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Light from Black Holes and Uncertainty in Quantum Gravity

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Abstract: A recent work of Ali, Khali and Vagenas has shown that the existence of a minimal length and the subsequent existence of a Generalized Uncertainty Principle (GUP) modifies the Hawking temperature of black holes (BHs). A previous work of one of us (C. Corda) introduced a BH effective state which takes into account the BH's back reaction for the emission of Hawking radiation obtaining a non-strictly black body spectrum from the tunnelling mechanism corresponding to the famous probability of emission of an outgoing particle found by Parikh and Wilczek. In the present work, we modify the effective temperature by taking into account the new result by Ali, Khali and Vagenas and, by using Hawking's periodicity arguments, we write down the corresponding modified (by the GUP) effective metric. Thus, we obtain further corrections to the non-strictly thermal BH radiation spectrum as the final distributions take into account both the BH dynamical geometry during the emission of the particle and the correction to the Hawking temperature due to the GUP.

Keywords: Quantum Tunnelling, Hawking radiation, GUP corrected effective temperature, non-thermal spectrum
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The existence of a minimal lenght at the Planck scale seems to be a model-independent feature of various theories of quantum gravity [1]. That existence leads, in turn, to a

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modification of the standard uncertainty principle of quantum mechanics to a GUP, see [2] and references within. Such a modification includes an additional quadratic term in momentum [1, 2] (we use Planck units $G = c = k_B = \hbar = \frac{1}{4\pi\epsilon_0} = 1$ in this paper)

$$\triangle x \triangle p \geq \frac{1 + \beta \triangle p^2}{2}, \tag{1}$$

where β is a dimensionless constant. Another interesting form of the GUP, which seems consisten with doubly special relativity, string theory, and black hole physics, has been recently proposed in [3, 4, 5]. This new approach to the GUP shows a linear term in momentum and leads to a maximum observable momentum in addition to a minimal length [2, 3, 4, 5]

$$[x_i p_j] = i \left[\delta_{ij} - \alpha \left(p\delta_{ij} + \frac{p_i p_j}{p} \right) + \alpha^2 \left(p\delta_{ij}^2 + 3p_i p_j \right) \right], \tag{2}$$

where α is a dimensionless constant which upper bounds have been calculated in [3]. What is important for the goals of the present work is that the GUP of eq. (2) generates a modification of the Hawking temperature (*GUP modified Hawking temperature*) from its standard value [6]

$$T_H = \frac{1}{8\pi M}, \tag{3}$$

to its GUP corrected value [2]

$$T_H^{(GUP)} = \frac{1}{8\pi M} \left[1 - \frac{\alpha}{8\pi M} + 5 \left(\frac{\alpha}{8\pi M} \right)^2 \right]. \tag{4}$$

We can introduce *a GUP modified BH mass and a GUP modified horizon radius* as

$$M^{(GUP)} \equiv \frac{M}{\left[1 - \frac{\alpha}{8\pi M} + 5\left(\frac{\alpha}{8\pi M}\right)^2\right]}, \quad r^{(GUP)} \equiv 2M^{(GUP)} \tag{5}$$

Thus, eq. (4) reads

$$T_H^{(GUP)} = \frac{1}{8\pi M^{(GUP)}}. \tag{6}$$

Let us consider Hawking radiation [6] in the tunnelling approach [7] - [18]. The particle creation mechanism caused by the vacuum fluctuations near the BH horizon can be descripted as follows. A virtual particle pair is created just inside the horizon and the virtual particle with positive energy can tunnel out the BH horizon as a real particle. Otherwise, the virtual particle pair is created just outside the horizon and the negative energy particle can tunnel inwards. Thus, for both the possibilities, the particle with negative energy is absorbed by the BH and as a result the mass of the BH decreases. The flow of positive energy particles towards infinity is considered as Hawking radiation. Earlier, this approach was limited to obtain only the Hawking temperature through a comparison of the probability of emission of an outgoing particle with the Boltzmann factor rather than the actual radiation spectrum with the correspondent distributions. This problem was formally addressed by Banerjee and Majhi [7]. By a slightly different formulation of the tunnelling formalism, they were able to directly reproduce the black body spectrum for either bosons or fermions from a BH with standard Hawking temperature. However, considering contributions beyond semiclassical approximation in the tunnelling process, Parikh and Wilczek [9, 10] found a probability of emission compatible with a non-thermal spectrum of the radiation from BH. This non precisely thermal character of the spectrum is important to resolve the BH information loss puzzle [19] because arguments that information is lost during Hawking's BH quantum evaporation partially rely on the assumption of strict thermal behavior of the radiation spectrum [10, 19]. Two interesting approaches to resolve the BH information puzzle have been recently proposed in [20, 21]

The important difference between the works [9, 10] and the work [7] is consideration or non-consideration of the energy conservation. As a result, there will be a dynamical [9, 10] or static [7] BH geometry. In fact, due to conservation of energy, in [9, 10] the BH horizon contracts during the radiation process which deviates from the perfect black body spectrum. In the language of the tunnelling mechanism, a trajectory in imaginary or complex time joins two separated classical turning points [10]. The key point is that the forbidden region traversed by the emitting particle has a *finite* size[10] from $r = r_{initial}$ to $r = r_{final}$ ($r_{initial}$ is the radius of the horizon of the BH initially and r_{final} is the radius of the horizon of the BH after particle emission). This finite size implies a discrete nature of the tunnelling mechanism, which is characterized by the physical state before the emission of the particle and that after the emission of the particle. As a result, the radiation spectrum is also discrete [16, 21]. Consequently, particle emission can be interpreted like a quantum transition of frequency ω between the two discrete states [16, 21]. It is the particle itself which generates a tunnel through the horizon [16, 21] having finite size. In thermal spectrum, the tunnelling points have zero separation, so there is no clear trajectory because there is no barrier [10, 16, 21].

The tunnelling probability in strictly thermal approximatipon is given by [6, 9, 10]

$$\Gamma \sim \exp\left(-\frac{\omega}{T_H}\right), \tag{7}$$

where T_H is the Hawking temperature of eq. (3) and ω is the energy-frequency of the emitted radiation. However, considering contributions beyond semiclassical approximation and taking into account the conservation of energy, the tunnelling probability can be reformulated as [9, 10]

$$\Gamma \sim \exp[-\frac{\omega}{T_H}(1-\frac{\omega}{2M})] \quad \Longrightarrow \quad \Gamma = \alpha \exp[-\frac{\omega}{T_H}(1-\frac{\omega}{2M})], \tag{8}$$

where $\alpha \sim 1$ and the additional term $\frac{\omega}{2M}$ is present. This non-thermal spectrum enables the introduction of an intriguing way to consider the BH dynamical geometry through the *BH effective state.* In fact, one introduces the *effective temperature* as [16, 21]

$$T_E(\omega) \equiv \frac{2M}{2M-\omega} T_H = \frac{1}{4\pi(2M-\omega)}, \tag{9}$$

which permits to rewrite the probability of emission (2) in Boltzmann-Hawking form as [16, 21]

$$\Gamma \sim \exp[-\beta_E(\omega)\omega] = \exp(-\frac{\omega}{T_E(\omega)}), \tag{10}$$

where the effective Boltzmann factor takes the form [16, 21]

$$\beta_E(\omega) \equiv \frac{1}{T_E(\omega)}. \tag{11}$$

One interpretes the effective temperature as the temperature of a black body emiting the same total amount of radiation [16, 21]. Hence, one replaces the Hawking temperature with the effective temperature in the equation of the probability of emission. The ratio $\frac{T_E(\omega)}{T_H} = \frac{2M}{2M-\omega}$ stands for the deviation of the radiation spectrum of a BH from the strictly thermal feature [16, 21]. It is better to further clarify the definition of effective temperature that has been introduced in BH physics in [22, 23] for the Schwarzschild BH, in [24] for the Kerr BH and in [25] for the Reissner-Nordstrm BH. The probability of emission of Hawking quanta found by Parikh and Wilczek, i.e. eq. (8), shows that the BH does NOT emit like a perfect black body, i.e. it has not a strictly thermal behavior. On the other hand, the temperature in Bose-Einstein and Fermi-Dirac distributions is a perfect black body temperature. Thus, when we have deviations from the strictly thermal behavior, i.e. from the perfect black body, one expects also deviations from Bose-Einstein and Fermi-Dirac distributions. One attacks this problem by analogy with other various fields of Science, also beyond BHs, for example the case of planets and stars. One defines the effective temperature of a body such as a star or planet as the temperature of a black body that would emit the same total amount of electromagnetic radiation [21, 26]. The importance of the effective temperature in a star is stressed by the issue that the effective temperature and the bolometric luminosity are the two fundamental physical parameters needed to place a star on the Hertzsprung–Russell diagram [21, 26]. Both effective temperature and bolometric luminosity actually depend on the chemical composition of a star, see again [21, 26].

Further, in analogy with the effective temperature, one can define the *effective mass* and the *effective horizon radius* as [16], [21] - [25]

$$M_E = M - \frac{\omega}{2} \qquad and \qquad r_E = 2M_E = 2M - \omega. \tag{12}$$

Note that these effective quantities are nothing but the average value of the corresponding quantities before (initial) and after (final) the particle emission (i.e., $M_i = M$, $M_f = M - \omega$; $r_i = 2M_i$ and $r_f = 2M_f$) [16], [21] - [25]. Accordingly, T_E is the inverse of the average value of the inverses of the initial and final Hawking temperatures [16], [21] - [25]. Hence, there is a discrete character (in time) of the Hawking temperature. Thus, the effective temperature may be interpreted as the Hawking temperature *during* the emission of the particle [16], [21] - [25].

Hawking's periodicity argument [27, 28], which enables to derive a corresponding line element from a corrected temperature, has been used in [16, 17] to obtain the *effective*

Schwarzschild line element

$$ds_E^2 = -(1 - \frac{2M_E}{r})dt^2 + \frac{dr^2}{1 - \frac{2M_E}{r}} + r^2(\sin^2\theta d\varphi^2 + d\theta^2), \tag{13}$$

which takes into account the BH *dynamical* geometry during the emission of the particle.

The BH effective state has been recently used one of us (C. Corda), in an important series of papers [21], [29] - [33] which improved the pioneering works [21, 22], to discuss a connection between Hawking radiation and the BH quasi-normal modes (QNMs) which is important in the route to quantize gravity because one can naturally interpret the BH QNMs in terms of quantum levels. Such a model of quantum BH is somewhat similar to the historical semi-classical model of the structure of a hydrogen atom introduced by Bohr in 1913 [34, 35]. In a certain sense, the BH QNMs "triggered" by Hawking quanta represent the "electron" which jumps from a level to another one and the absolute values of the QNMs frequencies represent the energy "shells" [21], [29] - [33]. In Bohr model electrons can only gain and lose energy by jumping from one allowed energy shell to another, absorbing or emitting radiation with an energy difference of the levels according to the Planck relation $E = hf$, where h is the Planck constant and f the transition frequency. In the Bohr-like BH model of [21], [29] - [33], the the BH QNMs can only gain and lose energy by jumping from one allowed energy shell to another, absorbing or emitting radiation (emitted radiation is given by Hawking quanta) with an energy difference of the levels according to eq. (12) of [29]. The similarity is completed if one notes that the interpretation of a QNM is of a particle, the "electron ", quantized on a circle of length inversely proportional to the introduced effective temperature [21], [29] - [33]. Another important result on this issue [21], proposes a new, independent solution to the BH information paradox. In fact, the time evolution of the model is governed by a time dependent Schrodinger equation for the system composed by Hawking radiation and BH QNMs. The physical state and the correspondent wave function are written in terms of an unitary evolution matrix instead of a density matrix [21]. Thus, the final state results to be a pure quantum state instead of a mixed one while emitted energies result entangled with BH QNMs [21].

Through the introduction of the above discussed BH effective state, a non-strictly black body spectrum from the tunnelling mechanism corresponding to the probability of emission of an outgoing particle found by Parikh and Wilczek can be obtained [16]. The final non-strictly thermal distributions which take into account the BH dynamical geometry are [16, 21, 29]

$$\begin{gathered} < n >_{boson} = \frac{1}{\exp[4\pi(2M-\omega)\omega]-1} \\ < n >_{fermion} = \frac{1}{\exp[4\pi(2M-\omega)\omega]+1}. \end{gathered} \tag{14}$$

Another intriguing result of the Bohr-like BH model in [21], [29] - [33] is that, although in general it is well approximated by 1, the pre-factor of the Parikh and Wilczek probability of emission of eq. (8) depends on the BH quantum level n if one assumes the unitarity of the BH quantum evaporation [29]. In that case, it has been rigorously shown in [29] that the correct value of the pre-factor in eq. (8) is

$$\alpha \equiv \alpha_n = \frac{1 - \exp\left[-2\pi\right]}{1 - \exp\left[-2\pi\left(n_{max} - n + 1\right)\right]}, \tag{15}$$

where [29]

$$n_{max} = 2(M^2 - 1) \tag{16}$$

is the maximum value of the principal quantum number n, see [29] for details. Thus, eq. (8) is replaced by

$$\Gamma = \left\{\frac{1 - \exp\left[-2\pi\right]}{1 - \exp\left[-2\pi\left(n_{max} - n + 1\right)\right]}\right\}\exp[-\frac{\omega}{T_H}(1 - \frac{\omega}{2M})], \tag{17}$$

which, together with eqs. (14) finalizes the analysis of [9, 10].

Now, we further modify the effective temperature by incorporating the GUP correction to the Hawking temperature of eq. (4). As a result, the quantum physics of BHs will be further modified. In fact, following [16, 17], one can again use Hawking's periodicity argument [27, 28] to modify the metric as follows. The euclidean form of the line element will be given by

$$\left[ds^{(GUP)}\right]^2 = x^2\left[\frac{d\tau}{4M\left(1 - \frac{\omega}{2M}\right)}\right]^2 + \left(\frac{r}{r^{(GUP)}}\right)^2 dx^2 + r^2(\sin^2\theta d\varphi^2 + d\theta^2), \tag{18}$$

which is regular at $x = 0$ and $r = r^{(GUP)}$. τ is treated as an angular variable with period $\beta^{(GUP)}(\omega)$ [16, 17, 28]. Replacing the quantity $\sum_i \frac{\beta_i}{M^{2i}}$ in [28] with the quantity $-\frac{\omega}{2M}$, if one follows step by step the detailed analysis in [28] at the end one easily gets the *GUP modified Schwarzschild line element*, which takes the form

$$\left[ds^{(GUP)}\right]^2 = -(1 - \frac{2M^{(GUP)}}{r})dt^2 + \frac{dr^2}{1 - \frac{2M^{(GUP)}}{r}} + r^2(\sin^2\theta d\varphi^2 + d\theta^2). \tag{19}$$

By recalling that $\kappa \equiv \frac{1}{4M}$ is the BH surface gravity, one can introduce the *GUP modified surface gravity*

$$\kappa^{(GUP)} \equiv \frac{1}{4M^{(GUP)}}, \tag{20}$$

that, using eq. (5) becomes

$$\kappa^{(GUP)} = \frac{\left[1 - \frac{\alpha}{8\pi M} + 5\left(\frac{\alpha}{8\pi M}\right)^2\right]}{4M}. \tag{21}$$

Eq. (19) enables one to define the *GUP corrected effective temperature*

$$T_E^{(GUP)}(\omega) \equiv \frac{2M^{(GUP)}}{2M^{(GUP)} - \omega}T_H^{(GUP)} = \frac{1}{4\pi(2M^{(GUP)} - \omega)}, \tag{22}$$

the *GUP corrected effective Boltzmann factor*

$$\beta_E^{(GUP)}(\omega) \equiv \frac{1}{T_E^{(GUP)}(\omega)} \tag{23}$$

and the *GUP corrected effective mass* and *effective horizon* radius

$$M_E^{(GUP)} = M^{(GUP)} - \frac{\omega}{2} \qquad \textit{and} \qquad r_E^{(GUP)} = 2M_E^{(GUP)} = 2M^{(GUP)} - \omega. \tag{24}$$

We will use again Hawking's periodicity argument [27, 28] following [16, 17]. Now, the euclidean form of the metric will be given by

$$\left[ds_E^{(GUP)}\right]^2 = x^2 \left[\frac{d\tau}{4M^{(GUP)}\left(1 - \frac{\omega}{2M^{(GUP)}}\right)}\right]^2 + \left(\frac{r}{r_E^{(GUP)}}\right)^2 dx^2 + r^2(\sin^2\theta d\varphi^2 + d\theta^2), \tag{25}$$

which is regular at $x = 0$ and $r = r_E^{(GUP)}$. Again, τ is treated as an angular variable with period $\beta_E^{(GUP)}(\omega)$ [16, 17, 28]. Replacing the quantity $\sum_i \frac{\beta_i}{M^{2i}}$ in [28] with the quantity $-\frac{\omega}{2M^{(GUP)}}$, if one again follows step by step the detailed analysis in [28], at the end the *GUP corrected effective Schwarzschild line element* is easily obtained

$$\left[ds_E^{(GUP)}\right]^2 = -(1 - \frac{2M_E^{(GUP)}}{r})dt^2 + \frac{dr^2}{1 - \frac{2M_E^{(GUP)}}{r}} + r^2(\sin^2\theta d\varphi^2 + d\theta^2). \tag{26}$$

One also easily shows that $r_E^{(GUP)}$ in eq. (25) is the same as in eq. (24). Thus, the line element of eq. (26) takes into account both the BH dynamical geometry during the emission of the particle and the GUP correction to the Hawking temperature.

Starting from the standard Schwarzschild line element [7, 16, 17]

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(\sin^2\theta d\varphi^2 + d\theta^2), \tag{27}$$

the analysis in [7] permitted to write down the (normalized) physical states of the system for bosons and fermions as [7]

$$|\Psi >_{boson} = (1 - \exp(-8\pi M\omega))^{\frac{1}{2}} \sum_n \exp(-4\pi n M\omega) |n_{out}^{(L)} > \otimes |n_{out}^{(R)} >$$

$$\tag{28}$$

$$|\Psi >_{fermion} = (1 + \exp(-8\pi M\omega))^{-\frac{1}{2}} \sum_n \exp(-4\pi n M\omega) |n_{out}^{(L)} > \otimes |n_{out}^{(R)} > .$$

Hereafter we focus the analysis only on bosons. In fact, for fermions the analysis is identical [7]. The density matrix operator of the system is [7]

$$\hat{\rho}_{boson} \equiv \Psi >_{boson} < \Psi|_{boson}$$

$$= (1 - \exp(-8\pi M\omega)) \sum_{n,m} \exp[-4\pi(n+m)M\omega] |n_{out}^{(L)} > \otimes |n_{out}^{(R)} > < m_{out}^{(R)}| \otimes < m_{out}^{(L)}|. \tag{29}$$

If one traces out the ingoing modes, the density matrix for the outgoing (right) modes reads [7]

$$\hat{\rho}_{boson}^{(R)} = (1-\exp(-8\pi M\omega))\sum_n \exp(-8\pi n M\omega)\,|n_{out}^{(R)}><n_{out}^{(R)}|. \tag{30}$$

This implies that the average number of particles detected at infinity is [7]

$$<n>_{boson} = tr\left[\hat{n}\hat{\rho}_{boson}^{(R)}\right] = \frac{1}{\exp(8\pi M\omega)-1}, \tag{31}$$

where the trace has been taken over all the eigenstates and the final result has been obtained through a bit of algebra, see [7] for details. The result of eq. (31) is the well known Bose-Einstein distribution. A similar analysis works also for fermions [7], and one easily gets the well known Fermi-Dirac distribution

$$<n>_{fermion} = \frac{1}{\exp(8\pi M\omega)+1}. \tag{32}$$

Both the distributions correspond to a black body spectrum with the Hawking temperature $T_H = \frac{1}{8\pi M}$. On the other hand, if one follows step by step the analysis in [7], but starting from the GUP corrected effective Schwarzschild line element (26) at the end obtains the correct physical states for boson and fermions as

$$|\Psi>_{boson} = \left(1-\exp\left(-8\pi M_E^{(GUP)}\omega\right)\right)^{\frac{1}{2}}\sum_n \exp\left(-4\pi n M_E^{(GUP)}\omega\right)|n_{out}^{(L)}>\otimes|n_{out}^{(R)}>$$

$$|\Psi>_{fermion} = \left(1+\exp\left(-8\pi M_E^{(GUP)}\omega\right)\right)^{-\frac{1}{2}}\sum_n \exp\left(-4\pi n M_E^{(GUP)}\omega\right)|n_{out}^{(L)}>\otimes|n_{out}^{(R)}> \tag{33}$$

and the correct distributions as

$$<n>_{boson} = \frac{1}{\exp\left(8\pi M_E^{((GUP)}\omega\right)-1} = \frac{1}{\exp\left[4\pi\left(2M^{(GUP)}-\omega\right)\omega\right]-1}$$

$$<n>_{fermion} = \frac{1}{\exp\left(8\pi M_E^{(GUP)}\omega\right)+1} = \frac{1}{\exp\left[4\pi\left(2M^{(GUP)}-\omega\right)\omega\right]+1}. \tag{34}$$

Using eq. (5) eqs. (34) read

$$<n>_{boson} = \frac{1}{\exp\left[4\pi\left(2\frac{M}{\left[1-\frac{\alpha}{8\pi M}+5\left(\frac{\alpha}{8\pi M}\right)^2\right]}-\omega\right)\omega\right]-1}$$

$$<n>_{fermion} = \frac{1}{\exp\left[4\pi\left(2\frac{M}{\left[1-\frac{\alpha}{8\pi M}+5\left(\frac{\alpha}{8\pi M}\right)^2\right]}-\omega\right)\omega\right]+1}, \tag{35}$$

which take into account both the BH dynamical geometry during the emission of the particle and the GUP correction to the Hawking temperature. We note that setting $\alpha = 0$ in eqs. (35) we find the results in [16], i.e. eqs. (14). In fact, in [16] only the BH dynamical geometry was taken into account. Here, we further improved the analysis by taking into account also the GUP correction to the Hawking temperature.

Concluding remarks

Considering the results in the recent work [2], which have shown that the existence of a minimal length and the subsequent existence of a GUP modifies the Hawking temperature of BHs, in the present paper we modified the BH effective state. By using Hawking's periodicity arguments [16, 17, 27, 28], we derived the corresponding GUP corrected effective metric. The important result is that we obtained further corrections to the non-strictly thermal BH radiation spectrum as the final distributions take into account both the BH dynamical geometry during the emission of the particle and the GUP correction to the Hawking temperature.

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References

[1] L. J. Garay, Int. J. Mod. Phys. A 10, 145 (1995).

[2] A. F. Ali, M. M. Khalil and E. C. Vagenas, to appear in EPL, pre-print in arXiv:1510.06365.

[3] A. F. Ali, S. Das and E. C. Vagenas, Phys. Rev. D 84, 044013 (2011).

[4] S. Das, E. C. Vagenas and A. F. Ali, Phys. Lett. B 690, 407 (2010).

[5] A. F. Ali, S. Das and E. C. Vagenas, Phys. Lett. B 678, 497 (2009).

[6] S. W. Hawking, Commun. Math. Phys. 43, 199 (1975).

[7] R. Banerjee and B. R. Majhi, Phys. Lett. B 675, 243 (2009).

[8] M. Arzano, A. J. M. Medved and E. C. Vagenas, JHEP 0509, 037 (2005).

[9] M. K. Parikh and F. Wilczek, Phys. Rev. Lett. 85, 5042 (2000).

[10] M. K. Parikh, Gen. Rel. Grav. 36, 2419 (2004).

[11] L. Vanzo, G. Acquaviva and R. Di Criscienzo, Class. Quant. Grav. 28, 183001 (2011).

[12] M. Angheben, M. Nadalini, L. Vanzo and S. Zerbini, JHEP 0505, 014 (2005).

[13] R. Banerjee and B. R. Majhi, JHEP 0806 095 (2008).

[14] R. Kerner and R. B. Mann, Phys. Rev. D 73, 104010 (2006).

[15] Q. Q. Jiang, S. Q. Wu and X. Cai, Phys. Rev. D 73, 064003 (2006). Erratum-ibid, Phys. Rev. D 73 069902 (2006).

[16] C. Corda, Ann. Phys. 337, 49 (2013), final version corrected by typos available in arXiv:1305.4529v3.

[17] S. Chakraborty, S. Saha and C. Corda, Galaxies 3, 103 (2015).

[18] S. Massar and R. Parentani, Nucl. Phys. B 575, 333 (2000).

[19] S. W. Hawking, Phys. Rev. D 14, 2460 (1976).

[20] B. Zhang, Q.-Y. Cai, M. S. Zhan, and L. You, Int. Journ. Mod. Phys. D 22, 1341014 (2013).

[21] C. Corda, Ann. Phys. 353, 71 (2015).

[22] C. Corda, JHEP 08, 101 (2011).

[23] C. Corda, Int. Journ. Mod. Phys. D 21, 1242023 (2012).

[24] C. Corda, S. H. Hendi, R. Katebi, N. O. Schmidt, JHEP 06, 008 (2013).

[25] C. Corda, S. H. Hendi, R. Katebi, N. O. Schmidt, Adv. High En. Phys. 527874 (2014).

[26] A. E. Roy and D. Clarke, Astronomy: Principles and Practice, Fourth Edition (2003, PBK).

[27] S. W. Hawking, "*The Path Integral Approach to Quantum Gravity*", in General Relativity: An Einstein Centenary Survey, eds. S.W. Hawking and W. Israel, (Cambridge University Press, 1979).

[28] R. Banerjee and B. R. Majhi, Phys. Lett. B 674, 218 (2009).

[29] C. Corda, Class. Quant. Grav. (2015).

[30] C. Corda, Eur. Phys. J. C 73, 2665 (2013).

[31] C. Corda, Adv. High En. Phys. 867601 (2015).

[32] C. Corda, Int. Journ. Theor. Phys. 54, 3841 (2015).

[33] C. Corda, AIP Conf. Proc. 1648, 020004 (2015).

[34] N. Bohr, Philos. Mag. 26 , 1 (1913).

[35] N. Bohr, Philos. Mag. 26 , 476 (1913).