Hamilton-Jacobi Formulation of Supermembrane

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\textbf{Abstract}: The Hamilton-Jacobi formalism of constrained systems is applied to Supermembrane system. The equations of motion for a singular system are obtained as total differential equations in many variables. These equations of motion are in exact agreement with those obtained by Dirac’s method.

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1. Introduction

The Hamiltonian Formulation of constrained systems was initiated by Dirac\cite{1}, he obtained the equations of motion of singular Lagrangian systems by using the consistency conditions. He also showed that the number of degrees of freedom of the mechanical system can be reduced, this formalism has a wide application in field theory \cite{2,3}. An alternative Hamilton-Jacobi scheme for constrained systems was proposed by Güler\cite{4,5}. Güler used the Hamilton-Jacobi formulation to obtain the equations of motion as total differential equations. This approach based on Carathéodory’s equivalent Lagrangian method \cite{6}. This method has been applied to very few physical examples \cite{7-21}. A better understanding of its features, in the study of constrained systems when compared to Dirac’s Hamiltonian formalism is still necessary. In this paper we wish to discuss the classical mechanics of the Supermembrane system using the general procedure of Hamilton-Jacobi formulation.

Now let us make a brief review on the Hamilton-Jacobi formulation of singular system

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(canonical method). The Lagrangian function of any physical system with \( n \) degrees of freedom is a function of \( n \) generalized coordinates, \( n \) generalized velocities and parameter \( t \). i.e \( L \equiv L(q_i, \dot{q}_i, t) \). The Hess matrix is defined as

\[
A_{ij} = \frac{\partial^2 L(q_i, \dot{q}_i, t)}{\partial \dot{q}_i \partial \dot{q}_j}, \quad i, j = 1, 2, \ldots, n. \tag{1}
\]

If the rank of this matrix is \( n \), then the Lagrangian is called regular, otherwise it is called singular. Moreover, systems which have singular Lagrangian are called singular systems or constrained system.

One starts from the singular Lagrangian \( L(q_i, \dot{q}_i, t) \) with Hess matrix of rank \((n - p)\), \( p < n \). The generalized momenta \( p_i \) corresponding to the generalized coordinates \( q_i \) are defined as

\[
p_a = \frac{\partial L}{\partial \dot{q}_a}, \quad a = 1, 2, \ldots, n - p, \tag{2}
\]
\[
p_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \quad \mu = n - p + 1, \ldots, n. \tag{3}
\]

Since the rank of the Hess matrix is \((n - p)\), one may solve eqn. (2) for \( \dot{q}_a \) as

\[
\dot{q}_a = \dot{q}_a(q_i, \dot{q}_\mu, p_a; t) \equiv \omega_a \tag{4}
\]

Substituting eqn. (4), into eqn. (3), we get

\[
p_\mu = \frac{\partial L}{\partial \dot{q}_\mu} \bigg|_{\dot{q}_a = \omega_a} \equiv -H_\mu(q_i, \dot{q}_\mu, p_a; t). \tag{5}
\]

The canonical Hamiltonian \( H_o \) is defined as

\[
H_0 = -L(q_i, \dot{q}_i, \dot{q}_a; t) + p_a \dot{q}_a + p_\mu \dot{q}_\mu \bigg|_{p_\nu = -H_\nu}. \tag{6}
\]

The set of Hamilton-Jacobi partial differential equations (HJPDE) is expressed as

\[
H'_\alpha(\tau, q_\nu, q_\alpha, P_i = \frac{\partial S}{\partial q_i}, P_0 = \frac{\partial S}{\partial \tau}) = 0, \tag{7}
\]

where

\[
H'_0 = p_0 + H_0, \tag{8}
\]
\[
H'_\mu = p_\mu + H_\mu. \tag{9}
\]

We define \( P_0 = \frac{\partial S}{\partial \tau} \), and \( P_i = \frac{\partial S}{\partial q_i} \), with \( q_0 = t \), and \( S \) being the action.

The equations of motion are obtained as total differential equations and take the form

\[
dq_a = \frac{\partial H'_\alpha}{\partial p_a} dt_\alpha, \tag{10}
\]
\[
dp_r = -\frac{\partial H'_\alpha}{\partial q_r} dt_\alpha, \quad r = 0, 1, \ldots, n. \tag{11}
\]
\[ dZ = (-H_\alpha + p_\alpha \frac{\partial H'_\alpha}{\partial p_\alpha}) dt_\alpha. \] (12)

where \( Z = S(t_\alpha, q_\alpha) \). These equations are integrable if and only if

\[ dH'_0 = 0, \] (13)

\[ dH'_\mu = 0, \quad \mu = n - r + 1, \ldots, n. \] (14)

If conditions (13) and (14) are not satisfied identically, one may considered them as a new constraints and again test the integrability conditions, then repeating this procedure, a set of conditions may be obtained.

In the last works [13,14], the superparticles and superstring were studied using Hamilton-Jacobi approach. In the recent work we will use the above formalism to discuss the classical mechanics of the supermembrane.

2. Hamilton-Jacobi Formulation of Supermembrane in Four Dimensions

Consider the following supermembrane action in four dimensions,[22]

\[ I = \int d^3\xi \left[ \frac{1}{2} \sqrt{\gamma} N^{-1} \Pi_0^2 \Pi_{0\mu} - \sqrt{\gamma} N^a N^{-1} \Pi_0^\mu \Pi_{a\mu} \right. \]
\[ \left. - \frac{1}{2} \sqrt{\gamma} (\gamma^{ab} - N^a N^b N^{-1}) \Pi_a^\mu \Pi_{b\mu} + \frac{1}{2} \sqrt{\gamma} N + 3 \varepsilon^{ab} \Pi_0^A \Pi_{a}^B \Pi_{b}^C B_{CBA} \right]. \] (15)

The Lagrangian density is

\[ \mathcal{L} = \frac{1}{2} \sqrt{\gamma} N^{-1} \Pi_0^2 \Pi_{0\mu} - \sqrt{\gamma} N^a N^{-1} \Pi_0^\mu \Pi_{a\mu} \]
\[ - \frac{1}{2} \sqrt{\gamma} (\gamma^{ab} - N^a N^b N^{-1}) \Pi_a^\mu \Pi_{b\mu} + \frac{1}{2} \sqrt{\gamma} N + 3 \varepsilon^{ab} \Pi_0^A \Pi_{a}^B \Pi_{b}^C B_{CBA}, \] (16)

where

\[ \varepsilon^{12} = -\varepsilon^{21} = 1, \quad \Pi_i^a = \partial_i X^\mu - i \bar{\psi} \Gamma^\mu \partial_i \psi, \quad \Pi_i^a = \partial_i \psi^\alpha, \quad \xi^i = (\tau, \sigma, \rho)(i = 1, 2, 3) \] are the coordinates, \( \psi^\alpha \) is a 32-components Majorana spinor, and \((X^\mu, \psi^\alpha)\) are the coordinates of the eleven-dimensional superspace. Using the definitions (2) and (3), the canonical momenta \((P_\mu, P_\alpha, \Pi, \Pi_a, \Pi_{ab})\) conjugated to canonical variables \((X^\mu, \psi^\alpha, N, N^a, \gamma^{ab})\) take the forms

\[ P_\mu = \frac{\partial \mathcal{L}}{\partial (\partial^0 X^\mu)} = \sqrt{\gamma} N^{-1} \Pi_{0\mu} - \sqrt{\gamma} N^a N^{-1} \Pi_{a\mu} + S_\mu, \] (17)

\[ P_\alpha = \frac{\partial \mathcal{L}}{\partial (\partial^0 \psi^\alpha)} = i (\bar{\Psi} \Gamma^\mu)_\alpha P_\mu + S_\alpha = -H_\alpha, \] (18)

\[ \Pi = \frac{\partial \mathcal{L}}{\partial (\partial^0 N)} = 0 = -H_\Pi, \] (19)

\[ \Pi_a = \frac{\partial \mathcal{L}}{\partial (\partial^0 N^a)} = 0 = -H_a, \] (20)
\[ \Pi_{ab} = \frac{\partial L}{\partial (\partial^0 \gamma_{ab})} = 0 = -H_{ab}, \]  

We can solve (17) for \( \dot{X}_\mu \) in terms of \( P_\mu \) and other coordinates as

\[ \dot{X}_\mu \equiv \partial_0 X_\mu = \frac{2P_\mu}{\sqrt{\gamma}N-1} + \frac{i\bar{\psi}\Gamma_\mu \psi}{\sqrt{\gamma}} + 2N^a \Pi_{a\mu} - \frac{2S_\mu}{\sqrt{\gamma}N-1} \]  

(22)

Now, we introduce the Hamiltonian density \( H_0 \) as

\[ H_0 = P_\mu (\partial^0 X^\mu) + P_\alpha (\partial^0 \Psi^\alpha) + \Pi (\partial^0 N) + \Pi_a (\partial^0 N^a) + \Pi_{ab} (\partial^0 N_{ab}) - L \]

\[ = \frac{N}{2\sqrt{\gamma}}(P_\mu - S_\mu)(P^\mu - S^\mu) + \frac{1}{2} N \sqrt{\gamma} \gamma^{ab} \Pi_a^\mu \Pi_b^\mu - \frac{1}{2} N \sqrt{\gamma} \]

\[ + N^a \Pi_a^\mu (P_\mu - S_\mu), \]  

(23)

and the canonical Hamiltonian may be written as

\[ H_0 = \int d\sigma d\rho H_0 \]

\[ = \int d\sigma d\rho \left[ \frac{N}{2\sqrt{\gamma}}(P_\mu - S_\mu)(P^\mu - S^\mu) + \frac{1}{2} N \sqrt{\gamma} \gamma^{ab} \Pi_a^\mu \Pi_b^\mu - \frac{1}{2} N \sqrt{\gamma} \right] \]

\[ + N^a \Pi_a^\mu (P_\mu - S_\mu) \]  

(24)

The set of HJPDEs defined in (8) and (9) reads as

\[ H'_0 = P_0 + H_0 = P_0 + \frac{N}{2\sqrt{\gamma}}(P_\mu - S_\mu)(P^\mu - S^\mu) + \frac{1}{2} N \sqrt{\gamma} \gamma^{ab} \Pi_a^\mu \Pi_b^\mu - \frac{1}{2} N \sqrt{\gamma} \]

\[ + N^a \Pi_a^\mu (P_\mu - S_\mu), \]

(25)

\[ H'_\alpha = P_\alpha + H_\alpha = P_\alpha + i(\bar{\Psi} \Gamma^\mu) \alpha P_\mu + S_\alpha = 0, \]

(26)

\[ H'_\Pi = \Pi = 0, \]

(27)

and

\[ H'_{ab} = \Pi_{ab} = 0 \]  

(28)

Using (10) and (11), the set of HJPDE (25)-(29) leads to the following total differential equations:

\[ dX_\mu = \left( \frac{2P_\mu}{\sqrt{\gamma}N-1} + \frac{i\bar{\psi}\Gamma_\mu \psi}{\sqrt{\gamma}} + 2N^a \Pi_{a\mu} - \frac{2S_\mu}{\sqrt{\gamma}N-1} \right) d\tau \]  

(29)

\[ dP_\mu = 0, \]

(30)

\[ dP_\alpha = 0, \]

(31)

\[ d\Pi = \left( \frac{1}{2\sqrt{\gamma}}(P_\mu - S_\mu)(P^\mu - S^\mu) + \frac{1}{2} \sqrt{\gamma} \gamma^{ab} \Pi_a^\mu \Pi_b^\mu - \frac{1}{2} \sqrt{\gamma} \right) d\tau, \]

(32)

\[ d\Pi_a = \Pi_a^\mu (P_\mu - S_\mu) d\tau, \]

(33)

and

\[ d\Pi_{ab} = \frac{1}{2} N \Pi_a^\mu \Pi_b^\mu d\tau \]  

(34)

The set of total differential equations (29) to (34) are integrable if they satisfy the integrability conditions (13) and (14). In fact the variations of \( H'_0 \) and \( H'_\alpha \) are not identically satisfied, therefore the system is not integrable.
3. Conclusion

In this work the supermembrane in four dimensions is studied as a dynamical constrained Hamiltonian system. The Hamilton-Jacobi method is examine to obtain the equations of motion as total differential equations. In Hamilton-Jacobi treatment one obtains the constraints in phase space directly in forming Hamilton-Jacobi partial differential equations. It is no need to classify the constraints as primary or secondary constraints, first class or second class as in Dirac’s method. Furthermore, in Dirac’s method we must reduce any constrained system to one with first class constraints only, which require to introduce an arbitrary variables, therefore we have made a gauge fixing. This is not necessary at all in Hamilton-Jacobi treatment. The final point in this conclusion is that the equations of motion are not consequence, since the integrability conditions are not identically satisfied, and the theory needs to add more constraints to the system.

References


