energy $\frac{|\nabla S|^2}{2m}$ of the particle while the term $\frac{\hbar^2}{2m}\left(\nabla^2 S_Q\right)$ can be interpreted as the quantum corrector of the potential energy V. On the basis of equations (2), (3) and (4), the quantum entropy emerges just as the informational line of the quantum potential, so describing the deformation of the geometry in the presence of quantum effects. We note that the logarithmic form of (1) "exalts" the correlations between "trajectories" or informational lines, which is exactly what it is expected from an entropy.

## 3 The Entropic Bell Length

The geometrical properties of the configuration space associated with the quantum entropy can be also characterized by introducing a quantum-entropic length given by the relation:

$$L_{quantum} = \frac{1}{\sqrt{(\nabla S_Q)^2 - \nabla^2 S_Q}} \tag{5}$$

Let us now consider the origin and the physical meaning of this quantity. In the papers [12, 7] it has been suggested that it is possible to reinterpret quantum mechanics as a manifestation of a noneuclidean breaking with respect to classical physics, and a variational principle has been used in order to obtain a geometrical interpretation of quantum phenomena. In the picture of the Weyl geometry, Novello, Salim and Falciano pointed out that the Bohmian quantum potential can be identified with the curvature scalar of the Weyl integrable space. The fundamental equation of Novellos, Salims and Falcianos approach is

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\nabla S \cdot \nabla S + V - \frac{\hbar^2}{2m}\frac{\nabla^2 \Omega}{\Omega} = 0 \tag{6}$$

where the scalar function $\Omega$ is linked to the curvature through the relation $R = 8\frac{\nabla^2\Omega}{\Omega}$. If one identifies the scalar function $\Omega$ with the amplitude of the wave function, equation (6) is analogous to the quantum Hamilton-Jacobi equation of de Broglie-Bohm. The quantum potential is identified with the curvature scalar in Weyl geometry, namely

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 \Omega}{\Omega} \tag{7}$$

In this model, the inverse square root of the curvature scalar defines a typical length (Weyl length) which can be used to provide a measure of the strength of quantum effects

$$L_W = \frac{1}{\sqrt{R}}, \tag{8}$$

in other words the quantity

$$L_W = \frac{1}{\sqrt{\frac{\nabla^2\Omega}{\Omega}}} \tag{9}$$

is the quantum length.

Taking into account the equivalence between the fundamental equation (6), based on Weyl integrable space, and the fundamental quantum Hamilton-Jacobi equation (3), one obtains:

$$\left[(\nabla S_Q)^2 - \nabla^2 S_Q\right] = \frac{\nabla^2\Omega}{\Omega}. \tag{10}$$

Therefore, the quantum length (8) (or (9), that is the same) may be written as

$$L_{quantum} = \frac{1}{\sqrt{(\nabla S_Q)^2 - \nabla^2 S_Q}} \tag{11}$$

and it depends on the quantum entropy (1). According to equation (5), the quantum entropy may be considered as the ultimate visiting card of the quantum length (9). Relation (8) defines a quantum length that can be used to evaluate the strength of quantum effects by Weyl Geometry. In this picture, Heisenberg's uncertainty principle derives from the fact that we are unable to perform a classical measurement to distances smaller than the quantum length (8). By expressing the quantum length (8) in the form (5), in function of the entropy, this relation tells us something about the quantum system dynamics. In fact, the presence of the two quantum correctors of the energy seems to suggest that (5) is an indicator of non-local correlation. The maximum value of (5) is obtained for $L^{\max}_{quantum} = 1$, which corresponds to the maximum de-localization of a quantum system, that's why the quantity (5) can be called Bell length, in honour of John S. Bell (1928-1990).

## 4 About the Dependence of the Bell Length on the Coordinates and its Physical Meaning

At first glance, the (5) could evoke an idea of "ultraviolet catastrophe". At second glance, indeed, the more general (2) let us see that when the quantum potential is "off" ( classical case), the problem doesnt exist. In particular, by using the Novello, Salim and Falciano model of the quantum potential in terms of the Weyl curvature scalar, the entropic Bell length (5) is equivalent to the geometric length (9), so the denominator can become zero only in the classical case. The denominator of the quantum length (9) may be equal to zero only if $\frac{\nabla^2\Omega}{\Omega} = 0$ namely $\nabla^2\Omega = 0$ and this corresponds, even here, to the classical case! This confirms that in EPR-like situations, the stronger the non-local correlation between two particles will be, the bigger the departure of distribution of the results from the classical one will be. This means that the quantum-entropic length (5) and the quantum length (9) can be considered as consistent and valid measures of the geometrical properties of a quantum system.

However, because the wave function depends on coordinates, also the Bell length depends on them. So we expect some supplementary conditions in order to guarantee its positive definite nature.

This property of the Bell length (5) can be clearly realized by considering, for example, the one-dimensional harmonic oscillator, with the potential $V = \frac{1}{2}m\omega^2x^2$. In this particular case, the quantum Hamilton-Jacobi equation of de Broglie-Bohm theory assumes the form

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S_1)^2 - \frac{\hbar^2}{2m}\left(\frac{\nabla^2 R_1}{R_1}\right) + \frac{1}{2}m\omega^2x^2 = 0, \tag{12}$$

the stationary states are given by $C\left(t\right) = u_n\left(x\right)e^{-iE_nt/\hbar}$ (where $u_n\left(x\right)$ are real functions proportional to Hermite polynomials and $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$, n=0,1,2, is the quantum number associated with each stationary state and the corresponding quantum potential is

$$Q = \left(n + \frac{1}{2}\right)\hbar\omega - \frac{1}{2}m\omega^2x^2 \tag{13}$$

namely

$$Q = -\frac{\hbar^2}{2m}\left[\frac{1}{2\hbar^2}m^2\omega^2x^2 - \left(n + \frac{1}{2}\right)\frac{m}{\hbar}\omega\right] \tag{14}$$

By comparing equations (13) and (2) one obtains

$$(\nabla S_Q)^2 = \frac{1}{2\hbar^2}m^2\omega^2x^2 \tag{15}$$

and

$$\nabla^2 S_Q = \left(n + \frac{1}{2}\right)\frac{m}{\hbar}\omega. \tag{16}$$

The geometrical properties of the configuration space can be characterized by introducing the quantum-entropic length given by the relation

$$L_{quantum} = \frac{1}{\sqrt{\frac{1}{2\hbar^2}m^2\omega^2x^2 - \left(n + \frac{1}{2}\right)\frac{m\omega}{\hbar}}}. \tag{17}$$

The quantum-entropic length (17) turns out to be positive definite for $x \geq \frac{2\hbar}{m\omega}\left(n + \frac{1}{2}\right)$ (which can be considered as its existence condition).

However, one can show here that the interval where the quantum-entropic length of the one-dimensional harmonic oscillator is positive definite actually covers all the physically relevant values of the coordinate x, in other words the Bell length of the one-dimensional harmonic oscillator is always positive definite in all the significant intervals for the coordinate x. In order to show this result, to be more explicit, let us consider a non-dispersive Gaussian-shaped packet given by the following superposition of the stationary wave-functions [13]:

$$\psi\left(x,t\right) = \sum_{n=0}^{\infty} A_n u_n\left(x\right)e^{-iE_nt/\hbar} \tag{18}$$

where

$$A_n = (m\omega/\hbar)^{n/2} a^n (2^n n!)^{-1/2} e^{-m\omega a^2/4\hbar}. \tag{19}$$

In this case the quantum state is

$$\psi(x,t) = (m\omega/\pi\hbar)^{1/4} \exp\left\{ -(m\omega/2\hbar)(x - a\cos\omega t)^2 - \frac{i}{2}\left[\omega t + (m\omega/\hbar)\left(2ax\sin\omega t - \frac{1}{2}a^2\sin 2\omega t\right)\right]\right\} \tag{20}$$

which represents a Gaussian packet centred around $x = a$ at $t = 0$ with half-width $\sigma_0 = \left(\frac{\hbar}{2m\omega}\right)^{1/2}$.

The amplitude function of the state (20) is

$$R(x,t) = (m\omega/\pi\hbar)^{1/4} \exp\left\{-(m\omega/2\hbar)(x - a\cos\omega t)^2\right\} \tag{21}$$

and thus the density is

$$\rho(x,t) = (m\omega/\pi\hbar)^{1/2} \exp\left\{-2(m\omega/2\hbar)(x - a\cos\omega t)^2\right\}. \tag{22}$$

The density of the ensemble of particles (22) describing the harmonic oscillator determines a deformation of the configuration space geometry described by a quantum entropy given by the relation

$$S_Q = -\frac{1}{2}\ln\left[(m\omega/\pi\hbar)^{1/2} \exp\left\{-2(m\omega/2\hbar)(x - a\cos\omega t)^2\right\}\right] \tag{23}$$

namely

$$S_Q = -\frac{1}{4}\ln\left[(m\omega/\pi\hbar)^{1/2}\right] + \frac{m\omega}{2\hbar}(x - a\cos\omega t)^2. \tag{24}$$

In this situation, the quantum potential (13) becomes

$$Q = -\frac{\hbar^2}{2m}\left[\left(\frac{m\omega}{\hbar}(x - a\cos\omega t)\right)^2 - \frac{m\omega}{\hbar}\right] \tag{25}$$

and thus the Bell length (17) may be written as

$$L_{quantum} = \frac{1}{\sqrt{\left(\frac{m\omega}{\hbar}(x - a\cos\omega t)\right)^2 - \frac{m\omega}{\hbar}}}. \tag{26}$$

Equation (26) shows clearly that the Bell length of the harmonic oscillator explicitly depends on the coordinate x. The quantum-entropic length (26) turns out to be positive definite if the coordinate x satisfies the conditions

$$x \le a\cos\omega t - \sqrt{\frac{\hbar}{m\omega}} \text{ or } x \ge a\cos\omega t + \sqrt{\frac{\hbar}{m\omega}}, \tag{27}$$

which can be considered as its existence conditions. Here, one can easily see that the intervals (27) cover all the physically relevant values of the coordinate x for the wave function (20).

The intervals (27) practically indicate that the coordinate $x$ cannot approach value $a\,\cos\omega\,t$ (the classical value) for a distance less than the quantity $\sqrt{\frac{\hbar}{m\omega}}$, which can be seen as a quantum corrector to the classical value. In the attempt to find an interpretation of this quantum corrector, it is convenient to express it in the form:

$$\sqrt{\frac{\hbar}{m\omega}} = \frac{1}{2\pi}\sqrt{\lambda\lambda_c} \tag{28}$$

where $\lambda_c = \frac{\hbar}{mc}$ is the Compton wave-length associated with the particle of mass m . If we consider this length as the physically minimum significant distance in the quantum domain, we can assume that $\lambda$ has to satisfy the inequality $\lambda \geq \lambda_c$. In particular, if we take into consideration the minimal case in which $\lambda = \lambda_c$ we obtain immediately that the quantum corrector (28) reduces to $\frac{\lambda_c}{2\pi}$.

The quantity (28) represents a sort of measure of the average amplitude of the quantum fluctuations around the classical solution $a\,\cos\omega\,t$. As a consequence, one can say that it is not possible to approach the classical solution $a\,\cos\omega\,t$ more than it is expected by the average amplitude (28) because of the uncertainty relations. In other words the quantity (28) measuring the average amplitude of the quantum fluctuations around the classical solution is directly related to the uncertainty in the measure of the position. In this regard, on the basis of Heisenbergs uncertainty principle by expressing $\Delta p$ as $\sqrt{\frac{\hbar\omega}{m}}$ and introducing equation (28), one easily obtains

$$\frac{1}{2\pi}\sqrt{\frac{\hbar\omega}{m}}\sqrt{\lambda\lambda_c} \geq \hbar \tag{29}$$

which yields

$$\lambda \geq \frac{4\pi^2\hbar m}{\omega\lambda_c} \tag{30}$$

namely

$$\lambda\omega \geq \frac{4\pi^2\hbar m}{\lambda_c}. \tag{31}$$

Equation (31) can be considered as a new version of the uncertainty relation, in terms of frequency and wavelength, applied to the one-dimensional harmonic oscillator in the approach based on the Bell length (26). In particular, in the minimal case in which $\lambda = \lambda_c$ the inequality (31) implies

$$\omega \geq \frac{4\pi^2\hbar m}{{\lambda_c}^2}. \tag{32}$$

Roughly speaking, such further condition on the semi-classical value means that the Heisenberg Uncertainty Principle is not just a vagueness on classical values, but indicates a specific and irreducible role of non-locality. Finally, it is that the authentic meaning of Bell Length.

## 5 The Quantum Rotator

As shown in the papers [14-16], quantum entanglement of a qubit pair of spin particles can be described by building an analogy between the quantum treatment of a rigid rotator and a typical system with spin and by studying the behaviour of the quantum potential, which is a formidable instrument of visualization as well as a very simple way to express globally the non-local aspects of the system. As usual, the wave function of such system is $\psi = Re^{iS}$, where $R(\xi)$ and $S(\xi)$ are real functions of Euler angles $\xi = (\alpha, \beta, \gamma)$ specifying the orientation of a rigid body.

The angular momentum $\vec{M}$ is given by a real three-dimensional vector:

$$\vec{M} = i\hat{\vec{M}}S. \tag{33}$$

The dynamics of the spherically symmetric rigid rotor is determined by the Hamilton-Jacobi-type equations with an additional quantum potential Q, namely

$$\hat{H} = \frac{\hat{\vec{M}}^2}{2I} + Q, \tag{34}$$

where

$$Q = \frac{\hat{\vec{M}}^2 R}{2IR} \tag{35}$$

is the quantum potential, I is the moment of inertia. The quantum potential (35) generates a quantum torque

$$\vec{T} = -i\hat{\vec{M}}Q \tag{36}$$

which rotates the angular momentum vector via the equation of motion

$$\frac{d\vec{M}}{dt} = \vec{T} \tag{37}$$

along the trajectory $\xi(t)$.

The quantum potential (35) may be written as

$$Q = \frac{1}{2I}\left[\left(\hat{\vec{M}}S_Q\right)^2 - \left(\hat{\vec{M}}^2 S_Q\right)\right] \tag{38}$$

With respect to equation (2) the interplay of the trajectories density distribution in the element $d^3\zeta$ along the trajectory $\xi(t)$ is here complicated by the presence of the angular momentum. The deformation of the configuration space is given by a quantum torque which has the following expression in terms of the entropy (1):

$$\vec{T} = -\frac{i}{2I}\left[\left(\hat{\vec{M}}S_Q\right)^2 - \left(\hat{\vec{M}}^2 S_Q\right)\right]\vec{M}. \tag{39}$$

From equation (39) one obtains the following equation of motion for the angular momentum:

$$\frac{d\vec{M}}{dt} = -\frac{i}{2I}\left[\left(\hat{\vec{M}}S_Q\right)^2 - \left(\hat{\vec{M}}^2 S_Q\right)\right]\vec{M} \tag{40}$$

along the trajectory $\xi(t)$.

In this case the Bell de-localization length is given by:

$$L_{quantum} = \frac{1}{\sqrt{\frac{1}{2I}\left(\left(\hat{\vec{M}}^2 S_Q\right) - \left(\hat{\vec{M}} S_Q\right)^2\right)}}. \tag{41}$$

Once the Bell length (41) becomes non-negligible the rigid rotator goes into a quantum regime.

## 6 Two Q-bit System Correlations

In order to analyse a two qubit system, let us start by considering the usual wave function:

$$|\psi\rangle = \cos\frac{\vartheta}{2}|\uparrow\downarrow\rangle + e^{i\phi}\sin\frac{\vartheta}{2}|\downarrow\uparrow\rangle \tag{42}$$

where $|\uparrow\downarrow\rangle$ corresponds to the state of the system when the first qubit is in the up state, namely in the direction of the z-axis, and the second qubit is in the down state, while $|\downarrow\uparrow\rangle$ corresponds to the state of the system when the first qubit is in the down state and the second qubit is in the up state.

In a Bohmian framework [17] the guiding wave function

$$\psi(\xi) = \cos\frac{\vartheta}{2}u_\uparrow(\xi_1)\,u_\downarrow(\xi_2) + e^{i\phi}\sin\frac{\vartheta}{2}u_\downarrow(\xi_1)\,u_\uparrow(\xi_2) \tag{43}$$

is given in a six-dimensional space $\xi = \{\xi_1, \xi_2\}$, where $\xi_1$, $\xi_2$ are the coordinates of the two rotators . The Hamiltonian for this system is:

$$H = \frac{\vec{M}_1^2 + \vec{M}_2^2}{2I} + Q \tag{44}$$

where

$$Q = \frac{\left(\hat{M}_1^2 + \hat{M}_2^2\right)R}{2IR} \tag{45}$$

is the quantum potential. By using the entropy (1), the quantum potential (45) may be expressed as

$$Q = \frac{1}{2I}\left[\left(\hat{\vec{M}}_1 S_Q\right)^2 + \left(\hat{\vec{M}}_2 S_Q\right)^2 - \left(\hat{\vec{M}}_1^{\,2} S_Q\right) - \left(\hat{\vec{M}}_2^{\,2} S_Q\right)\right]. \tag{46}$$

Moreover, the deformation of the geometry associated with the entangled qubit pair can be described by the following Bell length:

$$L_{quantum} = \frac{1}{\sqrt{\frac{1}{2I}\left(\left(\hat{\vec{M}}_1^{\,2} S_Q\right) - \left(\hat{\vec{M}}_1 S_Q\right)^2 + \left(\hat{\vec{M}}_2^{\,2} S_Q\right) - \left(\hat{\vec{M}}_2 S_Q\right)^2\right)}} \tag{47}$$

and can be seen as a consequence of the fact that the quantum entropy generates the quantum torques

$$\vec{T}_1 = -\frac{i}{2I}\left[\left(\hat{\vec{M}}_1 S_Q\right)^2 + \left(\hat{\vec{M}}_2 S_Q\right)^2 - \left(\hat{\vec{M}}_1^{\,2} S_Q\right) - \left(\hat{\vec{M}}_2^{\,2} S_Q\right)\right]\vec{M}_1 \tag{48}$$

and

$$\vec{T}_2 = -\frac{i}{2I}\left[\left(\hat{\vec{M}}_1 S_Q\right)^2 + \left(\hat{\vec{M}}_2 S_Q\right)^2 - \left(\hat{\vec{M}}_1^{\,2} S_Q\right) - \left(\hat{\vec{M}}_2^{\,2} S_Q\right)\right]\vec{M}_2. \tag{49}$$

The total angular momentum projection $M_{1z} + M_{2z}$ of the entangled qubit pair is zero while the angular momenta due to the action of the non-local quantum potential (45) and the corresponding quantum torques (48) and (49) exhibit a complex precessional motion.

In the paper [17] Ramsak shows that two relevant quantities which characterize in numerical terms the entanglement are the probability distribution

$$\frac{dP(\varphi)}{d\varphi} = \int \delta\left[\varphi - \varphi(\xi)\right] R^2(\xi)\, d\xi \tag{50}$$

of the ensemble average difference of azhimutal angles $\varphi\left[\xi(t)\right] = \varphi_2 - \varphi_1$ and the probability distribution of the average cosine $\frac{dP\langle\cos(\varphi-\phi)\rangle}{d\langle\cos(\varphi-\phi)\rangle}$. The probability distribution (50) is constant for unentangled qubits while grows with increasing entanglement. The probability distribution of the average cosine $\frac{dP\langle\cos(\varphi-\phi)\rangle}{d\langle\cos(\varphi-\phi)\rangle}$ ranges from zero for unentangled quibits to unity for maximal entanglement. In particular, in the entropic approach here proposed, the peaking of the probability distribution (50) is linked with the precessional motion of the quantum torques (48) and (49) and corresponds itself to the maximum value of the Bell length (47). In the light of the probability distribution of the average cosine, one can also introduce here another relevant parameter characterizing the entanglement defined by relation

$$C_B = 2\left|\langle\psi|\,\vec{T}_1 \cdot \vec{T}_2\,|\psi\rangle\right| = |\sin\vartheta|\,. \tag{51}$$

In fact, for $C_B \to 1$ there is the maximum correlation between the two qubits, whilst for $C_B \to 0$ the entanglement tends to disappear.

## 7 The Bell length as Non-Local Correlation

Let us analyse in detail in what sense the Bell length (47) of an entangled two qubits system provides a measure of the correlation between the two qubits. We will follow and deepen our analysis delineated in [18;19].

In this regard, Hall [20] recently suggested a qubit quantum correlation distance which also provides a direct entanglement criterion given by the relation

$$C(\rho_{AB}) > 2\sqrt{\left(1 - Tr\left[\rho_A^2\right]\right)\left(1 - Tr\left[\rho_B^2\right]\right)} \tag{52}$$

(where $\rho_{AB}$ is the joint density operator, $\rho_A$ is the density operator of the first qubit, $\rho_B$ is the density operator of the second qubit) which leads to the following lower bound for the quantum mutual information shared by the two qubits A and B

$$I\left(\rho_{AB}\right) \geq \begin{cases} \log 2 - H\left(\frac{1+C(\rho_{AB})}{2}, \frac{1-C(\rho_{AB})}{2}\right),\ C\left(\rho_{AB}\right) \leq 0,72654 \\ \log 4 - H\left(\frac{1}{4} + \frac{C(\rho_{AB})}{2}, \frac{1}{4} - \frac{C(\rho_{AB})}{6}, \frac{1}{4} - \frac{C(\rho_{AB})}{6}, \frac{1}{4} - \frac{C(\rho_{AB})}{6}\right),\ C\left(\rho_{AB}\right) > 0,72654 \end{cases} \tag{53}$$

(where H is the Shannon entropy of the probability distribution of the system of the two qubits under consideration) and showed that the lower bound of the quantum correlation distance (52) simulates a Bell inequality violation. Here, the correlation between the two qubits is the Bell length (47), which indeed plays the role of Halls quantum correlation distance (52). By using equation (52) and taking into account that, in terms of quantum entropy,$\rho_A = \exp\left(-2S_{Q_A}\right)$ and $\rho_B = \exp\left(-2S_{Q_B}\right)$, the strong condition for the entanglement between the two qubits here becomes

$$\frac{1}{\sqrt{\frac{1}{2I}\left(\left(\hat{\vec{M}}_1^{\,2} S_Q\right) - \left(\hat{\vec{M}}_1 S_Q\right)^2 + \left(\hat{\vec{M}}_2^{\,2} S_Q\right) - \left(\hat{\vec{M}}_2 S_Q\right)^2\right)}}$$
$$> 2\sqrt{\left(1 - Tr\left[\left(\exp\left(-2S_{Q_A}\right)\right)^2\right]\right)\left(1 - Tr\left[\left(\exp\left(-2S_{Q_B}\right)\right)^2\right]\right)} \tag{54}$$

which may be expressed also as

$$\frac{1}{\sqrt{\frac{1}{2I}\left(\left(\hat{\vec{M}}_1^{\,2} S_Q\right) - \left(\hat{\vec{M}}_1 S_Q\right)^2 + \left(\hat{\vec{M}}_2^{\,2} S_Q\right) - \left(\hat{\vec{M}}_2 S_Q\right)^2\right)\left(1 - Tr\left[\left(\exp\left(-2S_{Q_A}\right)\right)^2\right]\right)\left(1 - Tr\left[\left(\exp\left(-2S_{Q_B}\right)\right)^2\right]\right)}} > 2. \tag{55}$$

The Bell length (47) leads us to define a tight lower bound for the quantum mutual information shared by two qubits, which is given by the following relation:

$$I(L_{quantum}) \geq \begin{cases} \log 2 - S_Q\left(\frac{1+L_{quantum}}{2}, \frac{1-L_{quantum}}{2}\right),\ L_{quantum} \leq 0,72654 \\ \log 4 - S_Q\left(\frac{1}{4} + \frac{L_{quantum}}{2}, \frac{1}{4} - \frac{L_{quantum}}{6}, \frac{1}{4} - \frac{L_{quantum}}{6}, \frac{1}{4} - \frac{L_{quantum}}{6}\right),\ L_{quantum} > 0,72654 \end{cases} \tag{56}$$

here $I\left(L_{quantum}\right) = S_{Q_A} + S_{Q_B} - S_{Q_{AB}}$. For $L_{quantum} \leq 0,72654$ only entangled states can reach this lower bound, while for $L_{quantum} > 0,72654$, one obtains that only one of the reduced states is maximally mixed. In the light of equations (54)-(56) one can say that the Bell length (47) can be considered as the ultimate visiting card determining the quantum correlations, in analogy with Hall's quantum correlation distance (52).

In the entropic approach, on the basis of the usual assumptions characterizing quantum correlations (namely no signalling faster than light speed, free choice of measurement settings and independence of local outcomes), the standard Bell-CHSH inequalities

$$CHSH \equiv \langle AB \rangle + \langle AB' \rangle + \langle A'B \rangle - \langle A'B' \rangle \leq 2 \tag{57}$$

where A and B are of course two random-valued variables with values $\pm 1$ ,can be generalized as

$$\langle AB \rangle + \langle AB' \rangle + \langle A'B \rangle - \langle A'B' \rangle \leq \frac{4}{2 - L^{\max}_{quantum}} \tag{58}$$

where $L^{\max}_{quantum}$ is the maximum value of the Bell length for all A and B. The relation (58) leads to introduce an entropic length Bell inequality of the form

$$CHSH_{entropic} \leq 0 \tag{59}$$

where

$$CHSH_{entropic} \equiv \langle AB \rangle + \langle AB' \rangle + \langle A'B \rangle - \langle A'B' \rangle - \frac{4}{2 - L^{\max}_{quantum}}. \tag{60}$$

In correspondence to this entropic length Bell inequality (59), the observers share a minimum mutual information of

$$I_{\min} = \log 2 - S_Q \left( \frac{1 + L^{\max}_{quantum}}{2}, \frac{1 - L^{\max}_{quantum}}{2} \right) \geq \log 2 - S_Q \left( \frac{2 + 3V}{4 + 2V}, \frac{2 - V}{4 + 2V} \right). \tag{61}$$

for some $V > 0$, where

$$L^{\max}_{quantum} \geq \frac{2V}{2 + V}. \tag{62}$$

The mutual information (61) reduces to zero in the limit of no violation of Bell inequality, i.e. when V=0, and reaches a maximum of 1 bit of information in the limit of the maximum possible violation, V=2, namely for $L^{\max}_{quantum} = 1$, which is the limit value of the Bell length, beyond which the de-correlation between the two qubits begins.

The violation of the entropic length Bell inequality (59) essentially coincides with the one of the standard CHSH inequality. When V=0, there is no violation of the Bell inequality, and the mutual information becomes equal to 0. When V=2, namely $L^{\max}_{quantum} = 1$, one obtains the maximal violation of the entropic length Bell inequality (59), which corresponds to the maximum value of 1 bit of the mutual information (61). As regards the entangled state (42), this maximal violation of (59) is obtained for $\vartheta = \frac{\pi}{2}$, on which one gets $CHSH_{entropic} \approx +0,237$. For other values of $\vartheta$, the maximal violation of (59) when optimized over the measurements, is characterized by the exact same features as for the standard Bell-CHSH inequality (57). However, it must be emphasized that, in general, for the standard Bell-CHSH scenario, the violation of the standard inequality (57) is a necessary but not sufficient condition for the violation of (66). An important merit of the entropic length Bell inequality (59) is to derive directly its degrees of violations from the different values of the Bell length: in this approach, different types of non-local correlations exist as a consequence of different values of the Bell length, in agreement with the recent proposal that all entangled quantum states are non-local [21].

## 8 Conclusions

In 1964 John Stewart Bell, by following David Bohms footsteps, showed that nonlocality is not an unexpected host, but is rather the keystone of the quantum world [1]. Today, half a century since from then, that theoretical debate is the fundamental guide for the exploration of quantum information [6].

In this work we have taken into consideration the quantum correlation between two qubits in a Bohmian framework, in particular we have focused our attention towards the entropic aspects of the quantum potential expressed by equation (2). Bohms quantum potential offers a vivid image of the informational configuration of a quantum system through the trajectories. From the analysis of the quantum correctors in equation (2), it is possible to derive a characteristic length of entanglement phenomena expressed by the general equation (5) for non-relativistic quantum mechanics and by equation (41) in the case of the quantum torques which measures the range of a quantum informational gradient, or the maximum degree of de-localization of a system. We have entitled this measure of non-local effects to J. S. Bell.

The maximum correlation in the two qubits system described by the general state (42), is determined by the limit value 1 of the Bell length. On the other hand, the maximum degree of entanglement corresponds also to the limit value 1 of the probability distribution of the average cosine (53) (or (54), that is the same). This implies that there is a fundamental link between the Bell length and the probability distribution of the average cosine (53). In particular, on the basis of equation (54), one can say that the peaking of the entanglement corresponding to the maximum value of the Bell length $L^{\max}_{quantum} = 1$ is associated with the precessional motion of the quantum torques (48) and (49). In other words, one can say that the geometrical properties associated with the quantum potential (46), expressed by the Bell length (47), generate the quantum torques (48) and (49) whose precessional motions lead to a peaking of the probability distribution (59) of the ensemble average difference of azimuthal angles in correspondence of the maximum value of the Bell length $L^{\max}_{quantum} = 1$. Finally, through the Bell length it is possible to introduce a generalization of Bell inequalities, which takes explicitly account of the physical complexity of the interplay between information and entropy in a quantum system.

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## References

[1] J. S. Bell, On the Impossible Pilot Wave, Found. Phys. 12(10), 1982, 989-999, reprinted in *Speakable and Unspeakable in Quantum Mechanics*, Cambridge University Press; 2 edition (June 28, 2004).

[2] Robert S. Cohen, M. Horne, J.J. Stachel (eds) Experimental Metaphysics, *Quantum Mechanical Studies for Abner Shimony*, Volume One, Boston Studies in the Philosophy and History of Science, 193, Springer, 1997.

[3] I. Licata, D. Fiscaletti, *Quantum Potential. Physics, Geometry, Algebra*, Springer, Heidelberg, 2014.

[4] D. Drr, S. Goldstein, N. Zanghi, *Quantum Physics Without Quantum Philosophy*, Springer, Heidelberg, 2013

[5] X. Oriols, J. Mompart, *Applied Bohmian Mechanics: From Nanoscale Systems to Cosmology*, Pan Stanford Publishing, 2012

[6] V. Scarani (2012), The device-independent outlook on quantum physics, *Acta Physica Slovaca*, **62**, 4, 347387.

[7] G. Resconi, I. Licata and D. Fiscaletti (2013) Unification of Quantum and Gravity by Non Classical Information Entropy Space, *Entropy*, **15**, 3602-3619.

[8] D. Fiscaletti and I. Licata (2012), Weyl geometry, Fisher information and quantum entropy in quantum mechanics. *International Journal of Theoretical Physics*, **51**, 11, 3587-3595.

[9] I. Licata and D. Fiscaletti (2014), Bohm trajectories and Feynman paths in light of quantum entropy, *Acta Physica Polonica B*, **45**, 4, 885-904, 2014.

[10] V. I. Sbitnev (2008) Bohmian split of the Schrodinger equation onto two equations describing evolution of real functions, *Kvantovaya Magiya*, **5**, 1, 1101-1111.

[11] D. Fiscaletti (2012) The quantum entropy as an ultimate visiting card of de Broglie-Bohm theory, *Ukr. J. Phys.*, **57**, 9, 946-963.

[12] M. Novello, J. M. Salim, and F. T. Falciano (2011) On a geometrical description of Quantum Mechanics, *Int. J. Geom. Methods Mod. Phys.*, **08**, 87.

[13] L. I. Schiff, *Quantum Mechanics*, McGraw-Hill, New York, 1968.

[14] P. R. Holland, *The quantum theory of motion*, Cambridge University Press, 1995.

[15] P. R. Holland (1988) Causal interpretation of a system of two spin-1/2 particles, *Phys. Rep.*, **169**, 5, 293327.

[16] A. Ramsak (2011) Geometrical view of quantum entanglement, *Eur. Phys. Lett.*, **96**, 40004.

[17] A. Ramsak (2012) Spinspin correlations of entangled qubit pairs in the Bohm interpretation of quantum mechanics , *J. Phys. A: Math. Theor.*, **45**, 115310.

[18] I. Licata, D. Fiscaletti (2014) Bell Length as Mutual Information in Quantum Interference, *Axioms* 3(2), 153-165

[19] Fiscaletti, D., Licata, I. (2014) Bell Length in the Entanglement Geometry, *Int. Jour. Theor. Phys.*,**54**(7): 2362-2381

[20] M. W. Hall (2013) Correlation distance and bounds for mutual information, *Entropy*, **15**, 3698-3713

[21] F. Buscemi (2012) All entangled quantum states are nonlocal, *Physical Review Letters*, **108**, 200401-1.