

Spin Angular Momentum and the Dirac Equation

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Abstract: Quantum mechanical spin angular momentum density, unlike its orbital counterpart, is independent of the choice of origin. A similar classical local angular momentum density may be defined as the field whose curl is equal to twice the momentum density. Integration by parts shows that this spin density yields the same total angular momentum and kinetic energy as obtained using classical orbital angular momentum. We apply the definition of spin density to a description of elastic waves. Using a simple wave interpretation of Dirac bispinors, we show that Dirac's equation of evolution for spin density is a special case of our more general equation. Operators for elastic wave energy, momentum, and angular momentum are equivalent to those of relativistic quantum mechanics.

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1 Introduction

The spin angular momentum of elementary particles is often puzzling to students because it is not obviously related to an angular velocity. The particle velocity is taken to be the de Broglie wave velocity, and since the waves propagate in vacuum there is presumably nothing else to rotate. However, it is possible to interpret spin as a property of waves. [1] And it is well known that elastic waves in solids have two types of momentum: that of the medium and that of the wave (see e.g. Ref. [2]). Clearly there must also be two types of angular momentum in an elastic solid: "spin" associated with rotation of the medium, and "orbital" associated with rotation of the wave. However, spin angular momentum is not normally considered to be a classical physics concept.

A standard part of undergraduate physics education is the definition of angular momentum density as $\mathbf{r} \times \mathbf{p}$, where \mathbf{r} is the radius vector and $\mathbf{p} = \rho \mathbf{u}$ is the momentum

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density ($\rho =$ mass density, $\mathbf{u} =$ velocity). An obvious shortcoming of this definition is that it depends on the arbitrary choice of origin of the coordinates. Hence the computed angular momentum is not a local property of the physical system. Another limitation of this definition is that for a given field of angular momentum density $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, there is no simple way to reconstruct the associated rotational momentum density. For example,

$$\begin{aligned} \nabla \times (\mathbf{r} \times \mathbf{p}) &= \hat{\mathbf{e}}_k \epsilon_{ijk} \epsilon_{jlm} \partial_i (r_l p_m) = \hat{\mathbf{e}}_k (\delta_{kl} \delta_{im} - \delta_{il} \delta_{km}) \partial_i (r_l p_m) \\ &= \hat{\mathbf{e}}_k \partial_i (r_k p_i - r_i p_k) = \hat{\mathbf{e}}_k (-2p_k + r_k \partial_i p_i - r_i \partial_i p_k) \\ &= -2\mathbf{p} + \mathbf{r} \nabla \cdot \mathbf{p} - \mathbf{r} \cdot \nabla \mathbf{p}. \end{aligned} \quad (1)$$

Even for incompressible momentum fields, the last term containing derivatives of \mathbf{p} is problematic.

Coordinate-independent descriptions of rotational dynamics can be traced back to the nineteenth century. [3] MacCullagh modeled light as rotationally elastic shear waves in an isotropic medium with shear modulus μ and displacements $\mathbf{a}(\mathbf{r}, t)$, taking $\mu (\nabla \times \mathbf{a})^2 / 2$ as the energy density. [4] Requiring stationary variations of the associated Lagrangian yields the equation for elastic shear waves with speed $c = \sqrt{\mu/\rho}$:

$$\rho \partial_t^2 \mathbf{a} = -\mu \nabla \times (\nabla \times \mathbf{a}). \quad (2)$$

In terms of rotations, the force density is proportional to the curl of a torque density, which itself is proportional to the infinitesimal rotation angle $(\nabla \times \mathbf{a})/2$. Heaviside similarly interpreted force density as minus the curl of torque density. [5] We wish to clarify these definitions and extend them to arbitrarily large rotations.

It is not a simple matter to describe rotational motion in an elastic solid. The standard treatment of elastic waves (e.g. Ref. [6]) assumes infinitesimal derivatives of displacement $\partial_i a_j$, and decomposes them into symmetric strain $((\partial_i a_j + \partial_j a_i)/2)$ and anti-symmetric rotation $((\partial_i a_j - \partial_j a_i)/2)$ tensors. The strain tensor may be regarded as the deviation (to first order) from a rigid rotation. Stress is assumed to be proportional to strain. For density ρ and elastic constants λ and μ , the resultant equation of evolution of displacement is

$$\rho \partial_t^2 \mathbf{a} = (\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{a}) - \mu \nabla \times (\nabla \times \mathbf{a}). \quad (3)$$

The two terms on the right side of this equation describe compressional and shear waves, respectively. The equation for shear waves is of course equivalent to MacCullagh's equation for light (Eq. 2).

Feynman analyzed the shear resulting from variations of rotation angle φ_z along an axis ($\hat{\mathbf{z}}$), obtaining the formula: [7]

$$\partial_t^2 \varphi_z = c^2 \partial_z^2 \varphi_z. \quad (4)$$

Equation (3) is valid only for infinitesimal displacements. Generalization to finite rotations (rotational shear waves) destroys the neat separation between irrotational and incompressible waves. The basic difficulty is that finite rotations have zero divergence

of velocity, but non-zero divergence of displacement. For example, rotation in the $x - y$ plane by angle φ yields displacement (a_x, a_y) of

$$\begin{bmatrix} a_x \\ a_y \end{bmatrix} = \begin{pmatrix} \cos \varphi - 1 & -\sin \varphi \\ \sin \varphi & \cos \varphi - 1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x(\cos \varphi - 1) - y \sin \varphi \\ x \sin \varphi + y(\cos \varphi - 1) \end{bmatrix}. \quad (5)$$

The divergence of displacement is $\partial_x a_x + \partial_y a_y = 2(\cos \varphi - 1)$, which is not zero in general.

The theory of elastic waves could be improved by including higher-order derivatives, [8] but this does not solve the fundamental limitation to small displacements. Instead we use a different approach based on velocity rather than displacement. This approach leads directly to the concept of spin density.

According to Helmholtz's Theorem, any vector field may be decomposed into irrotational and incompressible components (see e.g. Ref. [9]). Since shear waves are incompressible, it is more natural to describe them in terms of rotational (incompressible) velocity rather than displacement. Recent work by this author attempts to utilize angular momentum and torque densities in place of momentum and force densities as the fundamental variables for rotational shear waves. [10–12] This description of elastic waves results in equations quite similar to those of relativistic quantum mechanics, thereby providing a tangible basis for understanding quantum mechanical spin angular momentum. In this paper we derive the relationships between spin density and the usual classical angular momentum in Section 2, apply the concept of spin density to a rigidly rotating cylinder in Section 3, and analyze elastic waves in Section 4.

2 Angular Variables

The relationship between coordinate-dependent and independent descriptions of angular variables may be seen as follows. Consider a locally rigid rotation with angular velocity \mathbf{w} around the z -axis. The velocity is given by $\mathbf{u} = -\mathbf{r} \times \mathbf{w}$, and the differential velocity is $d\mathbf{u} = -d\mathbf{r} \times \mathbf{w}$. Solving for w_z yields $w_z = \partial_x v_y = -\partial_y v_x = (\nabla \times \mathbf{u}_\varphi)/2$, which is the usual definition of vorticity.

We desire a similar local spin density \mathbf{S} whose curl is proportional to linear momentum. This definition would make the motion explicitly rotational (incompressible), distinguishing it from compressible (irrotational) motion.

Simply comparing equations $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and $\mathbf{u} = -\mathbf{r} \times \mathbf{w}$, we might expect the relationship to be $\rho\mathbf{u} = -\nabla \times \mathbf{S}/2$. However, the angular momentum density must be the same sign as vorticity in order to have positive kinetic energy density in the form of $(1/2)\mathbf{w} \cdot \mathbf{S}$. It must also fall to zero at infinity in order to have finite total angular momentum. These conditions require an angular momentum density maximal at the axis of rotation and decreasing with increasing radius. This requires $d\mathbf{S} = -d\mathbf{r} \times \rho\mathbf{u}$, or

$$\rho\mathbf{u} = +\frac{1}{2}\nabla \times \mathbf{S}. \quad (6)$$

Force and torque should be similarly related:

$$\mathbf{f} = +\frac{1}{2}(\nabla \times \boldsymbol{\tau}). \quad (7)$$

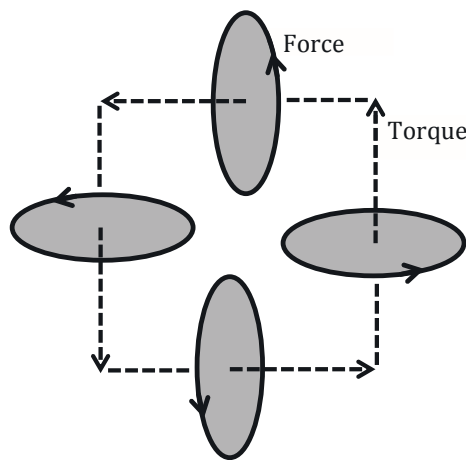


Fig. 1 Proportionality Between Force and the Curl of Torque.

The reason that the sign differs from that predicted from the relationship between \mathbf{w} and \mathbf{u} is that in the equation $\mathbf{u} = -\mathbf{r} \times \mathbf{w}$, the velocity \mathbf{u} is the circulating vector field, whereas in the equation $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{f}$, the force \mathbf{f} is the circulating vector field. Fig. 1 shows that at the center of the counter-clockwise torque loop the force points toward the reader, consistent with the positive curl of torque. Hence the positive signs in Eqs. (6) and (7) are correct.

Assuming differentiable functions and boundary terms of zero when integrating by parts, the total angular momentum is given by:

$$\begin{aligned} \mathbf{J} &= \int (\mathbf{r} \times \mathbf{p}) d^3r = \frac{1}{2} \int \mathbf{r} \times (\nabla \times \mathbf{S}) d^3r \\ &= \frac{1}{2} \hat{e}_k \int (\epsilon_{ijk} r_i \epsilon_{lmj} \partial_l S_m) d^3r = \frac{1}{2} \hat{e}_k \int (\delta_{kl} \delta_{im} - \delta_{km} \delta_{il}) (r_i \partial_l S_m) d^3r \\ &= \frac{1}{2} \hat{e}_k \int (r_i \partial_k S_i - r_i \partial_i S_k) d^3r = -\frac{1}{2} \hat{e}_k \int ([\partial_k r_i] S_i - [\partial_i r_i] S_k) d^3r \\ &= \int \mathbf{S} d^3r. \end{aligned} \quad (8)$$

The total kinetic energy is similarly:

$$\begin{aligned}
 K &= \frac{1}{2} \int \rho u^2 d^3r = \frac{1}{8\rho} \int [\nabla \times \mathbf{S}] \cdot [\nabla \times \mathbf{S}] d^3r \\
 &= \frac{1}{8\rho} \int \partial_i S_j \partial_i S_j - \partial_i S_j \partial_j S_i d^3r = -\frac{1}{8\rho} \int [S_j \partial_i \partial_i S_j - S_j \partial_j \partial_i S_i] d^3r \\
 &= -\frac{1}{8\rho} \int \mathbf{S} \cdot \nabla^2 \mathbf{S} - \mathbf{S} \cdot \nabla [\nabla \cdot \mathbf{S}] d^3r = \frac{1}{8\rho} \int \mathbf{S} \cdot [\nabla \times \nabla \times \mathbf{S}] d^3r \\
 &= \frac{1}{2} \int \mathbf{w} \cdot \mathbf{S} d^3r.
 \end{aligned} \tag{9}$$

Notice that the last step requires the positive sign in Eq. (6).

Hence the definition of spin density given in Eq. (6) yields the same total angular momentum and kinetic energy as the conventional definition $\mathbf{r} \times \mathbf{p}$. The new definition is independent of the choice of origin, is defined only by the motion in a local neighborhood, and completely determines the rotational momentum density $\mathbf{p}(\mathbf{r})$.

Next we show that spin density may be used to describe ordinary rigid rotations.

3 Rigid Rotation

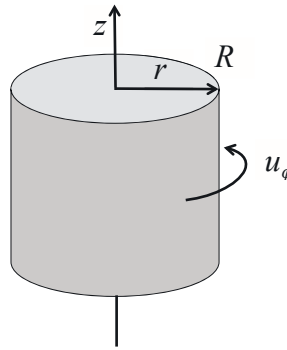


Fig. 2 A Rotating Cylinder.

We will use spin density to describe a cylinder aligned with the z -axis and rotating rigidly with angular velocity w_0 (Fig. 2). The non-zero variables are

$$S_z = \rho w_0 [R^2 - r^2] \text{ for } r \leq R; \text{ zero for } r > R; \tag{10}$$

$$u_\phi = \frac{1}{2\rho} \frac{\partial}{\partial r} S_z = r w_0 \text{ for } r \leq R; \text{ zero for } r > R; \tag{11}$$

$$w_z = \frac{1}{2r} \frac{\partial}{\partial r} r u_\phi = w_0 [1 - R\delta(r - R)/2] \text{ for } r \leq R; \text{ zero for } r > R. \tag{12}$$

The reader may verify that the delta-function in the vorticity yields the correct velocity jump at the boundary $r = R$.

The total angular momentum per unit height is

$$\begin{aligned} J &= 2\pi \int_0^R S_z r dr = 2\pi \int_0^R \rho w_0 [R^2 - r^2] r dr \\ &= \frac{1}{2} \pi \rho R^4 w_0 = \frac{1}{2} M R^2 w_0 \\ &= I w_0. \end{aligned} \tag{13}$$

where we have used the mass per unit height $M = \rho \pi R^2$ and moment of inertia per unit height $I = M R^2 / 2$.

The kinetic energy per unit height is

$$\begin{aligned} K &= \frac{1}{2} \int \mathbf{w} \cdot \mathbf{S} r dr d\phi = \pi \int_0^R w_0 [1 - R\delta(r - R)/2] \rho w_0 [R^2 - r^2] r dr \\ &= \pi \rho w_0^2 \left[\frac{R^4}{2} - \frac{R^4}{4} \right] = \frac{M R^2}{4} w_0^2 = \frac{1}{2} I w_0^2. \end{aligned} \tag{14}$$

Thus we see that spin density correctly describes rigid rotation of a cylinder about its axis, yielding the usual expressions for total angular momentum and kinetic energy. However, orbital angular momentum is likely simpler for describing arbitrary motion of rigid bodies. The main application for spin density is in continuous media where incompressible motion may be described as the curl of a vector potential.

4 Application to Elastic Waves

The velocity defined by Eq. (6) is explicitly divergence-free, making this a natural way to describe shear waves. Previous attempts have demonstrated that rotational shear waves share many properties with relativistic quantum mechanics. [10–12] Here we derive the wave equation for rotational shear waves and clarify the relationship to the Dirac equation.

4.1 Wave Equation

The equation of evolution for rotational shear waves is derived by relating torque density to the rate of change of spin density:

$$\tau = \frac{d\mathbf{S}}{dt} = \partial_t \mathbf{S} + \mathbf{u} \cdot \nabla \mathbf{S} - \mathbf{w} \times \mathbf{S}. \tag{15}$$

The final two terms subtract the contributions of convection and rotation to the partial time derivative of $\mathbf{S}(\mathbf{r}, t)$, since these result from motion of the medium rather than torque. The right-hand side of the equation is called the "total" time derivative of \mathbf{S} since it describes the change in angular momentum density of a moving piece of the solid.

We introduce an angular potential \mathbf{Q} defined by

$$\partial_t \mathbf{Q} = \dot{\mathbf{Q}} = \mathbf{S}. \tag{16}$$

The relationship between \mathbf{S} and \mathbf{u} implies that for infinitesimal motion

$$\frac{1}{2\rho}\nabla\times\mathbf{Q}\approx\mathbf{a}; \quad (17)$$

$$\frac{1}{4\rho}\nabla\times\nabla\times\mathbf{Q}\approx\boldsymbol{\varphi}. \quad (18)$$

The usual equation for infinitesimal shear waves is therefore equivalent to:

$$\frac{1}{2\rho}\nabla\times(\partial_t^2\mathbf{Q}+c^2\nabla\times\nabla\times\mathbf{Q})=0. \quad (19)$$

and Feynman's Eq. (4) for one-dimensional torsion waves is equivalent to:

$$\frac{1}{4\rho}\nabla\times\nabla\times(\partial_t^2\mathbf{Q}-c^2\nabla[\nabla\cdot\mathbf{Q}])=0. \quad (20)$$

Hence for infinitesimal motion, the torque density is:

$$\tau=c^2\nabla[\nabla\cdot\mathbf{Q}]-c^2\nabla\times\nabla\times\mathbf{Q}=c^2\nabla^2\mathbf{Q}. \quad (21)$$

This expression for torque density is based on the usual assumption of a linear relationship between stress and infinitesimal strain. However, we will simply take it to be the defining characteristic of the solid medium for arbitrary motion. Notice that we are not limited to small displacements because $\mathbf{Q}(\mathbf{r},t)$ is simply a time integral of $\mathbf{S}(\mathbf{r},t)$ at each fixed point \mathbf{r} .

We now have a consistent description of rotational variables:

$$\mathbf{S}=\dot{\mathbf{Q}}; \quad (22)$$

$$\rho\mathbf{u}=(\nabla\times\mathbf{S})/2; \quad (23)$$

$$\mathbf{w}=(\nabla\times\mathbf{u})/2; \quad (24)$$

$$\tau=c^2\nabla^2\mathbf{Q}. \quad (25)$$

Setting the total time derivative of angular momentum equal to torque and rearranging terms yields the equation of evolution: [11]

$$\partial_t^2\mathbf{Q}-c^2\nabla^2\mathbf{Q}+\mathbf{u}\cdot\nabla\dot{\mathbf{Q}}-\mathbf{w}\times\dot{\mathbf{Q}}=0. \quad (26)$$

The first two terms of this equation describe a wave-like response to torques in the medium. The last two terms in Eq. (26) represent nonlinear corrections due to finite motion of the medium. If the nonlinear terms do not cancel, they must be in phase with the other terms. Replacing the nonlinear terms by $M^2\mathbf{Q}$ yields :

$$\partial_t^2\mathbf{Q}-c^2\nabla^2\mathbf{Q}+M^2\mathbf{Q}=0. \quad (27)$$

If the coefficient M is a constant, then this is a vector Klein-Gordon equation. In general, however, M could be a function of position.

4.2 Dirac Equation

Richard Feynman referred to the Dirac equation as a simple and beautiful one "...which no one has really been able to understand in any direct fashion." [13] Dirac's equation is admittedly difficult to manipulate since the wave function has four complex components. However, the following analysis based on Ref. [11] shows that the Dirac equation may be interpreted simply as a factorization of an ordinary second-order vector wave equation.

Consider a single-polarization wave propagating in one-dimension with amplitude (not displacement) of $a(z, t)$. If the wave equation is

$$\partial_t^2 a = c^2 \partial_z^2 a, \quad (28)$$

then the general solution is

$$a = a_B(ct + z) + a_F(ct - z) \quad (29)$$

where $a_B(z, t)$ and $a_F(z, t)$ are arbitrary functions that propagate along the axis in the backward and forward directions, respectively. The two directions of wave propagation are clearly independent states, and they are separated in space by a 180° rotation. This property is the fundamental characteristic of spin one-half functions. Generalization to three dimensional space should therefore yield a Dirac wave function.

To demonstrate this, we write the wave equation as a matrix equation. The two wave solutions form an array:

$$\begin{bmatrix} a_B \\ a_F \end{bmatrix}. \quad (30)$$

Noting that temporal and spatial derivatives vary only by a factor of $\pm c$, the wave equation becomes

$$\partial_t \begin{bmatrix} \dot{a}_B \\ \dot{a}_F \end{bmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} c \partial_z \begin{bmatrix} \dot{a}_B \\ \dot{a}_F \end{bmatrix} = 0. \quad (31)$$

We have now reduced the second-order scalar wave equation to a first-order matrix equation. The next step is a bit unusual. We further divide each component of the wave into positive and negative regions ($\dot{a}_B = \dot{a}_{B+} - \dot{a}_{B-}$ and $\dot{a}_F = \dot{a}_{F+} - \dot{a}_{F-}$). Now each of the four wave components ($\dot{a}_{B+}, \dot{a}_{B-}, \dot{a}_{F+}, \dot{a}_{F-}$) is positive-definite, and only one of the components may be non-zero for each propagation direction. Some caution is warranted because these components may have discontinuous derivatives at sign transitions, but we will ignore that issue here. In higher dimensionality positive and negative components can coexist, indicating a different polarization direction.

We arrange the components in the following order, corresponding to the chiral repre-

sensation of the Dirac wave function:

$$\begin{bmatrix} \dot{a}_{B+} \\ \dot{a}_{F-} \\ \dot{a}_{F+} \\ \dot{a}_{B-} \end{bmatrix}. \quad (32)$$

We may now write the time derivative of a as a matrix product:

$$\dot{a} = \begin{bmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \end{bmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{bmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \end{bmatrix} = \psi^T \sigma_z \psi \quad (33)$$

where σ_z is the Dirac matrix for the z-component of spin. The temporal and spatial derivatives have the same sign for backward-propagating waves and opposite signs for forward-propagating waves. The spatial derivative is therefore given by:

$$c\partial_z a = - \begin{bmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \end{bmatrix}^T \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \end{bmatrix} = -\psi^T \gamma^5 \psi \quad (34)$$

where γ^5 is the Dirac matrix for chirality in the chiral representation. If the amplitude (a) represents rotation angle, then positive and negative chirality ($\partial_z a$) are analogous to left- and right-handed threads on a screw. Wave velocity (v) is obtained by combining the two matrices used above:

$$v\psi = c \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{bmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \end{bmatrix} = c\gamma^5 \sigma_z \psi. \quad (35)$$

The one-dimensional scalar wave equation may be written in the form:

$$\partial_t[\psi^T \sigma_z \psi] + c\partial_z[\psi^T \gamma^5 \psi] = 0. \quad (36)$$

Other matrices may be inserted between the wave functions, resulting in the following corresponding expressions, each of which equals zero for the wave solutions:

$$\partial_t[\psi^T \sigma_z \psi] + c \partial_z[\psi^T \gamma^5 \psi] = \partial_t^2 a - c^2 \partial_z^2 a; \quad (37)$$

$$\partial_t[\psi^T \psi] + c \partial_z[\psi^T \gamma^5 \sigma_z \psi] = \partial_t[|\partial_t a_F| + \partial_t|\partial_t a_B| + c^2[|\partial_z|\partial_z a_F| - \partial_z|\partial_z a_B|]]; \quad (38)$$

$$\partial_t[\psi^T \gamma^5 \sigma_z \psi] + c \partial_z[\psi^T \psi] = c[|\partial_t|\partial_z a_F| - \partial_t|\partial_z a_B| + \partial_t|\partial_z a_F| + \partial_t|\partial_z a_B|]; \quad (39)$$

$$\partial_t[\psi^T \gamma^5 \psi] + c \partial_z[\psi^T \sigma_z \psi] = \partial_t[-c \partial_z a] + c \partial_z[\partial_t a]. \quad (40)$$

Generalization to three dimensions is straightforward. The 3-D generalization of $\partial_z \partial_z a_z$ utilizes geometric algebra:

$$\begin{aligned} \nabla(\nabla \mathbf{a}) &= \nabla(\nabla \cdot \mathbf{a} + \mathbf{i} \nabla \times \mathbf{a}) \\ &= \nabla(\nabla \cdot \mathbf{a}) - \nabla \times (\nabla \times \mathbf{a}). \end{aligned} \quad (41)$$

The one-dimensional Dirac wave functions are real with positive-definite components. Generalization to three dimensions requires complex components and additional matrices. The one-dimensional wave equation has the bispinor form:

$$\psi^T \{ \sigma_z \partial_t \psi + c \gamma^5 \partial_z \psi \} + \text{Transpose} = 0. \quad (42)$$

We can separate a common factor of $\psi^\dagger \sigma_z$:

$$\psi^\dagger \sigma_z \{ \partial_t \psi + c \gamma^5 \sigma_z \partial_z \psi \} + \text{Transpose} = 0. \quad (43)$$

For arbitrary components and derivatives this becomes:

$$\psi^\dagger \sigma_j \{ \partial_t \psi + c \gamma^5 \sigma_i \partial_i \psi \} + \text{h.c.} = 0 \quad (44)$$

where "h.c." stands for "hermitian conjugate".

The matrices σ_j are the Dirac spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad (45)$$

$$\sigma_y = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}; \quad (46)$$

$$\sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (47)$$

The matrices $c\gamma^5\sigma_j$ are the Dirac velocity matrices. The matrix γ^5 was defined in Eq. (34).

Expanding the spatial derivative term in Eq. (44) yields the 3-D generalization of the wave equation (37):

$$\begin{aligned} 0 &= \partial_t [\psi^\dagger \boldsymbol{\sigma} \psi] + c \nabla [\psi^\dagger \gamma^5 \psi] - ic \{ [\nabla \psi^\dagger] \times \gamma^5 \boldsymbol{\sigma} \psi + \psi^\dagger \gamma^5 \boldsymbol{\sigma} \times \nabla \psi \} \\ &= \partial_t^2 \mathbf{a} - c^2 \nabla (\nabla \cdot \mathbf{a}) + c^2 \nabla \times (\nabla \times \mathbf{a}). \end{aligned} \quad (48)$$

Eq. (40) similarly generalizes to:

$$0 = \partial_t [\psi^\dagger \gamma^5 \psi] + c \nabla \cdot [\psi^\dagger \boldsymbol{\sigma} \psi] = \partial_t [-c \nabla \cdot \mathbf{a}] + c \nabla \cdot [\partial_t \mathbf{a}]; \quad (49)$$

Eqs. (38) and (39) are not easily generalized to vector equations, but in terms of bispinors they become:

$$0 = \partial_t [\psi^\dagger \psi] + c \nabla \cdot [\psi^\dagger \gamma^5 \boldsymbol{\sigma} \psi] \quad (50)$$

$$0 = \partial_t [\psi^\dagger \gamma^5 \boldsymbol{\sigma} \psi] + c \nabla [\psi^\dagger \psi]. \quad (51)$$

The foregoing analysis results in the following identifications between vectors and bispinors:

$$\partial_t \mathbf{a} \equiv [\psi^\dagger \boldsymbol{\sigma} \psi]; \quad (52)$$

$$[\nabla \cdot \mathbf{a}] \equiv -[\psi^\dagger \gamma^5 \psi]; \quad (53)$$

$$c^2 \{ \nabla \times \nabla \times \mathbf{a} \} \equiv -ic \{ [\nabla \psi^\dagger] \times \gamma^5 \boldsymbol{\sigma} \psi + \psi^\dagger \gamma^5 \boldsymbol{\sigma} \times \nabla \psi \}; \quad (54)$$

$$0 = ic \nabla \cdot \{ [\nabla \psi^\dagger] \times \gamma^5 \boldsymbol{\sigma} \psi + \psi^\dagger \gamma^5 \boldsymbol{\sigma} \times \nabla \psi \}. \quad (55)$$

These identifications provide seven constraints on the eight free parameters of the complex Dirac bispinor: three for the first, one for the second, two for the third (since a curl has only two independent components), and one for the fourth. There is also an arbitrary overall phase factor.

The last identification simply states that the divergence of a curl is zero. This condition is necessary for consistency. Note that if we attempt to define the curl as a single term (i.e. $c \nabla \times \mathbf{a} = \psi^\dagger \gamma^5 \boldsymbol{\sigma} \psi$, resulting in a "-" sign in Eqs. [54] and [55]), then it becomes impossible to write a Dirac equation for $\partial_t \psi$ because there is no common factor of $\psi^\dagger \sigma_j$ as in Eq. (44).

We now apply a similar interpretation of the Dirac wave function in terms of spin density (correcting a sign error in Ref. [11]):

$$\mathbf{S} = \partial_t \mathbf{Q} \equiv \frac{1}{2} [\psi^\dagger \boldsymbol{\sigma} \psi]; \quad (56)$$

$$c \nabla \cdot \mathbf{Q} \equiv -\frac{1}{2} [\psi^\dagger \gamma^5 \psi]; \quad (57)$$

$$c^2 \{\nabla \times \nabla \times \mathbf{Q}\} \equiv -\frac{ic}{2} \{[\nabla \psi^\dagger] \times \gamma^5 \boldsymbol{\sigma} \psi + \psi^\dagger \gamma^5 \boldsymbol{\sigma} \times \nabla \psi\}; \quad (58)$$

$$0 = \frac{ic}{2} \nabla \cdot \{[\nabla \psi^\dagger] \times \gamma^5 \boldsymbol{\sigma} \psi + \psi^\dagger \gamma^5 \boldsymbol{\sigma} \times \nabla \psi\}. \quad (59)$$

In terms of bispinors, the rotational wave equation (26) is

$$0 = \partial_t [\psi^\dagger \boldsymbol{\sigma} \psi] + c \nabla [\psi^\dagger \gamma^5 \psi] - ic \{[\nabla \psi^\dagger] \times \gamma^5 \boldsymbol{\sigma} \psi + \psi^\dagger \gamma^5 \boldsymbol{\sigma} \times \nabla \psi\} \\ + \mathbf{u} \cdot \nabla [\psi^\dagger \boldsymbol{\sigma} \psi] - \mathbf{w} \times [\psi^\dagger \boldsymbol{\sigma} \psi]. \quad (60)$$

For comparison, the Dirac equation for a free electron (with $M = m_e c^2 / \hbar$) is

$$\partial_t \psi + c \gamma^5 \boldsymbol{\sigma} \cdot \nabla \psi + iM \gamma^0 \psi = 0. \quad (61)$$

Multiplying this equation by $\psi^\dagger \sigma_j$ on the left and adding the hermitian conjugate yields:

$$\partial_t [\psi^\dagger \sigma_j \psi] + c \partial_j [\psi^\dagger \gamma^5 \psi] + ic \epsilon_{ijk} \{[\partial_i \psi^\dagger] \gamma^5 \sigma_k \psi - \psi^\dagger \gamma^5 \sigma_k \partial_i \psi\} = 0. \quad (62)$$

This is equivalent to:

$$\partial_t [\psi^\dagger \boldsymbol{\sigma} \psi] + c \nabla [\psi^\dagger \gamma^5 \psi] - ic \{[\nabla \psi^\dagger] \times \gamma^5 \boldsymbol{\sigma} \psi + \psi^\dagger \gamma^5 \boldsymbol{\sigma} \times \nabla \psi\} = 0. \quad (63)$$

Using our definitions, this is just the wave equation. It differs from our equation for the evolution of spin angular momentum density in an elastic solid only by the two nonlinear terms. This is interesting because many researchers have attempted to obtain particle-like solutions from the Dirac equation by adding nonlinear terms to Dirac's original equation. [14–21]

It is also instructive to write the equation for elastic waves in Dirac form. Expanding the derivatives yields

$$\psi^\dagger \sigma_j \left[\partial_t \psi + c \gamma^5 \boldsymbol{\sigma} \cdot \nabla \psi + \mathbf{u} \cdot \nabla \psi + \mathbf{w} \cdot \frac{i \boldsymbol{\sigma}}{2} \psi \right] + \text{h.c.} = 0. \quad (64)$$

The Hermitian conjugate wave function ψ^\dagger may be regarded as an independent variable (it may be combined with the original wave function to separate the real and imaginary parts). Validity for arbitrary ψ^\dagger requires the terms in brackets to sum to zero. This yields the equation

$$\partial_t \psi + c \gamma^5 \boldsymbol{\sigma} \cdot \nabla \psi + \mathbf{u} \cdot \nabla \psi + i \mathbf{w} \cdot \frac{\boldsymbol{\sigma}}{2} \psi + i \chi \psi = 0 \quad (65)$$

where χ may be any operator with the property

$$\text{Re} \{ \psi^\dagger \sigma_j i \chi \psi \} = 0. \quad (66)$$

Dirac's mass term is an example. However, we will set χ to zero, assuming that mass (along with any missing nonlinear terms in the Dirac equation) is derived from the convection and rotation terms.

4.3 Lagrangian and Hamiltonian

Now we construct a Lagrange density \mathcal{L} . Lagrange's equation of motion for a field variable ψ is

$$\partial_t \frac{\partial \mathcal{L}}{\partial [\partial_t \psi]} + \sum_j \partial_j \frac{\partial \mathcal{L}}{\partial [\partial_j \psi]} - \frac{\partial \mathcal{L}}{\partial \psi} = 0. \quad (67)$$

A similar equation holds with ψ^\dagger replacing ψ . It is possible to construct a Lagrangian with no derivatives of ψ^\dagger , in which case the equation of motion is simply $\partial \mathcal{L} / \partial \psi^\dagger = 0$. The nonlinear terms contain two factors of ψ^\dagger . In the rotation term, these may be interchanged using integration by parts, so a factor of 1/2 is required in the Lagrangian. Integration by parts of the convection term yields a term containing $\nabla \cdot \mathbf{u}$, which is zero. Therefore the factor of ψ^\dagger in \mathbf{u} does not contribute to the equation of evolution. The Lagrangian is therefore

$$\mathcal{L} = i\psi^\dagger \partial_t \psi + \psi^\dagger c\gamma^5 \boldsymbol{\sigma} \cdot i\nabla \psi + \mathbf{u} \cdot \psi^\dagger i\nabla \psi - \frac{1}{2} \mathbf{w} \cdot \psi^\dagger \frac{\boldsymbol{\sigma}}{2} \psi. \quad (68)$$

This Lagrangian is not real, but real-valued quantities may be regarded as the real part of complex expressions.

The conjugate momentum to the field ψ is p_ψ :

$$p_\psi = \frac{\partial \mathcal{L}}{\partial [\partial_t \psi]} = i\psi^\dagger. \quad (69)$$

The Hamiltonian is

$$\mathcal{H} = p_\psi \partial_t \psi - \mathcal{L} = -\psi^\dagger c\gamma^5 \boldsymbol{\sigma} \cdot i\nabla \psi - \mathbf{u} \cdot \psi^\dagger i\nabla \psi + \frac{1}{2} \mathbf{w} \cdot \psi^\dagger \frac{\boldsymbol{\sigma}}{2} \psi. \quad (70)$$

We recognize the last term in the Hamiltonian as the kinetic energy density $K = \mathbf{w} \cdot \mathbf{S}/2$. The first term involves only spatial derivatives, so we propose that it represents elastic potential energy. The second term represents convection of gradients by the motion of the medium. Since shear waves are transverse, this motion is perpendicular to the wave velocity (determined by the matrix $\gamma^5 \boldsymbol{\sigma}$). Therefore this term can be non-zero only if the wave velocity is not parallel to $-\psi^\dagger i\nabla \psi$ (which we shall see is the wave momentum). We hypothesize that this term integrates to zero, not contributing to the total energy. A prior attempt by this author to incorporate this term into the kinetic energy blurred the distinction between wave propagation and motion of the solid medium. [12]

The Hamiltonian operator is defined by $i\partial_t \psi = H\psi$, with

$$H = -c\gamma^5 \boldsymbol{\sigma} \cdot i\nabla \psi - \mathbf{u} \cdot i\nabla \psi + \frac{1}{2} \mathbf{w} \cdot \frac{\boldsymbol{\sigma}}{2} \psi. \quad (71)$$

4.4 Dynamical Variables

The Hamiltonian is a special case (T_0^0) of the energy-momentum tensor:

$$T_\nu^\mu = \frac{\partial \mathcal{L}}{\partial [\partial_\mu \psi]} \partial_\nu \psi - \mathcal{L} \delta_\nu^\mu. \quad (72)$$

Notice that in the Lagrangian, the kinetic energy term is negative. Therefore the conjugate momenta computed from the Lagrangian will also have the opposite sign of physical quantities. The dynamical (or wave) momentum density p_i is

$$p_i = -T_i^0 = -\frac{\partial \mathcal{L}}{\partial [\partial_t \psi]} \partial_i \psi = -\psi^\dagger i \partial_i \psi. \quad (73)$$

The wave angular momentum density is likewise

$$\mathbf{L} = -\frac{\partial \mathcal{L}}{\partial [\partial_t \psi]} \partial_\phi \psi = -i \psi^\dagger \partial_\phi \psi = -i \psi^\dagger \frac{\partial r_i}{\partial \phi} \partial_i \psi = -\mathbf{r} \times \psi^\dagger i \nabla \psi. \quad (74)$$

This expression assumes a particular origin for the axis of rotation of the angle ϕ , in contrast to the coordinate-independent spin angular momentum. One could attempt to express orbital angular momentum density as the field whose curl is twice the wave momentum density, but we will not pursue that here.

For total momentum and angular momentum, we must combine the wave and medium contributions (\mathbf{p} and \mathbf{q} , respectively):

$$\mathbf{P} = \mathbf{p} + \mathbf{q} = -\psi^\dagger i \nabla \psi + \frac{1}{2} \nabla \times \psi^\dagger \frac{\boldsymbol{\sigma}}{2} \psi; \quad (75)$$

$$\mathbf{J} = \mathbf{L} + \mathbf{S} = -\mathbf{r} \times \psi^\dagger i \nabla \psi + \psi^\dagger \frac{\boldsymbol{\sigma}}{2} \psi. \quad (76)$$

The expression for total momentum density was previously obtained by Ohanian using a symmetrized energy-momentum tensor. [1]

Interestingly, we could have obtained the results of Eqs. (75) and (76) by treating either velocity \mathbf{u} or vorticity \mathbf{w} as an independent variable in the Lagrangian above. Rewriting the kinetic energy density as $\rho \mathbf{u}^2/2$, the negative of the conjugate momentum would be

$$\mathbf{P} = -\frac{\partial \mathcal{L}}{\partial \mathbf{u}} = -\psi^\dagger i \nabla \psi + \rho \mathbf{u} = -\psi^\dagger i \nabla \psi + \frac{1}{2} \nabla \times \psi^\dagger \frac{\boldsymbol{\sigma}}{2} \psi. \quad (77)$$

And if \mathbf{u} includes a rotational component $\mathbf{r} \times \mathbf{w}$, then the negative of the conjugate angular momentum would be (treating \mathbf{S} as a function of \mathbf{w})

$$\mathbf{J} = -\frac{\partial \mathcal{L}}{\partial \mathbf{w}} = -\mathbf{r} \times \psi^\dagger i \nabla \psi + \psi^\dagger \frac{\boldsymbol{\sigma}}{2} \psi. \quad (78)$$

These results are identical to Eqs. (75) and (76).

When interpreting angular derivatives, the reader should be cautious to distinguish between active and passive rotations. The differential $\partial_\phi \psi = \mathbf{r} \times \nabla \psi$ refers to passive rotation, or rotation of the point of evaluation of the function. Active rotations rotate the function along with the evaluation point, as described by the operator U_ϕ satisfying the equations: [22]

$$\partial_\phi U_\phi \psi = -i(\mathbf{L} + \mathbf{S})\psi = -\mathbf{r} \times \nabla \psi - i \frac{\boldsymbol{\sigma}}{2} \psi; \quad (79)$$

$$U_\phi \psi = \exp \{-i(\mathbf{L} + \mathbf{S}) \cdot \phi\} \psi. \quad (80)$$

Thus we see that classical spin density applied to elastic waves yields equations and operators very similar to relativistic quantum mechanics.

5 Discussion

We have shown that classical spin density, which was originally derived as an interpretation of quantum mechanical spin density, is consistent with the usual classical description of arbitrary rotations. Spin density is therefore an important concept for a unified understanding of both classical and quantum physics. In particular, rotational elastic waves share properties of both classical and quantum systems.

Elastic waves have played an important role in the study not only of solids, but also of light and matter. Physicists have attempted to describe the universe as a solid since the 18th century, when Thomas Young explained polarization of light as analogous to shear waves. Young's idea was further developed by the likes of Fresnel, Navier, Cauchy, Rayleigh, Heaviside, Green, Thomson (Lord Kelvin), Riemann, Boussinesq, and many others. [3] An elastic solid model was the basis for MacCullagh's original derivation of an equation for light. [4] Maxwell developed the equations for electromagnetism by modeling a lattice of elastic cells, and questioned, "... what if these molecules, indestructible as they are, turn out to be not substances themselves, but mere affections of some other substance?" [23]

Unfortunately, introductory physics textbooks typically dismiss the idea of a universal wave medium, saying it was disproven by Michelson and Morley. That is of course nonsense, as Lorentz-invariant equations such as MacCullagh's and Maxwell's are quite commonly derived for a medium carrying a wave. What aether-drift experiments proved is that Earth does not move through space like a rock through water (or through oobleck, the corn starch solution that behaves like a solid for rapid vibrations but like a liquid for slower processes).

We now know that matter propagates through the vacuum in a wave-like manner. Equations describing these waves, such as the Dirac equation, may be interpreted as describing dynamics as well as probabilities. [24, 25] Although the properties of matter can be described without reference to an "aether", such models can still be useful for illuminating relationships between physical quantities. Dirac himself held this view, writing "It is necessary to set up an action principle and to get a Hamiltonian formulation of the equations suitable for quantization purposes, and for this the aether velocity is required." [26] Recently, there have been several investigations of solid crystalline models of the vacuum. [27–30]

Given the similarity between classical and quantum equations, it is interesting to ponder what the universe would be like if the vacuum were an elastic solid. Elementary particles would have to be standing or particle-like waves, subject to the wave uncertainty principle. Special relativity would be a consequence of the Lorentz invariance of the wave solutions, and not a property of the space-time in which the waves propagate. [31] The spatial reflection of any solution would also be a solution, so every particle would have an anti-particle that behaves like its mirror image. [32] Measurements would have to change the standing wave configuration from one stable state to another, implying quantization of measurement. Tension induced by twisting of the elastic medium would increase density

and decrease wave speed, similar to the way the presence of matter decreases wave speed in general relativity. Hence gravity could be described by an index of refraction. [33–35] In short, an elastic solid universe would be similar in many ways to the one we live in. And although some properties of matter may be impossible to explain using such a classical model, spin angular momentum is not one of them.

6 Conclusions

Classical spin angular momentum density is the field whose curl is equal to twice the momentum density for incompressible (rotational) motion. Compared with the usual classical definition of angular momentum density as $\mathbf{r} \times \mathbf{p}$, spin density is a local and complete description of rotational motion that yields the same total angular momentum and kinetic energy. A rotating cylinder constitutes a simple example for the application of spin density. Using spin density to describe elastic waves yields equations similar to those of fermions with identical operators for energy, momentum, and angular momentum.

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